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ABSTRACT VELOCITY FUNCTORS

by R.A. Bowshell

It is still uncertain which category differential geometry studies. One general definition of a structured manifold is given by the notion of a higher order G-structure, but the definition of admissible maps is elusive.

Since the theory of connections on groupoids plays a central role, it would be desirable that a category of structured manifolds admit many groupoids - a groupoid can, of course, be defined in any category with pull-backs. This requirement indicates that the proper domain for geometry is a category of groupoids. In any case all the familiar constructions on a manifold B can be considered as constructions on the trivial groupoid $\Pi(B) = B \times B$, and as such have immediate generalizations to any Lie groupoid.

Lacking any clear conception of what is required from a category of structured manifolds, especially because to date differential geometry has focussed on first order differential equations: existence of flows and the Frobenius theorem, it seems premature to attempt a definition of admissible maps. Nevertheless there seems some point in studying functors on the category of all manifolds which are likely to have significant analogies on a category of structured manifolds.

In this paper we define an abstraction of Ehresmann's velocity functors $T_n^k = J_o^k(R^n, -)$ called an extensor; it consists of a pair (τ, Ω), τ being a local product preserving functor into fibre bundles $\tau M \to M$ on which a Lie groupoid $\Omega(M)$ acts effectively, such that local isomorphisms of M are lifted by τ into $\Omega(M)$ with appropriate continuity conditions satisfied. The precise definition is given in (2.8). The assumption of product preserving is needed to ensure that τ preserves the structure of submanifolds; for example, the functor into vector bundles $M \to T_k(M) = J^k(M, R)_o^*$ introduced by Ambrose, Palais and Singer in [1] does not preserve products and does not even preserve diagonals.

In the final section of the paper we show that the lifting of local isomorphisms factors through a homomorphism $\Pi^k(M) \to \Omega(M)$ for some k, and thus that extensors are classified by sequences of smooth homomorphisms $L_n^k \to G_n$, where G_n is a Lie group acting effectively on E^n for some vector space E. For a category of locally flat pseudogroup structures we would expect the same thing to hold, except that the homomorphism $L_n^k \to G_n$ would be restricted. Similarly for general pseudogroup structures we would have a restriction of the homomorphisms $\Pi^k(M) \to \Omega(M)$, however in this case there is no obvious way of constructing an extensor (τ, Ω) from Ω. In fact the problem of constructing an extensor subordinated to the prolonging groupoids $\Omega(M)$ is closely related to the problem of admissible maps, and it depends basically on a study of the (nonlinear) representations of a sequence G_n of Lie groups on a vector space E.

We emphasize Ω rather than τ because, as will be shown in Section two, there is a notion of connection, on a Lie groupoid, associated to each extensor (τ, Ω), which coincides with Ehresmann's higher order connection for (T_n^k, Π^k), and depends only on the way in which the local isomorphisms are lifted into Ω.

In Section one, after presenting a few technical properties of Lie groupoids and fibre bundles, we sketch the basic properties of the vector bundle $A\Phi$ of infinitesimal translations of a groupoid. No essential use is made of $A\Phi$, but nevertheless it carries the obstruction theory of groupoids and connections, and as such plays the role of Lie algebra for a groupoid.

Section two begins with some consequences of the assumption of product preservation for a local functor into fibered manifolds. Next extensors are defined and the associated notion of connection is introduced. The main part of the section is devoted to the construction of a prolongation of a Lie groupoid Φ which generalizes the holonomic prolongations Φ^k. This notion of prolongation is at the heart of extensor connections and a fortiori of higher order connections in the sense of Ehresmann. The section ends with the proof of the existence of extensor connections.

In Section three we compare extensor connections with Ehresmann's higher order connections, and with an abstraction of the lifting of velocities, induced by an Ehresmann connection, that was introduced by Virsik.

The last section studies the structure of an extensor; it ends with a generalization of a theorem of Ngo Van Que.

I would like to express my thanks to Ph.D. supervisor Dr. J. Virsik for suggesting the topic of this paper, and for several useful discussions.

Finally a word on notation; we use throughout the terminology of Lang [6] , thus, for example, a projection of maximal rank is called a surjective submersion. In addition we denote the zero section of a vector bundle by i, and if $F: E \to E'$ is a map between fibre bundles we denote the restriction of F to the fibre E_x over x by F_x. A smooth map is differentiable to any order.

1. Connections and Groupoids

In this section we sketch the theory of connections on groupoids, and prove some results needed later.

1.1 DEFINITION. A *groupoid* is a category consisting of a set of objects B, and a set of invertible maps Φ. Each groupoid is equipped with right and left unit projections $a, b: \Phi \to B$, thus $f: af \to bf$, af being the right unit of f since it is traditional to compose maps backwards, and an injection $\sim: B \to \Phi$ of objects (units) onto identities.

A *Lie groupoid* is a groupoid for which Φ and B have the structures of smooth manifolds, $\sim$, a, b, the inverse $\sigma: \Phi \to \Phi$, and the composition $K: \Phi * \Phi \to \Phi$ are smooth maps; and in addition $\sim: B \to \Phi$ is an embedding, and $(a, b): \Phi \to B \times B$ is a surjective submersion.

In the above $\Phi * \Phi = (a \times b)^{-1}(\Delta)$ where Δ is the diagonal of $B \times B$; $\Phi * \Phi$ is a submanifold of $\Phi \times \Phi$ since $a \times b$ is transversal over Δ - in fact both a and b are submersions. We shall write $f.g$ for $K(f, g)$, f^{-1} for σf, and we shall identify the object x with the identity $\tilde{x}$, wherever convenient.

We use the following notations:

$$\Phi_x = a^{-1}\{x\} = \{f \in \Phi : af = x\},$$

$$\Phi_{xy} = (a, b)^{-1}\{(x, y)\} = \{f \in \Phi : af = x, \ bf = y\}.$$

It is clear by transversality that Φ_x is a submanifold of Φ, that Φ_{xy} is a submanifold of Φ_x, and that Φ_{xx} is a Lie group.

The following result is easily verified.

1.2 PROPOSITION. *If* Φ *is a groupoid over* B *for which* $\sim, a, b, \sigma,$ *and* K *are smooth maps between smooth manifolds, then* Φ *is a Lie groupoid iff, for some* x *in* B, $b: \Phi_x \to B$ *is a principal bundle with group* Φ_{xx}.

We shall view a fibre bundle as a fibred manifold, i.e. a surjective submersion, on which a Lie groupoid acts.

1.3 DEFINITION. Let Φ be a Lie groupoid over B, and let $\pi: E \to B$ be a fibred manifold; define $\Phi * E = (a \times \pi)^{-1}(\Delta)$; it is a submanifold of $\Phi \times E$ by transversality. We say that Φ *acts on* E if there is a smooth map $\Phi * E \to E$ written $(f, v) \to f.v$ which satisfies $\pi(f.v) = bf$, and $f.(g.v) = (f.g).v$ whenever $af = bg$.

The following result gives a useful characterization for fibre bundles.

1.4 LEMMA. *Let* Φ *be a Lie groupoid over* B *and let* $\mathfrak{E}$ *be a functor from* Φ *into the category of smooth manifolds (i.e.* $\mathfrak{E}$ *assigns to each* x *in* B *a smooth manifold* E_x *and to each* f *in* Φ_{xy} *a smooth map* $E_f: E_x \to E_y$*) such that for some* z *in* B *the map* $\Phi_{zz} \times E_z \to E_z$ *defined by* $(s, v) \to E(s)v$ *is smooth. Then* $E = \bigcup E_x$ *has a canonical fibre bundle structure.*

PROOF. Fix z in B so that $\Phi_{zz} \times E_z \to E_z$ is smooth. Construct a fibre bundle by the standard mixing process: namely take the orbit space of $\Phi_z \times E_z$ under the action of Φ_{zz} via $s.(h, v) = (hs, s^{-1}v)$. There is an obvious one to one function onto E; thus E inherits the structure of a fibred manifold on which the Lie groupoid Φ clearly acts.

Notice that by a theorem in Montgomery and Zippin, [8] p. 212, it suffices to assume continuity of $\Phi_{zz} \times E_z \to E_z$ for some z.

The following result is well known.

1.5 PROPOSITION. *If* Ω *is a subgroupoid over* B *of the Lie groupoid* Φ *such that, for some* z, Ω_{zz} *is a Lie subgroup of* Φ_{zz}*, and* B *is covered by local sections* $\phi_i: U_i \to \Phi_z$ *of* $b: \Phi_z \to B$ *which take values in* Ω_z*, then* Ω *is a Lie subgroupoid of* Φ.

The following result can be found in Ngo Van Que [13].

1.6 PROPOSITION. *If s is a section of the fibre bundle E associated to the Lie groupoid Φ over B, and the subgroupoid of Φ leaving s invariant is transitive, i.e. for each x and y in B there is an f in Φ such that $f.s(x)=s(y)$, then this subgroupoid is a Lie subgroupoid.*

Given a Lie groupoid (Φ, B), we define a groupoid Φ^k over B by letting Φ^k_x be the set of $j^k_x s$ where s is a local section of $a: \Phi \to B$ such that bs is a local isomorphism in B. We have right a^k, and left b^k, unit projections: $a^k j^k_x s = x$, $b^k j^k_x s = bs(x)$, and composition

$$j^k_y s \circ j^k_x t = j^k_x \{ z \to s(bt(z)).t(z) \}$$

whenever $a^k j^k_y s = b^k j^k_x t$.

Φ^k is the k-th holonomic prolongation of Φ. We have analogously the k-th semi-holonomic, and k-th non-holonomic prolongations: $\bar{\Phi}^k$, $\tilde{\Phi}^k$ respectively; for the local structure see Virsik [14]. They were introduced by Ehresmann in C.R. Acad. Sci. Paris, 240 (1955).

The basic invariant of a Lie groupoid Φ is the vector bundle $A\Phi$ of infinitesimal displacements. This was implicit in Ehresmann's original paper defining connections on a fibre bundle, but was first emphasized by Atiyah in [2].

$A\Phi = \bigcup T_x\Phi_x$ is a vector bundle associated to the groupoid Φ^1. Sections of $A\Phi$ correspond naturally to right invariant sections of $T\Phi_z$. Thus if V is a section of $A\Phi$, then $h \to V(bh).ih$ is a right invariant vector field on Φ_z, and given such a vector field W on Φ_z, then $x \to W(h).ih^{-1}$ where $bh = x$ defines a section of $A\Phi$.

It follows that there is a Lie bracket of sections of $A\Phi$. In fact if V is a section of $A\Phi$ and $\phi_t(h)$ is the local flow of the associated vector field $h \to V(bh).ih$ on Φ_z, we can define a local one-parameter group of automorphisms $\psi_t(kh^{-1}) = \phi_t(k)\phi_t(h)^{-1}$. It can then be shown that $[V, W](x) = \lim \frac{1}{t} \{ W(x) - T\psi_t [W(\psi_{-t}(x))] \}$ coincides with the bracket induced from the right invariant vector fields associated to V, W.

There is an exact sequence of vector bundles over B associated to each Lie groupoid:

$$0 \to L\Phi \to A\Phi \xrightarrow{Tb} TB \to 0$$

where $L\Phi = \bigcup T_x \Phi_{xx}$, and it is easily seen that

$$Tb\,[V, W] = [TbV, TbW],$$

and that, if the sections V, W of $A\Phi$ take values in $L\Phi$, then

$$[V, W](x) = [V(x), W(x)]$$

where the latter is Lie bracket, by right invariant vector fields, in the Lie algebra $T_x\Phi_{xx}$.

Another basic property of A is that $A\Phi^k \sim J^k A\Phi$, a natural equivalence under the twist map:

$$j_0^1\{t \to j_x^k[y \to s(t, y)]\} \to j_x^k\{y \to j_0^1[t \to s(t, y)]\}.$$

This was first noticed by P. Libermann in the case that Φ is the trivial groupoid $\Pi(B) = B \times B$.

A first order connection on Φ is defined to be a splitting of the exact sequence

$$0 \to L\Phi \to A\Phi \to TB \to 0,$$

i.e. a section $\lambda: TB \to A\Phi$ of $Tb: A\Phi \to TB$. Its curvature is the section of $L_a^2(TB, L\Phi)$ defined, on vector fields, by

$$\Lambda(X, Y) = \lambda[X, Y] - [\lambda X, \lambda Y].$$

It can be shown that this is just the standard curvature moved down from the principal bundle to the base manifold; it also coincides with the definition of curvature given by Ehresmann in [3].

Since $TB = A\Pi(B)$, where $\Pi(B) = B \times B$, the role of curvature as an obstruction is shown explicitly by the following result of Ngo Van Que [12].

1.7 THEOREM. *Let Φ and Φ' be two Lie groupoids over B. Each local homomorphism $f: U \to \Phi'$, defined in a neighbourhood U of the identity submanifold of Φ, induces a vector bundle map $F: A\Phi \to A\Phi'$ which satisfies $Tb.F = Tb$, Tb being the canonical projection of $A\Phi$ and $A\Phi'$ on TB, and which preserves Lie bracket: $F[V, W] = [FV, FW]$.*

Conversely any vector bundle map $F: A\Phi \to A\Phi'$ preserving Lie bracket and satisfying $Tb.F = Tb$ is induced by a local homomorphism $f: U \to \Phi'$; if both $f: U \to \Phi'$ and $f': U' \to \Phi'$ induce F, then f and f' agree on $U \cap U'$.

It is interesting to notice that, if Φ acts linearly on a vector bundle $E \to B$, then sections V of $A\Phi$ act as first order differential operators on sections s of E as follows:

$$(Vs)(x) = lim \frac{1}{t} \{ \theta_t(x)^{-1} s [b\theta_t(x)] - s(x) \},$$

where $h \to \theta_t(bh).h$ is the local flow of $h \to V(bh).ih$. It can be shown that $[V, W] s = V(Ws) - W(Vs)$; thus, if we define covariant derivative by $\nabla_X s = (\lambda X) s$, for X a vector field on B, then we have the classical formula:

$$-\Lambda(X,Y) s = \{\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\} s.$$

The minus arises because in the classical case left invariant vector fields are used to define the Lie algebra structure in each fibre of $L\Phi$; thus a different action of $L\Phi$ on E must be used to preserve the formula

$$[A, B](v) = A(Bv) - B(Av).$$

2. Extensors

In this section we study the properties of local product preserving functors from manifolds into fibred manifolds. In particular we introduce extensors: an abstraction of the velocity functors T^k_n. For each extensor we define a notion of connection on a Lie groupoid which coincides with Ehresmann's higher order connection [3], in the special case of a velocity functor. Finally we show that for any extensor such a generalized connection exists in any Lie groupoid over a paracompact manifold.

2.1 DEFINITION. A *natural fibering* is an endofunctor $\mathfrak{F}$ of the category of smooth manifolds with a natural retraction $p: \mathfrak{F} \to 1$ onto the identity functor, p having the natural section $i: 1 \to \mathfrak{F}$, such that:

a) Each $p_M: \mathfrak{F}M \to M$ is a submersion, necessarily surjective.

b) If $\{x\}$ is a one point manifold, then $\mathfrak{F}\{x\}$ is a one point manifold, thus $\mathfrak{F}\{x\} = \{ i_{\{x\}} x \}$.

c) $\mathfrak{F}$ preserves products; thus if $p_j: M_1 \times M_2 \to M_j$ is the projection ($j = 1, 2$), then

$$(\mathfrak{F}p_1, \mathfrak{F}p_2): \mathfrak{F}(M_1 \times M_2) \to \mathfrak{F}M_1 \times \mathfrak{F}M_2$$

is a smooth isomorphism.

d) $\mathfrak{F}$ is a local functor; thus if $q: U \to M$ is the inclusion of an open submanifold, then $\mathfrak{F}U = \mathfrak{F}M | U = p^{-1}(U)$ where $p_M: \mathfrak{F}M \to M$, and $\mathfrak{F}q: \mathfrak{F}U \to \mathfrak{F}M$ is the inclusion of an open submanifold.

This definition is adapted from Virsik's paper [14] in which he calls the functor a local regular connector, and uses it to generalize the lifting properties of connections. We will examine this problem briefly in section 3.

In the following results we suppose that $\mathfrak{F}$ is a given natural fibering. The properties of transversality used here may be found in Lang [6].

2.2 LEMMA. *If* $g: M \to N$ *is an immersion, then* $\mathfrak{F}g: \mathfrak{F}M \to \mathfrak{F}N$ *is an immersion. Furthermore, if* g *embeds* M *on a submanifold* A *of* N, *then* $\mathfrak{F}g$ *embeds* $\mathfrak{F}M$ *on the submanifold* $\mathfrak{F}A$ *of* $\mathfrak{F}N$.

PROOF. Suppose g is an immersion, and take x in M; then there is an open neighbourhood V of $g(x)$ and a map $h: V \to g^{-1}(V)$ such that $hgj = 1_{g^{-1}(V)}$ where $j: g^{-1}(V) \to M$ is the inclusion. It follows that

$$\mathfrak{F}h . \mathfrak{F}g | \mathfrak{F}(g^{-1}(V)) = 1_{\mathfrak{F}(g^{-1}(V))}$$

since $\mathfrak{F}$ is local. Hence $\mathfrak{F}g$ is locally a section; this implies that $\mathfrak{F}g$ is an immersion.

It is clear that g injective implies $\mathfrak{F}g$ injective. Suppose A is a submanifold of N; the proof will be complete if we show that $\mathfrak{F}A$ is a submanifold of $\mathfrak{F}N$.

Take x in A and a chart $\Phi: U \to V \times W$ for N with $\Phi(x) = (0,0)$, where V, W are open sets in vector spaces E, F, such that $\Phi(U \cap A) = V \times 0$. Then $\mathfrak{F}(U \cap A) = \mathfrak{F}A | (U \cap A)$ and $\mathfrak{F}\Phi | \mathfrak{F}(U \cap A) = \mathfrak{F}(V \times 0)$.

It is clear that $\mathfrak{F}(V \times 0)$ is a submanifold of $\mathfrak{F}V \times \mathfrak{F}W \simeq \mathfrak{F}(V \times W)$, and it follows that $\mathfrak{F}(U \cap A)$ is a submanifold of $\mathfrak{F}U$. Hence $\mathfrak{F}A$ is locally a submanifold of $\mathfrak{F}M$, and the result follows.

2.3 LEMMA. *If* $g: M \to N$ *is transversal over the submanifold* A *of* N, *then* $g^{-1}(A)$ *is a submanifold of* M, *and* $\mathfrak{F}(g^{-1}(A)) = (\mathfrak{F}g)^{-1}(\mathfrak{F}A)$.

PROOF. It is well known that $g^{-1}(A)$ is a submanifold of M, and that $gj: g^{-1}(A) \to A$ is a smooth map, where $j: g^{-1}(A) \to M$ is the inclusion. Hence $\mathfrak{F}g$ maps $\mathfrak{F}(g^{-1}(A))$ into $\mathfrak{F}A$, since Lemma 2.2. shows that $\mathfrak{F}j$ is an embedding. It follows that $\mathfrak{F}(g^{-1}(A)) \subset (\mathfrak{F}g)^{-1}(\mathfrak{F}A)$.

For the converse we work locally. Take x in $g^{-1}(A)$ and put $y = g(x)$; take a chart $\psi: V \to V_1 \times V_2$ at y, where V_i is an open set in the vector space E_i, $i=1,2$, such that $\psi(y)=(0,0)$, $\psi(V \cap A)=V_1 \times 0$

Since g is transversal over A, there is an open neighbourhood U of x for which $p_2 \psi g : U \to V \to V_1 \times V_2 \to V_2$ is a submersion. This means explicitly that there is a chart $\Phi : U \to U_1 \times V_2 \subset E_1 \times E_2$ such that

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & U_1 \times V_2 \\ & {\scriptstyle p_2 \psi g} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & V_2 & \end{array}$$

commutes (if necessary the chart ψ may be restricted).

Now suppose that X is in $(\mathfrak{F} g)^{-1}(\mathfrak{F} A)$ with $p_M X = x$, then $y = g(p_M X) = p_N \mathfrak{F} g X$ is in A, and so x is in $g^{-1}(A)$. Since $\mathfrak{F}$ is local, X is in $\mathfrak{F} U$, and $\mathfrak{F} g(X)$ is in $\mathfrak{F}(V \cap A)$. $\mathfrak{F}\Phi(X)$ is in $\mathfrak{F} U_1 \times i_{V_2} 0 = \mathfrak{F}(U_1 \times 0)$ since $\mathfrak{F} p_2 \mathfrak{F}\Phi(X) = \mathfrak{F}(p_2 \psi g)(X) = i_{V_2} 0$.

Notice that $\Phi : U \cap g^{-1}(A) \to U_1 \times 0$ is an isomorphism. It follows that $X = \mathfrak{F}\Phi^{-1}\mathfrak{F}\Phi(X)$ is in $\mathfrak{F}(U \cap g^{-1}(A))$ and thus in $\mathfrak{F}(g^{-1}(A))$. The result follows.

2.4 LEMMA. *If $\Delta \subset M \times M$ is the diagonal, then $\mathfrak{F}\Delta \subset \mathfrak{F} M \times \mathfrak{F} M$ is also the diagonal.*

PROOF. We have the isomorphism $(1,1) : M \to \Delta$, hence $(\mathfrak{F} 1, \mathfrak{F} 1) = \mathfrak{F}(1,1) : \mathfrak{F} M \to \mathfrak{F}\Delta$ is an isomorphism, and the result follows since $\mathfrak{F}\Delta$ is a submanifold of $\mathfrak{F} M \times \mathfrak{F} M$.

2.5 COROLLARY. *If $f, g : M \to N$ are such that $(f,g) : M \to N \times N$ is transversal over the diagonal Δ of $N \times N$, then the equalizer $e : E \to M$ of f and g exists and is given by $E = (f,g)^{-1}(\Delta)$. Furthermore $\mathfrak{F} e : \mathfrak{F} E \to \mathfrak{F} M$ is the equalizer of $\mathfrak{F} f$ and $\mathfrak{F} g$.*

2.6 COROLLARY. *If $f : M \to Q$, $g : N \to Q$ are such that $f \times g : M \times N \to Q \times Q$ is transversal over the diagonal Δ of $Q \times Q$, then the pullback P of f and g exists and is given by $(f \times g)^{-1}(\Delta)$. Furthermore $\mathfrak{F} P$ is the pullback of $\mathfrak{F} f$ and $\mathfrak{F} g$.*

Not even the tangent functor preserves arbitrary equalizers. Consider for example $f, g : R \to R^2$ given by $f(t) = (t, 0)$, $g(t) = (t, t^3)$.

If we call linear fibering a functor $\mathfrak{F}$ satisfying a), b) and d) of definition 2.1 and such that each fibre is a vector space and each map $\mathfrak{F} g$

is linear on fibres, and say that it preserves diagonals iff

$$\mathfrak{F}(\Delta)=(\mathfrak{F}p_1, \mathfrak{F}p_2)^{-1}(\Delta \mathfrak{F}M)$$

whenever Δ is the diagonal of $M\times M$, $\Delta\mathfrak{F}M$ is the diagonal of $\mathfrak{F}M\times\mathfrak{F}M$, and $p_1, p_2 : M\times M\to M$ are the projections, then we have

2.7 PROPOSITION. *A linear fibering preserves products iff it preserves diagonals.*

PROOF. Necessity follows from Lemma 2.4.

Suppose that $\mathfrak{F}$ is a linear fibering which preserves diagonals. Define $J:\mathfrak{F}M\times\mathfrak{F}N\to\mathfrak{F}(M\times N)$ by

$$J(Vx, Wy)=\mathfrak{F}(1,\overline{y})(Vx)+\mathfrak{F}(\overline{x}, 1)(Wy),$$

where $\overline{y}$ is the constant map $M\to N$ with value $y=p_N(Wy)$. (This definition is taken from Virsik [14] .)

It is clear that $(\mathfrak{F}p_1, \mathfrak{F}p_2)J=1_{\mathfrak{F}M\times\mathfrak{F}N}$, where p_1 and p_2 are the projections. Furthermore, if $f: M\to M'$, $g: N\to N'$, then

$$J(\mathfrak{F}f\times\mathfrak{F}g)=\mathfrak{F}(f\times g)J.$$

We first prove that, if $M=N$, $J_M=J:\mathfrak{F}M\times\mathfrak{F}M\to\mathfrak{F}(M\times M)$ satisfies $J(\mathfrak{F}1, \mathfrak{F}1)(V)=\mathfrak{F}(1,1)V$. Thus take V in $\mathfrak{F}M$, then $(\mathfrak{F}1, \mathfrak{F}1)(V)=(V, V)$ is in the diagonal of $\mathfrak{F}M\times\mathfrak{F}M$. Now $(\mathfrak{F}p_1, \mathfrak{F}p_2)J(V, V)=(V, V)$ and since $\mathfrak{F}$ preserves diagonals we have $J(V, V)$ in $\mathfrak{F}\Delta$. But $(1, 1):$ $M\to\Delta$ is an isomorphism, thus $J(V, V)=\mathfrak{F}(1, 1)(W)$ for some W in $\mathfrak{F}M$, and clearly we must have $W=V$. It follows that $J(\mathfrak{F}1, \mathfrak{F}1)(V)=\mathfrak{F}(1,1)V$.

For any smooth maps $f: A\to M$, $g: A\to N$, we have

$$J(\mathfrak{F}f, \mathfrak{F}g)=J(\mathfrak{F}f\times\mathfrak{F}g)(\mathfrak{F}1, \mathfrak{F}1)=\mathfrak{F}(f\times g)J_A(\mathfrak{F}1, \mathfrak{F}1)=$$

$$\mathfrak{F}(f\times g)\mathfrak{F}(1, 1)=\mathfrak{F}(f, g).$$

In particular take the projections $p_1: M\times N\to M$, $p_2: M\times N\to N$; then $J(\mathfrak{F}p_1, \mathfrak{F}p_2)=\mathfrak{F}(p_1, p_2)=\mathfrak{F}1$; it follows that J is an isomorphism, and thus $\mathfrak{F}$ preserves products.

We remark that the only product preserving functors into vector bundles are the first order velocities T^1_n. This is a very special case of a theorem of Epstein [4] , and it is quite easy to prove directly.

Suppose that Φ is a Lie groupoid on B with right and left unit projections a, b, with inverse $\sigma: \Phi \to \Phi$, and with composition $K: \Phi * \Phi \to \Phi$ where $\Phi * \Phi$ is the pullback of $a: \Phi \to B$ and $b: \Phi \to B$. Since a, b are submersions, it follows that $\mathfrak{F}(\Phi * \Phi)$ is the pullback of $\mathfrak{F} a$ and $\mathfrak{F} b$; we can thus prolong the composition to $\mathfrak{F} K: \mathfrak{F}\Phi * \mathfrak{F}\Phi \to \mathfrak{F}\Phi$, and since the axioms for a groupoid can be expressed by commuting diagrams, it is clear that $\mathfrak{F}\Phi$ is a differentiable groupoid over $\mathfrak{F} B$. $\mathfrak{F}\Phi$ is a Lie groupoid because it is locally trivial: if $g: U \to \Phi_z$ is a local section of b, then $\mathfrak{F} g: \mathfrak{F} U \to \mathfrak{F}\Phi_z$ is a local section of $\mathfrak{F}\Phi_z = (\mathfrak{F} a)^{-1}\{ i_B z \} = (\mathfrak{F}\Phi) i_B z$.

Finally it is easy to show that $p_\Phi : \mathfrak{F}\Phi \to \Phi$ is a homomorphism of groupoids.

We now proceed to the definition of an extensor and the connection associated with it.

2.8 DEFINITION. An *extensor* is a pair (τ, Ω) in which τ is a natural fibering and Ω assigns to each smooth manifold M a Lie groupoid $\Omega(M)$ over M, with right and left unit projections α, β, which acts effectively on the fibred manifold τM. In addition the pair (τ, Ω) should satisfy the following conditions:

a) $\Omega(M)$ leaves invariant the natural section $i: M \to \tau M$; thus $\xi(i\alpha\xi) = i\beta\xi$ for all ξ in $\Omega(M)$.

b) If U is an open submanifold of M, then

$$\Omega(U) = \Omega(M) \,|\, U = (\alpha, \beta)^{-1}(U).$$

There is an embedding $(\xi, \eta) \to \xi \times \eta$ of $\Omega(M) \times \Omega(N)$ onto a Lie subgroupoid of $\Omega(M \times N)$ for each pair of smooth manifolds M, N which satisfies $(\xi \times \eta) \times \mu = \xi \times (\eta \times \mu)$.

c) If $g: M \to N$ is a smooth isomorphism, then we have a smooth isomorphism of acting groupoids $\Omega(M) \to \Omega(N)$ written $\xi \to \tau_g \xi \tau_g^{-1}$ which satisfies $(\tau_g \xi \tau_g^{-1})(V) = \tau g[\xi(\tau g^{-1} V)]$.

A local isomorphism $f: U \to U'$ in M induces a local section of $\alpha: \Omega(M) \to M$ written $x \to \tau_x f$ which satisfies $\tau f(V) = \tau_x f \circ V$, the latter being the action of $\Omega(M)$ on $\tau(M)$.

If U is an open subset of $R \times M$ and if $U \to M$, $(t, x) \to f_t(x)$ is a flow on M, then the map $U \to \Omega(M)$ given by $(t, x) \to \tau_x f_t$ is smooth.

The principal examples of extensors are given by $\tau M = J^k_0(R^n, M)$ and $\Omega(M) = \Pi^k(M)$ and the analogous examples in which holonomic jets are replaced by semi-holonomic or non-holonomic jets. It is straightforward to check that (T^k_n, Π^k), $(\bar{T}^k_n, \bar{\Pi}^k)$, and $(\tilde{T}^k_n, \tilde{\Pi}^k)$ are in fact extensors. The projection $T^k_n(M) \to M$ is the target map, and the section $i: M \to T^k_n(M)$ takes x into $j^k_0\{y \to x\}$, the k-jet of the constant map.

Since, if V is an n-dimensional manifold, we have $J^k_x(V, M)$ isomorphic to $J^k_0(R^n, M)$, it is clear that $(J^k_x(V,-), \Pi^k)$ is an extensor. A similar remark holds in the semi-holonomic and non-holonomic cases.

Take a fixed smooth manifold B; we now define functors $\mathfrak{A}, \mathfrak{L}$ from the category of Lie groupoids over B to the category of fibre bundles over B. $\mathfrak{A}$ and $\mathfrak{L}$ are constructed from τ in the same way as the first order invariants A and L are constructed from T. Thus

$$\mathfrak{A}\Phi = \bigcup_{x \in B} \tau_x \Phi_x, \quad \mathfrak{L}\Phi = \bigcup_{x \in B} \tau_x \Phi_{xx} \quad (\text{where } \tau_x \Phi_x = \tau(\Phi_x)_x),$$

and, if $g: \Phi \to \Pi$ is a smooth homomorphism of groupoids, let $\mathfrak{A}g, \mathfrak{L}g$ be the appropriate restrictions of τg.

Notice that we have a projection $\tau b: \mathfrak{A}\Phi \to \tau B$ induced by the left unit projection $b: \Phi_x \to B$. Moreover,

$$\tau_x \Phi_{xx} = \tau_x(\Phi_x \cap b^{-1}\{x\}) = \tau_x \Phi_x \cap (\tau b)^{-1}\{ix\};$$

hence $\mathfrak{L}\Phi$ consists of all V in $\mathfrak{A}\Phi$ such that $\tau b V = i\pi V$, where $\pi: \mathfrak{A}\Phi \to B$ is the obvious projection. This is analogous to the exact sequence of vector bundles over B,

$$0 \to L\Phi \to A\Phi \to TB \to 0.$$

In order to complete the description of $\mathfrak{L}$ and $\mathfrak{A}$ and to define the bundle $\mathfrak{C}\Phi$ of elements of τ-connection, we must define the groupoids acting on $\mathfrak{L}\Phi$ and $\mathfrak{A}\Phi$

It will be shown that Φ acts on $\mathfrak{L}\Phi$ in a natural way.

2.9 DEFINITION. Let $\Omega^o(\Phi_z)$ consist of all ξ in $\Omega(\Phi_z)$ such that:

a) For all V in $\tau_{\alpha\xi}\Phi_z$ and A in $\tau_z \Phi_{zz}$, $\xi(V.A) = \xi(V).A$ where '.' indicates prolongation of composition in Φ.

b) There is an element ξ_0 in $\Omega(B)$ such that, for all V in $\tau_{\alpha\xi}\Phi_z$,

$$\tau b\xi(V)=\xi_0\tau b(V).$$

It is clear that $\Omega^o(\Phi_z)$ is a subgroupoid of $\Omega(\Phi_z)$. Moreover, since $\Omega(B)$ acts effectively on τB and $\tau b:\tau\Phi_z\to\tau B$ has local sections,it is easy to check that ξ_0 is uniquely determined by ξ, and that $\xi\to\xi_0$ is a homomorphism $\Omega^o(\Phi_z)\to\Omega(B)$ of groupoids.

If $h:U\to\Phi_z$ is a local section of $b:\Phi_z\to B$ on a neighbourhood of z such that $h(z)=\tilde{z}$, then we have an isomorphism $\bar{h}:\tau_z\Phi_z\to\tau_z B\times\tau_z\Phi_{zz}$ given by $\bar{h}(V)=(\tau bV,\ [\tau(hb)(V)]^{-1}.V)$

If k is another such local section at z, then it is easy to see that $\bar{k}^{-1}.\bar{h}=\tau_z f$, where f is a local isomorphism of Φ_z given by

$$f(q)=k(bq).h(bq)^{-1}.q\,.$$

Since $f(qs)=f(q)s$ for s in Φ_{zz}, and $bf(q)=bq$, it follows that $\tau_z f$ is in $\Omega^o{}_{zz}(\Phi_z)$ with $(\tau_z f)_0=1_z$ in $\Omega(B)$.

2.10 DEFINITION. The bundle of *elements of* τ *connection,* $\mathcal{C}\Phi$, is constructed as follows: $\mathcal{C}\Phi=\bigcup_{z\in B}\mathcal{C}_z\Phi$, where $\mathcal{C}_z\Phi$ consists of all smooth maps $\eta:\tau_z B\times\tau_z\Phi_{zz}\to\tau_z\Phi_z$ satisfying

$$\eta(X,A)=\eta(X,z).A \text{ and } \tau b\,\eta(X,A)=X,$$

and such that $\eta\bar{h}$ is in $\Omega^o{}_{zz}(\Phi_z)$ for some, and thus any, local section $h:U\to\Phi_z$ with $h(z)=\tilde{z}$.

It will be shown that $\mathcal{C}\Phi$ is a smooth fibre bundle over B that is associated to $\mathcal{Q}\Phi$.

2.11 DEFINITION. A *(τ,Ω)-connection* is a smooth section of $\mathcal{C}\Phi$.

The following propositions are devoted to showing that the objects constructed are in fact smooth fibre bundles.

2.12 PROPOSITION. *$\mathcal{L}\Phi$ is a fibre bundle over B on which the Lie groupoid Φ acts smoothly.*

PROOF. If f is in Φ_{xy}, define $\bar{f}:\mathcal{L}_x\Phi\to\mathcal{L}_y\Phi$ by $\bar{f}(A)=if.A.if^{-1}$.

Since $\Phi_{yy}=(a,b)^{-1}\{(y,y)\}$ we have

$$\mathcal{L}_y\Phi=(\tau a,\tau b)^{-1}\{(iy,iy)\},$$

as a submanifold of $\tau\Phi$, and it follows that $\bar{f}(A)$ is in $\mathcal{L}_y\Phi$.

It is clear that $\overline{f}\overline{g}=\overline{fg}$ whenever $af=bg$. Furthermore

$$\mu:\Phi_{xx}\times\mathfrak{L}_x\Phi\to\mathfrak{L}_x\Phi \quad \text{given by} \quad \mu(s,A)=is.A.is^{-1}$$

is a smooth action. The result follows.

In order to construct a Lie groupoid acting on $\mathfrak{A}\Phi$, we must first show that $\Omega\circ(\Phi_z)$ is a Lie subgroupoid of $\Omega(\Phi_z)$. We proceed to do this in two stages.

Let $\Omega'(\Phi_z)$ consist of all ξ in $\Omega(\Phi_z)$ such that

$$\tau b\xi(V)=\eta\tau b(V) \quad \text{for all } V \text{ in } \tau\Phi_{\alpha\xi} ,$$

where η, necessarily unique, is in $\Omega(B)$. $\Omega'(\Phi_z)$ is clearly a subgroupoid of $\Omega(\Phi_z)$.

2.13 LEMMA. *$\Omega'(\Phi_z)$ is a Lie subgroupoid of $\Omega(\Phi_z)$.*

PROOF. Put $G_z=\Omega'_{zz}(\Phi_z)$, $K_z=\Omega_{zz}(B)$, and let H_z consist of all ξ in G_z such that $\tau b\xi(V)=\tau b(V)$ for all V in $\tau_z\Phi_z$.

It is clear that H_z is a closed Lie subgroup of $\Omega_{zz}(\Phi_z)$. We will exhibit G_z as a semidirect product of H_z and K_z; it will follow that G_z is a Lie subgroup of $\Omega_{zz}(\Phi_z)$.

Choose a local section $h:U\to\Phi_z$ of $b:\Phi_z\to B$ such that $h(z)=\tilde{z}$. We have an induced isomorphism $f:\Phi_z\,|\,U\to U\times\Phi_{zz}$ given by

$$f(q)=(bq,h(bq)^{-1}q).$$

Define the homomorphisms $g:K_z\to G_z$ by

$$g(\eta)=(\tau_z f)^{-1}(\eta\times 1)(\tau_z f).$$

Since $(\eta,1)$ is in $\Omega(U)\times\Omega(\Phi_{zz})$, we have $\eta\times 1$ in $\Omega(U\times\Phi_{zz})$, and since f is an isomorphism we have $g(\eta)$ in $\Omega(\Phi_z)$; it is easy to check that $g(\eta)$ is in G_z, and that g is a homomorphism.

Define $\psi:K_z\to Aut(H_z)$ by $\psi(\eta)(\xi)=g(\eta)\xi g(\eta)^{-1}$. We have now the isomorphism $F:K_z\times_\psi H_z\to G_z$ given by $F(\eta,\xi)=\xi g(\eta)$, with inverse given by $F^{-1}(\xi)=(\xi_0,\xi g(\xi_0{}^{-1}))$, where $\tau b\xi=\xi_0\tau b$. It is clear that $K_z\times_\psi H_z$ is a Lie group, and that $F:K_z\times_\psi H_z\to\Omega_{zz}(\Phi_z)$ is a smooth map. It follows that G_z is a Lie subgroup of $\Omega_{zz}(\Phi_z)$.

To complete the proof of the lemma we must show that at each point of Φ_z there is a smooth local section of $\beta : \Omega_z(\Phi_z) \to \Phi_z$ taking values in $\Omega'_z(\Phi_z)$.

Take a chart (U_0, θ_0) at z in B such that $\theta_0(U_0) = R^n$, $n = dim(B)$, $\theta_0(z) = 0$, and such that there is a local section $\phi_0 : U_0 \to \Phi_z$ with $\phi_0(z) = \tilde{z}$. For some x in B take a chart (U, θ) with $\theta(x) = 0$, $\theta(U) = R^n$, and a local section $\phi : U \to \Phi_z$; put $h_0 = \phi(x)$.

For each v in R^n define $f_v : U \to U$ by $f_v(y) = \theta^{-1}(\theta y + v)$, then $f_v f_w = f_{v+w}$, and $f_v(x) = \theta^{-1}(v)$.

Define $F_v : \Phi_z | U \to \Phi_z | U$ by $F_v(q) = \phi f_v(bq) . \phi(bq)^{-1}. q$, then

$$F_v F_w = F_{v+w}, \text{ and } F_v(h_0) = \phi f_v(x) = \phi \theta^{-1}(v).$$

For each h in $\Phi_z | U$, define $P_h : \Phi_z | U \to \Phi_z | U$ by

$$P_h(q) = F_{\theta b h}(q) . \phi(bh)^{-1}. h .$$

Then $P_h(h_0) = h$. Define $H : \Phi_z | U_0 \to \Phi_z | U$, by

$$H(t) = \phi \theta^{-1} \theta_0(bt) . \phi_0(bt)^{-1}. t;$$

Then $bH(t) = \theta^{-1}\theta_0(bt)$.

It is now clear that $\Phi_z | U \to \Omega_z(\Phi_z)$ given by $h \to \tau_{h_0} P_h . \tau_z H$ is a local section of $\beta : \Omega_z(\Phi_z) \to \Phi_z$ with values in $\Omega'_z(\Phi_z)$; for $H(z) = \phi(x) = h_0$, $bP_h(q) = f_{\theta b h}(bq)$, and $\tau_x f_{\theta b h}$ is an element in $\Omega(B)$.

Put $g(h) = h^{-1}\phi(bh)$, then $P_h(q) = g(h)[F_{\theta b h}(q)]$, where to each s in Φ_{zz} corresponds the map $s : \Phi_z \to \Phi_z$ given by $s(q) = qs^{-1}$. It follows that $\tau_{h_0} P_h = \tau_{\phi b h} g(h) \tau_{h_0} F_{\theta b h}$.

Now $h \to \tau_{h_0} F_{\theta b h}$ is smooth since $v \to \tau_{h_0} F_v$, v in R^n, is smooth. The latter is true because, if $\{e_1, \ldots, e_n\}$ is the standard basis for R^n, then $F_v = F_{v_1 e_1} \cdots F_{v_n e_n}$ where $(t, q) \to F_{t e_i}(q)$ is a flow.

Furthermore $h \to \tau_{\phi b h} g(h)$ is smooth since $\Phi_{zz} \to \Omega_q(\Phi_z)$ given by $s \to \tau_q s$ is smooth. The latter is true because if a_t is any one-parameter subgroup of Φ_{zz}, then $t \to \tau_q a_t$ is smooth.

It follows that $h \to \tau_{h_0} P_h . \tau_z H$ is a smooth local section of $\Omega_z(\Phi_z)$ with values in $\Omega'_z(\Phi_z)$. This completes the proof.

2.14 REMARK. The above proof shows that $\Omega'(\Phi_z) \to \Omega(B)$, given by $\xi \to \xi_0$ where $\tau b \xi = \xi_0 \tau b$, is a smooth homomorphism of groupoids.

2.15 LEMMA. *$\Omega^o(\Phi_z)$ is a Lie subgroupoid of $\Omega'(\Phi_z)$.*

PROOF. Define a fibre bundle $E \to \Phi_z$ by setting E_b equal to the isotropy group of $\Omega(\Phi_z)\times\Phi_{zz}$ at (b,z). The Lie groupoid $\Omega'(\Phi_z)$ acts smoothly on E via $\xi_o\omega = (\xi\times 1)\omega(\xi\times 1)^{-1}$ where ω is in $E_{\alpha\xi}$; it follows that $E\to\Phi_z$ is locally trivial.

Define the smooth isomorphism $f:\Phi_z\times\Phi_{zz}\to\Phi_z\times\Phi_{zz}$ by $f(b,s) = (bs,s)$; then $b\to\tau_{(b,z)}f$ is a smooth section of $E\to\Phi_z$.

It is easily checked that $\Omega^o(\Phi_z)$ is the subgroupoid of $\Omega'(\Phi_z)$ leaving this section invariant; it follows that $\Omega^o(\Phi_z)$ is a Lie subgroupoid of $\Omega'(\Phi_z)$.

2.16 PROPOSITION. *Φ_{zz} acts smoothly on $\Omega^o(\Phi_z)$ as a group of groupoid automorphisms. The quotient $\Sigma = \Sigma(z) = \Omega^o(\Phi_z)/\Phi_{zz}$ is a groupoid over B.*

PROOF. Define for ξ in $\Omega^o(\Phi_z)$, s in Φ_{zz}, $\xi s = \tau_s{}^{-1}\xi\tau_s$ where $s: \Phi_z\to\Phi_z$, $s(b) = bs^{-1}$.

It is clear that $(\xi s)t = \xi(st)$, and, if $\alpha\xi = \beta\eta$, then $(\xi\eta)s = (\xi s)(\eta s)$. Furthermore $(\xi,s)\to\xi s$ is smooth since $s\to\tau_b s$ is smooth, as was shown in the proof of 2.13 .

Let $\Sigma(z)$ be the set of orbits under this action, and write $\bar\xi$ for the orbit of ξ. Define $\bar\alpha,\bar\beta:\Sigma(z)\to B$ by $\bar\alpha(\bar\xi) = b\alpha\xi$, $\bar\beta(\bar\xi) = b\beta\xi$. (Note that $\alpha(\xi s) = (\alpha\xi)s$.)

If $\bar\alpha(\bar\xi) = \bar\beta(\bar\eta)$ define $\bar\xi_o\bar\eta = \overline{\xi s\cdot\eta}$ where $s = (\alpha\xi)^{-1}(\beta\eta)$. It is straightforward to show that this composition is well defined, and that it turns $\Sigma(z)$ into a groupoid over B.

We prove next that $\Sigma(z)$ is a Lie groupoid over B. An equivalence relation R on a manifold M is said to be regular if the quotient M/R has a manifold structure necessarily unique, making the projection $M\to M/R$ a submersion.

2.17 THEOREM. *R is a regular equivalence relation on M iff R is a submanifold of $M\times M$ and the projection $p:R\to M$ given by $(x,y)\to x$ is a submersion.* A proof will be found in Serre [10] .

2.18 PROPOSITION. *$\Sigma(z)$ is a Lie groupoid over B and the projection $\Omega^o(\Phi_z)\to\Sigma(z)$ is a submersion.*

PROOF. Write Ω° for $\Omega^\circ(\Phi_z)$, and let R be the subset of $\Omega^\circ\times\Omega^\circ$ consisting of pairs (ξ,η) for which $\xi s=\eta$ for some s in Φ_{zz}.

The proof will be complete if we show that R is a regular equivalence relation on Ω°; for if p is a smooth surjective submersion, and f a function, then f is a smooth map (or submersion) iff fp is a smooth map (or submersion).

Consider $b\alpha\times b\alpha:\Omega^\circ\times\Omega^\circ\to B\times B$; it is clearly a submersion. Put $R_1=(b\alpha\times b\alpha)^{-1}(\Delta)$. In addition $\alpha\times\alpha:\Omega^\circ\times\Omega^\circ\to\Phi_z\times\Phi_z$ is a submersion. Put $S_1=(\alpha\times\alpha)^{-1}(\Delta)$; it is clear that $\Delta\subset\Omega^\circ\times\Omega^\circ$ is a submanifold of S_1.

Define $F:R_1\to S_1$ by $F(\xi,\eta)=(\xi(\alpha\xi)^{-1}(\alpha\eta),\eta)$; clearly $R\subset R_1$ and $R=F^{-1}(\Delta)$. Take (ξ,η) in R_1 and put $F(\xi,\eta)=(\xi s,\eta)$. Define $G:S_1\to R_1$ by $G(\mu,\lambda)=(\mu s^{-1},\lambda)$; then

$$G(\xi s,\eta)=(\xi,\eta)\quad\text{and}\quad FG(\mu,\lambda)=(\mu,\lambda).$$

It follows that F is a submersion, and thus that R is a submanifold of $\Omega^\circ\times\Omega^\circ$.

Take (ξ,η) in R, with $\xi s=\eta$ say, and define $g:\Omega^\circ\to R$ by $g(\lambda)=(\lambda,\lambda s)$. Then g is a section of $p:R\to\Omega^\circ:p(\mu,\lambda)=\mu$, and $g(\xi)=(\xi,\eta)$. It follows that $p:R\to\Omega^\circ$ is a submersion.

The result now follows from 2.17.

Let $\Sigma(Z)$ act on $\mathfrak{A}\Phi$ as follows: if V is in $\mathfrak{A}_x\Phi$ and $\bar{\alpha}(\xi)=x$,

$$\bar{\xi}(V)=\xi(V.i\alpha\xi).i(\beta\xi)^{-1}.$$

This is a well defined action, and since the projection $\Omega^\circ(\Phi_z)\to\Sigma(z)$ is a submersion, it is easy to check that it is a smooth action.

2.19 THEOREM. *$\mathfrak{A}\Phi$ is a smooth fibre bundle with base B, which is associated to the Lie groupoid $\Sigma(z)$*

We now show that $\Sigma(z)$ is essentially independent of z.

2.20 PROPOSITION. *For each x and y in B, $\Sigma(x)$ is canonically isomorphic to $\Sigma(y)$.*

PROOF. Take f in Φ_{xy} and define an isomorphism $\Omega^\circ(\Phi_x)\to\Omega^\circ(\Phi_y)$ by $\xi\to\xi f^{-1}=\tau_f\,\xi\,\tau_f^{-1}$, where $f:\Phi_x\to\Phi_y$ is given by $f(h)=hf^{-1}$. If s is in

Φ_{xx}, then $(\xi s)f^{-1} = \xi f^{-1}(fsf^{-1})$; but fsf^{-1} is in Φ_{yy}, hence we have the induced isomorphism $F(f): \Sigma(x) \to \Sigma(y)$ given by $F(f)(\xi) = \xi f^{-1}$. It is easy to check that $F(f)$ is independent of the choice of f in Φ_{xy}. Finally $F(f)$ is a smooth map since $\Omega\circ(\Phi_x) \to \Sigma(x)$ is a submersion for each x in B.

2.21 PROPOSITION. *The homomorphism of groupoids* $\rho: \Sigma(z) \to \Phi * \Omega(B)$ *given by* $\rho(\bar{\xi}) = ((\beta\xi)(\alpha\xi)^{-1}, \xi_0)$ *is a smooth surjective submersion.*

PROOF. ρ is clearly well defined. It is a smooth surjective submersion since the composition $\Omega\circ(\Phi_z) \to \Sigma(z) \to \Phi * \Omega(B)$ is.

To prove the latter statement it is enough to show that

$$\Omega\circ(\Phi_z) \to \Phi_z * \Omega_z(B)$$

is a smooth surjective submersion, and an adaptation of the proof of 2.13 will achieve this. We omit the details.

Let $\Sigma(z)$ act on $\mathcal{C}\Phi$ as follows; if λ is in $\mathcal{C}_x\Phi$ and ξ is in $\Sigma(z)$ with $\bar{\alpha}(\bar{\xi}) = x$, define

$$(\bar{\xi}\circ\lambda)(Y, A) = \bar{\xi}(\lambda\rho(\bar{\xi})^{-1}(Y, A)).$$

This is easily shown to induce a smooth action of an isotropy group of $\Sigma(z)$ on a fibre of $\mathcal{C}\Phi$, and so we have the

2.22 THEOREM. $\mathcal{C}\Phi$ *is a smooth fibre bundle over* B *which has the associated groupoid* $\Sigma(z)$.

We will show in the next section that, when (τ, Ω) is the velocity extensor (T^k_n, Π^k), then $\mathcal{C}\Phi$ with groupoid $\Sigma(z)$ is covariantly isomorphic to the bundle $Q^k\Phi$ with groupoid Φ^k. In particular this shows that two extensors (T^k_m, Π^k) and (T^k_n, Π^k) give rise to the same higher order connections.

2.23 DEFINITION. Two extensors (τ, Ω) and (τ', Ω') are *associated* if for each M there is an isomorphism $g_M: \Omega(M) \to \Omega'(M)$ such that

a) $g_{M\times N} | \Omega(M)\times\Omega(N) = g_M \times g_N$.

b) If $h: U \to V$ is a local isomorphism of M, then $g_M(\tau h) = \tau' h$.

2.24 PROPOSITION. *If* (τ, Ω) *and* (τ', Ω') *are associated extensors, and* *if Φ is a Lie groupoid over B, then the corresponding bundles of elements of connections $(\mathcal{C}\Phi, \Sigma(z))$ and $(\mathcal{C}'\Phi, \Sigma'(z))$ are covariantly isomorphic.*

PROOF. Define $f: \Phi_z \times \Phi_{zz} \to \Phi_z \times \Phi_{zz}$ by $f(h, s) = (hs, s)$; then f is an isomorphism. Hence $\tau f: \tau\Phi_z \times \mathcal{L}_z\Phi \to \tau\Phi_z \times \mathcal{L}_z\Phi$ induces an element of $\Omega(\Phi_z \times \Phi_{zz})$ for each h in Φ_z. If ξ is in $\Omega(\Phi_z)$, then $\xi(V.A) = \xi(V).A$ for all V in $\tau_{\alpha\xi}\Phi_z$ and all A in $\mathcal{L}_z\Phi$ iff $(\xi \times 1)\tau f = \tau f(\xi \times 1)$.

It follows from the definition of the associated extensors that:

$$\xi(V.A) = \xi(V).A \text{ for all } V \text{ and } A$$

iff

$$g(\xi)(V.A) = g(\xi)(V).A \text{ for all } V \text{ and } A.$$

Let (U, ϕ) be a chart at z in B. For each map $\theta: U \to \Phi_{zz}$ with $\theta(z) = z$, define $\bar{\theta}: \Phi_z | U \to \Phi_z | U$ by $\bar{\theta}(h) = h.\theta(bh)$; $\bar{\theta}$ is a smooth isomorphism. It follows that $\tau_h \theta$ is in $\Omega(\Phi_z)$ for each h in $\Phi_z | U$, in particular $\tau_z \theta$ is in $\Omega_{zz}(\Phi_z)$.

If ξ is in $\Omega_{zz}(\Phi_z)$, then $\tau b \xi = \tau b$ iff $\xi \tau\bar{\theta} = \tau\bar{\theta}\xi$ for all maps $\theta: U \to \Phi_{zz}$. This is clear because $\tau\bar{\theta}(V) = V.\tau\theta\tau b V$, and $\tau\theta\tau b V$ is in $\tau_z\Phi_{zz}$. Note that we can find a finite family $\theta_j: U \to G_z$ such that

$$(\theta_1, \ldots, \theta_n): U \to G_z^n$$

is an immersion, and thus such that $(\tau_x\theta_1, \ldots, \tau_x\theta_n)$ is injective. It follows that $\tau b \xi = \tau b$ iff $\tau' b g(\xi) = \tau' b$, because $g(\xi\tau\bar{\theta}) = g(\xi)\tau'\bar{\theta}$.

It is now straightforward to show that g induces an isomorphism $\Omega^{\circ}(\Phi_z) \to \Omega'^{\circ}(\Phi_z)$. It follows that g induces an isomorphism

$$\Sigma(z) \to \Sigma'(z),$$

and it is clear that there is an isomorphism $\mathcal{C}\Phi \to \mathcal{C}'\Phi$ which is covariant with respect to the isomorphism $\Sigma(z) \to \Sigma'(z)$. The details are omitted.

We have thus shown that a (τ, Ω)-connection depends only on the groupoids $\Omega(M)$ and the way in which local isomorphisms of M are lifted into $\Omega(M)$.

In the remainder of this section we prove that a (τ, Ω)-connection exists on any Lie groupoid over a para-compact manifold. We anticipate a result that will be proved in the last section: there exists a positive integer

k such that, if $j^k_x f$ is represented by a constant map, then $\tau_x f$ is a constant map.

2.25 LEMMA. *Let G be a Lie group with Lie algebra g, and let (τ, Ω) be an extensor. Then with the prolonged structures, $\tau_e G$ is a simply connected nilpotent Lie group, $\tau_0\, g$ is its Lie algebra, and $\tau(exp) : \tau_0\, g \to \tau_e G$ is the exponential map.*

PROOF. It is clear that $\tau_e G$ is a Lie group, that $\tau_0\, g$ is a Lie algebra, and that $\tau(exp)$ is a smooth isomorphism. We can thus identify $T_e(\tau_e G)$ as a vector space with $\tau_0\, g$: if $f(t)$ is a curve in $\tau_e G$ with $f(0) = e$, then its tangent at $t = 0$ is $lim \frac{1}{t}\, \tau(log) f(t)$.

Each A in $\tau_0\, g$ induces a one-parameter subgroup $\tau(exp)\, tA$ of $\tau_e G$, and it is clear from the above identification that all one-parameter subgroups are of this form. To prove that $\tau_0\, g$ is the Lie algebra of $\tau_e G$, we must check that the Lie bracket prolonged from g coincides with Lie bracket of right invariant vector fields.

Define $\phi_t : g \times g \to g$ by

$$\phi_t(A, B) = \frac{1}{t^2} log\ [\, exp(-tB)\, exp(-tA)\, exp(tA)\, exp(tB)\,]\ ,$$

if $t \neq 0$ and $\phi_0(A, B) = [A, B]$. It is clear that $(t, A, B) \to \phi_t(A, B) = \phi(t, A, B)$ is smooth, and it follows that

$$(t, V, W) \to \tau\phi_t(V, W) = \tau\phi(it, V, W)$$

is smooth; in particular $\tau\phi_0(V, W) = lim\ \tau\phi_t(V, W)$. But the latter is the Lie bracket of V, W defined by the right invariant vector fields that they induce.

It only remains to show that $\tau_0\, g$ is a nilpotent Lie algebra. Consider the special case $\tau' = J^k_0(R, -)$, It can be checked by a direct calculation that $\tau'_0\, g$ is nilpotent. Define $\mu_n : g^n \to g$ by

$$\mu_n(x_1, \dots, x_n) = [\, x_1\, [\, x_2, \dots [\, x_{n-1}, x_n\,] \dots]\ ;$$

then, for some n, $j^k_0 \mu_n$ is represented by the constant map with value zero. It follows that, for some n, $\tau_0\, \mu_n$ is the zero map, where τ is any extensor. Thus $\tau_0 g$ is nilpotent.

2.26 LEMMA. *Let G be a simply connected, nilpotent Lie group, let M be any smooth manifold, and let H be a connected Lie group acting effectively on $M \times G$ in such a way that each f in H satisfies*

$$f(x,a) = (x, \overline{f}(x).a)$$

for some $\overline{f}: M \to G$. Then H is a simply connected, nilpotent Lie group.

PROOF. It is clear that $\overline{f}$ is a smooth map for each f in H, and that $f \to \overline{f}$ is injective and satisfies $\overline{fk}(x) = \overline{f}(x).\overline{k}(x)$ for all x in M.

Let h, g be the Lie algebras of H, G respectively. For each F in h define a map $\overline{F}: M \to g$ by $exp\ \overline{F}(x) = \overline{exp\, F}(x)$. Since $exp: g \to G$ is a smooth isomorphism (Hochschild [5], Ch.12), $\overline{F}$ is a smooth map.

Evidently $\overline{F}(x)$ generates the one-parameter subgroup $\overline{exp\ tF}(x)$ of G. It follows that $F \to \overline{F}$ is injective and preserves the Lie algebra structure in the sense that

$$\overline{(\alpha F + \beta G)}(x) = \alpha \overline{F}(x) + \beta \overline{G}(x) \quad \text{and} \quad \overline{[F, G]}(x) = [\overline{F}(x), \overline{G}(x)].$$

Since g is nilpotent it follows that h is nilpotent, but $exp: h \to H$ is injective since $F \to \overline{exp\ F} = exp\ \overline{F}$ is; it follows that H is isomorphic to its universal covering group.

Unfortunately we can't work directly with the bundle $\mathcal{C}\Phi$ because its fibres are not necessarily even connected. We must first pick out a subbundle with connected fibres.

Let H_x be the component of the identity in the subgroup $\Omega_{xx}(\Phi_x)$ consisting of ξ such that $\tau b \xi(V) = \tau b(V)$ and $\xi(V.A) = \xi(V).A$ for all V in $\tau_x \Phi_x$, A in $\tau_x \Phi_{xx}$.

Let $\mathcal{C}'_x\Phi$ consist of all isomorphisms $\eta: \tau_x B \times \tau_x \Phi_{xx} \to \tau_x \Phi_x$ such that, for any local section θ of $b: \Phi_x \to B$ with $\theta(x) = \check{x}$, the isomorphism $\eta\overline{\theta}$ (cf. 2.10) is in H_x. To show that $\mathcal{C}'_x\Phi$ is well defined, we prove that, if ϕ is another section of $b: \Phi_x \to B$ with $\phi_x = \check{x}$, then $\overline{\phi}\overline{\theta}^{-1}$ is in H_x. It is more convenient to identify H_x with the connected component of the subgroup of the isotropy group at (x, x) of $\Omega(B \times \Phi_{xx})$ consisting of η such that $\eta(X, A) = (X, \overline{\eta}(X).A)$ for some map

$$\overline{\eta}: \tau_x B \to \tau_x \Phi_{xx}.$$

We must now show that $\overline{\phi}\overline{\theta}^{-1}$ is in H_x.

Now $\bar{\phi}\bar{\theta}^{-1}(X, A) = \bar{\phi}(\tau\theta X.A) = (X, \tau(\phi^{-1}.\theta)X.A)$. Define the local map $f: B \to \Phi_{xx}$ by $f(y) = \phi_y^{-1}.\theta_y$ and define another local map f^t by $f^t(y) = exp\ t\ log\ f(y)$, and finally the local map $F^t: B \times \Phi_{xx} \to B \times \Phi_{xx}$ by

$$F^t(y, a) = (y, f^t(y).a).$$

It is clear that $\tau_{(x,x)}F^t$ is a curve in $\Omega(B \times \Phi_{xx})$ joining $\bar{\phi}\bar{\theta}^{-1}$ to the identity; furthermore $\tau F^t(X, A) = (X, \tau f^t(X).A)$ for all t, X and A. It follows that $\bar{\phi}\bar{\theta}^{-1}$ is in H_x.

Consequently $\mathcal{C}'\Phi = \bigcup \mathcal{C}'_x\Phi$ is well defined and, since clearly the action of $\Omega^o(\Phi_z)/\Phi_{zz}$ on $\mathcal{C}\Phi$ leaves $\mathcal{C}'\Phi$ invariant, it follows that $\mathcal{C}'\Phi$ is a subbundle of $\mathcal{C}\Phi$.

Each fibre of $\mathcal{C}'\Phi$ is isomorphic to H_x, and by lemmas 3.25 and 3.26 H_x is a simply connected nilpotent Lie group, and thus is isomorphic to a vector space: its Lie algebra. We can now apply Steenrod's theorems on the existence of smooth sections [11]. Over a paracompact manifold $\mathcal{C}'\Phi$ has a section, and consequently so does $\mathcal{C}\Phi$.

2.27 THEOREM. *A (τ, Ω)-connection exists on any Lie groupoid over a paracompact manifold.*

3. Comparison Theorems

We show that a (T_n^k, Π^k)-connection is equivalent to Ehresmann's notion of a k-th order holonomic connection [3]; the same result holds with the same proof for the semi-holonomic and non-holonomic velocity extensors $(\bar{T}_n^k, \bar{\Pi}^k)$ and $(\tilde{T}_n^k, \tilde{\Pi}^k)$.

We then compare a (τ, Ω)-connection with the notion of $(\mathcal{V}, \tau)$-connection introduced by Virsik in [14] as an abstraction of first order connections. In the pair $(\mathcal{V}, \tau)$, τ is a natural fibering, called a local regular connector in [14], and $\mathcal{V}$ is a category of fibred manifolds containing all the maps $\tau f: \tau M \to \tau N$ between the fibred manifolds $\tau M \to M$ and $\tau N \to N$.

Recall that a first order connection on Φ is a section $\lambda: TB \to A\Phi$ of Tb; a $(\mathcal{V}, \tau)$-connection is the most natural generalization of this, namely a section $\mathfrak{X}: \tau B \to \mathfrak{A}\Phi$ of τb, where $\mathfrak{A}\Phi = \bigcup \tau_x \Phi_x$, such that $(-, j\mathfrak{X})$

as a map of fibred manifolds $\tau B \to B$ into $\tau\Phi \to \Phi$ is in $\mathcal{C}$, $\sim$ is the embedding of B onto the identities of Φ, and j is the inclusion of $\mathfrak{A}\Phi$ in $\tau\Phi$. All the familiar definitions of a first order connection: by horizontal subspaces, a Lie algebra valued form, etc..., have analogues for a $(\mathcal{C}, \tau)$-connection. Furthermore Virsik has shown that, if τ is the velocity functor $\tilde{T}^k_n$ and $\mathcal{C}$ the category of fibred manifolds $p: \tilde{T}^k_n(M) \to M$ with maps $(\beta Z, Zp)$, where Z is a section of $\tilde{J}(M, N) \to M$, then a $(\mathcal{C}, \tau)$-connection is equivalent to a k-th order non-holonomic connection in the sense of Ehresmann.

For our purposes $\mathcal{C}$ will be the category of fibre bundles and fibre preserving maps, and τ will be subordinate to an extensor (τ, Ω), then $\mathfrak{A}\Phi$ will carry a natural fibre bundle structure. We show that, if the $(\mathcal{C}, \tau)$-connection $\mathfrak{X}: \tau B \to \mathfrak{A}\Phi$ is, in a special sense, a fibre bundle map, and for some x, $\mathfrak{X}x = \xi \circ \tau_x s$ where s is a local section of $\Phi_x \to B$ with $s(x) = \tilde{x}$, and ξ an element of the groupoid of $\mathfrak{A}\Phi$, then $\mathfrak{X}$ is equivalent to a (τ, Ω)-connection.

Let (τ, Ω) be the velocity extensor (T^k_n, Π^k) described in 2.8, and let Φ be a Lie groupoid on B.

3.1 THEOREM. *There is an isomorphism $F: Q^k\Phi \to \mathcal{C}\Phi$, and an isomorphism $G: \Phi^k \to \Sigma(z)$ of groupoids such that $F(\xi \circ \eta) = G(\xi) \circ F(\eta)$ whenever ξ is in Φ^k, η is in $Q^k\Phi$, and $\xi \circ \eta$ is defined.*

PROOF. Define $F: Q^k\Phi \to \mathcal{C}\Phi$ as follows: if η is in $Q^k\Phi_x$, then

$$F(\eta): \tau_x B \times \mathfrak{L}_x\Phi \to \mathfrak{A}_x\Phi$$

is given by

$$F(\eta)(X, A) = \eta X . A,$$

where ηX is composition of jets. Note that, if α and β are respectively the source and target maps for jets, then $\beta X = x = \alpha \eta$ and $\beta \eta = \tilde{x}$. We now show that $F(\eta)$ is in $\mathcal{C}\Phi$.

In a neighbourhood of x take a local section θ of $b: \Phi_x \to B$ such that $\theta x = \tilde{x}$: We have the isomorphism $\bar{\theta}: \mathfrak{A}_x\Phi \to \tau_x B \times \mathfrak{L}_x\Phi$ given by

$$\bar{\theta}(V) = (\tau b V, (\tau\theta\, \tau b V)^{-1}. V).$$

It follows that

$$\xi(V) = F(\eta)\bar{\theta}(V) = \eta\,\tau b V.(\tau\theta\,\tau b V)^{-1}.\,V.$$

Let $K:\Phi * \Phi \to \Phi$, and $\sigma:\Phi \to \Phi$ be the composition and inverse, respectively, in Φ. We have $\xi = \tau K\{\eta\tau b,\ \tau K(\tau\sigma\tau\theta\tau b,\ \tau 1)\}$, and this is clearly an invertible k-jet from Φ_x to Φ_x, hence ξ is in $\Pi^k(\Phi_x) = \Omega(\Phi_x)$. It is easy to check that ξ is in fact in $\Omega^\circ(\Phi_x)$, and it follows that $F(\eta)$ is in $\mathcal{C}\Phi$.

The inverse of F is defined as follows: if $\lambda:\tau_x B \times \mathcal{L}_x\Phi \to \mathcal{A}_x\Phi$ is in $\mathcal{C}\Phi$, then put $F'(\lambda) = \lambda\tau j$, where $j: B \to B\times\Phi_{xx}$ takes y into (y, x). It is clear from the definition of $\mathcal{C}\Phi$ that λ is in $J^k_{(x,x)}(B\times\Phi_{xx}, \Phi_x)$; hence $F'(\lambda)$ is in $J^k_x(B, \Phi_x)$, and it is easy to check that in fact $F'(\lambda)$ is in $Q^k\Phi$. It is straightforward to show that F' is the inverse of F.

Define $G:\Phi^k \to \Sigma(z)$ by $G(\xi) = \bar{\eta}$, where η in $\Omega^\circ{}_{hk}(\Phi_z)$ is defined by $k = (\beta\xi).h$ and $\eta(V) = \xi\tau b V.V$. It is clear that

$$\eta = \tau K(\xi\tau b,\ \tau 1)$$

is in $\Omega(\Phi_z)$ and thus in $\Omega^\circ(\Phi_z)$. We show that G is well defined. Suppose that $\eta_1:\tau_{h_1}(\Phi_z) \to \tau_{k_1}(\Phi_z)$ in $\Omega^\circ(\Phi_z)$ is such that $k_1 = (\beta\xi).h_1$ and $\eta_1(W) = \xi\tau b W.W$; then there is a unique s in Φ_{zz} such that $h_1 = hs$ and $k_1 = ks$. Furthermore for W in $\tau_{h_1}(\Phi_z)$ we have

$$\eta s(W) = \xi\tau b(W.is^{-1}).\,W.is^{-1}.is = \eta_1(W).$$

It follows that $\bar{\eta} = \bar{\eta}_1$

We now show that G is a homomorphism. Suppose ζ and ξ are in Φ^k with $a^k(\zeta) = b^k(\xi)$, and that $G(\zeta) = \bar{\mu}$ and $G(\xi) = \bar{\lambda}$. We may clearly choose μ and λ in $\Omega^\circ(\Phi_z)$ so that $\alpha\mu = \beta\lambda$. We have

$$G(\zeta_\circ\lambda) = G(\zeta\tau b\xi.\,\xi) = \bar{\eta},$$

say, where η in $\Omega^\circ(\Phi_z)$ is given by:

$$\begin{aligned}\eta(V) &= (\zeta\tau b\xi.\,\xi)\tau b V.V = \zeta\tau b\xi\tau b V.\,\xi\tau b V.V\\ &= \zeta\tau b\lambda(V).\,\lambda(V) = \mu\,\lambda(V).\end{aligned}$$

It follows that

$$G(\zeta_\circ\,\xi) = \overline{\mu\,\lambda} = G(\zeta)_\circ\,G(\xi).$$

Furthermore it is clear that $aG(\xi) = a^k(\xi)$, and that $bG(\xi) = b^k(\xi)$.

The inverse of G is defined as follows: if $\eta:\tau_h\Phi_z \to \tau_k\Phi_z$ is in

$\Omega\circ(\Phi_z)$, then put $G'(\bar{\eta})=\tau K(\eta\tau\theta, \tau\sigma\tau\theta)$, where θ is a local section of $b:\Phi_z\to B$ satisfying $\theta(bh)=h$. The routine verification that G' is a well defined inverse of G is omitted.

Suppose now that ξ in Φ^k and λ in $Q^k\Phi$ are such that $a^k(\xi)=\alpha\lambda$, and put $G(\xi)=\bar{\eta}$, where $\eta:\tau_h\Phi_z\to\tau_k\Phi_z$ is in $\Omega\circ(\Phi_z)$. We have:

$$\begin{aligned}(G(\xi)\circ F(\lambda))(X,A) &= \eta\{F(\lambda)(\eta_0^{-1}X, ihk^{-1}.A.ikh^{-1}).ih\}.ik^{-1}\\ &= \eta\{\lambda\eta_0^{-1}X.ihk^{-1}.A.ikh^{-1}.ih\}.ik^{-1}\\ &= \xi\tau b\lambda\eta_0^{-1}X.\lambda\eta_0^{-1}X.ihk^{-1}.A\\ &= (\xi\eta_0^{-1}.\lambda\eta_0^{-1}.i(\beta\xi)^{-1})X.A\\ &= (\xi\circ\lambda)(X).A=F(\xi\circ\lambda)(X,A).\end{aligned}$$

In this calculation we have used several definitions in the following order: the action of $\Sigma(z)$ on Φ, the definition of F, the definition of G, the action of Φ^k on $Q^k\Phi$, and finally the definition of F again.

The proof is completed by a routine check that F and G are smooth isomorphisms; we omit this.

Thus (T_n^k,Π^k)-connections are equivalent to holonomic connections of order k. Similarly $(\bar{T}_n^k,\bar{\Pi}^k)$ and $(\tilde{T}_n^k,\tilde{\Pi}^k)$-connections are equivalent to semi-holonomic and non-holonomic connections of order k respectively; the proofs are the same. Notice that (2.24) shows that a (T_n^k,Π^k)-connection is independent of n.

We turn to $(\mathcal{C},\tau)$-connections.

3.2 DEFINITION. A $(\mathcal{C},\tau)$-connection on the Lie groupoid Φ is a fibre preserving map $\mathcal{X}:\tau B\to\mathfrak{A}\Phi$ such that $\tau b\mathcal{X}=1$.

$\mathcal{X}$ is *regular* if the subgroupoid Σ' of $\Sigma=\Omega\circ(\Phi_z)/\Phi_{zz}$ consisting of ξ such that $\xi\circ\mathcal{X}(X)=\mathcal{X}(\xi\circ X)$ is a Lie subgroupoid.

$\mathcal{X}$ is *normal* if $\mathcal{X}$ is regular and if for some x in B there is η in $\mathcal{C}_x\Phi$ such that $\eta(X,A)=\mathcal{X}(X).A$ for all (X,A) in $\tau_x B\times\mathcal{L}_x\Phi$.

3.3 THEOREM. *There is a one to one correspondence between normal $(\mathcal{C},\tau)$-connections and (τ,Ω)-connections on* Φ.

PROOF. Let $\mathcal{X}:\tau B\to\mathfrak{A}\Phi$ be a normal $(\mathcal{C},\tau)$-connection and suppose $\eta(X,A)=\mathcal{X}(X).A$ for η in $\mathcal{C}_x\Phi$. Define $\Lambda:B\to\mathcal{C}\Phi$ as follows: for each local section ξ of $\Sigma'(x)$ put $\Lambda y = \xi(y)_{\circ}\,\eta$. Then for (Y, C) in $\tau_y B \times \mathfrak{L}_y \Phi$ we have, if $\xi(y) = \overline{\zeta}$ where ζ is in $\Omega_{hk}(\Phi_z)$:

$$\Lambda y(Y, C) = \overline{\zeta}_{\circ}(\mathfrak{X}(\zeta_0^{-1} Y).ih.ik^{-1}.C.ik.ih^{-1})$$
$$= \zeta\{\mathfrak{X}(\zeta_0^{-1} Y).ih\}.ik^{-1}.C = \mathfrak{X}(Y).C$$

Since $\Omega(\Phi_z)$ acts effectively on $\tau\Phi_z$, this shows that Λy is independent of the local section of Σ' chosen. It is clear that Λ is a smooth section of $\mathcal{C}\Phi$.

Conversely, let $\Lambda : B \to \mathcal{C}\Phi$ be a (τ, Ω)-connection. Define $\mathfrak{X}$: $\tau B \to \mathfrak{A}\Phi$ by $\mathfrak{X}(X) = \Lambda x(X, ix)$ for X in $\tau_x B$; it is clear that $\tau b \mathfrak{X} = 1$. Furthermore it is easy to check that $\xi_{\circ}\,\Lambda x = \Lambda y$ for ξ in Σ_{xy} iff

$$\xi_{\circ}\,\mathfrak{X}(X) = \mathfrak{X}(\xi_0 X) \text{ for all } X \text{ in } \tau_x B.$$

It follows that Σ' is the subgroupoid of Σ leaving the section Λ invariant, and thus Σ' is a Lie subgroupoid if it is transitive; we prove now that it is transitive.

Take x and y in B; choose λ in $\Omega_{xy}(B)$, f in Φ_{xy}, and h in Φ_{zx}, and define $\xi : \tau_h \Phi_z \to \tau_k \Phi_z$, where $k = fh$, by

$$\xi(W) = \Lambda y(\lambda, f) \Lambda x^{-1}(W.ih^{-1}).ik,$$

where $(\lambda, f) : \tau_x B \times \mathfrak{L}_x \Phi \to \tau_y B \times \mathfrak{L}_y \Phi$ takes (X, A) into $(\lambda X, if.A.f^{-1})$. Using the definition of $\mathcal{C}\Phi$ it is straightforward although complicated to show that ξ is in $\Omega^{\circ}(\Phi_z)$, and that $\overline{\xi}_{\circ}\,\Lambda x = \Lambda y$.

It follows that $\mathfrak{X}$ is a regular $(\mathfrak{C}, \tau)$-connection and by its definition $\mathfrak{X}$ is also normal. Finally since the correspondence is given by

$$\Lambda(X, A) = \mathfrak{X}(X).A,$$

it is clearly one to one.

The regularity condition appears implicitly in the definition of a first-order connection, for the lifting $\lambda: TB \to A\Phi$ is required to be a section of $L(TB, A\Phi)$. Regularity should be implied by any reasonable local definition of a fibre bundle map.

Normality in the first-order case is implied by linearity.

The projection $\rho: \Sigma \to \Phi_* \Omega(B)$ (2.21) induces an isomorphism of Σ' with $\Phi_* \Omega(B)$, where Σ' is the Lie subgroupoid of Σ leaving invariant a regular $(\mathfrak{V}, \tau)$-connection. Thus:

3.4 PROPOSITION. *A regular $(\mathfrak{V}, \tau)$-connection is equivalent to a section of $\rho: \Sigma \to \Phi_* \Omega(B)$.*

4. Structure of extensors

In this section we prove that, for some k, if $j^k_x f$ is represented by a constant map, then $\tau_x f$ is a constant map. We can then give a partial structure theorem for extensors; we lack a means of constructing an extensor (τ, Ω) from a given Ω: We end the section by extending a theorem of Ngo Van Que on groupoids which are prolongations of manifolds in the sense that local isomorphisms lift into them.

We remark again that the only extensors taking manifolds into vector bundles are the first-order velocities T^1_n.

We use local methods, and, thus restrict an extensor (τ, Ω) to the category $\mathfrak{E}$ of Euclidean spaces R^n and smooth maps preserving the origin. Write E^n for the fibre of τR^n over the origin, let H_n be the isotropy group $\Omega(R^n)$ at the origin, and let h_n be its Lie algebra.

4.1 PROPOSITION. *E^n is a vector space with base point as origin. If f and g; $R^m \to R^n$ are in $\mathfrak{E}$, then, for ω in E^m,*

$$\tau(\mu f + \lambda g)(\omega) = \mu\, \tau f(\omega) + \lambda \tau g(\omega).$$

Furthermore τ restricted to linear maps is given by $L(V, -)$ for some vector space V.

PROOF. Since τ preserves products, all operations in R^n can be prolonged to operations in E^n, and all commutative diagrams in $\mathfrak{E}$ are taken into commutative diagrams. Finally it is clear that the functors $L(V, -)$ from vector spaces into vector spaces are the only ones that preserve products.

Each map $F: R^n \to R^n$ in $\mathfrak{E}$ induces a flow $f_t: R^n \to R^n$ in $\mathfrak{E}$ defined by $f_s f_t = f_{s+t}$, and $F(x) = \lim \frac{1}{t} [f_t(x) - x]$ as t tends to zero.

4.2 LEMMA. *For each v in E^n,*

$$\tau F(v) = \lim \frac{1}{t} [\tau f_t(v) - v].$$

PROOF. Put $g(t, x) = \frac{1}{t}(f_t(x) - x)$ if $t \neq 0$, $g(0, x) = F(x)$. Then it is easy to check that g is smooth; hence $(t,v) \to \tau g(t,-) = \tau g(it,v)$ is smooth, and since $\tau g(t,-) = \frac{1}{t}[\tau f_t - \tau 1]$, the result follows.

Each A in h_n induces a smooth map $E^n \to E^n$ given by

$$A(v) = \lim \frac{1}{t}(a_t v - v)$$

where $a_t = \exp tA$. The following properties are easy to check:

$(\alpha A + \beta B)(v) = \alpha A(v) + \beta B(v)$, $[A,B](v) = B'(v)A(v) - A'(v)B(v)$,

and $A(v) = 0$ for all v in E^n iff $A = 0$.

Write l_n for the vector space of germs of smooth local maps $R^n \to R^n$ in $\mathcal{E}$ at the origin, and write L_n for the group of germs of local isomorphisms $R^n \to R^n$ in $\mathcal{E}$. We have a homomorphism $\phi: L_n \to H_n$ because an extensor lifts local isomorphisms into the associated groupoid. There is a corresponding function $\phi_*: l_n \to h_n$ defined by $\exp t\phi_*(F) = \phi(f_t)$ where f_t is the germ of the flow induced by some representative of F.

l_n inherits the Lie bracket of local functions:

$$[F,G](x) = G'(x)F(x) - F'(x)G(x),$$

and if h is in L_n, F in l_n, we have $ad(h)(F)$ in l_n defined as the germ of $\bar{h}'(\bar{h}^{-1}(x))\bar{F}(\bar{h}^{-1}(x))$, where $\bar{h}$ and $\bar{F}$ are representatives of h, F respectively. Note that

$ad(h)[F,G] = [ad(h)(F), ad(h)(G)]$, $ad(h)ad(k)(F) = ad(hk)(F)$.

4.3 LEMMA. *$\phi_*: l_n \to h_n$ is linear and furthermore:*

(a) $\phi_*[ad(h)(F)] = ad\,\phi(h)(\phi_* F)$,

(b) $\phi_*[F,G] = [\phi_* F, \phi_* G]$.

PROOF. For each v in E^n, $\phi_*(F) = \tau F(v)$ by Lemma 4.2. Since, for A in h_n, $A(v) = 0$ for all v iff $A = 0$, it follows from Proposition 4.1 that ϕ_* is linear.

If f_t is the flow for F, then hf_th^{-1} is the flow for $ad(h)(F)$. It follows that $\phi_*[ad(h)(F)]$ is the tangent at the identity to the curve $\phi(hf_th^{-1}) = ad\,\phi(h)[\phi(f_t)]$, which proves (a).

Now it is well known that, for functions F and G,

$$[F,G](x) = \lim \frac{1}{t}[G(x) - ad(f_t)(G)(x)]$$

where f_t is the flow induced by F. It follows that, for each v in E^n,

$$\tau[F,G](v) = \lim \frac{1}{t}[\tau G(v) - \tau(ad(f_t)(G))(v)]$$

Since τ is local we can replace the functions F, G by germs to obtain

$$\phi_*[F,G](v) = \lim \frac{1}{t}[\phi_*(G) - \phi_*(ad(f_t)(G))](v)$$
$$= \lim \frac{1}{t}[\phi_* G - ad\,\phi(f_t)(\phi_* G)](v),$$

and (b) follows from the expression for the bracket of right invariant vector fields in h_n.

We now show that there exists some positive integer k such that $D^q f(0)=0$ $q \leqslant k$ implies $\phi_* f=0$. F is *flat* if $D^k F(0)=0$ for $k \geqslant 0$. The following device is taken from Peetre [9] . Let S_0 and S_1 be any disjoint closed subsets of the unit sphere in R^n; choose a smooth map Φ defined on the sphere such that $\Phi(S_0)=0$, $\Phi(S_1)=1$, and put $\theta(x)=\Phi(x/|x|)$ Then θ is smooth outside the origin, and if F is flat, then $\theta . F$ is smooth and zero is a limit point of the interior of each of the sets

$$\{x \mid \theta(x)F(x)=0\} \text{ and } \{x \mid \theta(x)F(x)=F(x)\}$$

- provided that neither S_0 nor S_1 is nowhere dense.

4.4 LEMMA. *If F in l_n is flat, then $\phi_*(F)=0$.*

PROOF. Let G be a representative of F and let U be the interior of $\{x \mid \theta(x)G(x)=0\}$; there is a sequence x_n in U tending to zero. Put $R_x(y)=(\theta.G)(x+y)$; then $\tau(R_{x_n})(v)=0$ for all v in E^n since R_{x_n} vanishes in a neighbourhood of the origin. It follows that

$$\tau(\theta . G)(v) = \tau R_0(v) = 0 \text{ for all } v;$$

hence $\phi_*(\theta F)=0$. Similarly $\phi_*[(1-\theta)F]=0$ and it follows that $\phi_*(F)=0$.

The sequence of pseudonorms $|F|_k = |D^k F(0)|$ defines a topology on l_k and the germs of flat functions form a closed ideal. If we factor out by this ideal, then by an extension of a theorem of Borel, Mirkil [7], we get the Frechet space of formal power series, here taken as Taylor series in Lang's sense.

4.5 LEMMA. *There exists a positive integer k such that, if A is any q-li-*

near map from R^n into R^n with $q > k$, and if A' is the germ of $x \to A x^q$, then $\phi_*(A') = 0$.

PROOF. If the lemma were false, then there would be an increasing sequence q_i of positive integers and for each i a q_i-linear map A_i such that

$$\phi_*(A'_i) \neq 0.$$

We show that the $\phi_*(A_i')$ are linearly independent; this invoves a contradiction because h_n is finite dimensional. Suppose that $\Sigma \lambda_i \phi_*(A_i') = 0$; then for each v in E^n and t in R we have

$$\Sigma t^{q_i} \lambda_i \tau A_i'(v) = \Sigma \lambda_i \tau A_i'(tv) = 0.$$

Hence $\lambda_i \tau A_i'(v) = 0$ for each v and each i; it follows that $\lambda_i \phi_*(A_i') = 0$ and thus $\lambda_i = 0$.

A combination of the two Lemmas and the remark proves that for each n there is a least integer k_n such that, if F is in l_n and $D^q F(0) = 0$ for $q \leqslant k_n$, then $\phi_*(F) = 0$, i.e. $\tau F = 0$.

4.6 LEMMA. *k_n is independent of n.*

PROOF. $k_n \leqslant k_{n+1}$ for all n because, if F is in l_n and $D^q F(0) = 0$ for $q \leqslant k_{n+1}$, then $D^q(F \times 0) = 0$ where $F \times 0$ is in l_{n+1}; hence $\phi_*(F \times 0) = 0$ and thus $\phi_*(F) = 0$.

Suppose that F in l_n satisfies $D^q F(0) = 0$ for $q \leqslant k_1$. Let

$$A : R \to R^n \text{ and } B : R^n \to R$$

be linear; then $D^q(BFA) = 0$ for $q \leqslant k_1$; hence $\tau(B)\tau(F)\tau(A) = 0$; and by 4.1, $B . \tau(F). A = 0$ for any linear B and A. It follows that $\phi_* F = 0$, and thus that $k_n \leqslant k_1$.

Call the common value k the order of the extensor τ. The Lie group L_n^k of k-jets of local isomorphisms preserving the origin of R^n has Lie algebra $l_n^k = J_0^k(R^n, R^n)_0$, $exp\ t j_0^k F = j_0^k f_t$ where f_t is the flow of F and $[j_0^k F, j_0^k G] = j_0^k(G' F - F' G)$.

It follows immediately that we have a Lie algebra homomorphism $\lambda_* : l_n^k \to h_n$ and it is easy to see that the underlying group homomorphism $\lambda : L_n^k \to H_n$ is globally defined, because $\lambda j^k = \phi$.

Notice that any local map $F : R^m \to R^n$ can be factored into the local maps $q : R^m \to R^{m+n}$, $F' : R^{m+n} \to R^{m+n}$, and $p : R^{m+n} \to R^n$ given by

$$q(x)=(x,0),\ F'(x,y)=(x,y+F(x)),\ \text{and}\ p(x,y)=y.$$

It is clear that F' is a local isomorphism. We employ this device to give a structure theorem for extensors.

4.7 THEOREM. *There is, up to isomorphism, a one to one correspondence between an extensor* (τ, Ω) *and the following list of objects: a positive integer* k, *a vector space* V, *a sequence of Lie groups* H_n *such that* H_n *acts effectively on* $L(V, R^n)$ *so that there is an embedding*

$$H_m \times H_n \to H_{m+n},$$

for each m *and* n, *compatible with this action, and finally for each* n *a smooth homomorphism* $\lambda_n : L_n^k \to H_n$ *of Lie groups which is compatible with the embeddings* $H_m \times H_n \to H_{m+n}$ *and* $L_m^k \times L_n^k \to L_{m+n}^k$, *and which does not factor through the projection* $L_n^k \to L_n^{k-1}$.

PROOF. We have already shown that an extensor gives rise to the above list of objects.

Given the objects listed we construct an extensor (τ, Ω) as follows.

Let M be a manifold of dimension n. Define $\tau_x M$ to be the set of equivalence classes of pairs (ϕ, v), where ϕ is a chart around x and v is in $L(V, R^n)$; (ϕ, v) is equivalent to (ψ, w) iff $\lambda_n j_0^k(\phi_x \psi_x^{-1})(w)=v$ where $\phi_x(y)=\phi(y)-\phi(x)$. Put $\tau M = \bigcup \tau_x M$; it is clear that τM is a manifold, since each chart $\phi: U \to R^n$ gives an isomorphism of τU with $U \times L(V, R^n)$ and such isomorphisms are smoothly compatible.

We define $\Omega(M)$ in the same way. Let $\Omega_{xy}(M)$ consist of triples (ψ, s, ϕ), where ψ, ϕ are charts around x, y respectively, s in H_n, and we identify (ψ, s, ϕ) with (ψ', s', ϕ') if

$$s' \lambda_n j_0^k(\phi_x' \phi_x^{-1}) = \lambda_n j_0^k(\psi_y' \psi_y^{-1}) s.$$

Put $\Omega(M) = \bigcup \Omega_{xy}(M)$; a pair of charts $\phi: U \to R^n$, $\psi: V \to R^n$ induces an isomorphism of $\Omega(M) \mid U \times V$ with $V \times H_n \times U$. These isomorphisms are smoothly compatible and thus define the structure of a manifold on $\Omega(M)$.

Write $\tau\phi_x^{-1}(v)$ for the equivalence class of (ϕ, v) at x, and $\tau(\psi_y^{-1}) s \tau(\phi_x)$ for the equivalence class of (ψ, s, ϕ) in $\Omega_{xy}(M)$. Define $(\alpha,\beta):\Omega(M)\to M\times M$ by $(\alpha,\beta)\{\tau(\psi_y^{-1})s\tau(\phi_x)\}=(x,y)$. This map is locally the projection $V\times H_n\times U\to U\times V$; hence it is a surjective submersion. Define composition in $\Omega(M)$ by

$$\tau(\psi_z^{-1})s\tau(\phi_y)\circ\tau(\omega_y^{-1})t\tau(\theta_x)=\tau(\psi_z^{-1})s\lambda_n j_0^k(\phi_y\omega_y^{-1})t\tau(\theta_x).$$

If we take $\omega=\phi$, then locally composition is given by $(z,s,y)(y,t,x)=(z,st,x)$; it is therefore a smooth map. Similarly the inverse map is smooth, and it follows that $\Omega(M)$ is a Lie groupoid over M. $\Omega(M)$ acts on τM as follows:

$$\tau(\psi_z^{-1})s\tau(\phi_y)\circ\tau(\theta_y^{-1})(v)=\tau(\psi_z^{-1})\{s\lambda_n j_0^k(\phi_y\theta_y^{-1})\}(v).$$

This is a smooth effective action because, after a choice of charts, the action of $\Omega_{xx}(M)$ on $\tau_x M$ is that of H_n on $L(V,R^n)$.

If $f:M\to N$ is a smooth map, then $\tau f:\tau M\to\tau N$ is defined as follows: take a chart ϕ around x and a chart ψ around $y=f(x)$ and define

$$\tau f\{\phi_x^{-1}(v)\}=\tau\psi_y^{-1}\{p_2\lambda_{n+p}(j_0^kF)(v,0)\},$$

where $n=dim\ M$, $p=dim\ N$, where $F:R^{n+p}\to R^{n+p}$ is given by

$$F(v,u)=(v,u+\psi_y f\phi_x^{-1}(v)),$$

and where $p_2:L(V,R^{n+p})\to L(V,R^p)$ is the projection. It is clear, using the isomorphisms $\tau M|U\to U\times L(V,R^n)$, that τf is smooth.

Checking of the outstanding properties of an extensor is straightforward. Finally it is clear that

local structure → extensor → local structure

is the identity; it remains to show that, if two extensors give rise to the same local structure, then they are isomorphic. Taking charts gives local isomorphisms and it is easy to check that these isomorphisms are compatible.

Note that an extensor of order k admits a smooth homomorphism $\lambda:\Pi^k(M)\to\Omega(M)$ of groupoids for each manifold M, through which the lifting of local isomorphisms factors. This is true in a more general context.

We sketch a proof of the following extension of a theorem of Ngo Van Que [12]

4.8 DEFINITION. *A prolongation of a manifold* M *is a Lie groupoid* Ω over M, and a lifting of local isomorphisms of M into Ω such that each isomorphism $f: U \to V$ induces a section $U \to \Omega$ of the source projection $\alpha: \Omega \to M$, written $x \to \rho_x f$, so that $\beta \rho_x f = f(x)$ and such that, if f_t is a local flow, then $(t, x) \to \rho_x f_t$ is smooth.

4.9 THEOREM. *Every prolongation Ω of M admits for some k a smooth homomorphism of groupoids $\lambda: \Pi^k \to \Omega$ through which the lifting of local isomorphisms factors.*

PROOF. We have only to prove that there exists $k \geq 1$ such that $j_x^k f = j_x^k 1$ implies $\rho_x(f) = \tilde{x}$.

Take a chart $\phi: U \to R^n$ with $f(U) = R^n$, such that there is a local section $\theta: U \to \Omega_x$ of $\beta: \Omega_x \to M$ where $x = \phi^{-1}(0)$. We have a homomorphism $\omega: L_n \to G = \Omega_{xx}$ taking germs of flows into one-parameter subgroups. Let g be the Lie algebra of G and define $\omega_*: l_n \to g$ by

$$exp\ t\omega_*(F) = \omega(f_t),$$

where f_t is the germ of the flow of a representative of F. As in [12] it can be shown that ω_* is R-linear and preserves bracket. Lemma 4.4 can be adapted to show that ω_* vanishes on flat functions by noting that, if f_t is the flow of $S_x: y \to \theta G(x+y) - \theta G(x)$, then $x \to \omega(f_t)$ is smooth and thus so is $x \to \omega_*(S_x)$. The adaptation needed in Lemma 4.5 is clear; we need only check that, if μ_t is the germ of multiplication by t, then

$$\omega_*(\mu_t^{-1} F \mu_t) = ad\ \omega(\mu_t)\, \omega_*(F).$$

We note that compactness is not required in the above theorem because k is from the start bounded by the dimension of G. We remark further that the really hard part of this theorem has been slurred over as it was in [12]. To prove that ω_* is R-linear and preserves the bracket, without additional continuity hypotheses on the lifting of local isomorphisms, requires quite subtle arguments of the kind used by Epstein [4] to prove the continuity of natural vector bundles.

It is worth noticing that a spray on a manifold M induces a first order connection on each groupoid Ω prolonging the manifold. For it induces a first order connection on $\Pi^k(M)$ for each k; thus define $\lambda: TM \to A\Pi^k(M)$ by

$$\lambda(X)=j_0^1\{t\to j_x^k[y\to exp_x(tX+log\,y)]\},$$

where X is in T_xM, $exp_x: T_xM\to M$ is the exponential map, and log is the local inverse of exp_x.

It can be shown that for $k=1$ this is precisely the linear connection without torsion associated to a spray by Ambrose, Palais, and Singer in [1] .

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