CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

H. APPELGATE
M. TIERNEY

Model induced adjoint functors

Cahiers de topologie et géométrie différentielle catégoriques, tome 11, nº 1 (1969), p. 1-21
<http://www.numdam.org/item?id=CTGDC_1969__11_1_1_0>

CAHIERS DE TOPOLOGIE Vol. XI, 1
ET GEOMETRIE DIFFERENTIELLE

MODEL INDUCED ADJOINT FUNCTORS

by H. APPELGATE and M. TIERNEY (1)

0. Introduction.

This paper should be considered as an addendum to [1]. There we developed a general theory of categories with models and treated a number of examples in detail. Here we discuss certain adjoint pairs that arise from a factorization of the models. Namely, given a diagram

$$\begin{tikzcd} \underline{M}_1 \arrow[rr, "J"] \arrow[dr, "I_1"'] & & \underline{M}_2 \arrow[dl, "I_2"] \\ & \underline{A} & \end{tikzcd}$$

we obtain an induced functor $\overline{J}$

$$\begin{tikzcd} \underline{A}_{\mathbf{G}_1} \arrow[rr, shift left, "\overline{J}"] \arrow[dr, "L_1"'] & & \underline{A}_{\mathbf{G}_2} \arrow[ll, shift left, dashed, "\hat{J}"] \arrow[dl, "L_2"] \\ & \underline{A} & \end{tikzcd}$$

compatible with the underlying functors L_1 and L_2 ($\mathbf{G}_1$ and $\mathbf{G}_2$ are the model induced cotriples corresponding to I_1 and I_2). Assuming that the lifted cotriples $\overline{\mathbf{G}}_1$ and $\overline{\mathbf{G}}_2$ are trivial (i.e. $\bar{\varepsilon}_i : \overline{\mathbf{G}}_i \xrightarrow{\sim} \underline{A}_{\mathbf{G}_i}$ for $i = 1, 2$, - a condition holding often in practice), we show in § 2 how to construct an adjoint $\hat{J}$ to $\overline{J}$. We also sketch a proof of the fact that $\overline{J}$ is cotripleable, meaning that, if $\overline{\mathbf{J}}$ is the cotriple generated by the adjoint pair $\overline{J} \dashv \hat{J}$, then $\underline{A}_{\mathbf{G}_1}$ is canonically equivalent to $(\underline{A}_{\mathbf{G}_2})_{\overline{\mathbf{J}}}$. Both of these are general results about triples - i.e. depend only on the existence of certain limits in $\underline{A}_{\mathbf{G}_1}$ and not on the fact that $\mathbf{G}_1$ and $\mathbf{G}_2$ are model induced - and their proof,

(1) The second named author was partially supported by the NSF under Grant GP-8618.

in a different form, is due originally to Jon Beck (unpublished). We are mainly interested in applications to manifolds, and at the end of §2 obtain a general adjoint to a forgetful functor between manifolds that generalizes the orientation cover of a topological manifold.

In order to make the paper as self-contained as possible, we have included in §1 a brief resumé of the basic facts about categories with models, together with a quick discussion of several examples; details and proofs should be sought in [1].

1. Categories with models.

A *category with models* is a category $\underline{A}$, together with a functor $I : \underline{M} \to \underline{A}$ where $\underline{M}$, the *model category*, is assumed small. We refer to $\underline{A}$ as the *ambient* category.

Any category with models defines a *singular functor* $s : \underline{A} \to (\underline{M}^*, \mathcal{S})$ given by

$$sA(M) = \underline{A}(IM, A), \qquad sA(\alpha) = \underline{A}(I\alpha, A)$$

for $A \in \underline{A}$, and M, α in $\underline{M}$. For $f \in \underline{A}(A_1, A_2)$, $sf : sA_1 \to sA_2$ is the natural transformation induced by composition with f, and we will generally drop the $\underline{A}$ from the notation $\underline{A}(A_1, A_2)$ when there is no danger of confusion.

If $\underline{A}$ has small colimits (direct limits) we can construct a coadjoint to s as follows. For $F \in (\underline{M}^*, \mathcal{S})$ let $\underline{M}_F$ be the (small) category whose objects are pairs (M, x) consisting of a model M and a point $x \in FM$, and whose morphisms $\alpha : (M, x) \to (M', x')$ are morphisms $\alpha : M \to M'$ in $\underline{M}$ such that $F\alpha(x') = x$. Let $\partial_o : \underline{M}_F \to \underline{M}$ be the projection

$$\partial_o(M, x) = M, \qquad \partial_o(\alpha) = \alpha,$$

and let rF be a colimit of the composite $\underline{M}_F \xrightarrow{\partial_o} \underline{M} \xrightarrow{I} \underline{A}$, i.e. $rF = \lim I\partial_o$. (This construction is due originally to D. Kan [3]).

For each $(M, x) \in \underline{M}_F$, let $i(M, x)$ - or more simply i_x if M is understood - denote the canonical injection of a colimit,

$$i_x : I\partial_o(M, x) = IM \to rF.$$

Then if $\gamma : F \to F'$ is a natural transformation, $r\gamma : rF \to rF'$ is the unique morphism such that

$$\begin{tikzcd} IM \arrow[r, "i_x"] \arrow[dr, "i_\gamma M(x)"'] & rF \arrow[d, "r\gamma"] \\ & rF' \end{tikzcd}$$

commutes for each $(M, x) \in \underline{M}_F$.

To show that r is coadjoint to s we need natural transformations

$$\varepsilon : rs \to \underline{A}, \qquad \eta : (\underline{M}^*, \mathcal{S}) \to sr$$

(the *counit* and *unit* respectively) such that the usual identities hold. These are defined as follows : For each $A \in \underline{A}$, $\varepsilon A : rsA \to A$ is that morphism satisfying

$$\begin{tikzcd} IM \arrow[r, "i_\varphi"] \arrow[dr, "\varphi"'] & rsA \arrow[d, "\varepsilon A"] \\ & A \end{tikzcd}$$

for each $M \in \underline{M}$ and $\varphi \in sA(M) = (IM, A)$; and if $F \in (\underline{M}^*, \mathcal{S})$ then $\eta F : F \to srF$ is the natural transformation whose value at the model M is the function

$$x \to \eta F(M)[x] = i_x ,$$

where $x \in FM$. It is easy to verify that $(\varepsilon, \eta) : r \dashv s$. We call r the *realization* (see example 1 below).

$\underline{M}$*-OBJECTS AND ATLASES.*

Let $A \in \underline{A}$. A *pre-atlas* $\mathfrak{A}$ for A is any subfunctor of the singular functor $sA : \mathfrak{A} \to sA$. A pre-atlas is then simply a functorial collection of $\underline{A}$-morphisms of the form $\varphi : IM \to A$. Letting $j : \mathfrak{A} \to sA$ denote the inclusion, we define $e : r\mathfrak{A} \to A$ to be the composite $r\mathfrak{A} \xrightarrow[rj]{} rsA \xrightarrow[\varepsilon A]{} A$. Note that e is characterized by the diagrams

$$\begin{tikzcd} IM \arrow[r, "i_\varphi"] \arrow[dr, "\varphi"'] & r\mathfrak{A} \arrow[d, "e"] \\ & A \end{tikzcd}$$

where $\varphi \in \mathfrak{A}(M) \subset (IM, A)$.

An $\underline{M}$*-object with atlas* $\mathfrak{A}$ is an object A together with a pre-atlas $\mathfrak{A} \to sA$ for which $e : r\mathfrak{A} \xrightarrow{\approx} A$ is an isomorphism. Intuitively, an $\underline{M}$-object is one which is «locally like the models». e epic means A is «covered» by the $\varphi : IM \to A$ in $\mathfrak{A}$, and e monic says that the φ's in $\mathfrak{A}$ are «compatible on overlaps». The meaning of this last phrase will become clear in the examples below. There the ambient category $\underline{A}$ will be equipped with a good underlying set-functor so that terms like «covering» will have an obvious meaning. The fact that e is not only epic and monic but an isomorphism says that in addition the «$\underline{A}$-structure», e.g. the topology of A, is determined by the morphisms φ in $\mathfrak{A}$.

Note that, by definition, an $\underline{M}$-object A is a colimit of a functor with small domain that factors through the models :

$$A \approx r\mathfrak{A} = \varinjlim \{ \underline{M}_{\mathfrak{A}} \xrightarrow{\partial_0} \underline{M} \xrightarrow{I} \underline{A} \} ,$$

Let I' be any such functor, i.e. suppose we are given a diagram

$$\begin{tikzcd} \underline{N} \arrow[rr, "D"] \arrow[dr, "I'"'] & & \underline{M} \arrow[dl, "I"] \\ & \underline{A} & \end{tikzcd}$$

with $\underline{N}$ small. Then $A = \varinjlim I'$ is an $\underline{M}$-object. Here an atlas $\mathfrak{A} \to sA$ is *generated* by the canonical morphisms of a colimit $I(DN) \xrightarrow{k(N)} \varinjlim I'$. That is, $\mathfrak{A}(M)$ consists of all composites

$$IM \xrightarrow{I\alpha} I(DN) \xrightarrow{k(N)} \varinjlim I',$$

with $\alpha : M \to DN$ in $\underline{M}$. In particular, each realization rF is an $\underline{M}$-object with atlas $\mathfrak{A}$ given by $\mathfrak{A}(M) = \{ i_x \mid x \in FM \}$. Thus $\underline{M}$-objects are, disregarding any additional structure given by the choice of $\mathfrak{A}$, images of functors $F \in (\underline{M}^*, \mathcal{S})$ under r. An $\underline{M}$-object may have distinct defining atlases $\mathfrak{A} \to sA$. Later we shall distinguish these using a certain «coalgebra» structure canonically associated with $\mathfrak{A}$.

EXAMPLES.

1. *Simplicial Spaces.* Let $\underline{M} = \underline{\Delta}$, the simplicial category. The

objects of $\underline{\Delta}$ are sequences $[n] = (0, 1, \dots, n)$, $n \geq 0$, and the morphisms are monotone functions $\alpha : [m] \to [n]$. Let $\underline{Top}$ be the category of topological spaces and continuous functions, and let $I : \underline{\Delta} \to \underline{Top}$ be the functor

$$I[n] = \Delta_n, \quad I\alpha = \Delta_\alpha,$$

where Δ_n is the standard n-simplex and where Δ_α is the unique affine map determined by α on the vertices. $\underline{Top}$ has enough colimits, so we get a realization r coadjoint to s

$$(\underline{\Delta}^*, \mathcal{S}) \underset{s}{\overset{r}{\rightleftarrows}} \underline{Top} .$$

The functor category here is just the category of simplicial sets (semi-simplicial complexes). The singular functor s is the usual functor which assigns to each topological space X its singular complex sX, and the realization r is the geometric realization of Milnor [5]. Given a simplicial set $K \in (\underline{\Delta}^*, \mathcal{S})$, $rK = |K|$ is defined as follows : Let $r_o K$ be the space $r_o K = \bigoplus_{n \geq 0} K_n \times \Delta_n$, where K_n is $K[n]$ with the discrete topology.Let $\sim$ be the equivalence relation on $r_o K$ generated by the relation :

$$(k, t) \simeq (k', t')$$

if there is an α in $\underline{\Delta}$ such that $K_\alpha(k') = k$ and $\Delta_\alpha(t) = t'$. Then rK is the quotient space $rK = r_o K/\sim$. If we denote by $|k, t|$ the equivalence class containing $(k, t) \in K_n \times \Delta_n$, then the canonical injections of the colimit rK, $i_k : \Delta_n \to rK$, are simply $i_k(t) = |k, t|$. The topology of rK is : $U \subset rK$ is open if and only if $i_k^{-1} U$ is open in Δ_n for all $k \in K_n$, $n \geq 0$. (The i_k for non-degenerate k give rK the structure of a CW-complex [5]).

We remark that, for any category with models $I : \underline{M} \to \underline{A}$, if $\underline{A}$ has a colimit preserving underlying set functor $U : \underline{A} \to \mathcal{S}$, then the underlying set of rF, for any $F \in (\underline{M}^*, \mathcal{S})$, is given by exactly the same construction as above - see [1] for details.

Let $X \in \underline{Top}$ and let $\mathfrak{A} \to sX$ be a pre-atlas for X. It is not hard to see that $e : r\mathfrak{A} \to X$ is just the function $e\,|\varphi, t| = \varphi(t)$, for

$\varphi \in \mathfrak{A}[n] \subset (\Delta_n, X)$ and $t \in \Delta_n$. Now if $\mathfrak{A}$ is a $\underline{\Delta}$-atlas for X, let $\mathbf{A}$ be any generating set for $\mathfrak{A}$. One can show that

(i) $\mathbf{A}$ covers X. That is, given $x \in X$ there is $\varphi \in \mathbf{A}$ and $t \in \Delta_n$ such that $\varphi(t) = x$.

(ii) X has the weak topology with respect to the family $\mathbf{A}$.

(iii) $\mathbf{A}$ is compatible on overlaps. That is, given $\varphi_1, \varphi_2 \in \mathbf{A}$ and points t_1, t_2 such that $\varphi_1(t_1) = \varphi_2(t_2)$, there are α_1, α_2 in $\underline{\Delta}$ and a point t_o such that

$$\begin{tikzcd} \Delta_{n_o} \arrow[r, "\Delta_{\alpha_1}"] \arrow[d, "\Delta_{\alpha_2}"'] & \Delta_{n_1} \arrow[d, "\varphi_1"] \\ \Delta_{n_2} \arrow[r, "\varphi_2"'] & X \end{tikzcd}$$

commutes, and $\Delta_{\alpha_1}(t_o) = t_1$, $\Delta_{\alpha_2}(t_o) = t_2$.

Conversely, given a collection $\mathbf{A}$ of maps $\varphi : \Delta_n \to X$ satisfying *(i)-(iii)* above, X is a $\underline{\Delta}$-object with atlas $\mathfrak{A} \to sX$ generated by $\mathbf{A}$. Thus spaces X for which there exists a family $\mathbf{A}$ satisfying *(i)-(iii)* are exactly the $\underline{\Delta}$-objects in $\underline{Top}$. These include the classical geometric simplicial complexes, but are somewhat more general in that affine identifications are allowed on the boundaries of simplices.

2. *Manifolds.* In order to have a unified treatment of manifolds we will consider manifolds defined by a pseudogroup of transformations. A pseudogroup Γ is a set of homeomorphisms into, $g : U \to V$, where U and V are open sets in Euclidean space E^n satisfying the following axioms:

(1) If $g_1, g_2 \in \Gamma$ and the composite $g_1 g_2$ is defined, then $g_1 g_2 \in \Gamma$.

(2) If $g \in \Gamma$ then $g^{-1} \in \Gamma$ (where the domain of g^{-1} is the image of g).

(3) If $i : U \to V$ is an inclusion, then $i \in \Gamma$.

(4) Γ is local, i.e. if $g : U \to V$ is a homeomorphism into such that each $x \in U$ has a neighborhood U_x such that $g \mid U_x \in \Gamma$, then $g \in \Gamma$.

The examples we have in mind for Γ are :

a) all homeomorphisms into;

b) orientation preserving homeomorphisms defined on oriented open sets (where the orientation is induced by choosing an orientation for E^n);

c) diffeomorphisms into g, such that the Jacobian matrix of g lies in some fixed subgroup G of $GL(n, \mathbf{R})$;

d) real or complex $(n = 2m)$ analytic isomorphisms into;

e) PL-homeomorphisms into.

A pseudogroup Γ defines a model category $\underline{E}_\Gamma$ in an obvious way: the objects are all open subsets of E^n and the morphisms are the elements of Γ. Let $I : \underline{E}_\Gamma \to \underline{Top}$ be the inclusion functor, which we drop from the notation. As in the previous example, there is a realization r and we can characterize the $\underline{E}_\Gamma$-objects in $\underline{Top}$ in much the same way as before. They are topological spaces X for which there exists a family $\mathbf{A}$ of maps $\varphi : U \to X$, where $U \in \underline{E}_\Gamma$ such that :

(i) $\mathbf{A}$ covers X as in example 1;

(ii) Each $\varphi \in \mathbf{A}$ is open;

(iii) $\mathbf{A}$ is compatible on overlaps (also as in example 1).

We remark that, in order to make this example more like example 1, we could replace (ii) by the equivalent (in the presence of (i) and (iii)) condition :

(ii') X has the weak topology with respect to $\mathbf{A}$.

Examples of $\underline{E}_\Gamma$-objects include ordinary Γ-manifolds, manifolds with boundary, manifolds of dimension less than n, etc. Although $\underline{E}_\Gamma$-objects are more general than manifolds, they do share some of their properties. For example, in the differentiable case they have «tangent bundles».

3. *Sheaves.* Let X be a topological space and denote by $\underline{X}$ the category of open sets of X and inclusions. Let $(\underline{Top}, X)$ be the category of spaces over X. Here the objects are morphisms $p : Y \to X$ and the morphisms are triangles

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

Define

$$I : \underline{X} \to (\underline{Top}, X) \text{ by } IU = {}_{i}\!\!\begin{array}{c} U \\ \downarrow \\ X \end{array} ,$$

where $i : U \to X$ is the inclusion. The category $(\underline{X}^*, \mathcal{S})$ is just the category of presheaves (of sets) over X. The singular functor is the section functor and the realization r,

$$r : (\underline{X}^*, \mathcal{S}) \to (\underline{Top}, X) ,$$

is the étalé-space functor (the sheafification of a presheaf). In this example the $\underline{X}$-objects are exactly the sheaves in $(\underline{Top}, X)$, i.e. objects $p : Y \to X$ where p is a local homeomorphism.

REGULAR M-OBJECTS.

As noted in the examples above, the $\underline{M}$-objects often include more than the classically defined objects one expects. This indicates the need for some «regularity» condition in order to distinguish the classical objects from among all $\underline{M}$-objects.

A set $\mathbf{A}$ of morphisms $\varphi : IM \to A$ (A fixed) is called *regular* if given any $\varphi_1, \varphi_2 \in \mathbf{A}$ there exists a pullback diagram in $\underline{A}$ of the form

$$\begin{tikzcd} IM_o \arrow[r, "I\alpha_1"] \arrow[d, "I\alpha_2"'] & IM_1 \arrow[d, "\varphi_1"] \\ IM_2 \arrow[r, "\varphi_2"'] & A \end{tikzcd}$$

where α_1, α_2 are in $\underline{M}$. An $\underline{M}$-object A is called *regular* if its defining atlas $\mathfrak{A}$ is generated by a regular set $\mathbf{A}$.

Using the notion of regularity one can show, for example, that the regular $\underline{\Delta}$-objects are the simplicial complexes and the regular $\underline{E}_\Gamma$-objects are precisely the Γ-manifolds. Although successful in examples such as the above, this notion of regularity is too simple and ad hoc to cover all cases (see the example of G-bundles in [1]) and so should be considered a temporary definition.

MODEL INDUCED COTRIPLES.

We would like to discuss now the connection between $\underline{M}$-objects and cotriples. First we recall some standard facts about cotriples [2].

A *cotriple* $\mathbf{G} = (G, \varepsilon, \delta)$ in a category $\underline{A}$ consists of a functor $G : \underline{A} \to \underline{A}$ and natural transformations $\varepsilon : G \to \underline{A}$, $\delta : G \to G^2$ such that

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[dr, "G"'] & G^2 \arrow[d, shift right, "\varepsilon G"'] \arrow[d, shift left, "G\varepsilon"] \\ & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & G^2 \arrow[d, "G\delta"] \\ G^2 \arrow[r, "\delta G"'] & G^3 \end{tikzcd}$$

commute. An adjoint pair

$$\begin{tikzcd} \underline{B} \arrow[r, shift left, "F"] & \underline{A} \arrow[l, shift left, "U"] \end{tikzcd}, \quad (\varepsilon, \eta) : F \dashv U$$

generates the cotriple $\mathbf{G} = (FU, \varepsilon, F\eta U)$ in $\underline{A}$. Conversely, any cotriple $\mathbf{G}$ in $\underline{A}$ is generated by an adjoint pair $\begin{tikzcd} \underline{A}_{\mathbf{G}} \arrow[r, shift left, "L"] & \underline{A} \arrow[l, shift left, "R"] \end{tikzcd}$, where $\underline{A}_{\mathbf{G}}$ is the category of $\mathbf{G}$-*coalgebras*. The objects of $\underline{A}_{\mathbf{G}}$ are pairs (A, θ), where $A \in \underline{A}$ and $\theta : A \to GA$ satisfies

$$\begin{tikzcd} A \arrow[r, "\theta"] \arrow[dr, "A"'] & GA \arrow[d, "\varepsilon A"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "\theta"] \arrow[d, "\theta"'] & GA \arrow[d, "\delta A"] \\ GA \arrow[r, "G\theta"'] & G^2 A \end{tikzcd} .$$

A morphism $f : (A, \theta) \to (A', \theta')$ in $\underline{A}_{\mathbf{G}}$ is a morphism $f : A \to A'$ in $\underline{A}$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\theta"'] & A' \arrow[d, "\theta'"] \\ GA \arrow[r, "Gf"] & GA' \end{tikzcd}$$

commutes. The functors L and R are given by

$$\begin{aligned} L(A, \theta) &= A, \\ L(f) &= f, \\ R(A) &= (GA, \delta A), \\ R(f) &= Gf. \end{aligned}$$

One checks easily that $L \dashv R$ and that this adjointness generates $\mathbf{G}$.

Now let $I : \underline{M} \to \underline{A}$ be a category with models, where $\underline{A}$ has small colimits. The adjoint pair

$$(\underline{M}^*, \mathcal{S}) \underset{s}{\overset{r}{\rightleftarrows}} \underline{A}$$

generates the *model induced cotriple*

$$\mathbf{G} = (rs = G,\ \varepsilon,\ \delta = r\eta s).$$

The realization r factors (since $(\varepsilon, \eta) : r \dashv s$ generates $\mathbf{G}$) through the coalgebras as

$$\begin{array}{ccc} & & \underline{A}_{\mathbf{G}} \\ & \overset{\bar{r}}{\nearrow} & \downarrow L \\ (\underline{M}^*, \mathcal{S}) & \xrightarrow[r]{} & A \end{array}$$

where the *lifted realization* $\bar{r}$ is given by

$$\bar{r}F = (rF, r\eta F), \qquad \bar{r}\varphi = r\varphi .$$

The basic functor $I : \underline{M} \to \underline{A}$ also factors through the coalgebras,

$$\begin{array}{ccc} & & \underline{A}_{\mathbf{G}} \\ & \overset{\bar{I}}{\nearrow} & \downarrow L \\ \underline{M} & \xrightarrow[I]{} & \underline{A} \end{array}$$

by defining $\bar{I}(M) = (IM, \theta_M)$, where $\theta_M : IM \to GIM$ is the canonical morphism $i(M, IM)$. Using $\bar{I}$ we obtain the *lifted singular functor*

$$\bar{s} : \underline{A}_{\mathbf{G}} \to (\underline{M}^*, \mathcal{S})$$

exactly as we obtained s from I.

For each $(\underline{A}, \theta) \in \underline{A}_{\mathbf{G}}$ there is a natural inclusion

$$j = j(A, \theta) : \bar{s}(A, \theta) \to sA,$$

and using the easily proven fact that

$$\bar{s}(A, \theta) \xrightarrow{j} sA \underset{\eta sA}{\overset{s\theta}{\rightrightarrows}} srsA$$

is an equalizer diagram, one can prove that $\bar{r} \dashv \bar{s}$. In fact, the counit $\bar{\varepsilon}(A, \theta) : \bar{r}\bar{s}(A, \theta) \to (A, \theta)$ is just the composite

$$r\bar{s}(A, \theta) \xrightarrow{rj} rsA \xrightarrow{\varepsilon A} A,$$

and the unit $\bar{\eta} F : F \to \bar{s}\,\bar{r} F$ is determined from the above equalizer diagram with $(A, \theta) = \bar{r} F$.

$$\begin{tikzcd} \bar{s}\,\bar{r} F \arrow[r, "j"] & s r F \arrow[r, shift left, "s r \eta F"] \arrow[r, shift right, "\eta s r F"'] \arrow[d, "\eta F"] & s r s r F \\ & F \arrow[ul, dashed, "\bar{\eta} F"] & \end{tikzcd}$$

Let A be an $\underline{M}$-object with atlas $j : \mathfrak{A} \to s A$. There is a uniquely determined coalgebra structure $\theta : A \to G A$, depending only on j and A, such that $e : \bar{r} \mathfrak{A} \underset{\sim}{\to} (A, \theta)$ is an isomorphism of coalgebras. Looking at the diagram

$$\begin{tikzcd} r \mathfrak{A} \arrow[r, "r j"] \arrow[d, "r \eta \mathfrak{A}"'] & G A \arrow[r, "\varepsilon A"] \arrow[d, "\delta A"] & A \arrow[d, "\theta"] \\ G r \mathfrak{A} \arrow[r, "G r j"'] & G^2 A \arrow[r, "G \varepsilon A"'] & G A , \end{tikzcd}$$

(where the right hand square is *not* commutative) and using the identity $G \varepsilon A . \delta A = G A$ plus the fact that $e = \varepsilon A . r j$ is an isomorphism, we obtain $\theta = r j e^{-1}$. Furthermore, θ is such that j always factors through the lifted singular functor,

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "j"] \arrow[dr, "\bar{j}"'] & s A \arrow[d, "j(A, \theta)"] \\ & \bar{s}(A, \theta) . \end{tikzcd}$$

Given any coalgebra (A, θ) we can prove that

$$\bar{\varepsilon}(A, \theta) : \bar{r}\,\bar{s}(A, \theta) \underset{\sim}{\to} (A, \theta)$$

is an isomorphism if, and only if,

$$r \bar{s}(A, \theta) \xrightarrow{r j(A, \theta)} G A \underset{\delta A}{\overset{G \theta}{\rightrightarrows}} G^2 A$$

is an equalizer diagram in $\underline{A}$. If A is an $\underline{M}$-object with atlas $\mathfrak{A} \underset{j}{\to} s A$ and coalgebra structure θ determined as above, then the diagram

$$\begin{tikzcd}
r\mathfrak{A} \arrow[d, "\bar{r}j"'] & & A \arrow[ll, "e^{-1}"'] \arrow[d, "\theta"] & \\
r\bar{s}(A,\theta) \arrow[rr, "rj(A,\theta)"'] & & GA \arrow[r, shift left, "G\theta"] \arrow[r, shift right, "\delta A"'] & G^2A
\end{tikzcd}$$

proves that the bottom line is an equalizer diagram if $rj(A,\theta)$ is a monomorphism. Assuming then that rj is a monomorphism for all inclusions of the type $j : \mathfrak{A} \to sA$, we get the result that $\bar{\varepsilon}(A,\theta)$ is an isomorphism for all coalgebras arising in the above way from $\underline{M}$-objects A. The converse is of course true. If (A,θ) is a coalgebra such that $\bar{\varepsilon}(A,\theta)$ is an isomorphism, then A is an $\underline{M}$-object with atlas

$$j(A,\theta) : \bar{s}(A,\theta) \to sA, \text{ and } \theta = rj(A,\theta)e^{-1}.$$

If $\bar{\varepsilon}(A,\theta)$ is an isomorphism for all coalgebras (A,θ), then the $\underline{M}$-objects of $\underline{A}$ are exactly the objects of $\underline{A}$ that admit a **G**-coalgebra structure. Note that $\bar{\varepsilon} : \bar{r}\,\bar{s} \xrightarrow{\sim} \underline{A}_{\mathbf{G}}$ is an interesting condition not only because it gives a convenient interpretation of $\underline{M}$-objects, but also because it makes $\underline{A}_{\mathbf{G}}$ a co-reflective subcategory of the functor category $(\underline{M}^*, \mathcal{S})$. In particular then $\underline{A}_{\mathbf{G}}$ has all small limits and colimits. In the case that $\underline{A}$ is equipped with a good underlying set functor $U : \underline{A} \to \mathcal{S}$, we can give a simple criterion for $rj(A,\theta)$ to be a monomorphism that depends only on $I : \underline{M} \to \underline{A}$ (and $U : \underline{A} \to \mathcal{S}$).

To complete the picture one can prove that, when $\bar{\varepsilon}$ is an isomorphism, then $\bar{\eta}$ is an isomorphism if, and only if, r reflects isomorphisms, or equivalently, r is faithful. In examples 1, 2, and 3 above, we have $\bar{\varepsilon}$ an isomorphism, and in example 1 we have the full equivalence.

2. Model induced adjoints.

As an illustration of the usefulness of the cotriple point of view, we shall prove a result on the existence of certain adjoint functors, which we can then apply to the example of manifolds.

So let $\underline{A}$ be a category with models in two ways connected by a functor, i.e. suppose we are given a diagram

$$\begin{tikzcd}
\underline{M}_1 \arrow[d, "j"'] \arrow[dr, "I_1"] & \\
\underline{M}_2 \arrow[r, "I_2"'] & \underline{A}
\end{tikzcd}$$

where $\underline{M}_1$ and $\underline{M}_2$ are small. Then we obtain two diagrams

$$\begin{tikzcd}
 & \underline{A}_{\mathbf{G}_\nu} \arrow[d, shift left, "L_\nu"] \arrow[r, shift right, "\bar{s}_\nu"'] & (\underline{M}^*_\nu, \mathcal{S}) \arrow[l, shift right, "\bar{r}_\nu"'] \arrow[dl, shift left, "s_\nu"] \\
\underline{M}_\nu \arrow[ur, "\bar{I}_\nu"] \arrow[r, "I_\nu"'] & \underline{A} \arrow[u, shift left, "R_\nu"] \arrow[ur, shift left, "r_\nu"] &
\end{tikzcd}$$

for $\nu = 1, 2$. Also, $J^* : \underline{M}^*_1 \to \underline{M}^*_2$ induces a functor $J^\bullet : (\underline{M}^*_2, \mathcal{S}) \to (M^*_1, \mathcal{S})$ defined by $J^\bullet F = F . J^*$. Moreover, J induces a natural transformation $j : G_1 \to G_2$ defined by requiring for each $\underline{A}$-morphism $\varphi : I_1 M \to A$, with $M \in \underline{M}_1$ and $A \in \underline{A}$, that the diagram

$$\begin{tikzcd}
I_1 M \arrow[r, phantom, "="] \arrow[d, "i_\varphi"'] & I_2 JM \arrow[d, "i_\varphi"] \\
r_1 s_1 A \arrow[r, "jA"] & r_2 s_2 A
\end{tikzcd}$$

commutes. j is obviously natural, and it is easy to see in fact that it is a morphism of cotriples $j : \mathbf{G}_1 \to \mathbf{G}_2$, meaning that the diagrams

$$\begin{tikzcd}
G_1 \arrow[rr, "j"] \arrow[dr, "\varepsilon_1"'] & & G_2 \arrow[dl, "\varepsilon_2"] \\
 & \underline{A} &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G_1 \arrow[rrr, "j"] \arrow[d, "\delta_1"'] & & & G_2 \arrow[d, "\delta_2"] \\
G_1 G_1 \arrow[r, "jG_1"] \arrow[dr, "G_1 j"'] & G_2 G_1 \arrow[rr, "G_2 j"] & & G_2 G_2 \\
 & G_1 G_2 \arrow[urr, "j G_2"'] & &
\end{tikzcd}$$

commute. Because of this, j induces a functor $\bar{J} : \underline{A}_{\mathbf{G}_1} \to \underline{A}_{\mathbf{G}_2}$ given by

$$\bar{J}(A, \theta_1) = (A, jA . \theta_1), \qquad \bar{J} f = f,$$

where (A, θ_1) and f are in $\underline{A}_{\mathbf{G}_1}$. Here one needs only check that, if $\theta_1 : A \to G_1 A$ is a $\mathbf{G}_1$-coalgebra structure for A, then $A \xrightarrow{\theta_1} G_1 \xrightarrow{jA} G_2 A$ is a $\mathbf{G}_2$-coalgebra structure, and this follows immediately from the above two diagrams.

Assuming that, in the adjoint pair

$$\underline{A}_{\mathbf{G}_1} \underset{\bar{r}_1}{\overset{\bar{s}_1}{\rightleftarrows}} (\underline{M}_1^*, \mathcal{S}),$$

we have $\bar{\varepsilon}_1 : \bar{r}_1 \bar{s}_1 \underset{\sim}{\to} \underline{A}_{\mathbf{G}_1}$, we indicate now how to construct an adjoint $\hat{J} : \underline{A}_{\mathbf{G}_2} \to \underline{A}_{\mathbf{G}_1}$ for $\bar{J}$. In this context there are several ways to do this. However, to highlight the cotriple methods, we use a technique due originally to Jon Beck, which applies to any functor between categories of coalgebras induced by a morphism of cotriples. A somewhat different proof may also be found in Linton [4]. In what follows we will often confuse a coalgebra with its underlying $\underline{A}$-object, omitting the $\mathbf{G}$-structure to simplify notation.

Since we have assumed that $\bar{\varepsilon}_1 : \bar{r}_1 \bar{s}_1 \underset{\sim}{\to} \underline{A}_{\mathbf{G}_1}$, it follows that the category $\underline{A}_{\mathbf{G}_1}$ has all small limits and colimits (see Mitchell [6]). Define $\hat{J}$ by requiring, for $(A, \theta_2) \in \underline{A}_{\mathbf{G}_2}$, that the diagram

$$\begin{tikzcd}
\hat{J}(A, \theta_2) \arrow[r] & G_1 A \arrow[rr, "G_1 \theta_2"] \arrow[dr, "\delta_1 A"'] & & G_1 G_2 A \arrow[dl, "G_1 j A"] \\
& & G_1 G_1 A &
\end{tikzcd}$$

be an equalizer in the category $\underline{A}_{\mathbf{G}_1}$. (In the general case it must be assumed that this equalizer exists). The behavior of $\hat{J}$ on $\mathbf{G}_2$-coalgebra morphisms is clear.

The unit and counit

$$\alpha : \underline{A}_{\mathbf{G}_1} \to \hat{J}\,\bar{J} \quad \text{and} \quad \beta : \bar{J}\,\hat{J} \to \underline{A}_{\mathbf{G}_2}$$

are defined by the diagrams

$$\begin{tikzcd}
& A \arrow[dl, dashed, "\alpha(A, \theta_1)"'] \arrow[d] & & \\
\hat{J}\,\bar{J}(A, \theta_1) \arrow[r] & G_1 A \arrow[r, shift left, "G_1 \theta_1"] \arrow[r, shift right, "\delta_1 A"'] & G_1 G_1 A \arrow[r, "G_1 j A"] & G_1 G_2 A
\end{tikzcd}$$

and

$$\begin{tikzcd}
\hat{J}(A,\theta_2) \arrow[r] \arrow[dd, dashed, "\beta(A,\theta_2)"'] & G_1A \arrow[rr, "G_1\theta_2"] \arrow[dd, "jA"'] \arrow[dr, "\delta_1 A"] & & G_1G_2A \arrow[dl, "G_1 jA"'] \arrow[dd, "jG_2A"] \\
& & G_1G_1A & \\
A \arrow[r, "\theta_2"] & G_2A \arrow[rr, shift left, "G_2\theta_2"] \arrow[rr, shift right, "\delta_2 A"'] & & G_2G_2A
\end{tikzcd}$$

(In defining β one must check that the bottom line is an equalizer in $\underline{A}_{\mathbf{G}_2}$, and that

$$\hat{J}(A,\theta_2) \to G_1A \xrightarrow{jA} G_2A$$

is a morphism of $\mathbf{G}_2$-coalgebras from $\bar{J}\hat{J}(A,\theta_2)$ to G_2A. But the first statement is a standard fact about coalgebras, and the second is easy, so we leave it to the reader).

To show $(\beta,\alpha) : \bar{J} \dashv \hat{J}$, we verify that both composites

$$\bar{J} \xrightarrow{\bar{J}\alpha} \bar{J}\hat{J}\bar{J} \xrightarrow{\beta\bar{J}} \bar{J} \quad \text{and} \quad \hat{J} \xrightarrow{\alpha\hat{J}} \hat{J}\bar{J}\hat{J} \xrightarrow{\hat{J}\beta} \hat{J}$$

are the respective identities. For the first, we have, by definition,

$$\begin{tikzcd}
& A \arrow[dl, "\alpha(A,\theta_1)"'] \arrow[dr, "\theta_1"] & \\
\hat{J}\bar{J}(A,\theta_1) \arrow[rr] \arrow[d, "\beta\bar{J}(A,\theta_1)"'] & & G_1A \arrow[d, "jA"] \\
A \arrow[r, "\theta_1"] & G_1A \arrow[r, "jA"] & G_2A ,
\end{tikzcd}$$

and, since $jA.\theta_1$ is monic, we are done. For the second, let us denote by $\nu : \hat{J}(A,\theta_2) \to G_1A$ the unnamed map in the definition of $\hat{J}$, and by

$$\hat{\theta}_1 : \hat{J}(A,\theta_2) \to G_1\hat{J}(A,\theta_2)$$

the $\mathbf{G}_1$-coalgebra structure on $\hat{J}(A,\theta_2)$. Then, again by definition, we have the diagram

$$\begin{tikzcd}
 & \hat{J}(A,\theta_2) \arrow[dl, "\alpha\hat{J}(A,\theta_2)"'] \arrow[dr, "\hat{\theta}_1"] & \\
\hat{J}\,\overline{J}\,\hat{J}(A,\theta_2) \arrow[rr] \arrow[d, "\hat{J}\beta(A,\theta_2)"'] & & G_1\hat{J}(A,\theta_2) \arrow[d, "G_1\beta(A,\theta_2)"] \\
\hat{J}(A,\theta_2) \arrow[rr, "\nu"] & & G_1A
\end{tikzcd}$$

and we are done if we show that $\nu = G_1\beta(A,\theta_2).\hat{\theta}_1$. For this, consider the diagram

$$\begin{tikzcd}
\hat{J}(A,\theta_2) \arrow[r, "\nu"] \arrow[d, "\hat{\theta}_1"'] & A \arrow[d, "\delta_1 A"] \\
G_1\hat{J}(A,\theta_2) \arrow[r, "G_1\nu"] \arrow[d, "G_1\beta(A,\theta_2)"'] & G_1G_1A \arrow[d, "G_1jA"] \\
G_1A \arrow[r, "G_1\theta_2"] & G_1G_2A\ .
\end{tikzcd}$$

The top square commutes since ν is a morphism of $\mathbf{G}_1$-coalgebras, and the bottom square by definition of β. Since $G_1jA.\delta_1A.\nu = G_1\theta_2.\nu$ we obtain

$$G_1\theta_2.\nu = G_1\theta_2.(G_1\beta(A,\theta_2).\hat{\theta}_1),$$

and composing on the left with $G_1\varepsilon A$ gives the result.

Having shown that $\hat{J}$ exists, we now derive a formula for it as follows. First notice that

$$\begin{tikzcd}
\underline{A}_{\mathbf{G}_1} \arrow[r, "\overline{J}"] & \underline{A}_{\mathbf{G}_2} \\
\underline{M}_1 \arrow[r, "J"] \arrow[u, "\overline{I}_1"] & \underline{M}_2 \arrow[u, "\overline{I}_2"']
\end{tikzcd}$$

commutes, since clearly

$$\begin{tikzcd}
I_1M \arrow[r, "\theta_M"] \arrow[dr, "\theta_{JM}"'] & G_1I_1M \arrow[d, "jI_1M"] \\
 & G_2I_1M
\end{tikzcd}$$

does for each $M \in \underline{M}_1$. Now let $(A, \theta_2) \in \underline{A}_{\mathbf{G}_2}$. Then

$$\bar{s}_1 \hat{J}(A, \theta_2)(M) = \underline{A}_{\mathbf{G}_1}(\bar{I}_1 M, \hat{J}(A, \theta_2)) \approx \underline{A}_{\mathbf{G}_2}(\bar{J}\bar{I}_1 M, (A, \theta_2))$$
$$= \underline{A}_{\mathbf{G}_2}(\bar{I}_2 JM, (A, \theta_2)) = J^\bullet \bar{s}_2(A, \theta_2)(M)$$

and thus $\bar{s}_1 \hat{J} \approx J^\bullet \bar{s}_2$, since the equivalence above is obviously natural in both M and (A, θ_2). Applying $\bar{r}_1$, and using the natural equivalence $\bar{\varepsilon}_1 : \bar{r}_1 \bar{s}_1 \underset{\sim}{\to} \underline{A}_{\mathbf{G}_1}$, we obtain the formula : $\hat{J} \approx \bar{r}_1 J^\bullet \bar{s}_2$. Alternatively, it is possible to show that $\bar{r}_1 J^\bullet \bar{s}_2(A, \theta_2)$ is an equalizer for the diagram defining $\hat{J}(A, \theta_2)$, for each $(A, \theta_2) \in \underline{A}_{\mathbf{G}_2}$, or one can simply verify directly that $\bar{r}_1 J^\bullet \bar{s}_2$ is adjoint to $\bar{J}$. In any case, it is useful to have the formula, since it gives an idea of how $\hat{J}$ works. Namely, take an $\underline{M}_2$-object A. If we assume also $\bar{\varepsilon}_2 : \bar{r}_2 \bar{s}_2 \underset{\sim}{\to} \underline{A}_{\mathbf{G}_2}$, then there is a canonical $\mathbf{G}_2$-coalgebra structure $\theta_2 : A \to G_2 A$ associated to A, and $\bar{s}_2(A, \theta_2)$ is a maximal $\underline{M}_2$-atlas for A (see [1]). Then to construct $\hat{J}(A, \theta_2)$, the formula says to reglue those overlaps coming from $\underline{M}_1$, and «stack» the rest. (To make sense of this remark, the reader should derive a formula for β using $\bar{r}_1 J^\bullet \bar{s}_2$ as $\hat{J}$).

To complete the above, recall that $J^\bullet : (\underline{M}_2^*, \mathcal{S}) \to (\underline{M}_1^*, \mathcal{S})$ has a coadjoint $\check{J}^\bullet : (\underline{M}_1^*, \mathcal{S}) \to (\underline{M}_2^*, \mathcal{S})$, and consider the diagram

$$\begin{tikzcd}
\underline{A}_{\mathbf{G}_1} \arrow[r, shift left, "\bar{s}_1"] \arrow[d, shift left, "\bar{J}"] & (\underline{M}_1^*, \mathcal{S}) \arrow[l, shift left, "\bar{r}_1"] \arrow[d, shift right, "\check{J}^\bullet"'] \\
\underline{A}_{\mathbf{G}_2} \arrow[u, shift left, "\hat{J}"] \arrow[r, shift right, "\bar{r}_2"'] & (\underline{M}_2^*, \mathcal{S}) \arrow[l, shift right, "\bar{s}_2"'] \arrow[u, shift right, "J^\bullet"']
\end{tikzcd}$$

Now $J^\bullet \bar{s}_2 \approx \bar{s}_1 \hat{J}$, so by the uniqueness of adjoints it follows that

$$\bar{r}_2 \check{J}^\bullet \approx \bar{J}\, \bar{r}_1, \text{ and thus } \bar{r}_2 \check{J}^\bullet \bar{s}_1 \approx \bar{J}.$$

Hence, considering $\underline{A}_{\mathbf{G}_1}$ and $\underline{A}_{\mathbf{G}_2}$ as coreflective subcategories of $(\underline{M}_1^*, \mathcal{S})$ and $(\underline{M}_2^*, \mathcal{S})$ respectively, one sees that the adjoint pair $\bar{J} \dashv \hat{J}$ is determined by the Kan adjointness $\check{J}^\bullet \dashv J^\bullet$.

An important property of the functor $\bar{J}$ is its cotripleability. That is, let $\mathbf{J} = (\bar{J}\hat{J}, \beta, \bar{J}\alpha\hat{J})$ be the cotriple on $\underline{A}_{\mathbf{G}_2}$ generated by the adjoint pair $(\beta, \alpha) : \bar{J} \dashv \hat{J}$. In the diagram

$$\begin{tikzcd}
(\underline{A}_{\mathbf{G}_2})_{\mathbf{J}} \arrow[d, shift left, "L"] & & \underline{A}_{\mathbf{G}_1} \arrow[ll, "\varphi"'] \arrow[dll, shift left, "\hat{J}"] \\
\underline{A}_{\mathbf{G}_2} \arrow[u, shift left, "R"] \arrow[urr, shift left, "\bar{J}"] & &
\end{tikzcd}$$

the Eilenberg-Moore comparison φ is defined by

$$\varphi(A, \theta_1) = (\bar{J}(A, \theta_1), \bar{J}\alpha(A, \theta_1))$$

for $(A, \theta_1) \in \underline{A}_{\mathbf{G}_1}$, with a corresponding effect on morphisms. What is asserted is that φ is an equivalence of categories. This observation is also due to Beck, and again holds for any adjoint pair induced by a morphism of cotriples. We sketch a short direct proof, since none is available in the literature.

First of all, φ is clearly faithful since $\bar{J}$ is. Now suppose

$$f : \varphi(A, \theta_1) \to \varphi(A', \theta_1')$$

is a morphism of $\mathbf{J}$-coalgebras, i.e.

$$f : \bar{J}(A, \theta_1) \to \bar{J}(A', \theta_1')$$

is a morphism of $\mathbf{G}_2$-coalgebras such that

$$\begin{tikzcd}
\bar{J}(A, \theta_1) \arrow[r, "f"] \arrow[d, "\bar{J}\alpha(A, \theta_1)"'] & \bar{J}(A', \theta_1') \arrow[d, "\bar{J}\alpha(A', \theta_1')"] \\
\bar{J}\hat{J}\bar{J}(A, \theta_1) \arrow[r, "\bar{J}\hat{J}f"] & \bar{J}\hat{J}\bar{J}(A', \theta_1')
\end{tikzcd}$$

commutes. But then consider the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha(A, \theta_1)"] \arrow[dd, bend right=60, "\theta_1"'] & A' \arrow[d, "\alpha(A', \theta_1')"'] \arrow[dd, bend left=60, "\theta_1'"] \\
\hat{J}\bar{J}(A, \theta_1) \arrow[r, "\hat{J}f"] \arrow[d] & \hat{J}\bar{J}(A', \theta_1') \arrow[d] \\
G_1 A \arrow[r, "G_1 f"] & G_1 A'
\end{tikzcd}$$

The outside triangles commute by definition of α, the top square by the previous diagram, and the bottom since $f : \overline{J}(A, \theta_1) \to \overline{J}(A', \theta_1')$ is a morphism of $\mathbf{G}_2$-coalgebras. Hence f is a morphism of $\mathbf{G}_1$-coalgebras and φ is full. Let

$$((A, \theta_2), \tau) \in (\underline{A}_{\mathbf{G}_2})_{\mathbf{J}}, \text{ where } \tau : (A, \theta_2) \to \overline{J}\,\hat{J}(A, \theta_2)$$

is a $\mathbf{J}$-coalgebra structure on (A, θ_2), and define $\theta_1 : A \to G_1A$ to be the composite

$$A \xrightarrow{\tau} \hat{J}(A, \theta_2) \xrightarrow{\nu} G_1A.$$

From the diagram

$$\begin{tikzcd}
A \arrow[r, "\tau"] \arrow[ddr, "A"'] & \hat{J}(A, \theta_2) \arrow[r, "\nu"] \arrow[dd, "\beta(A, \theta_2)"] & G_1A \arrow[dd, "jA"] \arrow[dr, "\varepsilon_1 A"] & \\
 & & & A \\
 & A \arrow[r, "\theta_2"'] & G_2A \arrow[ur, "\varepsilon_2 A"'] &
\end{tikzcd}$$

we obtain $\varepsilon_1 A \, . \, \theta_1 = A$, and from

$$\begin{tikzcd}
A \arrow[r, "\tau"] \arrow[dd, "\tau"'] & \hat{J}(A, \theta_2) \arrow[r, "\nu"] \arrow[d, "\hat{J}\tau"] & G_1A \arrow[dd, "G_1\tau"] \\
 & \hat{J}\,\overline{J}\,\hat{J}(A, \theta_2) \arrow[dr, "\nu"] & \\
\hat{J}(A, \theta_2) \arrow[ur, "\alpha\hat{J}(A, \theta_2)"] \arrow[rr, "\hat{\theta}_1"'] \arrow[d, "\nu"'] & & G_1\hat{J}(A, \theta_2) \arrow[d, "G_1\nu"] \\
G_1A \arrow[rr, "\delta_1 A"'] & & G_1G_1A
\end{tikzcd}$$

we see that

$$G_1\theta_1 . \theta_1 = \delta_1 A \, . \, \theta_1, \text{ so } (A, \theta_1) \in \underline{A}_{\mathbf{G}_1}.$$

Omitting the triangle involving ε_1 and ε_2 from the first diagram shows that $\overline{J}(A, \theta_1) = (A, \theta_2)$, and considering only the top two squares of the second diagram gives the result that τ is a morphism of $\mathbf{G}_1$-coalgebras (with respect to the $\mathbf{G}_1$-structure θ_1 on A). But now $\alpha(A, \theta_1)$ is determined as a morphism of $\mathbf{G}_1$-coalgebras (since ν is an equalizer) by requiring that

$$\begin{tikzcd} A \arrow[r, "\alpha(A,\theta_1)"] \arrow[dr, "\theta_1"'] & \hat{J}\bar{J}(A,\theta_1) \arrow[d] \\ & G_1 A \end{tikzcd}$$

commutes. On the other hand, τ is also a morphism of $\mathbf{G}_1$-coalgebras that makes this diagram commute. Hence $\alpha(A,\theta_1) = \tau$ and

$$\varphi(A,\theta_1) = (\bar{J}(A,\theta_1), \bar{J}\alpha(A,\theta_1)) = ((A,\theta_2),\tau),$$

making φ an equivalence.

The reader will be able to supply many examples of such adjoint pairs, but we shall consider only one here. Namely, let us consider the case of manifolds. If $\Gamma_1 \subset \Gamma_2$ is a pair of pseudogroups, then the inclusion induces the obvious diagram

$$\begin{tikzcd} \underline{E}_{\Gamma_1} \arrow[d, "J"'] \arrow[dr, "I_1"] & \\ \underline{E}_{\Gamma_2} \arrow[r, "I_2"'] & \underline{Top} \end{tikzcd}$$

and thus, since both $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ are equivalences, we have an adjoint pair

$$\underline{Top}_{\mathbf{G}_1} \underset{\hat{J}}{\overset{\bar{J}}{\rightleftarrows}} \underline{Top}_{\mathbf{G}_2} \quad \text{with } (\beta,\alpha) : \bar{J} \dashv \hat{J}.$$

One verifies immediately that $\bar{J}$ is the forgetful functor from $\underline{E}_{\Gamma_1}$-objects to $\underline{E}_{\Gamma_2}$-objects, and that $\hat{J}$ takes genuine Γ_2-manifolds into genuine Γ_1-manifolds, although it will in general destroy the property of being Hausdorff.

A well-known example of this adjoint pair arises by choosing an orientation for E^n and giving all open subsets the induced orientation. Then take

$$\Gamma_1 = \{\text{ orientation preserving homeomorphisms into}\},$$

$$\Gamma_2 = \{\text{ all homeomorphisms into}\}.$$

In this case, if X is a genuine topological manifold, then $\hat{J}X$ is the orientation cover of X. This follows by uniqueness of adjoints, since it is not hard to verify that the orientation cover is adjoint to the forgetful functor from oriented manifolds. Here one can see that the morphisms of Γ-manifolds that arise as coalgebra morphisms (these are continuous maps that are locally like elements of Γ') are useful for some purposes. Namely, the class of local homeomorphisms seems to be about the largest natural class of maps on which the orientation cover is even a functor.

Cotripleability of $\bar{J}$ says that, to put a Γ_1-structure on a Γ_2-manifold, it is necessary and sufficient that there exists a section

$$\tau : X \to \bar{J}\hat{J}X$$

to $\beta X : \bar{J}\hat{J}X \to X$ which is co-associative - i.e. is such that

$$\begin{tikzcd}
X \arrow[r, "\tau"] \arrow[d, "\tau"'] & \bar{J}\hat{J}X \arrow[d, "\bar{J}\hat{J}\tau"] \\
\bar{J}\hat{J}X \arrow[r, "\bar{J}\alpha\hat{J}"] & (\bar{J}\hat{J})^2 X
\end{tikzcd}$$

commutes. This is well-known for the orientation cover, where the coassociativity condition is vacuous. In the future we hope to develop an obstruction theory, expressible in cotriple terms, for the existence of such sections.

References.

[1] H. APPELGATE and M. TIERNEY, *Categories with models*, to appear in Springer Lecture Notes, Vol. 80.

[2] S. EILENBERG and J. MOORE, Adjoint functors and triples, *Ill. J. Math.* 9 (1965), 381-398.

[3] D. KAN, Functors involving *css*-complexes, *Trans. Amer. Math. Soc.* 87 (1958), 330-346.

[4] F.E.J. LINTON, *Coequalizers in categories of algebras*, to appear in Springer Lecture Notes, Vol. 80.

[5] J. MILNOR, The geometric realization of a semi-simplicial complex, *Ann. of Math.* 65 (1957), 357-362.

[6] B. MITCHELL, *Theory of Categories*, Academic Press, N.Y. 1965.

M. TIERNEY

Department of Mathematics,

Rutgers University,

New Brunswick, N. J. 08903 - (U. S. A)