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ON THE HIGHER ORDER CONNECTIONS

by Bohumil CENKL

The non-holonomic connection of order r on a principal fibre bundle H which is geometrically defined in this paper is considered in relation to that one studied in [5] . A given connection gives rise to $r-1$ tensor forms on the considered principal bundle H. Some characterisation of these forms is given.

After this paper has been finished, there appeared quite a few papers on differential geometry of higher order and particularly on the higher order connections. For example, the papers of C. Ehresmann, E.A. Feldman, P. Libermann, Ngo-Van-Que and others. Then some of the results, or similar theorems, have been proved by other mathematicians in the mean time.

1. Non-holonomic jets.

Let $\mathfrak{M}$ be the set of the C^s-mappings $(s \geq 1)$ f of R into the C^∞-manifold V, such that $f(0) = x$ for some fixed $x \in V$. Two such mappings $f, g \in \mathfrak{M}$ are said to be equivalent if $f(0) = g(0)$ and if the partial derivatives of the first order of the functions which give the mappings f, g in some local coordinates in the neighborhood of x on V are equal at the point 0. We shall denote by $T_x(V)$ the set of all so defined equivalence classes from $\mathfrak{M}$. This vector space is called the tangent (vector) space to V at x and $T(V) = \bigcup_{x \in V} T_x(V)$ the tangent bundle of the manifold V.

Let $p : W \to V$ be a projection of a manifold W onto a manifold V. (Throughout this paper will be considered only C^s-manifolds for s sufficiently large). If $Hom(T_x(V), T_y(W))$ denotes the set of homomorphisms of vector spaces, we shall consider the sets

$$hom(T_x(V), T_y(W)) = \{ \lambda \in Hom(T_x(V), T_y(W)) \mid p_* \circ \lambda = id \}$$

and

Part of this work was done during the author's stay at the Tata Institute of Fundamental Research, Bombay.

$$hom(T(V),\ T(W)) = \bigcup_{py = x \in V} hom(T_x(V),\ T_y(W)).$$

If we consider the product $V \times W$ of any manifolds V and W with the canonical projections $\tau_1 : V \times W \to V$, $\tau_2 : V \times W \to W$, we shall take

$$hom(T(V),\ T(V \times W)) = J^1.$$

We have well defined projections

$$\alpha : J^1 \to V,\ \ \beta_1 : J^1 \to V \times W,\ \ \beta = \tau_2 \circ \beta_1 : J^1 \to W.$$

Analogously can be defined by induction $\tilde{J}^r = hom(T(V),\ T(\tilde{J}^{r-1})), r > 0$, $J^1 \equiv \tilde{J}^1$. Let us denote $V \times W = J^o \equiv \tilde{J}^o$. For each $r \geq 0$ there are the projections $\alpha : \tilde{J}^r \to V$, $\beta_r : \tilde{J}^r \to \tilde{J}^{r-1}$, $\beta = \tau_2 \circ \beta_1 \circ \beta_2 \circ \ldots \circ \beta_r$.

DEFINITION 1.1. *The elements of the manifold* $\tilde{J}^r = hom(T(V),\ T(\tilde{J}^{r-1}))$ *are called the non-holonomic jets of order* r *of the manifold* V *into* W.

We shall denote by $\tilde{J}^r(V, W)$ or briefly $\tilde{J}^r$ the manifold of the non-holonomic r-jets of V into W. For $X \in \tilde{J}^r(V, W)$, $x = \alpha(X)$ is called the source and $y = \beta(X)$ the target of the jet X. The set of non-holonomic r-jets of $\tilde{J}^r$ with source x (resp. target y, resp. with source x and target y) is denoted by $\tilde{J}^r_{x,}$ (resp. $\tilde{J}^r_{,y}$, resp. $\tilde{J}^r_{x,y}$).

DEFINITION 1.2. *The elements of the manifold*

$$\bar{J}^r(V, W) = \bar{J}^r = \{\ X \in hom(T(V),\ T(\bar{J}^{r-1}))\ |\ \beta_r X = (\beta_{r-1})_* \circ X\ \},$$

$$for\ r \geq 2;\ \bar{J}^r = J^r,\ \ r = 0, 1\ .$$

are called the semi-holonomic jets of order r *of the manifold* V *into* W.

It is not difficult to prove that there exists a local mapping $\mathfrak{A}$ of the manifold $\bar{J}^r$ into the vector bundle $\sum_{s=1}^{r} T(W) \otimes (\overset{s}{\otimes} T^*(V))$ which is injective. A map of an element from $\bar{J}^r$ is called *its tensorial representation*. Let us denote by $\odot^r(T^*_x(V))$ the r-tuple symmetric product of $T^*_x(V)$. It is clear that

$$\sum_{s=1}^{r} T_y(W) \otimes \odot^s(T^*_x(V)) \subset \sum_{s=1}^{r} T_y(W) \otimes (\overset{s}{\otimes} T^*_x(V)).$$

The injection $\mathfrak{A}$ depends on the coordinates chosen in the respective neighborhoods. But if for some $X \in \bar{J}^r$, $\mathfrak{A}(X) \in \sum_{s=1}^{r} T_y(W) \otimes \odot^s(T^*_x(V))$, then the fact that $\mathfrak{A}(X)$ belongs to the manifold

$$\sum_{s=1}^{r} T_y(W)\otimes \bigcirc^s(T_x^*(V))$$

is independent of the coordinate systems. Or briefly, the symmetry of the tensorial representation of an element $X \in \overline{J}^r$ does not depend on the coordinate system.

DEFINITION. *The elements of*

$$J^r = \{\, X \in \overline{J}^r \mid \mathfrak{A}(X) \in \sum_{s=1}^{r} T(W)\otimes \bigcirc^s(T^*(V))\}$$

are called the holonomic jets of order r of the manifold V into W.

It is not difficult to see that each element from J^r can be given by some local mapping f of V into W. The element from $J^r_{x,y}$, if f is defined in the neighborhood of a point $x \in V$ and $f(x) = y \in W$, which is given by f, is denoted by $j^r_x f$. An element $X \in \tilde{J}^r$ is said to be *regular* if the corresponding linear mapping

$$\tau_2(\beta_2(\beta_3(\dots(\beta_r(X)\dots) \in Hom(T_x(V), T_y(W)), x = \alpha(X), y = \beta(X)$$

is of the maximal rank. It is not difficult to show that there exists a tensorial representation (a local mapping) of the manifold $\tilde{J}^r$ into the space

$$\sum_{s=1}^{r} T(W)\otimes(\bigotimes_{\substack{\alpha=1 \\ 1\leq i_1<\dots<i_\alpha\leq r}}^{s} T^*(V_{i_\alpha})),$$

V_{i_α} being different copies of the manifold V. The composition of non-holonomic jets is defined analogously to that one defined in [3]. Let V, N, W be three manifolds. Let $X \in \tilde{J}^r_{x,z}(V, N)$, $Y \in \tilde{J}^r_{z,y}(N, W)$. An element $YX \in \tilde{J}^r_{x,y}(V, W)$ is called *composition of X and Y* and is defined as follows : let us denote $X' = \beta_r(X)$, $Y' = \beta_r(Y)$. There exists a neighborhood of the point x in V and a mapping f of this neighborhood into some neighborhood of X' in $\tilde{J}^{r-1}(V, N)$ such that $\alpha \circ f$ is the identity on V and $f_* = X$. Analogously there exists a mapping g of a neighborhood of a point z in N into some neighborhood of a point Y' in $\tilde{J}^{r-1}(N, W)$ such that $\alpha \circ g$ is the identity on N and $g_* = Y$. It is easy to see that the composition YX is trivially defined for $r = 1$ as the composition of the linear mappings. Let us assume that the composition is well defined for jets of order $r-1$. Then there exists the composition $g \circ f$ which is a mapping defined on the neighborhood of the point x on V into some neighborhood of a point $Y'X'$ such that $\alpha \circ g \circ f$ is the identity on V. Let us define $YX = (g \circ f)_*$. It is clear that $YX \in \tilde{J}^r(V, W)$. The composition of semi-holonomic jets is a semi-holonomic jet and the composition of holonomic jets is a holonomic jet. Let n be the dimension of the manifold V. The regular non-holonomic r-jets $\in J^r_{o,x}(R^n, V)$ are called *non-holonomic r-frames at a point* $x \in V$ and we shall denote their set by $\tilde{H}^r_x(V)$ and further $\tilde{H}^r(V) = \bigcup_{x \in V} \tilde{H}^r_x(V)$. $\tilde{L}^r_{n,m}$ denotes the set of non-holonomic n^r-*velocities* of the manifold R^m at the point O, i.e. the elements of $\tilde{J}^r_{o,o}(R^n, R^m)$. The regular jets of order r of $\tilde{J}^r_{o,o}(R^n, R^n)$ form the group $\tilde{L}^r_n$, so called non-holonomic prolongation of order r of the linear group $L_n = GL(n, R)$. Let us denote further

$$\tilde{T}^r_n(V) = \tilde{J}^r_{o,}(R^n, V) \text{ and } \tilde{T}^{r*}_n(V) = \tilde{J}^r_{,o}(V, R^n).$$

We shall take similar notations for semi-holonomic and holonomic jets. It follows easily from the definition, that the manifold $\tilde{J}^r(V, W)$, where $dim\ V = n$, $dim\ W = m$, has three natural structures of fibre bundle [2], namely

$$\tilde{J}^r[V \times W, \tilde{L}^r_{n,m}, L^r_n \times L^r_m, H^r(V) \times H^r(W)],$$

$$\tilde{J}^r[V, \tilde{T}^r_n(W), L^r_n, H^r(V)], \tilde{J}^r[W, \tilde{T}^{r*}_m(W), L^r_m, H^r(W)].$$

The groupoid $\tilde{\Pi}^r(V)$ contained in $\tilde{J}^r(V, V)$ is a groupoid acting on $\tilde{J}^r(V, W)$. The *class of intransitivity* of the element $z \in \tilde{J}^r(V, W)$ with respect to $\tilde{\Pi}^r(V)$ is the set of all the elements $z\Theta \in \tilde{J}^r(V, W)$, $\Theta \in \tilde{\Pi}^r(V)$. To the class of intransitivity of $z \in \tilde{J}^r(V, W)$ with respect to $\tilde{\Pi}^r(V)$ there corresponds in $\tilde{T}^r_n(V)$ the class YL^r_n, $Y = zh$, $h \in H^r(V)$. The class YL^r_n is called the *non-holonomic element of contact* associated to Y or z. We speak also about a non-holonomic n^r- element of contact of W at the point $\beta(z) = y$. A non-holonomic element of contact X of W at y is said to be regular if all the non-holonomic n^r-velocities in the class X are regular, i.e. the corresponding n^1-velocities are regular 1- jets of dimension n.

REMARK. Let V and W be two differentiable manifolds and let X be a non-holonomic r-jet of V into W, $\alpha(X)=x$, $\beta(X)=y$. The element X gives rise to a unique linear mapping X_* of the vector space $\tilde{T}^r_x(V)$ into $\tilde{T}^r_y(W)$ and a unique linear mapping X^* of $\tilde{T}^{r*}_y(W)$ into $\tilde{T}^{r*}_x(V)$, where

$$\tilde{T}^{r*}(V)=\tilde{J}^r_{,o}(V,R),\ \tilde{T}^r(V)=(\tilde{T}^{r*}(V))^*$$

PROPOSITION 1.1. *Let $H(B,G)$ be a principal fibre bundle. The set*

$$D^r=\{\,X\in\tilde{J}^r(B,H)\mid p_*X=j^r_{\alpha(X)}\,\}$$

has a structure of a fibre bundle with the base B and the fibre

$$G\times G^r_n,\ G^r_n\equiv\tilde{J}^r_{o,e}(R^n,G).$$

On the fibre $G\times G^r_n$ acts the r-th prolongation of the operation $L_n\times G$ on $R^n\times G$.

PROOF. We shall identify first $D^r(R^n,R^n\times G)$ with $R^n\times G\times G^r_n$. Let $X\in D^r(R^n,R^n\times G)$, $\alpha(X)=x$, $\beta(X)=(x,a)$. Let us denote by

$$\tau_1: R^n\times G\to R^n,\quad \tau_2: R^n\times G\to G$$

the canonical projections. The isomorphism $D^r(R^n,R^n\times G)\cong R^n\times G\times G^r_n$ is given by the identification $X\longleftrightarrow(x,\,a,\,\tau_2(j^r_{x,a}t_{x,a})X(j^r_o t^{-1}_x))$, where

$$t_x: R^n\to R^n,\quad t_x(y)=x-y.$$

The operation of the pseudogroup ψ_n of operations on $R^n\times G$ is given by the formula $\overline{\psi}:(x,a)\to(\psi(x),ga)$, $\overline{\psi}\equiv(\psi,g)\in\psi_n$, $(x,a)\in$ $\in R^n\times G$. Let us consider the prolongation ψ^r_n of this pseudogroup on $R^n\times G\times G^r_n$.

The prolongation of the atlas $\mathfrak{A}\times\mathfrak{H}$ of $R^n\times R^n\times G$ onto $B\times H$ on the atlas $\overline{\mathfrak{A}}\times\overline{\mathfrak{H}}$ of $D^r(R^n,R^n\times G)$ onto $D^r(B,H)$ is given for the above chosen $X\in D^r(R^n,R^n\times G)$ and $(g,h)\in\mathfrak{A}\times\mathfrak{H}$ by the formula $(g,h):\ X\to(j^r_{x,a}h)(j^r_{o,e}t^{-1}_{x,a})(j^r_{x,a}t_{x,a})X(j^r_o t^{-1}_x)(j^r_x t_x)(j^r_{g(x)}g^{-1})$. This prolonged atlas is compatible with the operation of the pseudogroup ψ^r_n on $D^r(R^n,R^n\times G)$.

$$X\in D^r(R^n,R^n\times G),\ \overline{\psi}\equiv(\psi,g)\in\psi_n,\ X\to(j^r_{x,a}\overline{\psi})X(j^r_{\psi(x)}\psi^{-1}).$$

The operation of the group $G\times L^r_n$ on the fibre $G\times G^r_n$ is given in a natural way: $(g,s)(a,w)=(ga,ws^{-1})$, $(g,s)\in G\times L^r_n$, $(a,w)\in G\times G^r_n$.

The operation of $\psi_n \times G \times L_n^r$ on $R^n \times G \times G_n^r$ is in fact the operation of the pseudogroup ψ_n^r. The prolongation of the operation R_g, $g \in G$ on H is the right translation on $\tilde{J}^r(B,H)$ given by the elements of G. Then $D^r/_G$ as a quotient has a structure of fibre bundle with the base B, fibre G_n^r and structural group L_n^r ($s \in L_n^r$, $w \in G_n^r$, $s : w \to ws^{-1}$). Let us now consider some global section σ^r of the fibre bundle $D^r/_{\tilde{G}}$ over B. We shall consider the restriction $\overline{\mathfrak{D}}^r$ of the tangent bundle $T(\tilde{J}^r(B,H))$ on $D^r(B,H)$. The section σ^r can be looked at as a section of $D^r(B,H)$ over H which is invariant under the transformations of G. The restriction of $\overline{\mathfrak{D}}^r$ to that section σ^r is a manifold $\mathfrak{D}^r$ and because $\mathfrak{D}^r \times G \to \mathfrak{D}^r$ is a natural mapping we have the vector bundle $Q^r = \mathfrak{D}^r/G$ with the base B. Analogously let us consider the sub-bundle $F(\tilde{J}^r(B,H))$ of $T(\tilde{J}^r(B,H))$ of vertical tangent vectors on $\tilde{J}^r(B,H)$. If we consider the restriction $\overline{\mathfrak{F}}^r$ of F onto $D^r(B,H)$ and the restriction $\mathfrak{F}^r$ of $\overline{\mathfrak{F}}^r$ on the section $\sigma^r : B \to D^r$, we have a well defined vector-bundle $R^r = \mathfrak{F}^r/G$ over B. Then holds the following

THEOREM 1.1. *Let H be a principal fibre bundle with the base B, structural group G. Let σ^r be a global section of the fibre bundle D^r/G over B. Then there exists a canonical exact sequence $\mathfrak{A}(H,\sigma^r)$*

$$0 \to R^r \to Q^r \to T(B) \to 0$$

of vector bundles over B.

It is immediately clear, that *the splittings of the exact sequence $\mathfrak{A}(H,\sigma^r)$ are in 1-1 correspondence with the global sections σ^{r+1} of D^{r+1}/G over B*. Let us take B as a section of the respective fibre bundle over B for $r=0$. $\mathfrak{A}(H,\sigma^0)$ is the well known Atiyah's sequence [1]. We shall use later the following.

PROPOSITION 1.2. *Let V, W, B be three differentiable manifolds. Let $\overline{\alpha}$ be a mapping of V into B, $\overline{\beta}$ a mapping of $V \times W$ into B so that $\overline{\alpha}(x) = \overline{\beta}(x,y), \forall (x,y) \in V \times W$. If $X \in \tilde{J}^r(B,V)$, $Y \in \tilde{J}^r(B,W)$, $\alpha(X) = \alpha(Y) = u$, $\beta(X) = x$, $\beta(Y) = y$, then $(j_x^r \overline{\alpha})X = (j_{(x,y)}^r \overline{\beta})(X,Y)$.*

PROOF. Let τ_1 be the canonical projection of the product $V \times W$ onto V. Then

$$j^r_{(x,y)}\tau_1(X,Y)=X$$

and

$$j^r_{(x,y)}\overline{\alpha}j^r_{(x,y)}\tau_1=j^r_{(x,y)}\overline{\alpha}\tau_1=j^r_{(x,y)}\overline{\beta},$$

so that

$$j^r_x\overline{\alpha}X=j^r_x\overline{\alpha}j^r_{(x,y)}\tau_1(X,Y)=j^r_{(x,y)}\overline{\beta}(X,Y),$$

as was to be shown.

2. The non-holonomic connections of order r.

Let $E(B,F,G,H)$ be a fibre bundle with the structural group G, basis B, $\dim B=n$, standard fibre F and projection p. Let $H(B,G)$ be the associated principal bundle to E. Basis B is a differentiable manifold of the dimension n. A vector τ_z of a non-holonomic tangent vector space of order r, $\tilde{T}^r_z(H)$ is said to be vertical if $p_*\tau_z$ is a zero vector of $\tilde{T}^r_{p(z)}(B)$.

DEFINITION 2.1. *We say that a non-holonomic connection of order r on a principal bundle H is given if:*

1) to each $z\in H$ there is a vector space $\tilde{\mathcal{H}}^r_z$ (subspace of $\tilde{T}^r_z(H)$), assigned so that this assignement is C^∞.

2) The field of spaces $\tilde{\mathcal{H}}^r_z$ is invariant under the right translation on H, i.e. $\tilde{\mathcal{H}}_{zg}=R_{g*}\tilde{\mathcal{H}}_z$, $g\in G$, $R_g z=zg$.

3) There exists just one element $Z_z=Z\in\tilde{J}^r(B,H)$, $\alpha(Z)=$ $=p(z)$, $\beta(Z)=z$, $pZ=j^r_{p(z)}$, where $j^r_{p(z)}$ is the r-jet of the identical mapping of B onto itself with source $p(z)$, such that $\tilde{\mathcal{H}}^r_z=Z_*\tilde{T}^r_{p(z)}(B)$. The space $\tilde{\mathcal{H}}^r_z$ is called the horizontal space at the point $z\in H$.

THEOREM 2.1. *Let H be a principal bundle. The set of non-holonomic connections of order r on H is in 1-1 correspondence with the fields of regular non-holonomic elements of contact on H such that:*

1°) X_z being the regular n^r-element of contact at $z\in H$, $(j^r_z p)X_z$ (i.e. the set $(j^r_z p)Y_z L^r_n$, $Y_z L^r_n=X_z$) is a regular n^r-element of contact of B at $p(z)$.

2°) $X_{zg}=(j^r_z R_g)X_z$.

An element of such a field is called a horizontal n^r-element of contact at a point z of the manifold H.

PROOF. We shall prove presently that the field of horizontal non-holonomic n^r-elements of contact gives rise to the non-holonomic connection of order r on H. Let Y_z be a representative of the class X_z. An r-jet Y_z defines just one linear mapping Y_{z*} of $\tilde{T}^r_o(R^n)$ into $\tilde{T}^r_z(H)$. It is clear that the space $\tilde{\mathcal{H}}^r_z = Y_{z*}\tilde{T}^r_o(R^n)$ is independent of the choice of Y_z in the class X_z. X_z is namely the set of the elements $Y_z s$, $s \in L^r_n$ and L^r_n is the group of transformations on $\tilde{T}^r_o(R^n)$. The mapping Y_* is clearly C^∞. On the basis of 2° follows that $\tilde{\mathcal{H}}_{zg} = R_{g*}\tilde{\mathcal{H}}_z$, $g \in G$. We have $(j^r_z p)Y_z \in \in \tilde{J}^r(R^n, B)$, $\alpha\{(j^r_z p)Y_z\} = 0$. Let h be an element of $\tilde{H}^r(B)$, then $g = h^{-1}(j^r_z p)Y_z \in \tilde{L}^r_n$. We show that $Z_z = Y_z g^{-1}h^{-1}$ has the property 3 from the definition of connection. If we consider an element $Y_z s$, $s \in L^r_n$ instead of Y_z, then $Y_z s s^{-1}g^{-1}h^{-1} = Y_z g^{-1}h^{-1}$ and it is easy to see that Z_z depends only on the non-holonomic n^r-element of contact $\tilde{X}_{\tilde{z}}$. If we take namely instead of h any other isomorphism $k = ha$, $a \in \tilde{L}^r_n$, of $\tilde{T}^r_o(R^n)$ onto $\tilde{T}^r_{p(z)}(B)$ we have $Y_z\{a^{-1}h^{-1}(j^r_z p)Y_z\}^{-1}a^{-1}h^{-1} = Y_z g^{-1}h^{-1}$. And further $(j^r_z p)Z_z = h h^{-1}(j^r_z p)Z_z = h h^{-1}(j^r_z p)Y_z g^{-1}h^{-1} = j^r_{p(z)}$.

And now let us suppose on the contrary that there is given a non-holonomic connection of order r on H. Let $h \in H^r(B)$, $\gamma(h) = p(z)$, $z \in H$, γ being a projection of the fibre bundle H^r onto B, then $Z_z h \in \tilde{J}^r(R^n, H)$. Let us define an equivalence relation $Z_z h \sim Z_z hs$, $s \in L^r_n$. A class defined by Z_z (the connection is given) is a non-holonomic regular n^r-element of contact of H at the point z. The property 1 is satisfied on the basis of $(j^r_z p)Z_z h = \tilde{j}^r_{p(z)}h = h$. By the definition there is $\tilde{\mathcal{H}}^r_z = Z_{z*}\tilde{T}^r_{p(z)}(B)$, $\tilde{\mathcal{H}}^r_{zg} = Z_{zg*}\tilde{T}^r_{p(zg)}(B) = Z_{zg*}\tilde{T}^r_{p(z)}(B) = R_{g*}\tilde{\mathcal{H}}^r_z = R_{g*}Z_{z*}\tilde{T}^r_{p(z)}(B)$, i.e. $Z_{zg} = (j^r_z R_g)Z_z$. And this finishes the proof of the theorem.

Let Φ be the groupoid associated to the principal bundle H, i.e. $\Phi = HH^{-1}$. If we denote by $\hat{B}$ the units of Φ, there are two projections a, b of Φ onto $\hat{B}$. $a\Theta$ is a right unit of $\Theta \in \Phi$ and $b\Theta$ is a left unit of $\Theta \in \Phi$. We have now a mapping $\varphi: H \times H \to HH^{-1}$ so that $\varphi(h, h') = h'h^{-1}$. The targets of the elements of the diagonal Δ of $H \times H$ by the mapping φ,

$\varphi(h,h)=hh^{-1}$, belong to $\hat{B}$. hh^{-1} is a target of all elements $(hg, hg) \in \Delta$, $g \in G$. We can in a natural way identify Δ and H. We can also to each $\tilde{x} = hh^{-1} \in \hat{B}$ associate a point $x = p(h) \in B$ and on the contrary. Let $\hat{a}$ be the projection of Φ onto B defined as follows :

$$\hat{a} : (h'h^{-1}) \to a(h'h^{-1}) = hh^{-1} \to p(h).$$

In the same way can be defined a projection $\hat{b} : \Phi \to B$. Let $X \in \tilde{J}^r(B, \Phi)$, $\alpha(X) = x$, $\beta(X) = \tilde{x}$, $\hat{a}(X) = \hat{j}^r_x$, ($\hat{j}^r_x$ is an r-jet of the retraction of B to the point $x \in B$; this notation is used throughout this paper), $\hat{b}(X) = j^r_x$. Let $\tilde{Q}^r$ be a set of all these r-jets. $\tilde{Q}^r$ is a fibre bundle with basis B [5] . There exists a cross-section of $\tilde{Q}^r$ over B.

THEOREM 2.2. [5]. *The cross-sections in $\tilde{Q}^r$ are in 1-1 correspondence with the non-holonomic connections of order r on the principal bundle H. An element X of this cross-section over a point $x \in B$ is said to be an element of the non-holonomic connection of order r (or an element of the connection at x).*

PROOF. Let be given a connection on H and let Z_z be the element of $\tilde{J}^r(B, H)$ mentioned in 3. Let k be a mapping of the neighborhood of $x = p(z)$ in B into the point $z \in H$. Then $(j^r_x k, Z_z) \in \tilde{J}^r(B, H \times H)$. Let $W = (j^r_x k) \bullet Z_z$, where $\bullet$ is the non-holonomic prolongation of order r of the composition rule $\varphi: H \times H \to HH^{-1}$. By the definition [3] we have $(j^r_x k) \bullet Z_z = (j^r_{(z,z)} \varphi)(j^r_x k, Z_z)$. Obviously

$$W \in \tilde{J}^r(B, \Phi), \alpha(W) = \alpha(j^r_x k \bullet Z_z) = x, \beta(W) = \beta(j^r_{(z,z)} \varphi) = \tilde{x}.$$

$\hat{a}(W)$ is an abreviated notation for $(j^r_{\tilde{x}} \hat{a}) W = (j^r_{(z,z)} \hat{a} \circ \varphi)(j^r_x k, Z_z)$. Because of $(\hat{a} \circ \varphi)(h, h') = p(h)$, $(h, h') \in H \times H$, we have on the basis of proposition 1.2 the following :

$$\hat{a}(W) = (j^r_{(z,z)} \hat{a} \circ \varphi)(j^r_x k, Z_z) = (j^r_z p) \quad (j^r_x k) = \hat{j}^r_x.$$

Analogously we get the relation

$$\hat{b}(W) = (j^r_{(z,z)} \hat{b} \circ \varphi)(j^r_x k, Z_z) = (j^r_z p) Z_z = j^r_x.$$

We have to show now that W is independent on the choice of a point z on the fibre H_x over x. Because by the definition $Z_{zg} = (j_z^r R_g) Z_z$, we have

$$(j_x^r(R_g \circ k)) \bullet Z_{zg} = \{(j_z^r R_g)(j_x^r k)\} \bullet \{(j_z^r R_g) Z_z\} =$$
$$= (j^r_{(zg,zg)} \varphi)\{(j_z^r R_g)(j_x^r k), (j_z^r R_g) Z_z\} =$$
$$= (j^r_{(zg,zg)} \varphi)(j_z^r R_g)\{j_x^r k, Z_z\} = (j^r_{(z,z)} \varphi)\{j_x^r k, Z_z\} = W,$$

realizing $\varphi \circ R_g = \varphi$.

Let now on the contrary $X \in \tilde{J}^r(B, \Phi)$ be an element of a cross-section in $\tilde{Q}^r$ over B; then $\alpha(X) = x$, $\beta(X) = \tilde{x}$. Consider the mapping $\psi : \Phi \times H \to H$, $\psi(\theta, z) = \theta z$. The groupoid acts thus on H. Let k be a mapping of the neighborhood of a point x on B into the point $z \in H$. The prolongation of the composition rule ψ gives rise to the composition $X \bullet z = X \bullet (j_x^r k) = (j^r_{(\tilde{x},z)} \psi)(X, j_x^r k) \in \tilde{J}^r(B, H)$. On the basis of the relation $p(\psi(\theta, z)) = \hat{b}(\theta)$ and by the proposition 1.2 we get

$$(j_z^r p)(X \bullet z) = (j_z^r p)(j^r_{(\tilde{x},z)} \psi)(X, j_x^r k) = (j^r_{(\tilde{x},z)} p \circ \psi)(X, j_k^r k) =$$
$$= (j^r_{\tilde{x}} \hat{b}) X = j_x^r.$$

A non-holonomic n^r-element of contact belonging to $X \bullet z$ is then regular. We have further

$$X \bullet zg = (j^r_{(\tilde{x},zg)} \psi)(X, (j_z^r R_g)(j_x^r k)) = (j_z^r R_g)(j^r_{(\tilde{x},z)} \psi)(X, j_x^r k) =$$
$$= (j_z^r R_g)(X \bullet z),$$

because $\psi(\theta, zg) = R_g\{\psi(\theta, z)\}$. The theorem is then proved.

Let W be any m-dimensional manifold and let $Z \in \tilde{J}^r(W, H)$.

DEFINITION 2.2. *A horizontal projection of Z with respect to an element X of the non-holonomic connection of order r on H at a point $p(z)$ is a non-holonomic r-jet $X^{-1}Z = (X^{-1} p Z) \bullet Z$, where $\bullet$ is a non-holonomic prolongation of order r of the composition rule $\psi : \Phi \times H \to H$; $X \to X^{-1}$ is a prolongation of mapping $\theta \to \theta^{-1}$ defined on Φ.*

Obviously $X^{-1} Z \in \tilde{J}^r(W, H)$. A horizontal projection is called by C. Ehresmann [5] an absolute differential. It can be shown that the following proposition holds.

PROPOSITION 2.1. [5]. *An element* $X^{-1}Z$ *belongs to* $\tilde{J}^r(W, H_x)$, H_x *being the fibre of* H *over* $x \in B$.

PROOF. We have by the definition

$$(j^r_z p)X^{-1}Z = (j^r_{(\tilde{x},z)} p \circ \psi)(X^{-1}(j^r_z p)Z, Z) = (j^r_{\tilde{x}} \hat{b})X^{-1}(j^r_z p Z).$$

Let k be the mapping of the neighborhood of the point $\alpha(Z)$ on W into the point $x \in B$. On the basis of the relation $(j^r_{\tilde{x}} \hat{b})X^{-1} = \hat{j}^r_x$, $x = p(z)$, we have $\hat{a}X = \hat{j}^r_x$ and then $(j^r_z p)X^{-1}Z = \hat{j}^r_x(j^r_z p)Z = j^r_{\alpha(Z)} k$.

Let $W_z \in \tilde{H}^r(H)$, $\beta(W_z) = z$, then $(X^{-1}W_z)W_z^{-1} = X^{-1}j^r_z$, X being an element of connection at the point $x = p(z)$. W_z^{-1} is well defined because W_z is a regular element. Using the composition rule of non-holonomic r-jets, we have $W_z W_z^{-1} = j^r_z$. Now we have

$$\begin{aligned}(X^{-1}W_z)W_z^{-1} &= (j^r_{(\tilde{x},z)}\psi)(X^{-1}(j^r_z p)W_z, W_z)W_z^{-1} = \\ &= (j^r_{(\tilde{x},z)}\psi)(X^{-1}(j^r_z p)W_z W_z^{-1}, W_z W_z^{-1}) = \\ &= (j^r_{(\tilde{x},z)}\psi)(X^{-1}(j^r_z p)j^r_z, j^r_z) = X^{-1}j^r_z.\end{aligned}$$

Let $j^r_z re$ be an r-jet of the injective mapping of the fibre H_x into H, with source $z \in H_x$. Let us consider now

$$\begin{aligned}X^{-1}(j^r_z re) &= (j^r_{(\tilde{x},z)}\psi)(X^{-1}(j^r_z p)(j^r_z re), j^r_z re) = \\ &= (j^r_{(\tilde{x},z)}\psi)(X^{-1}\hat{j}^r_z, j^r_z re).\end{aligned}$$

If x is a fixed point, then the mapping $\psi(\tilde{x}, z) = z$ is the projection $(\tilde{x}) \times H_x \to H_x$ and by the proposition 1.2 we have $X^{-1}(j^r_z re) = j^r_z re$.

We have now a linear mapping P_{z*} of $\tilde{T}^r_z(H)$ into $\tilde{T}^r_z(H_x)$ defined by the non-holonomic r-jet $X^{-1}j^r_z = P_z$. $\omega = \{(j^r_z z^{-1})P_z\}_*$ is then a linear mapping of $\tilde{T}^r_z(H)$ into $\tilde{T}^r_e(G)$.

Let φ_g be an inner endomorphism of G associated with $g \in G$, $Ad(g)$ the linear mapping $(j^r_e \varphi_g)_*$ of $\tilde{T}^r_e(G)$ onto itself and by

$$\tilde{T}^r(V) = \bigcup_{x \in V} \tilde{T}^r_x(V)$$

the non-holonomic tangent bundle of order r over V. A cross-section in $\tilde{T}^r(V)$ over V is called a non-holonomic vector field of order r. It is easy to see that the set of non-holonomic r-jets of V into $\tilde{T}^s(V)$ with source x, denoted by $\tilde{S}^{r,s}_x(V)$, is a vector space.

REMARK. Let L_g be a left translation defined on G by the element $g \in G$. The structural group G is a right transformation group on H which is simply transitive on each fibre $H_x = p^{-1}(x)$. We shall now define, analogously as it is for an infinitesimal connection of the 1[st]- order, a non-holonomic fundamental vector field of order r on H associated to a vector $Y = {}_eY \in \tilde{T}^r_e(G)$ (e is a unit of G). Let ${}_gY = (j^r_e L_g)_* \, {}_eY$. Let h_x be a homomorphism of G onto H_x so that $h_x e = z$. Consider the element ${}_eY_z = (j^r_e h_x)_* \, {}_eY$ at the point $z \in H_x$ and the element ${}_gY_{zg} = (j^r_g h_x)_* \, {}_gY$ at the point $zg \in H_x$. It is easy to see that ${}_gY_{zg} = (j^r_e h_x g)_* \, {}_eY = {}_eY_{zg}$. We have now on H a vector field which corresponds to the left invariant vector field on G, and this correspondence does not depend on the choice of $h_x \in H_x$. We shall speak about the non-holonomic *fundamental vector field* Y *of order* r *associated to* Y (or briefly about a fundamental vector field only). It is clear that $R_{g*} Y_z$ is a vector of the vector field associated to $Ad(g^{-1})Y$.

DEFINITION. *Let* V *be a differentiable manifold, and* φ_x *a linear mapping of* $\overset{m}{\Lambda}\, \tilde{S}^{r,s}_x(V)$ *into a vector space* M. *A differentiable field* $x \to \varphi_x$ *is called an* M*-valued differentiable form* φ *on* V *of degree* $m > 0$, *order* (r, s).

THEOREM 2.3. *A non-holonomic connection of order* r *on* H *can be defined by a* $\tilde{T}^r_e(G)$*-valued differential form* ω *on* H *of the* 1[st]*-degree, order* $(0, r)$ *so that the following conditions are satisfied :*

1) $\omega(Y_z) = Y,\ Y \in \tilde{T}^r_e(G),\ z \in H,$

2) $\omega(R_{g*}X) = Ad(g^{-1})\omega(X),\ X \in \tilde{T}^r_z(H),$

3) There exists a non-holonomic r-jet $Z \in \tilde{J}^r(H, H_x),\ \alpha(Z) = \beta(Z) = z$ *so that* $(j^r_e z)_* \omega = Z_*$.

PROOF. First let us suppose that a non-holonomic connection on H is given. We have a linear mapping $\omega = \{(j^r_z z^{-1})(X^{-1} j^r_z)\}_*$ of $\tilde{T}^r_z(H)$ into $\tilde{T}^r_e(G)$. We shall prove presently that ω is a form considered in the theorem. We know that H_x is a submanifold of H. Let $V_z \in \tilde{H}^r(H_x)$ and $W_z \in \tilde{H}^r(H)$, be the r-frames at the point z. Let Y_z be any vertical vector at $z \in H$, i.e. a vector of $\tilde{T}^r_z(H_x)$. Let m be the dimension of H and n the

dimension of H_x. Then $Y' = (V_z^{-1})_* Y_z \in \tilde{T}_o^r(R^n), Y'' = (W_z^{-1})_* Y_z \in \tilde{T}_o^r(R^m)$.
It is clear that

$$(X^{-1}W_z)_* Y'' = (j^r_{(\tilde{x},z)}\psi)_* \{(X^{-1})_* p_* W_{z*} Y'', W_{z*} Y''\} =$$
$$= (j^r_{(\tilde{x},z)}\psi)_* \{(X^{-1})_* p_* V_{z*} Y', V_{z*} Y'\} = (X^{-1}V_z)_* Y'.$$

We have then $\{(X^{-1}W_z)W_z^{-1}\}_* Y_z = \{(X^{-1}V_z)V_z^{-1}\}_* Y_z$. But $(X^{-1}W_z)W_z^{-1} = j_z^r$ and analogously $(X^{-1}V_z)V_z^{-1} = j_z^r re$. On the basis of the relation

$$X^{-1}(j_z^r re) = j_z^r re$$

we see that the mappings

$$\omega = \{(j_z^r z^{-1})(X^{-1}j_z^r)\}_* \text{ and } \omega_1 = \{(j_z^r z^{-1})(j_z^r re)\}_*$$

are linear mappings of $\tilde{T}_z^r(H_x)$ onto $\tilde{T}_e^r(G)$ so that $\omega_1(Y_{\tilde{z}}) = Y$, Y_z being a vector of a fundamental vector field belonging to $Y \in T_e^r(G)$. We have proved then that 1) is satisfied for ω.

Let $A_z \in \tilde{T}_z^r(H)$ and let $A_{zg} = R_{g*}A_z$. We have then $\omega(A_{zg}) =$ $= \{(j_{zg}^r(zg)^{-1})(X^{-1}j_{zg}^r)(j_z^r R_g)\}_* A_z$. Let us prove first that

$$(X^{-1}j_{zg}^r)(j_z^r R_g) = (j_z^r R_g)(X^{-1}j_z^r).$$

We have namely the relations

$$(j_z^r R_g)(X^{-1}j_z^r) = (j_z^r R_g)(j^r_{(\tilde{x},z)}\psi)(X^{-1}p\, j_z^r, j_z^r) =$$
$$= (j^r_{(\tilde{x},zg)}\psi)(j_z^r R_g)(X^{-1}p\, j_z^r, j_z^r) = (j^r_{(\tilde{x},zg)}\psi)(X^{-1}p\, j_{zg}^r, j_{zg}^r)(j_z^r R_g) =$$
$$= (X^{-1}j_{zg}^r)(j_z^r R_g).$$

Then $\omega(A_{zg}) = \{(j_{zg}^r(zg)^{-1})(j_z^r R_g)(X^{-1}j_z^r)\}_* A_z$. Denoting by L_g a left translation on G defined by $g \in G$ we have

$$\omega(A_{zg}) = \{(j_g^r L_{g^{-1}})(j_{zg}^r z^{-1})(j_z^r R_g)(j_e^r z)(j_z^r z^{-1})(X^{-1}j_z^r)\}_* A_z =$$
$$= \{(j_g^r L_{g^{-1}})(j_e^r R_g)\}_* \{(j_z^r z^{-1})(X^{-1}j_z^r)\}_* A_z =$$
$$= (j_e^r \varphi_{g^{-1}})_* \omega(A_z) = Ad(g^{-1})\omega(A_z).$$

An element $X^{-1}j_z^r$ is just a non-holonomic r-jet Z mentioned in 3. The mapping $(j_e^r Z)_* \omega$ is the mapping associated with it.

Let now on the contrary ω be a $\tilde{T}_e^r(G)$- valued differential form of degree 1, order $(0, r)$ on H so that the properties 1, 2, 3 are fulfilled.

Let σ be a cross-section in H over a neighborhood of the point $x \in B$, $\sigma(x) = z$. Let $W = j^r_x \sigma$. Let Z be a non-holonomic r- jet from $\tilde{J}^r(H, H_x)$ with the properties contained in the theorem. Let $X = ZW \bullet W$, $\bullet$ being a prolongation of the composition rule $\varphi: H \times H \to HH^{-1}$. It is clear that $X \in \tilde{J}^r(B, \Phi)$. We have further $\alpha(X) = x$, $\beta(X) = \tilde{x}$. On the basis of $\hat{a}(b'b^{-1}) = p(b)$, $\hat{b}(b'b^{-1}) = p(b')$, $b, b' \in H$, we obtain

$$(j^r_{\tilde{x}}\hat{a})(j^r_{(z,z)}\varphi)(ZW, W) = (j^r_z p)ZW = j^r_x,$$
$$(j^r_{\tilde{x}}\hat{b})(j^r_{(z,z)}\varphi)(ZW, W) = (j^r_z p)W = j^r_x.$$

X is then an element of a cross-section in $\tilde{Q}^r$ over B. We prove now the independence of X on the choice of the cross-section σ over a neighborhood of the point $x \in B$. Let σ' be another lifting, $\sigma'(y) = \sigma(y)g(y)$, $g(y) \in G$ for each y from the considered neighborhood of the point $x \in B$, $g(x) = e$. Let us notice that we have defined a holonomic prolongation of a composition rule [6]. Using this operation we have $j^r_x\sigma' = (j^r_x\sigma)(j^r_x g)$. Let us prove that the identity $Z\{W(j^r_x g)\} = \{ZW\}(j^r_x g)$ holds. Let first $r = 1$. Let f be a mapping of the neighborhood $U_1(z) \subset H$ onto the neighborhood $U_2(z) \subset H_x$ so that $f(z) = z$ and $Z = j^1_z f$. Let σ be a cross-section in H over the neighborhood $V(x) \subset B$, $\sigma(V(x)) \subset U_1(z)$ and be $W = j^1_x\sigma$. Let σ' be another cross-section in H over $V(x)$ so that $\sigma'(y) = \sigma(y)g(y)$, $y \in V(x)$. Then $j^1_x(f \circ \sigma') = (j^1_x f)\{(j^1_x\sigma)(j^1_x g)\}$ and further

$$j^1_x(f \circ \sigma') = j^1_x\{(f \circ \sigma)g\} = (j^1_x(f \circ \sigma))(j^1_x g) =$$
$$= \{(j^1_z f)(j^1_x\sigma)\}(j^1_x g),$$

because $f \circ \sigma'$, $f \circ \sigma$ are the cross-sections in H over $V(x)$. Let s be a cross-section in $\tilde{J}^{r-1}(B, H)$ over $V(x)$ defined as follows : $s(y) = j^{r-1}_y\sigma$, $y \in V(x)$. Let s' be a cross-section in $\tilde{J}^{r-1}(H, H_x)$ over $U_1(x)$ so that $j^1_z s' = Z$. We have then a mapping $s'' = s's$ of $V(x)$ into H_x so that $s''(x) = z$. Let λ be a cross-section in $\tilde{J}^{r-1}(B, G)$ over $V(x)$ so that $\lambda(y) = j^{r-1}_y g$, $y \in V(x)$. We have then $j^1_x\lambda = j^r_x g$. By the assumption

$$s'(w)\{s(y)\lambda(y)\} = \{s'(w)s(y)\}\lambda(y), \quad y \in V(x), \quad \sigma(y) = w.$$

We have then

$$j^1_x\{s'(s\lambda)\} = (j^1_z s')\{(j^1_x s)(j^1_x\lambda)\},$$
$$j^1_x\{(s's)\lambda\} = \{(j^1_z s')(j^1_x s)\}(j^1_x\lambda)$$

and, on the basis of the equality $j^1_x\{(s's)\lambda\} = j^1_x\{s'(s\lambda)\}$, the result

$$Z\{W(j^r_x g)\} = (ZW)(j^r_x g).$$

But by the definition $ZW \bullet W = (j^r_{(z,z)}\varphi)(ZW, W)$. Denote $W' = W(j^r_x g)$. It is then

$$ZW' \bullet W' = (j^r_{(zg,zg)}\varphi)(ZW(j^r_x g), W(j^r_x g)) =$$
$$= (j^r_{(z,z)}\varphi)(ZW, W).$$

The last equality may be proved by induction. We must show now that $\{Z(j^r_z R_g)W\} \bullet \{(j^r_z R_g)\tilde{W}\} = ZW \bullet \tilde{W}$ holds. We know that $\tilde{Z}_{z*} = (j^r_e z)_* \omega$ is a linear mapping of $T^r_z(H)$ onto $\tilde{T}^r_z(H_x)$. Let $A_{zg} \in T^r_{zg}(H)$, $A_{zg} = R_{g*}A_z$; then

$$Z_{zg*}(A_{zg}) = (j^r_e zg)_*\omega(A_{zg}) = \{(j^r_g z)(j^r_e L_g)\}_* Ad(g^{-1})\omega(A_z) =$$
$$= \{(j^r_z R_g)(j^r_e z)\}_*\omega(A_z) = (j^r_z R_g)_* Z_{z*}(A_z)$$

because of $zgg^{-1}ag = z(Lg(\varphi_g\text{-}1(a))) = R_g(z(a))$. Then

$$\{Z(j^r_z R_g)W\} \bullet \{(j^r_z R_g)W\} = \{(j^r_z R_g)ZW\} \bullet \{(j^r_z R_g)W\}.$$

On the basis of the relation $\varphi \circ R_g = \varphi$ we have the above result and so is the theorem completely proved.

THEOREM 2.4. *Let $\mathfrak{A}(H, \sigma^k)$, $0 \leqq k \leqq r-1$, be the exact sequence associated to a global section σ^k of the bundle D^k/G over B, such that σ^k is given by a splitting ρ^k of the exact sequence $\mathfrak{A}(H, \sigma^{k-1})$, for $k \geqq 1$ and σ^0 is B itself.*

The non-holonomic connections of order r on the principal fibre bundle H are in 1-1 correspondence with the splittings ρ^s, $1 \leqq s \leqq r$, of the exact sequence $\mathfrak{A}(H, \sigma^{s-1})$ of vector bundles.

PROOF. First, let be given a non-holonomic connection of order r on H. Then to each $z \in H$ there is associated by definition a non-holonomic r-jet $Z \in \tilde{J}^r_{x,z}(B, H)$, s.t. $p_* Z = j^r_x$, $Z_{zg} = R_g Z_z$, $g \in G$. The projection $j^1 Z$ into $J^1(B, H)$ uniquely gives the section σ^1 of D^1/G over B. Assuming that the section σ^{r-1} of D^{r-1}/G is given uniquely by the projection $j^{r-1}Z$ we shall prove that the section $\sigma^r : B \to D^r/G$ is given by the element Z. We know already that σ^{r-1} can be considered as a section of D^r over H which is invariant under the transformations of G. The jet Z is then defined as $j^1_x\sigma$, where σ is some section of D^{r-1} over B such that $\sigma(x) = z$. Further let us take $Z_{zg} = j^1_{p(zg)}(R_g\sigma)$, then $Z_{zg} = R_g j^1_x\sigma = R_g Z_z$. We have thus a well defined section σ^r of D^r over H, which is invariant under the transformations of G, namely

$$\sigma^r : a \to j^1_{p(a)}\sigma \, , \; a = \beta(Z) \in H,$$

σ being the section mentioned above. And now on the contrary, let there be given r splittings ρ_s $(1 \leq s \leq r)$ of the exact sequences $\mathfrak{A}(H, \sigma^{s-1})$ of vector bundles. We have to prove that there is given exactly one non-holonomic connection of order r on H by these splittings. This holds for $r = 1$. Let us assume that the statement be true for $s = r-1$. To the splitting ρ_{r-1} is uniquely associated the section σ^r of D^r/G over B or, what is the same, the G-invariant section σ^r of D^r over H and so we have the non-holonomic connection of order r on H (straight by the definition). From similar reasons as in [1] follows that a non-holonomic connection of order r on H in the real case always exists.

3. Induced connection and prolongation.

Let H be a principal bundle with the structural group G and let M be a vector space and R a representation of G in M. Let $\tilde{S}^{r,s}_z(H)$ be the vector space of all non-holonomic r-jets of H into $\tilde{T}^s(H)$ with source $z \in H$. A vector $X \in \tilde{S}^{r,s}_z(H)$ is said to be vertical if $p_*\beta(X)$ is a zero vector of $\tilde{T}^s_{p(z)}(B)$. Denote by $R_{g*}X$ the element $(j^{r+s}_z R_g)_* X$ of $\tilde{S}^{r,s}_z(H)$. The operation $(\tilde{j}^{r+s}_z R_g)_* X$ is defined as follows: we know that $X \in \tilde{S}^{r,s}_z(H)$ is an element of $J^r(H, T^s(H))$. If $r = 1$, then $X = j^1_z\sigma$, σ being a cross-section in $\tilde{T}^s(H)$ over a neighborhood of the point $z \in H$. We have now a cross-section $\tilde{\sigma} : z \to (j^s_z R_g)_*\sigma(z)$ in $\tilde{T}^s(H)$ over a neighborhood of the point $zg \in H$. We denote then $j^1_{zg}\tilde{\sigma} = (j^{s+1}_z R_g)_* j^1_z\sigma$. It is clear now how is the mapping $(j^{r+s}_z R_g)_*$ defined for $r > 1$. It is the prolongation of the composition rule defined by the transformations of $(j^s_z R_g)_*$ on $\tilde{T}^s(H)$.

DEFINITION 3.1. *An M- valued differential form φ of degree $m > 0$, order (r, s) on a principal bundle H is said to be a tensorial form of degree $m > 0$, order (r, s), type $\mathfrak{R}(G)$, if the following conditions are satisfied :*

a) if at least one of the vectors $X_1, \dots, X_m$ is vertical, then $\varphi(X_1, \dots, X_m) = 0$.

b) $\varphi(R_{g*}X_1, \dots, R_{g*}X_m) = \mathfrak{R}(g^{-1})\varphi(X_1, \dots, X_m)$.

PROPOSITION 3.1. *Let $Y \in \tilde{J}^r(V, N)$, $Z \in \tilde{J}^r(N, W)$ and let j^k be the projection of the non-holonomic r-jets into the non-holonomic k-jets. Then $(j^k Z)(j^k Y) = j^k(ZY)$.*

PROOF. We know that

$$j^{k+l} = j^k \circ j^l.$$

Let σ_1 be a cross-section in $\tilde{J}^{r-1}(V, N)$ over a neighborhood of the point $x \in V$ and let σ_2 be a cross-section in $\tilde{J}^{r-1}(N, W)$ over a neighborhood of the point $\beta(\sigma_1(x)) = y \in N$ and let $Y = j^1_x \sigma_1$, $Z = j^1_y \sigma_2$. Let

$$\sigma : u \to \sigma_2(v)\sigma_1(u), \quad v = \beta(\sigma_1(u))$$

be a cross-section in $\tilde{J}^{r-1}(V, W)$ over a neighborhood of $x \in V$. We have then $ZY = j^1_x \sigma$ and then $j^{r-1}(j^1_x \sigma) = \sigma(x)$, $j^{r-1}Z = \sigma_2(y)$, $j^{r-1}Y = \sigma_1(x)$. Then $\sigma(x) = \sigma_2(y)\sigma_1(x)$. We have proved then the theorem for the case $k = r-1$, but it is clear that by induction one can easily prove that the theorem is true for an arbitrary $k < r$.

THEOREM 3.1. [5]. *Let C be a cross-section in $\tilde{Q}^r$ over B, i.e. a non-holonomic connection of order r. Denoting by X the element $C(x)$, $x \in B$, we have the cross-section $x \to j^1(X) = C_1(x)$ in Q^1 over B. Let*

$$X' = (j^1_{(X, \check{x})} \psi)(j^1_x C, j^1 X).$$

The mapping $x \to X'$ is a cross-section in $\tilde{Q}^{r+1}$ over B, i.e. a non-holonomic connection of order $r+1$ which is called the prolongation of the non-holonomic connection C. ψ is the composition rule

$$\psi : \tilde{J}^r(B, \Phi) \times \Phi \to \tilde{J}^r(B, \Phi).$$

PROOF. We first show what ψ, the composition rule, is looking like. Let $\lambda : \Phi * \Phi \to \Phi$ be the composition of the groupoid Φ. Let $A \in \tilde{J}^r(B, \Phi)$,

$\alpha(A) = x \in B, \beta(A) = \theta' \in \Phi, D \in \tilde{J}^r(B,\Phi), \alpha(D) = x, \beta(D) = \theta$. Let $D = j^r_x k$, k being a mapping of a neighborhood of $x \in B$ into the point $\theta \in \Phi$. We have then $A \bullet D = (j^r_{(\theta',\theta)}\lambda)(A,D) \in \tilde{J}^r(B,\Phi)$.

We can identify D with the point θ and write then $\psi(A,\theta) = A \bullet D = A \bullet \theta$. On the basis of $(a \circ \lambda)(\theta',\theta) = a(\theta)$ we have

$$\{(j^r_{(\theta',\theta)}\hat{a}) \circ \psi\}(A,D) = (j^r_\theta \hat{a})D.$$

We have further $j^1_x C \in \tilde{J}^{r+1}(B,\Phi)$, $j^1 X \in J^1(B,\Phi)$ and then

$$(j^{r+1}_{\check{x}}\hat{a})(j^1_{(X,\check{x})}\psi)(j^1_x C, j^1 X) = (j^{r+1}_{\check{x}}\hat{a})j^1 X.$$

But we know that $(j^r_{\check{x}}\hat{a})X = \hat{j}^r_x$ and using the operation of projection j^1we have $(j^1_{\check{x}}\hat{a})j^1 X = \hat{j}^1_x$. Let l be a mapping of a neighborhood of the point x on B into the point $j^1 X \in J^1(B,\Phi)$ and let us identify $j^r_x l$ with the point $j^1 X$. We have then $(j^{r+1}_{\check{x}}\hat{a})j^1 X = \hat{j}^{r+1}_x$. Analogously on the basis of $(b \circ \lambda)(\theta',\theta) = b(\theta')$ the relation $\{(j^r_{(\theta',\theta)}\hat{b}) \circ \psi\}(A,D) = (j^r_{\theta'}\hat{b})A$ holds and then

$$(j^{r+1}_{\check{x}}\hat{b})(j^1_{(X,\check{x})}\psi)(j^1_x C, j^1 X) = (j^{r+1}_{\check{x}}\hat{b})(j^1_x C) = j^{r+1}_x.$$

THEOREM 3.2. *Let X' be a prolongation of order k of the element X of a non-holonomic connection of order r, C with respect to C. Then $j^r X' = X$.*

PROOF. Denote by τ the operation of prolongation of the 1st order of X with respect to C and further

$$\tau^k X = \underbrace{\tau(\tau(\ldots(\tau}_{k \text{ times}}(X))\ldots) = X';$$

$\tau^2 X$ is prolongation of the 1st order of the element τX with respect to the prolongation C' of the first order of C. It is sufficient to prove the theorem in the case $k = 1$. By the definition $X' = (j^1_{(X,\check{x})}\psi)(j^1_x C, j^1 X)$. Then

$$j^r X' = j^r(j^1_{(X,\check{x})}\psi)j^r(j^1_x C, j^1 X) = \psi(C(x),\check{x}) = X.$$

THEOREM 3.3. *Let be given a non-holonomic connection of order r on H by the form ω. This connection uniquely gives rise to the non-holonomic connection of order k, $(k \leq r)$ on H with the form $\omega_{(k)} = j_k \omega$, j_k being the projection defined by the canonical projection j^k for non-holonomic r-jets into k-jets. The following diagram is commutative.*

$$\begin{array}{ccc} \tilde{T}^k_z(H) & \xrightarrow{\quad \omega_{(k)} \quad} & \tilde{T}^k_e(G) \\ \Big\downarrow {\scriptstyle (j^k)_*} & & \Big\downarrow {\scriptstyle (j^k)_*} \\ \tilde{T}^r_z(H) & \xrightarrow{\quad \omega \quad} & \tilde{T}^r_e(G) \end{array}$$

$(j^k)_*$ is the linear mapping associated to the projection j^k.

PROOF. Let $X \in \tilde{Q}^r$ be an element of the non-holonomic connection of order r and let $X_{(k)} = j^k X$. It is clear that

$$\alpha(X_{(k)}) = x,\ \beta(X_{(k)}) = \tilde{x},$$

$$(j^k_x \hat{a})X_{(k)} = \{j^k(j^r_x \hat{a})\}\{j^k X\} = j^k\{(j^r_{\tilde{x}} \hat{a})X\} = j^k(\hat{j}^r_x) = \hat{j}^k_x.$$

Analogously we have

$$(j^k_x \hat{b})X_{(k)} = \{j^k(j^r_x \hat{b})\}\{j^k X\} = j^k\{(j^r_{\tilde{x}} \hat{b})X\} = j^k(j^r_x) = j^k_x.$$

Further

$$j^k\{X^{-1}j^r_z\} = j^k[j^r_{(\tilde{x},z)}\psi\ (X^{-1}p\,j^r_z, j^r_z)] =$$
$$= (j^k_{(\tilde{x},z)}\psi)(X^{-1}_{(k)}p\,j^k_z, j^k_z) = X^{-1}_{(k)}j^k_z,$$

ψ being the mapping $\psi : \Phi * H \to H;\ \psi(\theta, z) = \theta z$. We have $j^k(j^r_z z^{-1}) = j^k_z z^{-1}$. Then $\omega_{(k)} = \{(j^k_z z^{-1})(X^{-1}_{(k)} j^k_z)\}_* = j_k\,\omega$ the form of the non-holonomic connection of order k associated to ω and to the projection j^k. Let further $L \in \tilde{J}^r_e(G)$, then $j^k L \in \tilde{J}^k_e(G)$, $j^k L(j^k_z z^{-1})(X^{-1}_{(k)} j^k_z) \in \tilde{J}^k_z(H)$. But it is easy to see that $j^k L(j^k_z z^{-1})(X^{-1}_{(k)} j^k_z) = j^k L\{(j^r_z z^{-1})(X^{-1} j^r_z)\}$. If we take the dual vector spaces, we see then that the diagram is commutative.

Let $\omega^{(k)} = i^k \omega$ be the form of the non-holonomic connection of order $r+k$, which is the prolongation of order k of the non-holonomic connection of order r given by the form ω. The operator i^0 is the identity. On the basis of the theorem 3.3 we see that the diagram

$$\begin{array}{ccc}
\tilde{T}^{r+k}_z(H) & \xrightarrow{\omega^{(k)}} & \tilde{T}^{r+k}_e(G) \\
{\scriptstyle (j^r)_*}\downarrow & & \downarrow{\scriptstyle (j^r)_*} \\
T^r_z(H) & \xrightarrow{\omega} & \tilde{T}^r_e(G)
\end{array}$$

is commutative.

Let ω be the form of the non-holonomic connection C of order r on H. This connection gives rise to $(r-1)$ connections of order r on H. These connections are given by the forms $\pi_\alpha = i^{r-\alpha} j_\alpha \omega$; $\alpha = 1, 2, \ldots, r-1$. The forms $\varkappa_\alpha = \omega - \pi_\alpha$ are the n $\tilde{T}^r_e(G)$-valued tensorial forms of degree 1, order $(0, 1)$, type $Ad(G)$. In the notation introduced above $i^1\omega$ is the form of a connection of order $s+1$ if ω is the form of a connection of order s. If $\varkappa_\alpha$ is a zero form, then $\omega = i^{r-\alpha} j_\alpha \omega$ and on the basis of the theorem 3.3 we have $j_\beta \omega = i^{\beta-\alpha} j_\alpha \omega$, $\beta = \alpha, \alpha+1, \ldots, r-1$. Then $\pi_\beta = i^{r-\beta} j_\beta \omega = i^{r-\alpha} j_\alpha \omega = \omega$ and then $\varkappa_\beta$ is a zero form for each $\beta = \alpha, \ldots, r+1$. We have proved then the theorem.

THEOREM 3.4. *Let C be a non-holonomic connection of order r on H. This connection gives rise to $r-1$ non-holonomic connections of order s $(s = 1, 2, \ldots, n-1)$ on H and to $r-1$ $\tilde{T}^r_e(G)$-valued tensorial forms $\varkappa_\alpha$ of degree 1, order $(0, 1)$, type $Ad(G)$. The form $\varkappa_\alpha$ is a zero form if and only if there exists a non-holonomic connection C_α so that the connection C is a prolongation of order $r-\alpha$ of the connection C_α.*

Let $H'(B', G')$, $H(B, G)$ be two principal bundles. Let $\varphi: B' \to B$ be an imbedding of B' into B, ρ a homomorphism of G' into G and f a mapping of H' into H compatible with ρ, i.e.

(1) $f(z'g') = f(z')\rho(g')$,

(2) $p(f(z')) = \varphi(p'(z'))$, $z' \in H'$, $g' \in G'$.

The mapping f is called an immersion of the principal bundle H' into H. A linear mapping π of $\tilde{T}^r_e(G)$ into $\tilde{T}^r_e(G')$ is called an invariant projection if

a) $\pi(\rho_*(X')) = X'$, $X' \in \tilde{T}^r_{e'}(G')$,

b) $\pi(Ad(\rho(g'))X') = Ad(g')\pi(\rho_*(X'))$, $g' \in G'$, $X' \in \tilde{T}^r_{e'}(G')$.

c) There exists a non-holonomic r-jet $Z \in \tilde{J}^r(G, G')$ so that $\alpha(Z) = e \in G$, $\beta(Z) = e' \in G'$, $Z_* = \pi$.

THEOREM 3.5. *Let ω be the form of a connection of order r on H. Let $f : H' \to H$ be a homomorphism of the principal bundle H' into H and*

$$\pi : \tilde{T}^r_e(G) \to \tilde{T}^r_{e'}(G')$$

an invariant projection. Then the form $\omega' = \pi \omega f_*$ is the form of a non-holonomic connection of order r on $H'(B', G')$. We shall speak about the induced non-holonomic connection of order r. The induction of the non-holonomic connection is invariant under prolongation and projection of the connection.

PROOF. Let $Y_{z'}$ be the vector of the fundamental vector field on H' at the point z', which belongs to $Y' \in \tilde{T}^r_{e'}(G')$. Then $f_*(Y_{z'})$ is a vertical vector of H at the point $f(z')$. From the definition of an immersion f follows that $f_*(Y_{z'})$ is a vector of the fundamental vector field on H associated to $\rho_*(Y') \in \tilde{T}^r_e(G)$. Denoting by $h_{z'} : G' \to H'_{p'(z')}$, $h_z : G \to H_{p(z)}$, $h_{z'}(e') = z'$, $h_z(e) = z = f(z')$ the respective homomorphisms we see that the mapping f is identical with $h_z^{-1} \circ \rho \circ h_{z'}$. Let $X_{z'} \in \tilde{T}^r_{z'}(H')$, $R_{g'*} X_{z'} = X_{z'g'} \in \tilde{T}^r_{z'g'}(H')$. We have $f_*(X_{z'g'}) = R_{\rho(g')} * f_*(X_{z'})$. On the basis of the relation

$$\omega(R_g * X_z) = Ad(g^{-1})\omega(X_z),\ X_z = f_*(X_{z'})$$

we have

$$\pi\omega(R_g * X_z) = \pi(Ad(g^{-1})\omega(X_z)) = Ad(g'^{-1})\pi\omega(X_z) = $$
$$= Ad(g'^{-1})\omega'(X_{z'}).$$

Let K be the r-jet associated to ω. Let Z be an r-jet with the property c from the definition of an invariant projection. Let

$$W = (j^r_{e'} z')Z(j^r_z z^{-1})K(j^r_{z'} f).$$

It is easy to see that $W \in \tilde{J}^r(H', H'_{p'(z')})$, $\alpha(W) = \beta(W) = z'$ and $W_* = (j^r_{e'} z')_* \omega'$. Because $\omega = \{(j^r_z z^{-1})(X^{-1} j^r_z)\}_*$, $\omega_{(k)} = \{(j^k_z z^{-1})(X^{-1}_{(k)} j^k_z)\}_*$ and $\omega' = \{Z(j^r_z z^{-1})(X^{-1} j^r_z)(j^r_{z'} f)\}_*$ we have

$$j_k \omega' = \{(j^k Z)(j^k_z z^{-1})(X^{-1}_{(k)} j^k_z)(j^k_{z'} f)\}_* = (j^k Z)_* j_k \omega (j^k_{z'} f)_* = \pi(j_k \omega) f_*.$$

On the basis of this condition it is immediately clear that the induction is invariant with respect to the prolongation.

REMARK. It is possible to show that the space $G/\rho(G')$ being of a certain special type (generalized weak reductivity) we have an invariant projection uniquely given.

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