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Harmonic Analysis and Unitary Group Representations:

The Development from 1927 to 1950

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1. Introduction.

The organizers of this conference, on the development of harmonic analysis between 1930 and 1950, asked me to speak on the infinite dimensional case. Insofar as one identifies modern harmonic analysis with the theory of group representations and their decomposition into irreducibles one can interpret the phrase "infinite dimensional" in two different ways. The question is as to whether it is the representations or their irreducible constituents that are to be infinite dimensional. I suspect that it is the second interpretation which the organizers had in mind. However, in either case, a paper published by Hermann Weyl in 1927 played a key role and I propose to treat both interpretations to some extent and to begin my account with the year 1927 rather than 1930.

To set the stage let us recall that, after an abortive beginning in the mid-eighteenth century, harmonic analysis (in the sense of expanding functions in Fourier series or as Fourier integrals) began in earnest with a celebrated memoir of Fourier completed in 1807 and Fourier's book on heat conduction published in 1822. The theory of group representation began almost a century later in 1896 with Frobenius' beautiful extension of the group character notion from finite commutative groups to finite non commutative groups. This theory developed as a new and exciting branch of the theory of finite groups until Frobenius' student, I. Schur, noticed in 1924, that much of the theory could be extended to a special class of of compact Lie groups – the orthogonal groups in n dimensions. This work of Schur stimulated Hermann Weyl to make the observation that one could do the same for all compact Lie groups and to publish a celebrated three-part paper in 1925 and 1926 determining the irreducible representations of all such groups.

2. The synthesis of Peter and Weyl and the stimulus of quantum mechanics (1927-1931).

In 1927 Hermann Weyl published two quite different papers each having considerable bearing on the subject of this lecture. One appeared in *Mathematische Annalen*, was written in collaboration with F. Peter, and was entitled "Die Vollständigkeit der primitiven Darstellungen einer geschossenen kontinuerlichen Gruppe". The other appeared in

Zeischrift der Physik and bore the title, "Quantenmechanik und Gruppentheorie".

The first paper contains the celebrated Peter-Weyl theorem which has several equivalent formulations. For our purposes it is best regarded as continuing the program of Schur and Weyl begun in 1924 by proving a generalization to compact Lie groups of the following fundamental theorem about finite groups: The regular representation of any finite group G is a direct sum of irreducible representations and each irreducible representation of G occurs with a multiplicity equal to its dimension. The regular representation of a connected compact Lie group is necessarily infinite dimensional and infinite dimensional representations enter representation theory here (in a sense) for the first time. However, Peter and Weyl noticed that when the theorem is applied to the special case of the additive group of all real numbers modulo 2π, the irreducible representations are all one dimensional and are of the form $x \rightarrow e^{nix}I$ for $n = 0, \pm 1 \pm 2$, etc., and the theorem reduces to the statement that every complex valued function of period 2π which is square summable on finite intervals is of the form $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ where convergence is in the mean. In other words, the Fourier expansion theorem for square summable functions is a special case of the Peter-Weyl theorem, and one sees that in a sense harmonic analysis was concerned with infinite dimensional group representations from its beginnings. On the other hand, in recognizing the essentially group theoretical character of Fourier analysis and unifying it with the theory of group representations, Peter and Weyl greatly expanded the scope of harmonic analysis itself. Indeed one can replace the reals modulo 2π by much more general groups and seek analogous theorems and hope for similarly far reaching applications.

It is interesting to note that if one decomposes the regular representation of a finite commutative group G one can interpret the result as stating that every complex valued function f on G can be written uniquely in the form $f(x) = \sum c_\chi \chi(x)$ where the c_χ are complex numbers and the sum is over all "characters" of G, i.e.., is over all complex valued functions on G which are nowhere zero and satisfy the identity $\chi(xy) = \chi(x)\chi(y)$ for all x and y in G. One shows without difficulty that the c_χ can be computed from f by the formula $c_\chi = \frac{1}{o(G)} \sum_{x \in G} f(x)\overline{\chi(x)}$. The analogy with the theory of Fourier series is striking and it is a fact that such "finite Fourier analysis" played an important role in number theory throughout the nineteenth century. Moreover, it seems to have been a problem in number theory that led Frobenius, at the instigation of Dedekind, to seek a generalization of characters in the above sense and thus to invent group representations. The word character goes back to Gauss who made use of characters whose squares are the identity in his celebrated *Disquisitiones Arithmeticae* of 1801. However it should not be forgotten that mathematicians did not begin to "think group theoretically" until late in the nineteenth century and it did not become popular to do so until the middle of the twentieth century. Thus these early uses of finite Fourier analysis were not recognized as such but appeared as *ad hoc* devices.

Weyl's paper, "Quantenmechanik und Gruppentheorie" contained the second application of the theory of group representations to the new new quantum mechanics. The first was made a few months earlier by E. Wigner in his paper, "Über nicht kombinierende Terme in der Neueren Quantentheorie II". Weyl cited Wigner's paper but asserted that his application was of quite a different nature – as indeed it was. Wigner showed how the diagonalization of self adjoint operators could be simplified when these operators commuted with a group representation. Weyl wished to clarify the foundations of quantum mechanics by explaining the fact that the self adjoint operators P and Q corresponding to the x, y amd z components of position and momentum satisfy the celebrated Heisenberg commutation relations $QP = PQ = hiI$ where h is Planck's constant and $i^2 = -1$.

Weyl's suggestive but not completely satisfying explanation was based on the observation that (to within unitary equivalence) the equation $QP - PQ = hiI$ has a unique irreducible solution. In order to work with bounded operators instead of the necessarily unbounded P and Q Weyl introduced the one parameter families $U_t = e^{itQ}$, $V_s = e^{isP}$ and pointed out that (on a formal level at least) the Heisenberg commutation relations hold if and only if $U_tV_s = V_sU_te^{isth}$ for all real s and t. Now it turns out that the one parameter families $t \rightarrow e^{itA}$ where A is self adjoint are precisely the same as the unitary representations $t \rightarrow U_t$ of the additive group of the real line (provided that one includes unbounded self adjoint operators and includes the appropriate continuity conditions in the definition of unitary representation). Thus the uniqueness of the irreducible solutions of the Heisenberg commutation relations is equivalent to a theorem about pairs of unitary representations of the real line.

As Weyl observed, this theorem about pairs of representations can be reformulated in a much more striking way. Given any pair U, V define $W_{t,s} = U_t V_s$. A trivial calculation shows that U and V satisfy the identity written down above if and only if W satisfies the identity

$$W_{(t_1,s_1)+(t_2,s_2)} = e^{ihs_1t_2} W_{t_1,s_1} W_{t_2,s_2}.$$

This identity implies that W is "almost" a unitary representation of the group R_2 of all pairs of real numbers under component wise addition. It fails to be such because the defining identity is "spoiled" by the numerical factor $e^{ihs_1t_2}$. However, it is a so-called projective or ray representation. Quite generally one calls a linear operator valued function $x \rightarrow A_x$ defined on a group G a projective representation if it satisfies the identity $A_{xy} = \sigma(x,y)A_xA_y$ for all x and y in G where σ is some scalar valued function defined on $G \times G$. For finite groups, projective representations were studied systematically by I. Schur in papers published in 1904 and 1907.

It follows from the above that pairs of unitary representations of the real line satisfying the identity $U_tV_s = V_sU_te^{isth}$ correspond one-to-one in a natural way to the unitary projective representations of R_2 with multiplier σ where $\sigma(t_1, s_1; t_2, s_2) = e^{ihs_1t_2}$. The more striking form in which Weyl stated his global or integrated form of the uniqueness theorem for solutions of the Heisenberg commutation relations is as follow: With respect to the multiplier σ for R_2, defined above, the group R_2 has a unique equivalence class of irreducible projective unitary representation. The significance of this result increases when one realizes (as Weyl showed) that to within a certain natural notion of equivalence there is only one non trivial multiplier for R_2. It increases still further when one realizes that its generalization to R_{2n} is also true provided that one replaces "non trivial" by "non degenerate". To Weyl the relationship of solutions of the Heisenberg commutation to such a simple result about the representation theory of even dimensional vector groups was the key to understanding why quantum mechanical operators take the form that they do. He expressed it as follows: "The kinematical structure of a physical system is expressed by an irreducible group of unitary representation in system spaces".

Weyl was writing for physicists and gave only heuristic proofs of the theorems about unitary group representations and self adjoint operators which he used. Since these theorems were of major importance for the further development of harmonic analysis it is fortunate that the task of giving rigorous proofs was soon taken up by Marshall Stone and completed a year later by J. von Neumann. In 1930 Stone published a note in the Proceedings of the National Academy of Sciences of the U.S.A. announcing two theorems and referring to Weyl's paper for an explanation of their significance for physics. One of these theorems was the uniqueness theorem for irreducible solutions of $U_tV_s = V_sU_te^{isth}$. The other was a version of the spectral theorem for self adjoint operators in which unitary representations of the additive group of the real line replace self adjoint operators. In slightly modernized language the latter can be stated as follows. Let $t \to U_t$ be an arbitrary unitary representation of the additive group of the real line. Then there exists a unique mapping $E \to P_E$ of Borel subsets of the line E into projection operators P_E in the Hilbert space $\mathcal{H}(U)$ in which U acts such that the following conditions hold:

(1) When E is the empty set $P_E = 0$ and when E is the whole real line $P_E = I$ the identity operator

(2) $P_{E\cap F} = P_EP_F = P_FP_E$ for all Borel sets E and F.

(3) If $E_1, E_2, \ldots$ are disjoint Borel sets and $E = E_1 \cup E_2 \ldots$ then $P_E = P_{E_1} + P_{E_2} + \cdots$.

(4) For all $\phi \in \mathcal{H}(U)$ and all real t

$$U_t(\phi) \cdot \phi = \int e^{ixt} d(P_x(\phi) \cdot \phi)).$$

By way of explanation of (4) we remark that for each fixed ϕ, $(P_E(\phi), \phi)$ as a function of E is a measure α_ϕ and the right hand side of (4) is defined to be the integral of $x \to e^{ixt}$ with respect to the measure α_ϕ.

Functions from Borel sets to projection operators are called *spectral measures* or *projection valued* measures when they satisfy (1), (2), and (3). We note for future reference that this definition makes sense when the real line is replaced by any topological space or, more generally, for any space in which an abstract notion of Borel set can be defined.

Stone's theorem says that for every unitary representation of the additive group of the real line there is a projection valued measure on the real line from which U can be reconstructed using the formula in (4). Conversely, it is relatively easy to show that given any projection valued measure P on the line, the equation in (4) may be used to define a unitary representation U having P as its associated projective valued measure. Thus Stone's theorem sets up a one-to-one correspondence between *all* projection valued measures on the real line on the one hand and *all* unitary representations of the real line on the other. Now the spectral theorem of Hilbert as generalized to unbounded self adjoint operators by Stone and von Neumann has a very similar structure. It sets up a one-to-one correspondence between all self adjoint operators H on the one hand and all projection valued measures on the real line on the other such that, when P and H correspond, one has the following analogue of the equation in (4). For all ϕ in the domain of H, $(H(\phi) \cdot \phi) = \int x d(P_x(\phi) \cdot \phi)$.

We now have *two* one-to-one correspondences each having the sets of all projection valued measures on the line as one term. Combining them yields a one-to-one correspondence between all self-adjoint operators on the one hand and all unitary representations of the real line on the other. One shows without difficulty that when U and H correspond they are related by the formula $U_t = e^{iHt}$ for all real t. This corollary of Stone's spectral theorem is what is popularly known as Stone's theorem and whose presumed truth played a key role in Weyl's paper.

Stone's paper contains an outline of the proof of his spectral theorem but not of the uniqueness theorem for pairs satisfying $U_t V_s = V_s U_t e^{isth}$. In fact, he never published his proof and the gap was remedied by a proof which von Neumann published in 1931. (According to Dixmier, Stone once told him that he had found an error in his original proof.) In any event, we now speak of the Stone-von Neumann theorem.

In 1929 E. Cartan published a slight generalization of the Peter-Weyl theorem in which the compact Lie group G was replaced by a Riemannian space S on which the group G acts transitively and so as to preserve the metric. This was interesting not only because it indicated a further extension of the scope of harmonic analysis but also because, when S is the surface of a sphere in three space and G is the three dimensional rotation group, Cartan's theorem reduces to a theorem about the expansion of functions in surface harmonics. Such expansions were introduced by Laplace and Legendre at the very end of the eighteenth century a decade or so before Fourier's celebrated memoir. It is gratifying to have such an elegant unification of these two theories.

Although this was not explicitly brought out at the time, the two theorems suggested by Weyl's work on the foundations of quantum mechanics, and proved by Stone and von Neumann, had considerable significance for the newly emerging non commutative harmonic analysis. First of all, Stone's theorem connecting self adjoint operators with unitary representation of the additive group of the real line showed that the spectral theory of Hilbert and his students, including spectral multiplicity theory, could be reinterpreted as a more or less complete working out of the Frobenius theory of group representations for one particular non compact group. The additive group of the real line is locally compact and commutative but is *not* compact. In the second place the uniqueness theorem, as already explained, is equivalent to the determination of all projective unitary representations of R_2 whenever the multiplier σ is non trivial. This is the first such determination for a non compact group and also the first in which the irreducible representations may be infinite dimensional. Moreover, if one insists on examples not involving projective multipliers one has only to use Schur's idea of relating projective representations of a group G to the ordinary representations of the central extension of G defined by the multiplier σ. In the case at hand the resulting two step nilpotent group is the now well known "Heisenberg group".

3. Haar measure, Pontrjagin duality and commutative Banach algebras (1933-1940).

The stimulus for the extension of the Frobenius theory of group representations from finite groups to compact Lie groups was Schur's observation that the key tool of summing over the elements of a finite group could be replaced by integration over the group manifold. The integration process (due to Hurwitz) made use of differential forms and was thus applicable only to Lie groups. The way was cleared for extending the Frobenius theory to all separable locally compact groups in a remarkable paper by Alfred Haar published in 1933. Haar proved that every separable locally compact group admits a measure defined on all Borel sets, finite on compact sets and invariant under right translation. That this measure is unique up to a multiplicative constant was proved by von Neumann in 1936.

Of course there is also a left invariant measure with the same properties but the two need not coincide. When they do the group is said to be unimodular. Commutative locally compact groups of course are unimodular but so are compact groups and semi simple Lie groups. As Haar himself observed, using his invariant measure one can extend the Peter-Weyl theorem and many other theorems about compact Lie groups to arbitrary separable compact groups.

The next step was not motivated by the extension of harmonic analysis made possible by the discoveries of Peter, Weyl, Haar, etc., but by the needs of algebraic topology. This was the development of the duality theory of locally compact commutative groups by Pontrjagin and von Kampen in 1934 and 1935. Let G be any locally compact commutative group and let us define a character χ of G to be a continuous function from G to the complex numbers of modulus one which satisfies the identity $\chi(xy) = \chi(x)\chi(y)$ for all x and y in G. (This notion for finite commutative groups was introduced by Weber in 1881 as an abstraction of a definition given three years earlier by Dedekind for the ideal class group of an algebraic number field. Dedekind's use of the word character goes back to Gauss who introduced characters of order two to distinguish binary quadratic forms with the same discriminant). Let $\widehat{G}$ denote the set of all characters of G. It is easy to see that the pointwise product of any two members of $\widehat{G}$ is again such and that with respect to this operation $\widehat{G}$ itself is a group. It is called the group *dual* to G. It is of course a commutative group. In addition, it is locally compact with respect to a natural topology – essentially the topology of uniform convergence on compact subsets. The use of the word "dual" is suggested and justified by the following. Consider $\chi(x)$ where $x \in G$ and $\chi \in \widehat{G}$ and consider $\chi(x)$ as a function of χ for fixed x. In other words, for each x in G define a function f_x on $\widehat{G}$ by setting $f_x(\chi) = \chi(x)$. One verifies at once that for each x, f_x is a character of $\widehat{G}$, i.e., a member of $\widehat{\widehat{G}}$. One also checks easily that the mapping $x \to f_x$ is a homomorphism of G into $\widehat{\widehat{G}}$. The celebrated duality theorem of Pontrjagin and von Kampen says that more is true: this homomorphism is an isomorphism onto and it and its inverse are both continuous. Thus G and $\widehat{\widehat{G}}$ are isomorphic as topological groups and G is just as much the dual of $\widehat{G}$ as $\widehat{G}$ is the dual of G. Locally compact commutative groups occur in dual pairs. In particular, the dual of every separable compact commutative group is countable and the dual of every countable commutative group is compact and separable. This much was proved by Pontrjagin. The extension to the non separable case and to general locally compact commutative groups was made by van Kampen.

The connection of this theory with harmonic analysis comes about because for each character χ the mapping $x \to \chi(x)I$ where I is the identity operator in a Hilbert space is a unitary representation of G which is evidently irreducible if and only if the Hilbert space is one dimensional. Moreover one can show that every irreducible unitary representation of G is of the indicated form for some character χ. Thus $\widehat{G}$ can be thought of as the set of all equivalence classes of unitary irreducible representations of G. Using Haar's invariant measure one can define an analogue of the regular representation for any separable locally compact group G. Its space is $\mathcal{L}^2(G,\mu)$ where μ is a right invariant measure and the regular representation R is defined by $R_x(f)(y) = f(yx)$. In the special case in which G is compact and separable the members χ of $\widehat{G}$ are mutually orthogonal and every member f of $\mathcal{L}^2(G,\mu)$ may be written uniquely in the form $f(x) = \sum_{\chi\in\hat{G}} c_\chi \chi(x)$ where convergence is in the Hilbert space norm and $c_\chi = \frac{1}{\mu(G)} \int_G f(\chi)\overline{\chi(x)}d\mu(x)$. In particular, one deduces that R is the direct sum of *all* irreducible unitary representations of G each occurring just once.

So much has already been noted for the special case in which G is a compact commutative Lie group. We show next how the duality idea allows us to view the formulae $f(x) = \sum_{\chi\in\widehat{G}} c_\chi \chi(x)$ and $c_\chi = \frac{1}{\mu(G)} \int_G f(x)\overline{\chi(x)}d\mu(x)$ in a form which is more symmetrical between G and $\widehat{G}$ and suggests a generalization to the case in which neither G nor $\widehat{G}$ is compact. First of all let us write the set of coefficients $\{c_\chi\}$ as a function g on $\widehat{G}$; that is, let g be the function on $\widehat{G}$ defined by the formula $g(\chi) = c_\chi$. Secondly let us note that since $\widehat{G}$ is countable and discrete the "counting measure" is a Haar measure ν for $\widehat{G}$ and sums over $\widehat{G}$ may be written as integrals over $\widehat{G}$ with respect to ν. With these notational changes the two formulae take the form:

$$f(x) = \int_{\widehat{G}} g(\chi)\chi(x)d\nu(\chi) \quad g(\chi) = \frac{1}{\mu(G)} \int f(x)\overline{\chi(x)}d\mu(x).$$

If we choose μ, as we may, so that $\mu(G) = 1$ they become essentially identical. Moreover one sees easily that

$$\int_G |f(x)|^2 = \int_{\widehat{G}} |g(\chi)|^2 d\nu(\chi).$$

Thus the mapping from square summable functions to their Fourier coefficients is a unitary map from $\mathcal{L}^2(G,\mu)$ to $\mathcal{L}^2(\widehat{G},\nu)$ whose inverse map has exactly the same form. In this formulation there is no need for either G or $\widehat{G}$ to be compact and the obvious conjecture is true. However it has to be formulated with care. Let G be any locally compact commutative group and for each f which is both summable and square summable let $\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}d\mu(x)$ where μ is some Haar measure on G. Then $\widehat{f}$ is a complex valued function on $\widehat{G}$. Moreover there exists a unique choice ν of the Haar measure (independent of f) such that

$$\int_{\widehat{G}} |\widehat{f}(\chi)|^2 d\nu(\chi) = \int_G |f(x)|^2 d\mu(x).$$

Thus the map $f \to \widehat{f}$ is a norm preserving linear map from a subspace of $\mathcal{L}^2(G,\mu)$ to $\mathcal{L}^2(G,\nu)$. Both the domain and the range of this map are dense in their respective Hilbert space. Hence it has a unique unitary extension from all of $\mathcal{L}^2(G,\mu)$ to all of $\mathcal{L}^2(\widehat{G},\nu)$. In the special case in which G is the additive group of the real line the most general character is $x \to g_y(x) = e^{ixy}$ so $\widehat{f}(y) = \widehat{f}(g_y) = \int_{-\infty}^{\infty} f(x)e^{ixy}dx$. This is just the classical Fourier transform. One carries over this terminology to the general case and speaks of $\widehat{f}$ as the Fourier transform of f. The theorem about the unitary character of the Fourier transform is called the Plancherel theorem in honor of the man who proved it in the classical case.

One can think of the Plancherel theorem as giving a decomposition of the regular representation of G into irreducibles. However it is a direct sum in the usual discrete sense only when $\widehat{G}$ is discrete, i.e., when G is compact. In the non compact case one has a "continuous direct sum decomposition" into "infinitesimal" components. This intuitive picture was later to lead to a precisely defined notion of "direct integral".

The above described application of the Pontrjagin duality theory to harmonic analysis on locally compact commutative groups appeared for the first time in a book by André Weil, "L'Intégration dans les Groupes Topologiques et ses Applications" whose publication was delayed for various reasons until 1940, but of which a complete typed manuscript was in the hands of a publisher at the end of 1936.

In 1934 a considerable stir was caused by the publication of a paper by von Neumann showing that H. Bohr's theory of almost periodic functions on the line could be generalized to functions defined on general (not necessarily commutative) topological groups. The very next year Weil published a short note showing how von Neumann's theory could be derived from the earlier theory of compact groups. Specifically he showed how to define, in a canonical way, a continuous homomorphism ϕ of any topological group G onto a dense subgroup of a compact group K in such a manner that the functions $x \rightarrow f(\phi(x))$ where f varies over the continuous complex valued functions on K are precisely the almost periodic functions on G. In the special case in which G is commutative and locally compact K can be taken to be the dual of the "discretified" dual of G. This result of course was included in Weil's book.

Weil's book was a comprehensive treatise giving a complete account of the Peter-Weyl theorem, the Pontrjagin-van Kampen duality theory and various theorems about the structure of locally compact commutative groups. In 1939, the year before Weil's book was finally published, the Princeton University Press published an English translation of a book by Pontrjagin entitled "Topological Groups", originally published in Russian: the two books overlapped considerably and both were extremely influential. They differed chiefly in that Pontrjagin went into detail about Lie groups and said almost nothing about connections with harmonic analysis. For Weil, on the other hand, harmonic analysis was the central point. In addition to his contributions to this subject already alluded to we shall mention one more for later use. This is his generalization of the notion of positive definite function and the theorems of Herglotz and Bochner. Weil defined a complex valued function f on a general topological group G to be *positive definite* if it is continuous and if $\sum_{i,j} c_i \bar{c}_j f(x_i x_j^{-1}) \geq 0$ for every choice of complex numbers $c_1, c_2, \cdots, c_n$ and group elements $x_1 \cdots x_n$. He then proved that if G is commutative and locally compact then f is positive definite if and only if it is of the form $f(x) = \int_{\widehat{G}} \chi(x) d\mu(\chi)$ for some finite positive measure on $\widehat{G}$. The special case in which G is the group of integers was proved by Herglotz in 1911 and that in which G is the additive group of R^n was proved by Bochner in his 1932 book on the classical Fourier transform.

In the interval between the completion of Weil's book in late 1937 and its publication in 1940 many of the facts about extending the Fourier transforms to locally compact commutative groups were discovered independently by I.E. Segal in the U.S.A. and I.M. Gelfand and D. Raikov in the USSR. These authors based their analysis on the fact that $\mathcal{L}^1(G)$ the set of all summable functions on G with respect to Haar measure μ is a linear associative algebra with respect to "convolution". Here one defines the convolution $f * g$ of f and g in $\mathcal{L}^1(G)$ to be the function such that $f * g(x) = \int f(xy^{-1})g(y)d\mu(y)$ and the integral is defined only almost everywhere. This algebra reduces to the classical group algebra when G is finite and the convolution concept is well known in classical Fourier analysis where the fact that $\widehat{f * g} = \widehat{f}\widehat{g}$ plays an important role. The convolution concept is also discussed in Weil's book. Unlike Segal, Gelfand and Raikov considered only the commutative case. On the other hand, Gelfand put the whole problem in the commutative case in a broader setting by beginning with an abstract object which he called a normed ring and is now called a Banach algebra. This is a Banach space in which an associative multiplication is defined having the property that $\|fg\| \leq \|f\|\|g\|$ for all f and f. Given any commutative Banach algebra $\mathcal{A}$ with an identity element the set $\mathcal{M}$ of all maximal ideals or equivalently the set of all homomorphisms ϕ of $\mathcal{A}$ into the algebra of all complex numbers can be given the structure of a compact topological space in such a manner that for each f in $\mathcal{A}$ the mapping $\phi \to \phi(f)$ regarded as a function on $\mathcal{M}$ is continuous. Denoting this function by $\tilde{f}$ one obtains a continuous homomorphism $f \to \tilde{f}$ of $\mathcal{A}$ into the commutative Banach algebra $\mathcal{C}(\mathcal{M})$ of all continuous complex valued functions on $\mathcal{M}$ with $\|\tilde{f}\| = \underset{\phi \in m}{\text{lub}}|f(\phi)|$. One speaks today of the homomorphism $f \to \tilde{f}$ of $\mathcal{A}$ into $\mathcal{C}(\mathcal{M})$ as the Gelfand map. It turns out that when G is a discrete commutative group the maximal ideals of $\mathcal{L}^1(G)$ correspond one-to-one to the members of $\widehat{G}$ and the Gelfand map reduces to the Fourier transform. When G is locally compact and commutative but not discrete, $\mathcal{L}^1(G)$ has no identity but can be embedded as a maximal ideal in a uniquely determined Banach algebra which does, and then the maximal ideals other than $\mathcal{L}^1(G)$ itself correspond one-to-one to the members of $\widehat{G}$, and the Gelfand map reduces once again to the Fourier transform. The advantage of the Gelfand approach is in providing a far reaching generalization of commutative harmonic analysis in which $\mathcal{L}^1(G)$ is replaced by a general commutative Banach algebra. The basic idea underlying the Gelfand map is due to Stone who studied it in the special case of the Banach algebra $\mathcal{C}(\mathcal{M})$ in a paper published in 1937.

Gelfand announced his results on abstract Banach algebras and their connection with *classical* commutative harmonic analysis in a series of three notes published in Russian in 1939. The extension to more general locally compact commutative groups was done in collaboration with D. Raikov and announced in a further note published in 1940. Slightly later in the same year Raikov published another note announcing the generalized Bochner theorem already found by Weil but not yet known in the USSR.

Segal's work is an extension of parts of his 1940 Yale Ph.D. thesis and was announced in a note in the Proceedings of the National Academy of Sciences of the USA in 1941. Most of the results in the thesis itself were announced in 1940 in Abstract 46-7-366 of the American Mathematical Society Bulletin. According to Segal he became aware of Gelfand's first 1939 announcement rather early but did not know about the Gelfand-Raikov work on applications to group algebras until his own work on the commutative case was complete.

4. Miscellaneous results in a period of transition (1936-1946).

The books of Weil and Pointrjagin marked the end of an era presenting as they did an almost complete generalization of the Frobenius-Schur theory of group representations to locally compact groups which are either compact or commutative. An obvious next step was to "close the gap", so to speak, by extending the theory to locally compact groups which are neither compact nor commutative. Haar's paper of 1933, of course, cleared away one obstacle but (due no doubt to the war of 1939-1945) a full scale systematic attack on this problem did not get under way until 1947. Nevertheless three more or less unrelated but extremely important steps beyond that of Haar were made before 1945, in 1936, 1939 and 1943, respectively, and in 1943 and 1944 both the compact and commutative theories were completed in an important point. We devote this section to a description of these five contributions.

We begin with the last two. While it is not difficult to deduce from the Peter-Weyl theorem that every unitary representation of a compact group is a discrete direct sum of finite dimensional irreducible representations, this was first spelled out explicitly in 1943 in a paper by A. Hurevitsch. The corresponding result for locally compact commutative groups is a bit more subtle since continuous rather than discrete direct sums are involved. However the proper formulation suggests itself as soon as one confronts Stone's spectral theorem for unitary representations of the additive group of the real line with Pontrjagin-van Kampen duality and remembers that the additive group of the real line is self dual. The key point is that the unitary representations of a commutative locally compact G are described by projection valued measure in the *dual* group $\widehat{G}$. The formula connecting the representation U of G with the corresponding projection valued measure in $\widehat{G}$ is an obvious generalization of that found by Stone: $(U_x(\phi) \cdot (\phi) = \int_{\widehat{G}} \chi(x) d(P_E(\phi) \cdot \phi)$. This theorem was stated and proved in independent papers published in 1944 by Ambrose in the U.S.A., Godement in France and Neumark in the USSR. Among physicists it is sometimes referred to as the SNAG theorem.

It will be convenient to discuss the three remaining papers in inverse chronological order. In 1943 Gelfand and Raikov proved a fundamental existence theorem stating that every locally compact group admits non trivial irreducible unitary representations and indeed sufficiently many so that given any x in G distinct from the identity there exists an irreducible unitary representation U such that U_x is not the identity. The proof made essential use of the notion of positive definite function on a group discussed earlier by Weil and Raikov and of a simple but important theorem connecting unitary representations of a group G with positive definite functions. Let U be such a representation and let ϕ be a unit vector in $\mathcal{H}(U)$. Then a straightforward computation shows that $x \to (U_x(\phi) \cdot \phi)$ is positive definite and one at the identity. Conversely if p is a positive definite function on G it is not difficult to show that one can use p to construct a unitary representation U such that $p(x) \equiv (U_x(\phi) \cdot \phi)$ for some ϕ. In addition one can show that the smallest closed invariant subspace containing ϕ is *irreducible* under U if and only in p is an *extreme point* in the convex set of all positive definite p with $p(e) = 1$. This connection established, the authors obtained their result as an application of the recently proved theorem (1940) of Krein and Milman on the existence of extreme points.

In 1939, the theoretical physicist, Eugene Wigner, already mentioned in section 1, published a celebrated paper in the *Annals of Mathematics* entitled "On Unitary Representations of the Inhomogeneous Lorentz Group". The inhomogeneous Lorentz group is a semi-direct product of the Lorentz group and the four dimensional commutative group of all translations in space-time. It is the group of all symmetries of space-time and is nowadays often referred to as the Poincaré group. If $\mathcal{H}$ is the Hilbert space of states of any relativistically invariant quantum mechanical system it follows from the general principles of quantum mechanics that there is associated with the system a canonical projective unitary representation of the symmetry group of space-time; that is, of the Poincaré group. Wigner wished to study and classify these representations in order to have a corresponding classification of physical systems.

Let G be a finite group which is a semi-direct product of a commutative subgroup N and some other subgroup H. Then every member h of H defines an automorphism α_h of the commutative group N and hence an automorphism α_h^* of the dual group $\widehat{N}$. Given χ and $\widehat{N}$ let $\mathcal{O}_\chi$, the "orbit of χ" be defined to be the set of all $\chi' \in \widehat{N}$ such that $\chi' = \alpha_h(\chi)$ for some χ. It is evident that two orbits are either identical or disjoint. As shown many years before by Frobenius, every irreducible representation L of G is associated with a unique orbit $\mathcal{O}$ in the sense that its restriction to N is a direct sum of all members of this orbit, each constituent occurring with the same multiplicity. Thus one may obtain a partial classification of the equivalence classes of irreducible representations by studying the action of H on $\widehat{N}$ and finding all possible orbits. Having done this, the problem remains of classifying the irreducible representations associated with each orbit $\mathcal{O}$. Let χ_0 be any member of $\mathcal{O}$ and let H_{χ_0} denote the subgroup of H, consisting of all h such that $\alpha_h^*(\chi_0) = \chi_0$. Frobenius showed that classifying the irreducible representations of G associated with $\mathcal{O}_{\chi_0}$ can be completely reduced to classifying the irreducible representations of H_{χ_0}. This procedure is sometimes called the Frobenius "little group method". The "little groups" are the groups H_χ.

Wigner showed that the "little group method" of Frobenius could be applied to the Poincaré group and led to only four different conjugacy classes of "little groups". Two of these were associated with classes of representations which could be ruled out on physical grounds and, of the other two, one was the rotation group in three space whose unitarity representations were well known and the other was a semi-direct product whose "little groups" were all commutative. Thus Wigner found all equivalence classes of irreducible unitary representations except those with "unphysical" properties. Moreover, he found the projective representations as well for the unique non trivial multiplier class. On the other hand, he made no attempt to deal with the analytical and topological questions which come up when one tried to give rigorous proofs. He confessed to this gap and said that it would bc taken care of by later work of von Neumann.

This work was of considerable mathematical interest for several reasons. Above all it was the first example (aside from that implicit in the uniqueness of thc solutions of the Heisenberg commutation relations) of a non compact, non commutative locally compact group, a large number of whose irreducible unitary representations had been explicitly determined – especially since these all turned out to be infinite dimensional. Secondly, it stimulated the later developmcnt of the unitary representation theory of the non compact semi-simple Lie groups. Indeed the two unphysical "little groups" were the Lorentz groups in 3 and 4 space-time dimensions which differ only slightly from $SL(2,R)$ and $SL(2,C)$ respectively. Thus to complete Wigner's work by finding *all* irreducible unitary representations, physical or not, one had to analyze the unitary irreducible representations of these groups.

In 1936 Murray and von Neumann published their now famous paper, "Rings of Operators", inaugurating a systematic study of what are now known as von Neumann algebras. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{A}$ be any family of bounded linear operators which contains A^* whenever it contains A. Let $\mathcal{A}'$, the *commutant* of $\mathcal{A}$, be the set of all bounded operators B such that $AB = BA$ for all A in $\mathcal{A}$. One sees easily that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ implies $\mathcal{A}'_2 \subseteq \mathcal{A}'_1$ and that $\mathcal{A}'' \supseteq \mathcal{A}$. Moreover, a simple argument shows that $\mathcal{A}''' = \mathcal{A}'$ for all $\mathcal{A}$. Every $\mathcal{A}'$ is of course a ring and in fact an algebra. The algebras of this special form are precisely the von Neumann algebras. One checks easily that the correspondence $\mathcal{A} \rightarrow \mathcal{A}'$ is a one-to-one inclusion inverting correspondence of the set of all von Neumann algebras defined on $\mathcal{H}$ onto itself.

Of course the set $B(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ is a von Neumann algebra whose commutant is the one-dimensional algebra of all complex number multiples of the identity. To get more general examples let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be a factorization of the Hilbert space $\mathcal{H}$ as a tensor product of two other Hilbert spaces. Let $\mathcal{A}_1$ denote the set of all operators in $\mathcal{H}$ of the form $T \times I$ where T is any bounded linear operator in $\mathcal{H}_1$. Then $\mathcal{A}_1$ is a von Neumann algebra and $\mathcal{A}'_1$ is equal to $\mathcal{A}_2$, the set of opeators in $\mathcal{H}$ of the form $I \times T$ where T is a bounded linear operator in $\mathcal{H}_2$. Von Neumann and Murray call any von Neumann algebra which can be constructed in this way from a factorization of $\mathcal{H}$ a *direct factor.* Evidently any direct factor $\mathcal{A}$ has the property that $\mathcal{A} \cap \mathcal{A}'$ is one-dimensional or equivalently that the center of $\mathcal{A}$ is one-dimensional. When $\mathcal{H}$ is finite-dimensional the converse is true. Any von Neumann algebra with $\mathcal{A} \cap \mathcal{A}'$ one-dimensional is a direct factor.

The theory of von Neumann algebras owes much if not all of its interest to the fact that this converse is *not* true when $\mathcal{H}$ is infinite-dimensional. Murray and von Neumann defined a *factor* to be a von Neumann algebra $\mathcal{A}$ with a one-dimensional center and gave examples of factors which are not direct factors. They did this in the course of developing a general theory of factors which resulted in a five-fold classification with five types to which they gave the labels $I_n, I_\infty, II_1, II_\infty$ and III. The type I factors are the direct factors, the index $n = 1, 2, 3 \cdots$ and ∞ denoting the dimension of $\mathcal{H}_1$ in the factorization. The other three types were interesting new objects whose exact nature is still under investigation over half a century later. The first example of factors of type II_1 were constructed using properly ergodic actions of discrete groups and the existence of type III was not confirmed until 1940.

The relevance of this work for the theory of group representations and non commutative harmonic analysis lies in the fact that the commuting algebra of any unitary group representation is a von Neumann algebra. Conversely every von Neumann algebra occurs in this way. Moreover, a unitary group representation has a commuting algebra which is a direct factor, i.e., a factor of type I if and only if the representation is a direct sum of mutually equivalent irreducible representations. This suggests that group representations having commuting algebras which are non type I factors may have interesting new properties with no parallel in the classical theory. This turns out to be the case. However none of this was made clear and explicit for a decade and a half.

In closing we mention the fact that von Neumann and Murray developed their theory in three more papers published in 1937, 1940, and 1943, respectively, the 1940 paper being by von Neumann alone. We also note that in 1943 Gelfand and Neumark published a paper connecting the von Neumann-Murray theory with Gelfand's theory of abstract commutative Banach algebras. Specifically they introduced a class of abstract algebras satisfying all the axioms of a commutative Banach algebra except the commutative law and endowed with an involuntary operator $A \to A^*$ with the properties of the adjoint operator for bounded operators on Hilbert space. In particular, the algebra $\mathcal{B}(\mathcal{H})$ of *all* bounded operators in a Hilbert space $\mathcal{H}$ satisfies the axioms and Gelfand and Neumark proved that every abstract Banach *-algebra has a canonical homomorphism into some $\mathcal{B}(\mathcal{H})$. This homomorphism coincides with the Gelfand map when the abstract *-algebra is commutative.

5. Beginnings of a systematic treatment of the general case (1947-1950).

In extending the theory of unitary group representations from compact groups and locally compact commutative groups to locally compact groups which are neither compact nor commutative one encounters at least three major difficulties. First of all, one is forced to consider irreducible representations which are infinite dimensional and consequently more difficult to find and classify. In particular, the trace of an infinite dimensional unitary operator never exists so that group characters must be defined indirectly when they exist at all. Secondly, just as in the locally compact but not compact commutative case, one cannot usually decompose the regular representation as a discrete direct sum of irreducibles but only as some kind of continuous direct sum. Stone's spectral theorem for unitary representations of the real line suggests the answer in the commutative case but it is by no means obvious how to go further. Thirdly, after one has the answer to the direct integral or continuous direct sum problem one finds that these decompositions do not always make it possible to reduce the classification problem to classifying the irreducibles. In fact, locally compact groups fall into two classes – those for which this reduction is not possible and those for which it is possible but non trivial. Those for which it is not possible turn out to be precisely those groups admitting non type I factor representations.

A full scale attack on these difficulties and more generally on the problem of extending the classical theory rather thoroughly to the general locally compact case began somewhat suddenly at the end of 1946. The three isolated contributions of von Neumann and Murray in 1936, Wigner in 1939, and Gelfand and Raikov in 1943 were augmented in 1947 by more than six contributions from six authors. Moreover, this year of intense activity marked the beginning of a long development which continues to this day.

Four of the six authors concerned themselves with determining the irreducible unitary representations of the simplest non compact semi-simple Lie groups $SL(2,R)$ and $SL(2,C)$. There were independent papers of Gelfand and Neumark, V. Bargmann and Harish-Chandra. Both Gelfand and Neumark and Bargmann studied both groups but their detailed papers complemented one another nicely, in that Gelfand and Neumark gave a complete account only for $SL(2,C)$, whereas Bargmann did so only for $SL(2,R)$. Harish-Chandra classified the irreducible representations of the Lie algebra of $SL(2,C)$, thus determining the representations of $SL(2,C)$ in "infinitesimal form". The connection between the actual and infinitesimal representation of Lie groups is considerably more subtle and difficult when the representations are infinite dimensional and the needed theory did not exist in 1947. Bargmann formulated his results on the group but used Lie algebra methods and an unproved conjecture to "prove" his final results. Only Gelfand and Neumark worked globally. Gelfand and Neumark also did more than find all irreducible unitary representations of $SL(2,C)$. They showed how to decompose every unitary representation as a sort of continuous direct sum of irreducible representations and in the special case of the regular representation proved a generalization of the Plancherel theorem. This implies finding a canonical measure in the set of all equivalence classes of irreducible representations which plays the role of the Haar measure in $\widehat{G}$ when G is commutative.

Gelfand and Neumark published two further joint papers in 1947. One of these gave a complete determination of all irreducible unitary representations of the so-called $ax+b$ group; that is the group of all transformations of the real line into itself of the form $ax+b$ where b is is an arbitrary real number and a is a positive real number. The other was a start on the program of generalizing the results on $SL(2,C)$ to $SL(n,C)$ for $n = 2,3,4\cdots$.

The other two contributors in 1947 were Segal and Godement. Godement published four short notes two of which studied a rather special class of groups called "central groups" and developed a theory of "characters" for these. This work was soon superseded by more general work by the same author to be described below. The other two notes were inspired by Bargmann's paper on the irreducible unitary representations of $SL(2, R)$. Bargmann showed that the regular representation of $SL(2, R)$ is a direct sum of two parts one of which has no irreducible subrepresentation and must be decomposed as a continuous dual sum or direct integral. The other part is a discrete dual sum of irreducible representations which may be explicitly described and constitute the so-called "discrete series" of representations of $SL(2, R)$. The members of the discrete series are characterized by the fact that their matrix elements with respect to any orthonormal basis are square integrable functions on the group. Moreover, the well known orthogonality relations of Frobenius and Schur have analogues for the members of the discrete series. Godement's two notes develop a theory of such "square integrable" representations for general locally compact groups. In addition to these four notes contributing to the general theory Godement published two full length papers in 1947 concerning the commutative case. One of these written jointly with H. Cartan gave an alternative exposition of the whole commutative theory using the theory of positive definite functions as a central tool. The other showed how some of the deep work of Norbert Weiner on generalized harmonic analysis could be generalized and better understood in terms of the ideal theory of the group algebra.

Segal published two full length papers in 1947. One of these had most of its results summarized in the 1941 PNAS note of Segal already mentioned in #3. As stated in #3 but not further discussed Segal defined $\mathcal{L}^1(G, \mu)$ for G non commutative and thus made an early contribution to the non compact non commutative case which might have been included in #4. However most results for the case in which the group is neither compact nor commutative are alluded to only very sketchily in the early note. A feature of the present much fuller treatment is an independent discussion of the connections with the work of Wiener also found by Godement but generalized further to a theorem about $\mathcal{L}^1(G)$ for general non compact non commutative G. Segal's other 1947 paper is closely related to an independent paper of Gelfand and Neumark which appeared in 1948. Both papers study the representation theory of what are now called C^* algebras and by definition are algebras of bounded operators on a Hilbert space closed under the operation $A \to A^*$ and in the topology defined by the norm. They include von Neumann algebras as a special case and are relevant to the unitary representation theory of locally compact groups because every such group has an associated C^* algebra where representation theory is related to the unitary representation theory of the group just as that of $\mathcal{L}^1(G)$. It forms an alternative group algebra with certain advantages over $\mathcal{L}^1(G)$.

We have already spoken of the need for a suitable notion of continuous direct sum or direct integral to augment the direct sum notion in reducing the study of general unitary representations to that of the irreducible ones. Such a notion was found by von Neumann and used by him to show that every von Neumann algebra can be canonically decomposed into factors. However his paper although written in 1937-38 was not published until 1949. According to a note at the beginning, "Various other commitments prevented the author from effecting some changes, which he had intended to carry out before publishing the paper. This delayed the publication until the present time. The paper is now published in the 1938 form, with only minor modifications $\cdots$".

In the meantime von Neumann had given F.I. Mautner access to the paper and Mautner had written a Princeton thesis on the application of von Neumann's notion to the decomposition of unitary group representations. Mautner's results were announced in a note in the Proceedings of the U.S. National Academy in 1948, and developed in detail in three papers published in 1950. Among other things Mautner discussed the fact that the regular representation of a locally compact group has a canonical decomposition into factor representations and that these need not be of type I. He found a class of countable discrete groups for which almost all components of the canonical decomposition of the regular representation are of type II. On the other hand, he showed that $SL(2,C)$ admits only factor representations of type I and made the conjecture that this is true for all semi-simple Lie groups.

Inspired by the papers of Mautner and von Neumann both Segal (1950) and Godement (1949 and 1951) wrote papers refining and improving the theory of direct integral decompositions in various ways. Segal in particular formulated and proved an "abstract" Plancherel theorem. The difference between Segal's "abstract" Plancherel theorem and the "concrete" Plancherel theorem that Gelfand and Neumark proved for $SL(2,C)$ is the difference between exhibiting an explicit measure in an explicit family of irreducible represntations of a particular group and proving an existence theorem for such a measure in a broad class of groups. More or less simultaneously with his work on direct integrals Godement made a serious study of how one might define "character" for unitary representations of locally compact group which are neither compact nor commutative. In particular he made it clear that in order to have anything like a complete theory one would have to allow characters to be measures or some more inclusive type of generalized function. Both in Segal's work on an abstract Plancherel theorem and Godement's work on characters account is taken of the possible occurrence of non type I factor representations. In general, the contributions of Segal, Godement and Mautner overlap and interrelate in a complicated fashion to which it is impossible to do justice without throwing this paper out of balance. One major work of Godement which the course of our exposition has kept us from mentioning until now is his long paper on positive definite functions published in 1948. This represented in part an independent discovery of some of the results of Gelfand and Raikov and in part an elaboration going beyond their work in certain respects. Godement's results were announced in six notes in the Paris *Comptes Rendus* published in volume 221 (1945) and 222 (1946). In the sixth note he cited the work of Gelfand and Raikov and said that he had just become aware of it. His 1948 paper contains a unified treatment of the contributions of all three authors.

The work of Gelfand and Neumark on the unitary representation theory of $SL(n, C)$ was continued with three further announcements in 1948 and culminated in a book published in 1950. This book not only treats $SL(n, C)$ itself in some detail but indicates how to generalize the results to the other complex classical simple Lie groups. These results include a description of enough irreducible unitary representations to make a Plancherel theorem possible and a proof of this Plancherel theorem.

In 1949 the author of this essay stumbled upon a general way of constructing unitary representations of locally compact groups out of unitary representations of closed subgroups which includes many of the constructions used by Gelfand, Neumark, and Bargmann in dealing with the irreducible unitary representations of $SL(2, C), SL(2, R)$ and the $ax + b$ group as special cases. I use the phrase "stumbled upon" because I was not seeking such a construction at all. I had just become aware of the 1930 paper of Stone cited in #2 and noticed that the fundamental uniqueness theorem of that paper could at least be formulated for an arbitrary locally compact commutative group G. One considers a pair U, V consisting of a unitary representation U of G and a unitary representation V of the dual $\widehat{G}$ of G and assumes that these representations satisfy the commutation relation $U_x V_\chi = \chi(x) V_\chi U_x$ for all x in G and all charcters χ in $\widehat{G}$. When G is the additive group of the real line $G = \widehat{G}$ and the commutation relation reduces to that of the Stone-von Neumann theorem. I wondered if the obvious analogue of that theorem were still true and discovered that it was. My approach was via the Neumark, Ambrose, Godement spectral theorem of 1944 stated in #4. Using that theorem one can replace the representation V of $\widehat{G}$ by a projection valued measure P^V defined on $\widehat{\widehat{G}} = G$ and I was delighted to find that the commutation relation for U and V translates into a very simple commutation relation between U and P^V. This may be stated as $U_x P^V_E = P^V_{[E]x} U_x$ for all x in G and all Borel subsets E of G where $[E]x$ denotes the translate of E by x. I noticed next that having transformed the problem in this way the dual group $\widehat{G}$ is no longer involved. Our question now concerns uniqueness for a pair U, P consisting of a unitary representation of G and a projection valued measure also defined on G and is in no way dependent upon the commutativity of G. I can still recall my excitement on realizing that my proof also did not depend upon the commutativity of G. The Stone-von Neumann theorem had turned out to be a very special case of a theorem about arbitrary locally compact groups. I published this theorem in volume 16 of the Duke Mathematical Journal (1949) under the title, "A theorem of Stone and von Neumann".

While this paper was in the process of being published I noticed that the commutation relation $U_x P_E = P_{[E]x} U_x$ could be formulated under even more general circumstances. P does not need to be defined on G itself but only on some space S in which G acts as a group of transformations. The obvious question then arose. Does uniqueness hold in this even greater generality? The answer is no! However all is not lost. In the special case in which $S = G/H$, the set of all right cosets Hy for some closed subgroup H of G, one can give a complete classification of all solutions of the commutation relation $U_x P_E = P_{[E]x} U_x$. To within unitary equivalence they correspond one-to-one to the equivalence classes of unitary representations of the subgroup H. Moreover the pair U, P is irreducible as a pair if and only if the associated representation L of H is an irreducible representation of H. This result together with an explicit rule for computing the pair U, P from L I called the "imprimitivity theorem" and published it in vol. 35 (1949) of the *Proceedings of the U.S. National Academy of sciences* under the title, "Imprimitivity for representations of locally compact groups I". Of course it carries with it a procedure $L \to U^L$ for taking unitary representations L of H into unitary representations U^L of G. One calls U^L the representation of G *induced* by the representation L of H. The associated projection valued measures P^L is called the *canonically associated system of imprimitivity.* At least for group characters of finite groups the inducing construction was already known to Frobenius and Weil in his book defined it for compact groups.

When H is an open subgroup so that the coset space G/H is discrete, the projection valued mesure P^L defines a direct sum decomposition of the space of U^L not into invariant subspaces but into subspaces which are permuted among themselves by the U^L_x. Such a decomposition is known classically as a system of imprimitivity and our definition is simply the obvious generalization to which one is led when admitting direct integral decompositions as well as direct sum decompositions.

The 1949 note under discussion contains a sketch of a proof of the imprimitivity theorem and notes the following application. Let N be a closed commutative normal subgroup of the locally compact group G and let U be any unitary representation of G. By the spectral theorem for unitary representations of commutative locally compact groups the restriction of U to N is described by a projection valued measure P on the dual $\widehat{N}$ of N. On the other hand, G acts on N by inner automorphisms and by duality also acts on $\widehat{N}$. It is easy to check that: *P is a system of imprimitivity for U relative to the action of G on $\widehat{N}$.* This observation together with the imprimitivity theorem can be used to give a thoroughgoing analysis of the irreducible unitary representations of G – at least whenever the action of G on $\widehat{N}$ is regular enough so that there exists a Borel set which meets each orbit just once. The analysis is most straightforward when G is a semi-direct product of N and another closed subgroup H. Indeed this is the only case that was examined at the time. Then one can construct a member of *every* equivalence class of irreducible unitary representation of G as follows. Choose an element χ of $\widehat{N}$. Let H_χ denote the subgroup of H leaving χ fixed. Choose an irreducible unitary representation L of H_χ. Then $nh \to \chi(n)L_h$ defines an irreducible unitary representation χL of the closed subgroup NH_χ of G and the induced representation $U^{\chi L}$ is irreducible and is the representation we set out to construct. One can show that $U^{\chi L^1}$ and $U^{\chi L^2}$ are equivalent if and only if L^1 and L^2 are equivalent, that $U^{\chi_1 L^1}$ and $U^{\chi_2 L^2}$ are inequivalent whenever χ_1 and χ_2 lie in distinct orbits and that every $U^{\chi_2 L}$ is equivalent to some $U^{\chi_1 M}$ when χ_1 and χ_2 lie in the same orbit.

Applying the theorem just stated to the $ax+b$ group one obtains the results of Gelfand and Neumark on this group. Applying it to the inhomogeneous Lorentz group one obtains the results of Wigner's 1939 paper cited in #4. In the case of a finite group it is the Frobenius "little group" method.

Although lack of suitable normal subgroups prevents the above analysis from being applied to $SL(2,R)$, $SL(n,C)$ and other semi-simple Lie groups one can still use the inducing construction to describe many and even most of their irreducible unitary representations. Let B_n for example denote the subgroup of $SL(n,C)$ consisting of all matrices which are zero above the main diagonal and let ϕ be the obvious homomorphism of B_n onto the subgroup A_n of all diagonal matrices. Then for each character χ of the commutative group A_n, $x \to \chi(\phi(x))$ is a one-dimensional unitary representation χ' of B_n. It can be shown that the induced representations $U^{\chi'}$ are all irreducible and constitute precisely what Gelfand and Neumark call the principal series of representations of $SL(n,C)$. There are irreducible unitary representations of $SL(n,C)$ not in the principal series but the set of all of these has measure zero with respect to the Plancherel measure and the Plancherel formula may be stated and proved without knowing them. Analogous results hold for the complex orthogonal groups and the complex symplectic groups. In 1950 the only real semi-simple Lie groups that had been studied from this point of view was $SL(2,R)$. Once again one has a principal series got by replacing B_n by the group of all $\begin{pmatrix} \lambda & 0 \\ a & 1/\lambda \end{pmatrix}$ with $\lambda \neq 0$ and D_n by the group of all $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ with $\lambda \neq 0$ and all but one of the members of this series are irreducible. However one can no longer decompose the regular representation using only the members of the principal series. This series has to be supplemented by the "discrete series". The countably many members of the discrete series may be constructed by a modification of the inducing process which will not be discussed here.

6. Afterwards. The innovations of Harish-Chandra.

The advances outlined in #5 continued to develop for a long time after 1950 and that date is by no means a natural place at which to stop the exposition. However my assigned topic covers only that period and to continue would make the paper much too long. The reader who is interested in what happened later is referred to the author's survey, "Infinite dimensional group representation" which appears in volume 69 (1963) of the *Bulletin of the American Mathematical Society*, pp. 628-686 and to the appendix to his book, "Unitary group representations" published in 1976 by the University of Chicago Press.

On the other hand, at the very beginning of 1951 a long paper appeared by Harish-Chandra whose analysis (by Godement) in *Mathematical Reviews* was almost five columns long and began as follow: "Cet article étant exceptionnellement important par l'originalité des resultats qu'il contient, on espère qu'on voudra bien excuser le rapporteur de publier une analyse exceptionnellement longue". This paper was followed by five notes in the *Proceedings of the National Academy of Sciences* published in 1951 announcing results whose proofs were published in detail in four long papers published in 1953 and 1954. This outpouring was the beginning of a steady stream of papers which continued unabated for almost thirty years until shortly before Harish-Chandra's premature death at the end of 1983. Because of its great importance and because it broke new ground it seems important to give a few indications about its nature in spite of our omission of other work done in 1951 and after.

Although both Bargmann and Harish-Chandra used infinitesimal methods in their 1947 papers on $SL(2, C)$ and $SL(2, R)$, this was exceptional, and the rest of the work described in #5 was carried out using global and functional analytical methods and neglecting the Lie algebra. Moreover, in their systematic work on the unitary representation theory of $SL(n, C)$ and the other classical complex groups Gelfand and Neumark gave a case by case analysis and did not treat the exceptional groups. Harish-Chandra introduced powerful new methods into Lie algebra theory that made it possible to discuss the infinite dimensional representations of semi-simple Lie algebras in a general and systematic way. He then supplemented these considerations with new and improved techniques for relating infinite dimensional representation of Lie algebras to those of the corresponding Lie groups and using the two together was able to prove Mautner's conjecture about the type I'ness of *all* semi-simple Lie group and to extend the work of Gelfand and Neumark by proving the Plancherel theorem for all complex semi-simple groups, both classical and exceptional, by one general argument. His principal tool in Lie algebra theory was systematic use of the so-called universal enveloping algebra $B_{\mathcal{L}}$ of a given Lie algebra $\mathcal{L}$. This is an infinite dimensional associative algebra which as an algebra is generated by a replica of $\mathcal{L}$; more precisely by the elements of a subset which is a Lie algebra isomorphic to $\mathcal{L}$ under $x, y \to xy - yx$.

The Plancherel theorem for the real semi-simple Lie groups turned out to be enormously more difficult than that for the complex groups. Harish-Chandra soon saw that the key to dealing with them lay in finding a suitable analogue of the Bargmann discrete series for all those real groups admitting a compact Cartan subgroup. Using this in conjunction with inducing he saw his way to constructing all the irreducible unitary representations that were needed. However, finding the generalized discrete series, more precisely their distribution characters, turned out to be unexpectedly difficult. The theorems describing the answer were not announced until 1963 and their lengthy proofs appeared in papers published in 1964, 1965 and 1966. Passing from the discrete series to the final answer, that is, to the Plancherel formula, also presented difficult problems and the results were first announced in the author's 1969 American Mathematical Society Colloquium lectures. The publication of the detailed proof was delayed for various reasons and finally appeared in three long papers in 1975 and 1976.

According to Harish-Chandra's Colloquium lectures a key role was played by "the philosophy of cusp forms" which he says he learned by studying the work of R.P. Langlands during the academic year, 1966-1967, and which derives from earlier work of Gelfand and Selberg. This refers to the fact that the classical theory of modular forms can be translated into the language of unitary group representations and, when so translated, is intimately connected with the decomposition of the unitary representations of $SL(2,R)$ induced by the identity representation of certain discrete subgroups Γ of $SL(2,R)$. The classical decomposition of modular forms into Eisenstein series and cusp forms is strongly reflected in this translation and generalizable into useful techniques for decomposing much more general induced representations – the regular representation in particular. This connection also led Harish-Chandra to study the Plancherel theorem for the p-adic analogues of the semi-simple Lie group and this program dominated the closing years of his productive life.

7. Final remarks and bibliographical indications.

We have deliberately blurred the distinction between locally compact groups which have a countable basis for their open sets (and thus are "separable") and those which do not. Some theorems have only been proved in the separable case and quite often the generalization is both less elegant and more costly an effort than seems worthwhile to this author. In reading this paper, the safest thing to do is to assume separability everywhere but realize that it is not always necessary.

In the 1950 volume of the *American Mathematical Society Bulletin* the author published a survey article entitled "Functions on locally compact groups". The bibliography for this article is offered as a substitute for any formal bibliography here.