ON THE *L* -STABILIZATION OF THE DOUBLE INTEGRATOR SUBJECT TO INPUT SATURATION

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Abstract. We consider a finite-dimensional control system (Σ) $\dot{x}(t) = f(x(t), u(t))$, such that there exists a feedback stabilizer k that renders $\dot{x} = f(x, k(x))$ globally asymptotically stable. Moreover, for (H, p, q) with H an output map and $1 \leq p \leq q \leq \infty$, we assume that there exists a $\mathcal{K}_\infty$-function α such that $\|H(x_u)\|_q \leq \alpha(\|u\|_p)$, where x_u is the maximal solution of $(\Sigma)_k$ $\dot{x}(t) = f(x(t), k(x(t)) + u(t))$, corresponding to u and to the initial condition $x(0) = 0$. Then, the gain function $G_{(H,p,q)}$ of (H, p, q) given by

$$G_{(H,p,q)}(X) \overset{\text{def}}{=} \sup_{\|u\|_p = X} \|H(x_u)\|_q,$$

is well-defined. We call profile of k for (H, p, q) any $\mathcal{K}_\infty$-function which is of the same order of magnitude as $G_{(H,p,q)}$. For the double integrator subject to input saturation and stabilized by $k_L(x) = -(1\ 1)^T x$, we determine the profiles corresponding to the main output maps. In particular, if σ_0 is used to denote the standard saturation function, we show that the L_2-gain from the output of the saturation nonlinearity to u of the system $\ddot{x} = \sigma_0(-x - \dot{x} + u)$ with $x(0) = \dot{x}(0) = 0$, is finite. We also provide a class of feedback stabilizers k_F that have a linear profile for (x, p, p), $1 \leq p \leq \infty$. For instance, we show that the L_2-gains from x and $\dot{x}$ to u of the system $\ddot{x} = \sigma_0(-x - \dot{x} - (\dot{x})^3 + u)$ with $x(0) = \dot{x}(0) = 0$, are finite.

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1. INTRODUCTION

Let (Σ) be the finite dimensional control system of the form:

$$(\Sigma) \quad \dot{x}(t) = f(x(t), u(t)), \tag{1.1}$$

where states $x(t) \in \mathbb{R}^n$ for all $t \geq 0$ (the integer n is the dimension of (Σ)), inputs $u : [0, \infty) \to \mathbb{R}^m$ are measurable locally essentially bounded maps (the integer m is the dimension of the input space), and f is locally Lipschitz in (x, u) with $f(0, 0) = 0$. Assume that (Σ) admits a feedback stabilizer, *i.e.*, a map $k : \mathbb{R}^n \to \mathbb{R}^m$ locally bounded such that $k(0) = 0$ and the closed loop system given by

$$\dot{x} = f(x, k(x)), \tag{1.2}$$

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is globally asymptotically stable (GAS for short) with respect to the origin. For basic terminology on control theory, we refer to [9]. We are interested in the robustness of the feedback k, *i.e.*, in the stability of k with respect to measurement and actuator noise. We first recall the following definition: the class of $\mathcal{K}_\infty$-functions consists of all $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ which are continuous, strictly increasing, unbounded and satisfy $\alpha(0) = 0$. For simplicity of the exposition, we also assume that k is locally Lipschitz. We hence consider the control system $(\Sigma)_k$ given by

$$(\Sigma)_k \quad \dot{x}(t) = f(x(t), k(x(t)) + u(t)). \tag{1.3}$$

We use x_u to denote the maximal solution of (1.3) corresponding to u and to the initial condition $x(0) = 0$. For $p, q \in [1, \infty]$, $p \le q$, and an output map $H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$ locally Lipschitz with $H(0, 0) = 0$, we then ask if $(\Sigma)_k$ verifies the following properties:

$\left(P^1_{(H,p,q)}\right)$ the assignment $u \mapsto H(x_u, u)$ defines a map $F_{(H,p,q)} : L^p(\mathbb{R}^+, \mathbb{R}^m) \to L^q(\mathbb{R}^+, \mathbb{R}^l)$;

$\left(P^2_{(H,p,q)}\right)$ there exists $\alpha_{(H,p,q)} \in \mathcal{K}_\infty$ such that

$$\left\| F_{(H,p,q)}(u) \right\|_q \le \alpha_{(H,p,q)}\left(\|u\|_p \right). \tag{1.4}$$

(Indices such as (H, p, q) are dropped when the context is clear.) The main output maps H are x, $f(x, u)$, (the integer l is then equal to n), $h^T x$, $h^T f(x, u)$ where h is a fixed nonzero vector of $\mathbb{R}^n$ (the integer l is then equal to 1). We also have local versions of $\left(P^2_{(H,p,q)}\right)$ when (1.4) only holds in a neighborhood of 0 or a neighborhood of ∞. The feedback k is said to be of *shape* α for (H, p, q) (at 0 or at ∞ respectively) or α *is a shape of k for* (H, p, q) (*at* 0 or *at* ∞ respectively) if (H, p, q) satisfies $\left(P^2_{(H,p,q)}\right)$ (in a neighborhood of 0 or ∞ respectively) with $C\alpha \in \mathcal{K}_\infty$ in equation (1.4) and $C > 0$ is a constant independent of u. Remark that if there exists a shape α of k for (H, p, q), we can define the *gain function* $G_{(H,p,q)} : \mathbb{R}^+ \to \mathbb{R}^+$ associated to (H, p, q) by

$$G_{(H,p,q)}(X) \overset{\text{def}}{=} \sup_{\|u\|_p = X} \left\| F_{(H,p,q)}(u) \right\|_q. \tag{1.5}$$

Then G is a nondecreasing function, continuous at 0 with $G(0) = 0$. In other words, k is of shape α for (H, p, q) (at 0 or at ∞ respectively) if $G_{(H,p,q)} = O(\alpha)$ (at 0 or ∞ respectively). We say that k is of *profile* α *for* (H, p, q) *at* 0 (*at* ∞ respectively) or α is a *profile* of k for (H, p, q) at 0 (at ∞ respectively) if there exists $c_1, c_2, X_0 > 0$ such that for every $X < X_0$ ($X > X_0$ respectively),

$$c_1 G_{(H,p,q)}(X) \le \alpha(X) \le c_2 G_{(H,p,q)}(X).$$

This can be denoted by $\alpha \asymp_0 G_{(H,p,q)}$ $\left(\alpha \asymp_\infty G_{(H,p,q)} \text{ respectively} \right)$. When $u \mapsto f(0, u)$ and $u \mapsto H(0, u)$ are not identically zero, and if the gain function G is defined and continuous at 0, then a profile for G exists at 0. If, in addition, G is unbounded, then it also exists at ∞ (see proof in Lem. 2.2 below). Note that a profile is uniquely determined up to the multiplication by a positive bounded function. A profile of k for (H, p, q) both at 0 and at ∞ is simply called a profile of k for (H, p, q).

A natural way to quantify the robustness of a feedback stabilizer k satisfying $\left(P^2_{(H,p,q)}\right)$ for some (H, p, q) is to determine (when possible) the asymptotic behaviors of a corresponding profile α when $\|u\|_p$ tends firstly to 0 and secondly to ∞. Moreover, given two feedback stabilizers k_1 and k_2 for (Σ) and (H, p, q), we can compare k_1 and k_2 by comparing the asymptotic behaviors at 0 and ∞ of their respective profiles (when they exist). In particular we say that k_1 is better than k_2 at 0 (at ∞ respectively) for (H, p, q) if we have, for their corresponding profiles α_1, α_2 (when they exist), that $\alpha_1(X) = o(\alpha_2(X))$ as X tends to 0 (to ∞ respectively). Therefore, the notion of profile is a weaker generalization of the concept of finite-gain stability as it is defined in input/output

studies and related works on computation of norms for nonlinear systems in state-space form (*cf.* [2,15] and the references given therein). The definition of profile is also closely linked to the concept of input-to-state stability (ISS) and related notions (*e.g.* integral input-to-state stability etc.) see [10]. For instance, if $H(x, u) = x$ and $p = q = \infty$, having a shape for (x, ∞, ∞) is equivalent to have the (ISS) property (*cf.* Th. H in [10]). To illustrate the previous definitions, we consider a simple case namely the linear one. We assume that (Σ) is a linear system $\dot{x} = Ax + Bu$ such that the uncontrollable modes have negative real parts. Then, there exists a linear feedback k such that for every $p, q \in [1, \infty]$, $p \leq q$ and main output map H, k is of *linear profile* (*i.e.* the corresponding α is a linear function for (H, p, q)). We can see with the previous example that that the notion of linear profile is less precise than the notion of finite gain.

Under which conditions on f, k and on (H, p, q), the control system $(\Sigma)_k$ satisfies the properties $(P^1)_{(H,p,q)}$ or $(P^2)_{(H,p,q)}$ are general issues one can address. In addition, other questions can be considered such as the relationships between the two previous properties and (in the spirit of ISS results) the links between the fact of admitting a profile and admitting a control Lyapunov function satisfying some dissipative inequality. In this paper, we rather focus on a particular control system (SI_2) (defined below) and we address through that example the two following issues: given a control system admitting a feedback stabilizer k and a main output map (H, p, q), then find first the profile of k for (H, p, q) and second, a feedback stabilizer with the best profile for (H, p, q).

The control system (SI_2) belongs to the class of linear control systems subject to input saturation:

$$\dot{x} = Ax + B\sigma(u), \tag{1.6}$$

where A is an $n \times n$ matrix, B an $n \times m$ matrix and σ is of "saturation" type. For instance, arctan, tanh and the standard saturation function $\sigma_0(s) = \frac{s}{\max(1, |s|)}$ are 1-dimensional examples of saturation functions. For this class of systems, examples of feedback stabilizers are well-known under the necessary and sufficient condition that the controllable modes are of non positive real part and the uncontrollable ones of negative real part (*cf.* [8,11–14], and references therein). The interesting problems for control systems (1.6) occur when A admits at least one eigenvalue λ with zero real part and more particularly when the multiplicity of λ is larger than one (we refer to the latter as the critical case). The feedback stabilizer is necessarily nonlinear in general: in [3] and [14], it is shown that, if (A, B) is controllable, $m = 1$ and A is equal to the nth-Jordan matrix, then the corresponding control system (SI_n), called the n-th-integrator with saturated input, cannot be globally stabilized by means of a linear feedback if $n \geq 3$. In [5], the authors consider the case when (A, B) is controllable and A is just neutrally stable (*i.e.* the eigenvalues of A have non positive real parts and there are no nontrivial Jordan blocks). It is shown there, that for the linear feedback stabilizer given by $k(x) \stackrel{\text{def}}{=} -B^T x$, we have for every $p \leq q \in [1, \infty)$ or $p = q = \infty$ and $H(x, u) = x$, a linear profile for (H, p, q). Moreover, for $(H, p, q) = (x, p, \infty)$ with p finite, the profile of k is linear at 0 and equal to $X \mapsto X^{p/p+1}$ at ∞. Finally, as a first result regarding the critical case, it is shown in [5] that for (SI_2), the linear feedback stabilizer $k_L(x) \stackrel{\text{def}}{=} -(1\ 1)^T x$ is not of linear profile at ∞ for (x, p, p) with $p \in [1, \infty]$. For related work, in [6], the authors show that any linear system subject to input saturation with poles of non positive real parts can be semi-globally stabilized (that is, on compact sets) by means of linear feedback. In addition, they show in [7] that for every $D > 0$, there exists a linear feedback stabilizer which is of linear profile (depending on D) under the restriction that all the inputs considered are essentially bounded by D.

In the present paper, we consider the control system (SI_2) given by

$$(SI_2) \begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \sigma(u). \end{cases}$$

In a first part we determine the profiles of the linear feedback stabilizer k_L for the main output maps. We show that, at 0, all the profiles considered are linear and, at ∞, they are of the type $X \mapsto X^r$, where r depends on (H, p, q) (*cf.* Th. 2.5 below). Let us illustrate the aforementioned results. For that purpose, (x_1, x_2) is used to denote the trajectory of $(SI_2)_{k_L}$ starting at 0 and corresponding to $u \in L^p(0, \infty)$. Then, we prove that there

exist $a_i, b_i > 0$ and $C > 0$ such that, for every input $u \in L^2(0, \infty)$ with $\|u\|_2 > C$, we have

$$\|x_1\|_2 \;\; \leq \;\; a_1 \|u\|_2^2, \qquad \|x_1\|_\infty \leq b_1 \|u\|_2^{\frac{4}{3}}, \tag{1.7}$$

$$\|x_2\|_2 \;\; \leq \;\; a_2 \|u\|_2^{\frac{4}{3}}, \qquad \|x_2\|_\infty \leq b_2 \|u\|_2^{\frac{2}{3}}, \tag{1.8}$$

$$\|\dot{x}_2\|_2 \;\; = \;\; \|\sigma(-x_1 - x_2 + u)\|_2 \leq a_3 \|u\|_2. \tag{1.9}$$

We also prove that the exponents of $\|u\|_2$ in the above equations are best possible. As a corollary, we partially solve part 2 of Problem 36 as defined in [1]: equation (1.9) shows that the L_2-gain from the output of the saturation nonlinearity to u of the system $\ddot{x} = \sigma_0(-x - \dot{x} + u)$ with $x(0) = \dot{x}(0) = 0$ is finite. However, we are not able to compute the value of that L_2-gain.

If one uses C^1 feedback stabilizers for (SI_2), the best profile one can get for the output map $H(x, u) = x$, is a linear profile. In the second part of the paper, we exhibit a set of nonlinear feedback stabilizers for (SI_2) such that each of them admits a linear profile for H (*cf.* Th. 2.8 below). We also provide additional results regarding nonzero initial states and the perturbed control system $(PSI_2)_k$:

$$(PSI_2)_k \begin{cases} \dot{x}_1 & = & x_2 + v_1, \\ \dot{x}_2 & = & \sigma(k(x) + u) + v_2, \end{cases}$$

with $v = (v_1, v_2)$, $v_1, v_2 \in L^{p'}(0, \infty)$, $p' \geq 1$ and $\|v\|_\infty \leq C_0$, for some constant C_0 only depending on (SI_2) (*cf.* Ths. 2.6 and 2.9 below). We conclude this introduction with a remark regarding the techniques used to obtain the results. In [5] and [7], the passivity technique is essentially generalized: for $p \in [1, \infty]$, the main argument of the proofs consists of finding a suitable "storage function" V_p and establishing for V_p a "dissipation inequality" of the form

$$\frac{\mathrm{d}V_p(x_u(t))}{\mathrm{d}t} \leq -\|x_u(t)\|^p + \lambda_p \|u(t)\|^p, \tag{1.10}$$

for some constant $\lambda_p > 0$. We recall that in [5] a non-smooth V_p was needed. For more discussion on passivity, see for instance [4] and [15]. In our situation, we are not able to find such storage functions. We just have at our disposal a control Lyapunov function V (in the case of the feedback k_L, the saturation function σ has to be increasing and V is not even $(iISS)$-Lyapunov – see [10] –). Therefore the "dissipation inequality" satisfied by V does not lead directly to the desired estimates. However, on a time interval $I = [t_0, t_1]$ where the variation of V is positive and "large", we estimate the measures of subsets of I where the input u is "big". We can then bound integrals on I of various quantities in terms of $\int_I |u|^p$ and get global results.

2. NOTATIONS AND STATEMENT OF THE RESULTS

In this section we define the class of saturation functions to be considered and state the main results we obtain for (SI_2). In addition, we provide simple remarks.

Definition 2.1. We call $\sigma : \mathbb{R} \to \mathbb{R}$ a saturation function (or an S-function) if there exist two real numbers $0 < a \leq K_\sigma$ such that for all $t, t' \in \mathbb{R}$

(i) $|\sigma(t) - \sigma(t')| \leq K_\sigma \inf(1, |t - t'|)$;
(ii) $|\sigma(t) - at| \leq K_\sigma t \sigma(t)$.

A constant K_σ defined as above is called an S-bound for σ. When the context is clear, we simply use K to denote an S-bound. Note that (i) is equivalent to the fact that σ is bounded and globally Lipschitz. On the other hand, (ii) is equivalent to

$$t\sigma(t) > 0 \text{ for } t \neq 0, \; \liminf_{|t| \to \infty} |\sigma(t)| > 0, \; \limsup_{t \to 0} \frac{\sigma(t) - \sigma'(0)t}{t^2} < \infty.$$

We say that σ is an iS-function if σ is an increasing S-function. If m is a positive integer, σ is an $\mathbb{R}^m$-valued S-function if $\sigma = (\sigma_1, \ldots, \sigma_m)$, where each component σ_i is an S-function and

$$\sigma(t) = (\sigma_1(t), \ldots, \sigma_m(t)),$$

for $x = (x_1, \ldots, x_m)^T$. Here, we use $(\cdots)^T$ to denote the transpose of $(\cdots)$. Throughout this paper, if $x \in \mathbb{R}^n$, we use $\|x\| \stackrel{\text{def}}{=} (\Sigma_{i=1}^n x_i^2)^{1/2}$ to denote the usual Euclidean norm. If E is a measurable set of $\mathbb{R}$, $|E|$ is its Lebesgue measure. If S is a finite set, we use $S^{\#}$ to denote its cardinality. For $p \in [1, \infty]$ and $T > 0$, we use L^p to denote $L^p(0, \infty)$ and we let $\|x\|_p$ ($\|x\|_{p,[0,T]}$ respectively) denote the L^p-norm:

$$\|x\|_p \stackrel{\text{def}}{=} \left(\int_0^\infty \|x(t)\|^p \mathrm{d}t \right)^{1/p}, \left(\|x\|_{p,[0,T]} \stackrel{\text{def}}{=} \left(\int_0^T \|x(t)\|^p \mathrm{d}t \right)^{1/p} \text{ respectively} \right),$$

and

$$\|x\|_\infty \stackrel{\text{def}}{=} \operatorname{ess} \sup_{0 \le t < \infty} \|x(t)\|, \left(\|x\|_{\infty,[0,T]} \stackrel{\text{def}}{=} \operatorname{ess} \sup_{0 \le t \le T} \|x(t)\| \text{ respectively} \right).$$

Various constants depending only on (SI_2) and (H, p, q) will be considered. We will use the symbol const. to denote those that depend on (SI_2) and (H, p, q). Sometimes, a constant term may depend on an extra parameter R. In this case, instead of const., we use const.$_R$. For $a \in \mathbb{R}$, let $a^+ \stackrel{\text{def}}{=} \sup(a, 0)$ and we use sgn a to denote the sign of a $\left(\frac{a}{|a|} \text{ if } a \ne 0 \text{ and } 0 \text{ otherwise} \right)$. For $a, b > 0$, the symbol $a \ll b$ means that b is much larger than a. If $a, b \in \mathbb{R}$, we use $\Delta f \big|_a^b$ or $\Delta f \big|_I$ to denote $f(b) - f(a)$ where f is a function on $\mathbb{R}$ and I is the interval between a and b.

Let us consider a control system of the form (Σ_1) together with a feedback stabilizer k that gives rise to a control system of the form $(\Sigma_1)_k$. For (H, p, q) with H a main output map and $1 \le p \le q \le \infty$, assume that $\left(P_{(H,p,q)}^1 \right)$ holds. Suppose that there exists $X_0 > 0$ such that $\sup_{\|u\|_p = X_0} \|F_{(H,p,q)}(u)\|_q$ is infinite. In this case, we say that k has *infinite gain for* (H, p, q). Otherwise, the gain function $G_{(H,p,q)} : \mathbb{R}^+ \to \mathbb{R}^+$ is well-defined and we say that k has *finite gain* $G_{(H,p,q)}$ *(or simply G) for* (H, p, q). The function G is not necessarily continuous but we have:

Lemma 2.2. *Let $(\Sigma_1)_k$ and (H, p, q) be defined as above. Suppose that k has finite gain G for (H, p, q). Then,*

(a) *$G(X) = \sup_{\|u\|_p \le X} \|F_{(H,p,q)}(u)\|_q$ and thus G is nondecreasing;*

(b) *the gain function $G : \mathbb{R}^+ \to \mathbb{R}^+$ is well-defined and continuous at 0 if and only if $\left(P_{(H,p,q)}^2 \right)$ holds. In addition, the shape α given by (1.4) can be taken as a profile of k for (H, p, q) if,*

(b1) *for $p < \infty$, G is unbounded;*

(b2) *for $p = \infty$, G is unbounded and $\|f(0, u_n)\| > 0$ for some sequence $(u_n)_{n \ge 0}$, $u_n \in \mathbb{R}^m$, such that $\lim_{n \to \infty} u_n = 0$.*

Proof of Lemma 2.2. We assume that $X > 0$. Let $A = G(X)$ and $B = \sup_{\|u\|_p \le X} \|F_{(H,p,q)}(u)\|_q$. We clearly have $A \le B \le \infty$. Assume first that $B = \infty$. Then, for every $R > 0$, there exists $u_R \in L^p$ such that $\|u_R\|_p \le X$ and $2R < \|F_{(H,p,q)}(u_R)\|_q < \infty$. If $\|u_R\|_p < X$, we change u_R outside an interval $(0, T_R)$ in order to construct some $v_R \in L^p$ such that $\|v_R\|_p = X$ and $\|F_{(H,p,q)}(v_R)\|_q \ge R$. The time T_R is chosen as follows: if q is finite, we should have $\int_0^{T_R} |F_{(H,p,q)}(u_R)|^q \ge R^q$ and if $q = \infty$, $|F_{(H,p,q)}(u_R)(T_R)| = R$. Then, $A \ge R$ for every $R > 0$, which implies that $A = \infty$. We now assume that B is finite. For every $\varepsilon > 0$, there exists $u_\varepsilon \in L^p$ such that $\|u_\varepsilon\|_p \le X$ and $B - \varepsilon < \|F_{(H,p,q)}(u_\varepsilon)\|_q \le B$. Exactly as before, if $\|u_\varepsilon\|_p < X$, we change u_ε outside an interval $(0, T_\varepsilon)$ in order to construct some $v_\varepsilon \in L^p$ such that $\|v_\varepsilon\|_p = X$ and $B - 2\varepsilon \le \|F_{(H,p,q)}(v_\varepsilon)\|_q$. Then $A \ge B - 2\varepsilon$ for every $\varepsilon > 0$, which implies that $A \ge B$.

We now prove item (b). We have two alternatives; either $u \mapsto f(0, u)$ is the zero function or there exists $X_0 \geq 0$ such that $G(X) > 0$ if $X > X_0$ and $G(X_0) = 0$. In the first case, $x_u \equiv 0$ for every locally essentially bounded measurable u. Any $\alpha \in \mathcal{K}_\infty$ is a shape of k for (H, p, q). In the second case, let $G^+(X_0) \stackrel{\text{def}}{=} \lim_{X \to X_0^+} G(X)$. If $G^+(X_0) > 0$, replace G by the function defined by G on (X_0, ∞) and $X \mapsto G^+(X_0)\frac{X}{X_0}$ on $[0, X_0]$. We can then assume without loss of generality that G is continuous at X_0. Let $g \stackrel{\text{def}}{=} G(X_0 + 1)$. The argument that follows is standard. Since $G(X_0) = 0$ and G is increasing, we can find a sequence of times $(t_n)_{n \geq 0}$ such that $t_0 = X_0 + 1$, $G(t_n) = g/2^n$ and for $n \geq 0$, $G(t_{n+1}) \leq G(t) \leq G(t_n)$ for $t \in [t_{n+1}, t_n]$. Then $(t_n)_{n \geq 0}$ is a decreasing sequence to X_0. Let α be the piecewise linear function defined on $[X_0, X_0 + 1]$ such that for $n \geq 0$ and $t \in [t_{n+1}, t_n]$, α is the line segment between $(t_{n+1}, 2G(t_{n+1}))$ and $(t_n, 2G(t_n))$. Then α is continuous, increasing, $\alpha(X_0) = 0$ and for all $t \in [X_0, X_0 + 1]$ we have $\frac{\alpha(t)}{4} \leq G(t) \leq \alpha(t)$. If $p < \infty$, then $X_0 = 0$ since there exist $u \in L^p$ with arbitrary small L^p-norm and arbitrary large L^∞-norm. If $p = \infty$, take $\bar{\alpha}(t) \stackrel{\text{def}}{=} t + \alpha(t)$. Then, for every $p \in [1, \infty]$, we have constructed a shape of k at 0 for (H, p, q). Remark that the extra hypothesis on f in $(b2)$ implies that we have $X_0 = 0$.

For the behavior at ∞, we distinguish whether G is unbounded or not. When G is unbounded, the procedure described as above holds in a similar way, by considering now a sequence of times $(s_n)_{n \geq 0}$ such that $s_0 = 1$, $G(s_n) = 2^n g$ and, for $n \geq 0$, $G(s_n) \leq G(t) \leq G(s_{n+1})$ for $t \in [s_n, s_{n+1}]$. It is clear that the function α previously defined is a $\mathcal{K}_\infty$-function and is a profile of k for (H, p, q) at ∞. If G is bounded, there exists n_0 such that $2^{n_0} g \leq \sup_{t \in \mathbb{R}^+} G(t) < 2^{n_0+1} g$. Then for $t \geq s_{n_0+1}$, we take for α a half-line of positive slope starting at $(s(t_{n_0+1}), 2G(s_{n_0+1}))$. $\qquad\square$

We give next a criterion for a shape to be a profile. This result will be used repeatedly in the sequel.

Lemma 2.3. *Let* $(\Sigma_1)_k$ *and* (H, p, q) *be defined as above. Suppose that* α *is a shape of* k *for* (H, p, q) *such that there exists* $K > 1$ *and*

(a) *there exist* $r > 0, X_0 > 1$ *and* $A \geq 1$ *such that we have* $\alpha(X/K) \geq \alpha(X)/AK^r$ *for* $X \leq 1/X_0$ *(*$\alpha(KX) \leq AK^r \alpha(X)$ *for* $X \geq X_0$ *respectively)*;

(b) *there exist* $a > 0$ *and a sequence* $(u_n)_{n \geq 0}$ *(*$(v_n)_{n \geq 0}$ *respectively) such that* $\|u_n\|_p \to 0$, $\|u_{n+1}\|_p \geq \|u_n\|_p/K$ *(*$\|v_n\|_p \to \infty$, $\|v_{n+1}\|_p \leq K\|v_n\|_p$*) and*

$$\frac{\|F(u_n)\|_q}{\alpha\big(\|u_n\|_p\big)} \geq a \quad \left(\frac{\|F(v_n)\|_q}{\alpha\big(\|v_n\|_p\big)} \geq a \text{ respectively}\right).$$

Then α *is a profile of* k *for* (H, p, q) *at 0 (at* ∞ *respectively)*.

Condition (a) simply says that if $\beta(X) \stackrel{\text{def}}{=} \frac{\alpha(X)}{X^r}$, then $\beta(KX)/\beta(X)$ is bounded below (above respectively) for X in a neighborhood of 0 (of ∞ respectively). This is obvious if β is constant.

Proof of Lemma 2.3. We just prove the lemma in the neighborhood of ∞. We may also assume that $(v_n)_{n \geq 0}$ is increasing. For X large enough, there exists n_0 such that $\|v_{n_0}\|_p \leq X < \|v_{n_0+1}\|_p$. Then, we get

$$\frac{G(X)}{\alpha(X)} \geq \frac{G(\|v_{n_0}\|_p)}{\alpha(\|v_{n_0}\|_p)} \frac{\alpha(\|v_{n_0}\|_p)}{\alpha(X)} \geq \frac{1}{AK^r} \frac{F(\|v_{n_0}\|_p)}{\alpha(\|v_{n_0}\|_p)} \frac{AK^r \alpha(\|v_{n_0}\|_p)}{\alpha(\|v_{n_0+1}\|_p)} \geq \frac{a}{AK^r}. \qquad\square$$

At the light of Lemma 2.3, one can follow the two-step strategy described below in order to show that the profile of k for (H, p, q) is a certain function $\alpha \in \mathcal{K}$:

Step 1: Establish that some $\alpha \in \mathcal{K}_\infty$ verifies the inequality (1.4).

Step 2: Exhibit a sequence of inputs such that the hypothesis of Lemma 2.3 are satisfied.

Remark 2.4. Let $(\Sigma_1)_k$ and (H, p, q) be defined as above, with H a main output map and q finite. Suppose that $(x, u) \mapsto f(x, k(x) + u)$ is differentiable at $(0, 0)$ and k has finite gain for (H, p, q). If $\dot{x} = Ax + Bu$ is the

linearized control system at $(0,0)$, assume for simplicity that A is Hurwitz. Then, a profile of k at 0 is linear. In addition, there exist $\gamma > 0$ and a sequence $(u_n)_{n\geq 0}$ with $\|u_n\|_p \to \infty$ such that

$$\frac{\|F(u_n)\|_q}{\|u_n\|_p} \geq \gamma. \tag{2.1}$$

(Simply consider inputs of small L^∞-norm but large L^p-norm. The corresponding trajectories have small L^∞-norm.) Then, G is unbounded and, if k is of profile α at ∞, we have $X = O(\alpha(X))$. Therefore, the real issue is to determine α and the best profile of k for (H, p, q) at ∞ is at least linear.

We now state the main results of the paper. In Section 3, we study the linear feedback $k_L = -(x_1 + x_2)$ for (SI_2) and get the following two theorems:

Theorem 2.5. *Let us consider the control system* $(SI_2)_{k_L}$, *where* σ *is an* (iS)-*function. For* (H, p, q) *with* H *a main output map and* $p \leq q \in [1, \infty]$, *the profile of* k_L *at 0 is linear and, at ∞, is as follows:*

  (1) *for* $H(x, u) = x$ *or* x_1, $X \mapsto X^{r_1(p,q)}$ *with* $r_1(p, q) = \frac{2p(q+1)}{q(p+1)}$;
  (2) *for* $H(x, u) = x_2$, $X \mapsto X^{r_2(p,q)}$ *with* $r_2(p, q) = \frac{p(q+2)}{q(p+1)}$;
  (3) *for* $H(x, u) = \sigma(-x_1 - x_2 + u) = \dot{x}_2$, $X \mapsto X^{p/q}$ *with* $q < \infty$.

In the previous results, we define $r_i(p, \infty)$ and $r_i(\infty, \infty)$ as

$$r_i(p, \infty) \stackrel{\text{def}}{=} \lim_{q\to\infty} r_i(p, q) \text{ if } p \text{ finite and } r_i(\infty, \infty) \stackrel{\text{def}}{=} \lim_{p\to\infty} r_i(p, \infty).$$

In order to state results relative to $(PSI_2)_{k_L}$, we consider the following Lyapunov function for (SI_2) (introduced in [14]):

$$V(x_1, x_2) = x_2^2 + \int_0^{x_1} \sigma(s)\mathrm{d}s + \int_0^{x_1+x_2} \sigma(s)\mathrm{d}s. \tag{2.2}$$

If σ is an (iS)-function, then V is a strict Lyapunov function for the closed loop system (1.2) associated to $(SI_2)_{k_L}$. In addition, V is not a $(iISS)$-Lyapunov function for the system $(SI_2)_{k_L}$. Note though that $V^{\frac{1}{2}}$ is a $(iISS)$ Lyapunov (but not (ISS)) for the system $(SI_2)_{k_L}$. We have:

Theorem 2.6. *Let us consider* $(PSI_2)_{k_L}$. *There exists* $\varepsilon_0 > 0$ *such that for* $p \in [1, \infty]$, $p_1 \leq p$ *and* p_1 *finite,* $u \in L^p$, $v \in L^{p_1}$ *and* $\|v\|_\infty \leq \varepsilon_0$, $\bar{x} \in \mathbb{R}^2$, *if* (x_1, x_2) *is the trajectory of* $(PSI_2)_{k_L}$ *corresponding to* (u, v) *starting at* $\bar{x}$, *we get*

  (1) *for* $p_1 \in [1, 2)$,

$$\|x_1\|_p \leq \text{const.}\left(L(v, u, \bar{x}) + \|u\|_p^2 + \|v\|_{p_1}^{\frac{p+1}{p}c_{p_1}} + V(\bar{x})^{\frac{p+1}{p}}\right), \tag{2.3}$$

$$\|x_2\|_p \leq \text{const.}\left(L(v, u, \bar{x}) + \|u\|_p^{\frac{p+2}{p+1}} + \|v\|_{p_1}^{\frac{p+2}{2p}c_{p_1}} + V(\bar{x})^{\frac{p+2}{2p}}\right), \tag{2.4}$$

$$\sup(\|x_2\|_\infty, \|x_2\|_\infty^2) \leq \text{const.}\|x_1\|_\infty \leq \text{const.}\left(L(v, u, \bar{x}) + \|u\|_p^{\frac{2p}{p+1}} + \|v\|_{p_1}^{c_{p_1}} + V(\bar{x})\right), \tag{2.5}$$

$$\|\dot{x}_2\|_p \leq \text{const.}\left(L(v, u, \bar{x}) + \|v\|_{p_1}^{\frac{c_{p_1}}{2p}} + V(\bar{x})^{\frac{p+2}{2p}}\right), \tag{2.6}$$

with $L(v, u, \bar{x}) \stackrel{\text{def}}{=} \|v\|_{p_1} + \|u\|_p + V(\bar{x})^{1/2}$ *and* $c_{p_1} = \frac{2p_1}{2-p_1}$;

(2) *for $p_1 = 2$, set $Q(v, u, \bar{x}) \stackrel{\text{def}}{=} \left(\|u\|_p^{\frac{2p}{p+1}} + V(\bar{x}) + 1 \right) e^{\text{const.} \|v\|_2^2} - 1$. We have*

$$\|x_1\|_p \leq \text{const.} \left(L(v, u, \bar{x}) + \|u\|_p^2 + V(\bar{x})^{\frac{p+1}{p}} + Q(v, u, \bar{x})^{\frac{p+1}{p}} \|v\|_2^{\frac{2}{p}} \right), \tag{2.7}$$

$$\|x_2\|_p \leq \text{const.} \left(L(v, u, \bar{x}) + \|u\|_p^{\frac{p+2}{p+1}} + V(\bar{x})^{\frac{p+2}{2p}} + Q(v, u, \bar{x})^{\frac{p+2}{2p}} \|v\|_2^{\frac{2}{p}} \right), \tag{2.8}$$

$$\sup(\|x_2\|_\infty, \|x_2\|_\infty^2) \leq \text{const.} \|x_1\|_\infty \leq \text{const.} \left(L(v, u, \bar{x}) + Q(v, u, \bar{x}) \right), \tag{2.9}$$

$$\|\dot{x}_2\|_p \leq \text{const.} \left(L(v, u, \bar{x}) + V(\bar{x})^{\frac{p+1}{2p}} + (Q(v, u, \bar{x}) \|v\|_2)^{\frac{2}{p}} \right); \tag{2.10}$$

(3) *for $p_1 > 2$ and $\varepsilon > 0$, there exist u, v with u continuous and compactly supported, $v \in L^{p_1}$ and $\|v\|_\infty \leq \varepsilon$, such that the trajectory of $(PSI_2)_{k_L}$ corresponding to (u, v) starting at $(0, 0)$ is unbounded.*

In the statement of Theorem 2.6, the exponents corresponding to the case $p = \infty$ are the limits of the exponents when p tends to ∞.

In Section 4, we consider nonlinear feedbacks for (SI_2). For that purpose, we define the class of $\mathcal{F}$-functions as follows:

Definition 2.7. A function $F : \mathbb{R} \to \mathbb{R}$ is an $\mathcal{F}$-function if F is C^1, odd, $F'(0) = 0$ and there exist $r \geq 1$ such that, for $|x_2| \geq 1$, we have

$$F'(x_2) \geq \sup \left(3|x_2|, r \frac{F(x_2)}{x_2} \right). \tag{2.11}$$

The next theorems describe the performances of the feedback k_F defined below:

Theorem 2.8. *Let us consider the control system (SI_2) with σ an S-function, and the feedback $k_F(x)$ given by*

$$k_F(x) \stackrel{\text{def}}{=} -\left(x_1 + x_2 + F(x_2) \right), \tag{2.12}$$

where F an $\mathcal{F}$-function. For $p \in [1, \infty]$, k_F is a feedback stabilizer for (SI_2) and is of linear profile for (x, p, p).

For instance $-(x_1 + x_2 + 3x_2|x_2|)$ and $-(x_1 + x_2 + x_2^3)$ are examples of k_F-feedbacks. The function $\bar{F} : \mathbb{R} \to \mathbb{R}$ defined by $\bar{F}(x_2) \stackrel{\text{def}}{=} x_2 + F(x_2)$ is a C^1-diffeomorphism and let $K : \mathbb{R} \to \mathbb{R}$ be defined by $K(y) \stackrel{\text{def}}{=} \bar{F}^{-1}(y)$. In order to state the results for the perturbed system $(PSI_2)_{k_F}$, we need to consider a Lyapunov function V_F defined in (4.5) below, for which there exist $a, b > 0$ such that, for all $(x_1, x_2) \in \mathbb{R}^2$, we have

$$a \left(x_1^2 + \bar{F}(x_2)^2 \right) \leq V_F(x_1, x_2) \leq b \left(x_1^2 + \bar{F}(x_2)^2 \right). \tag{2.13}$$

We get the following theorem (compare to Th. 2 of [5]):

Theorem 2.9. *Let us consider $(PSI_2)_{k_F}$. There exists $\varepsilon_0 > 0$ such that, for $p \in [1, \infty]$, $u, v \in L^p$ and $\|v\|_\infty \leq \varepsilon_0$, $\bar{x} \in \mathbb{R}^2$, if (x_1, x_2) is the trajectory of $(PSI_2)_{k_F}$ corresponding to (u, v) starting at $\bar{x}$, the following estimate holds*

$$\|x\|_p \leq \text{const.} \left(\|v\|_p + \|u\|_p + L_p(V_F(\bar{x})^{1/2}) \right), \tag{2.14}$$

where

$$L_p(X) \stackrel{\text{def}}{=} \left(\int_0^X \frac{s^p}{K(s)} \mathrm{d}s \right)^{1/p} \text{ for } p \text{ finite and } L_\infty(X) \stackrel{\text{def}}{=} X. \tag{2.15}$$

3. The linear feedback k_L

We rewrite $(SI_2)_{k_L}$ as follows

$$\begin{cases} \dot{z} &= y - \sigma(z + u), \\ \dot{y} &= -\sigma(z + u), \end{cases}$$

where $z = x_1 + x_2$, $y = x_2$ and $\sigma(\cdot)$ ($u(\cdot)$ respectively) stands for $-\sigma(-\cdot)$ ($-u(\cdot)$ respectively). Moreover, up to a time reparameterization and a linear change of variables, we can assume that $\sigma'(0) = 1$. Let us define $\sigma_+ = \lim_{+\infty} \sigma$, $\sigma_- = -\lim_{-\infty} \sigma$ and, for $r > 0$,

$$K_r \stackrel{\text{def}}{=} \inf\{|\sigma(z)|, \ |z| > r\} \text{ and } \delta_r \stackrel{\text{def}}{=} \sup_{|z|,|z'| > \frac{2}{3}r, zz' > 0} |\sigma(z) - \sigma(z')|.$$

Let K be an S-bound for σ and $\sigma_m \stackrel{\text{def}}{=} \min(\sigma_+, \sigma_-)$, $\sigma_M \stackrel{\text{def}}{=} \max(\sigma_+, \sigma_-)$. Then we have

$$K \geq \max(1, \sigma_M, \|\sigma'\|_\infty).$$

Fix $C, \rho > 0$ and ρ_p for p finite such that

$$\begin{aligned} \rho &\ll& 1 \ll C, \\ \varepsilon\sigma(\varepsilon C) &>& \frac{3}{4}\sigma_\varepsilon \text{ if } \varepsilon = \pm, \\ 2KC\rho_p &<& \frac{1}{2}\min\left(1, \frac{K_\rho^p}{2}\right), \\ \rho_p &<& \frac{\rho}{2}, \\ \delta_C &\ll& \min\left(1, K_C, \frac{3}{16}\sigma_m\right). \end{aligned}$$

In addition, let $\theta \stackrel{\text{def}}{=} \frac{C}{2(C+K)} \in (0, 1/2)$. The Lyapunov function defined in (2.2) is written now

$$V(z, y) = y^2 + \int_0^{z-y} \sigma(s)\mathrm{d}s + \int_0^z \sigma(s)\mathrm{d}s. \tag{3.1}$$

For $r > 0$, set $V_r \stackrel{\text{def}}{=} \max\{V(z, y), \ \|(z, y)\| \leq r\}$. We use $\dot{V}$ to denote the time derivative of V along trajectories of $(SI_2)_{k_L}$ and we have

$$\dot{V} = -2y\Delta\sigma\Big|_z^{z+u} + y\Delta\sigma\Big|_z^{z-y} - \sigma(z)\sigma(z + u). \tag{3.2}$$

We sometimes consider a classical Lyapunov function for (SI_2) given by

$$W(z, y) \stackrel{\text{def}}{=} \frac{y^2}{2} + \int_0^z \sigma(s)\mathrm{d}s. \tag{3.3}$$

Remark 3.1. When $yz < 0$, $z(z + u) \leq 0$ and $|y| \gg \max(|z|, |u|) \gg 1$, an easy computation shows that $\dot{V} \geq \frac{K}{2}|y|$. This situation explains why V cannot give rise to a dissipation inequality of the kind (1.10). However, in the case described previously, we also have that $|\dot{z}| \geq |y| - K$, which implies that the previous situation firstly cannot last more than a time interval of length 1 and secondly is preceded and followed by time intervals where $|z| \geq |y|$ and whose lengths are of same order as $|y|$. Therefore, by integrating (3.2) on appropriate time intervals, one hopes to achieve the desired results.

Remark 3.2. The above remark is related to the following fact: assume that, at some point t_0, we have $|y(t_0)| > K$ and $z(t_0) = 0$. Then, t_0 is an isolated zero of z and, if t_1 is an another zero of z, then

$$|t_1 - t_0| \geq 2\left(\frac{|y(t_0)|}{K} - 1\right). \tag{3.4}$$

Indeed, without loss of generality, we can assume that $y(t_0) > 0$ and $t_1 > t_0$. For $t \geq t_0$, we have $y(t) \geq y(t_0) - (t - t_0)K$, which implies that

$$\dot{z}(t) \geq y(t_0) - K - (t - t_0)K.$$

A simple integration yields equation (3.4).

3.1. **Proof of Theorem 2.5**

Thanks to Remark 2.4, for the rest of the paper, we look for profiles at ∞. Therefore, we can assume that all the inputs considered in the sequel satisfy $\|u\|_p \geq C$, for $p \in [1, \infty]$.

The next proposition is a preliminary step towards the proof of Theorem 2.5 for $p = \infty$ and its proof is given in the Appendix:

Proposition 3.3. Let $p \in [1, \infty]$, $T > 0$ and $u \in L^p$. Let (z, y) be the corresponding trajectory of $(PSI_2)_{k_L}$. Then, we have

(a) $\|y\|^2_{\infty,[0,T]} \leq \mathrm{const.} \, \|z\|_{\infty,[0,T]}$;

(b) either $\|y\|^2_{\infty,[0,T]} \geq \frac{\sigma_m}{2}\|z\|_{\infty,[0,T]}$ or

$$\|z\|_{\infty,[0,T]} \leq \mathrm{const.} \, \|u\|_{p,[0,T]}^{\frac{2p}{2p+1}}, \tag{3.5}$$

where $\frac{2p}{2p+1}$ is taken to be equal to 1 for $p = \infty$.

We now prove (1.4) for $p = \infty$. Taking into account Proposition 3.3, it is enough to show that:

Proposition 3.4. For $u \in L^\infty$ and $\|u\|_\infty \geq C$ we have

$$\|z\|_\infty \leq \mathrm{const.} \, \|u\|_\infty^2.$$

Proof of Proposition 3.4. Let $M = \|u\|_\infty \geq C$. For $T > 0$, we pick $[t_0, t_1] \in [0, T]$ such that we have, for $t \in [t_0, t_1]$,

$$W(t_0) = \frac{W_{\max}}{2} \leq W(t) \leq W(t_1) = W_{\max}.$$

Define $\lambda > 0$ such that $W_{\max} = W(t_1) = \lambda M^2$. Without loss of generality, we can assume that $\lambda > 2$. Then, proving Proposition 3.4 amounts to show that λ is actually bounded independently of u. This is what we do next.

Thanks to Proposition 3.3, we can suppose that

$$\text{const. } \lambda M^2 \leq y_{\max}^2 \leq \text{const. } z_{\max} \leq \text{const. } \lambda M^2, \tag{3.6}$$

where $y_{\max}, z_{\max}$ are defined as before.

The conclusion will be the consequence of two lemmas given below and whose proofs are given in the appendix. First, we prove that:

Lemma 3.5. *With the previous notations, either λ is bounded independently of u or there exists, on $[t_0, t_1]$, a sequence of times $(t^j)_{1 \leq j \leq 2N+1}$ such that*

(i) on $[t^1, t^{2N+1}]$, $|z(t)| = M + C$ exactly for t equal to some t^j, $1 \leq j \leq 2N + 1$;

(ii) $|z(t)| \geq M + C$ on $I_{2l-1} \overset{\text{def}}{=} [t^{2l-1}, t^{2l}]$ and $|z| \leq M + C$ on $I_{2l} = [t^{2l}, t^{2l+1}]$ for $l = 1, \cdots, N$;

(iii) $\Delta W \Big|_{t^1}^{t^{2N+1}} \geq \frac{\lambda}{16} M^2$.

In a second step, we evaluate the variations of W along I_{2l-1} and I_{2l}. Set $\Delta W \Big|_j \overset{\text{def}}{=} \Delta W \Big|_{t^j}^{t^{j+1}}$, for $j = 1, \cdots, 2N + 1$. We get the following key estimate:

Lemma 3.6. *With the previous notations, we have for $l = 1, \cdots, N$,*

$$\Delta W_{2l-1} + \Delta W_{2l} \leq -\text{const. } \sqrt{\lambda} M + \text{const. } M. \tag{3.7}$$

Finally, by adding all the inequalities (3.7) for $l = 1, \cdots, N$, we obtain that

$$\frac{\lambda}{16} M^2 \leq \Delta W \Big|_{t^1}^{t^{2N+1}} \leq NM(-\text{const. } \sqrt{\lambda} + \text{const.}).$$

Then, λ has to be bounded above by some const. and the proof of Proposition 3.4 is finished. □

We now consider (3) of Theorem 2.5, in the case where $p = q < \infty$. The first step consists of showing the next proposition:

Proposition 3.7. *Let $p \in [1, \infty)$, $u \in L^p$ and $x_u = (z, y)$ be the trajectory of $(SI_2)_{k_L}$ corresponding to u and starting at 0. Then, we have*

$$\|\sigma(z + u)\|_p \leq \text{const. } \|u\|_p. \tag{3.8}$$

Proof of Proposition 3.7. Equation (3.8) will be obtained from the integration of (3.2) on non-overlapping measurable sets S, whose countable union is equal to $\mathbb{R}^+$. We will distinguish several cases for the S's and, for each case, we aim at establishing the inequality

$$\int_S (\dot{V} + \text{const. } |\sigma(z + u)|^p) \leq \text{const. } \int_S |u|^p, \tag{3.9}$$

either by the corresponding differential inequality $\dot{V} + \text{const. } |\sigma(z+u)|^p \leq \text{const. } |u|^p$ or by a direct computation.

We will also use repeatedly a result given in [5]:

Remark 3.8. *The trajectories of $(SI_2)_{k_L}$ corresponding to inputs $u \in L^p$ with $p < \infty$ converge to 0 at infinity.*

For the rest of the proof, we assume that $\|u\|_p \geq C$. We will consider the following cases:

(A) $|y| \leq C$; we have two sub-cases defined by

(A1) $|z| \geq \rho$:

(A2) $|z| < \rho$;
(B) $|y| > C$.

Before starting the study of cases (A) and (B), we provide the following remark in order to motivate the technical details of the proof of Proposition 3.7:

Remark 3.9. We have defined the previous cases in such a way that we can reduce the core of the argument to the proof of equation (3.9) when $S = I$, an open interval on which $|z| - |y| - C$ has a constant sign and, more importantly, where z is monotone. As usual, we introduce the subset E of I defined as

$$E = \left\{ t \in I, \quad |u(t)| \geq \frac{|z(t)|}{3} \text{ a.e.} \right\}.$$

Next, we write $\dot{V}$ as a sum of several terms (see Eq. (3.14)), whose integrals over I can be easily estimated in terms of $\int_I |u|^p$ and $|E|$, except for one of them (a quantity J_1 defined below in Eq. (3.15)). Similarly to the integral J defined in equation (5.13), J_1 is handled after being written first, by using z as a parameter instead of the time t, and finally as a double integral. Note that, in most of the estimations, the hypothesis of σ being increasing is crucial.

If $(A1)$ holds, we have clearly have

$$\dot{V} + \text{const.} \, |\sigma(z+u)|^p \leq \text{const.} \, |u|^p, \tag{3.10}$$

whether $|u| \geq \rho_p$ or not.

If $(A2)$ holds, we consider the sets

$$\begin{aligned}
S_0 &= \{t \geq 0 \mid |y(t)| \leq C, \ V(z(t), y(t)) < 4\rho_p^2\}, \\
S_0' &= \{t \geq 0 \mid |y(t)| \leq C, \ |z(t)| < \rho\}.
\end{aligned}$$

As ε tends to 0^+, S_0 and S_0' are the limits of the open sets

$$\begin{aligned}
S_\varepsilon &= \{t \geq 0 \mid |y(t)| < C + \varepsilon, \ V(z(t), y(t)) < 4\rho_p^2\}, \\
S_\varepsilon' &= \{t \geq 0 \mid |y(t)| < C + \varepsilon, \ |z(t)| < \rho\}.
\end{aligned}$$

Note that $S_0 \subset S_0'$ and $S_\varepsilon \subset S_\varepsilon'$. Thanks to Remark 3.8, the open sets S_ε/S_0 and S_ε'/S_0' are bounded and their measures tend to zero as ε tends to 0^+. Since $\dot{V} + \text{const.} |\sigma(z+u)|^p$ is bounded over $\mathbb{R}^+$, in order to establish (3.9) for $E = S_0'$, it is enough to do it for $E = S_\varepsilon'$ with constants independent of ε.

On S_ε, we have

$$\dot{V} \leq \text{const.} \, |u|\sqrt{V} - \text{const.} \, V. \tag{3.11}$$

Set $V_p \overset{\text{def}}{=} V^{p/2}$. Then, using (3.11), we get

$$\dot{V}_p \leq \text{const.} \, V_p^{\frac{p-1}{p}} - \text{const.} \, V_p. \tag{3.12}$$

On each finite interval (t, t') of S_ε, we have $V_p(t) = V_p(t')$, which implies that $\Delta V \big|_t^{t'} = 0$. By integrating (3.12) (and applying Hölder's inequality for $p > 1$), we get

$$\Delta V \Big|_t^{t'} + \text{const.} \int_t^{t'} |\sigma(z+u)|^p \leq \text{const.} \int_t^{t'} |u|^p. \tag{3.13}$$

If $(t, \infty) \subset S_0$, we have

$$\int_t^\infty V_p^p \leq V_p^p(t) + \text{const.} \int_t^\infty |u|^p,$$

which leads to (3.13) with $t' = \infty$. On $S_\varepsilon'/S_\varepsilon$, we clearly have (3.10) since the dominant term in $\dot{V}$ is $\frac{1}{2}y\sigma(-y)$ if $|u| \leq \rho$. Thus, we have established (3.9) for $E = S_\varepsilon'$ and $\varepsilon > 0$ small enough.

We assume that (B) $|y| > C$ holds. The open set $S_2 = \{t \geq 0 | \ |y(t)| > C\}$ is a union (at most countable) of open intervals (I_s). Since $u \in L^p$, the trajectories of $(SI_2)_{k_L}$ converge to 0 at infinity, each interval I_s has finite length.

We now focus on an interval $I = (t_0, t_1)$ where $y(t_0) = y(t_1) = C$ and $y > C$ on I. We will prove (3.9) for $S = I$. Let $T \stackrel{\text{def}}{=} t_1 - t_0$. We assume that $T \geq \frac{10C}{\sigma_m}$, otherwise we are back to case (A). We say that condition (R) is satisfied at $t \in I$ if

$$(R) \quad y(t) + C \leq |z(t)|.$$

Moreover, z is strictly increasing from I to $[z_0, z_1]$, where $z_0 = z(t_0)$ and $z_1 = z(t_1)$. Then, z is equal to 0 at most once. We have the following situation:

Lemma 3.10. *With the notations above, one of the two next cases occurs:*

$(B1)$ *z has a constant sign on I and (R) is satisfied on I except (maybe) on an interval (t_0', t_1') such that*
if $z \leq 0$, $t_1' = t_1$ and there is equality in (R) at t_0';
if $z \geq 0$, $t_0' = t_0$ and there is equality in (R) at t_1';

$(B2)$ *z has a unique zero at $\bar{t}$ and (R) is satisfied on I, except on an interval $I' = (t_0', t_1')$ such that $\bar{t} \in I'$ and there is equality in (R) at t_0' and t_1'.*

Proof of Lemma 3.10. Consider the function f defined on I by

$$f(t) \stackrel{\text{def}}{=} |z(t)| - y(t) - C = \text{sgn } z(t) z(t) - y(t) - C.$$

For all $t \in I$, except maybe for at most one point, we have

$$(\text{sgn } z)\dot{f} = y(t) - (\text{sgn } z(t) + 1)\sigma(z(t) + u(t)) > 0.$$

Depending on the sign of z, $f(t_0)$ and $f(t_1)$, we easily get the conclusion. $\quad\square$

In each above case, we will establish (3.9) first on I' and second on I/I'. We also need to rewrite (3.2) as

$$\dot{V} = -2\dot{z}\Delta\sigma\Big|_{z-\theta y}^{z+u} - \dot{z}\Delta\sigma\Big|_z^{z-\theta y} - \dot{z}\Delta\sigma\Big|_{z-y}^{z-\theta y} - \sigma(z+u)\Delta\sigma\Big|_{z-y}^{z+u} - \sigma^2(z+u). \tag{3.14}$$

We get

$$\Delta V\Big|_{t_0}^{t_1} = J_1 + J_2 + J_3,$$

where

$$J_1 = -\int_{t_0}^{t_1} \dot{z}\Delta\sigma\Big|_z^{z-\theta y} - \int_{t_0}^{t_1} \dot{z}\Delta\sigma\Big|_{z-y}^{z-\theta y}, \tag{3.15}$$

$$J_2 = -\int_{t_0}^{t_1} \left(2\dot{z}\Delta\sigma\Big|_{z-\theta y}^{z+u} + \sigma(z-y)\Delta\sigma\Big|_{z-y}^{z+u} \right), \tag{3.16}$$

$$J_3 = -\int_{t_0}^{t_1} \left(\left[\Delta\sigma\Big|_{z-y}^{z+u}\right]^2 + \sigma^2(z+u) \right) \leq 0. \tag{3.17}$$

We now suppose that $(B1)$ holds. We will only treat the case when $z \geq 0$ and indicate the minor modifications occurring, when $z \leq 0$, before treating case $(B2)$. Then, we assume that $z \geq 0$ and $I' = [t_0, t_1')$. Let $T' = t_1' - t_0$. We have

$$T' \leq \frac{2}{1 - \frac{2K}{C}}, \ |\Delta V\big|_{I'}| \leq CT', \ \Delta z\big|_{I'} \leq CT' \text{ and } \Delta y\big|_{I'} \leq T',$$

by simply taking into account that $\dot{z} \geq C - K$ and $\dot{y} \geq K$. Let us consider the set E given by

$$E = \left\{ t \in I', \ |u(t)| > \frac{z(t)}{3} \text{ a.e.} \right\}. \tag{3.18}$$

Then, we have:

Lemma 3.11. *With the previous notations,* $|E| \geq$ const. T' *with* const. *independent of* C.

Proof of Lemma 3.11. From (5.8) written on I' and the fact that $\Delta y\big|_{I'} \geq 0$, we get

$$0 \leq -\sigma \left(\frac{C}{3} \right) (T' - |E| - |F|) + 2K|E|,$$

where $F = \{ t \in I', \ z \leq 1 \}$. We clearly have

$$|F| \leq \frac{\frac{C}{3}}{C + \frac{2}{1 - \frac{2K}{C}}} \leq \frac{1}{3},$$

and, if F is not empty, we have

$$T' - |F| \geq \frac{z(t_1') - \frac{C}{3}}{C + \frac{2}{1 - \frac{2K}{C}}} \geq \frac{5}{4}. \tag{3.19}$$

We conclude easily. □

In the case where F is not empty, E contains a subset E' of E of measure $\frac{|E|}{2} \geq$ const. T' such that $z \geq$ const. C on E'. Then $|u|$ will be larger than const. C on E' of E and we have

$$\int_{I'} |u|^p \geq \int_{E'} |u|^p \geq \text{const. } C^p T' \geq |\Delta V\big|_{I'}| + \text{const.} \int_{I'} |\sigma(z+u)|^p.$$

In the case where F is empty, we simply take $E' = E$, and now

$$\int_{I'} |u|^p \geq \int_E z \geq \text{const. } \frac{C}{2} T' \geq \text{const. } C(|\Delta V\big|_{I'}| + \text{const.} \int_{I'} |\sigma(z+u)|^p).$$

Then (3.9) is obtained.

We next establish (3.9) on $I/I' = [t_1', t_1]$. This is a consequence of the following lemma, whose proof is delayed to the Appendix:

Lemma 3.12. *Let* J_1, J_2 *and* J_3 *be defined in (3.15, 3.16) and (3.17) respectively. we have*

(J1) $J_1 \leq$ const. $\sup(y(t_1), C)\delta_C \leq$ const. $\delta_C T$;

(J2) $J_2 \leq$ const. $\delta_C T +$ const. $\int_{I/I'} |u|^p$;

(J3) $J_3 \leq -$const. $T +$const. $\displaystyle\int_{I/I'} |u|^p$;

where const. *is independent of C.*

Equation (3.9) is obtained on I and that concludes case (B1). When $z \leq 0$, the only differences with the other case occur in the proof of (J2) given in the Appendix.

We now suppose that case (B2) holds. The case where $z \leq 0$ can be deduced from the case where $z \geq 0$ similarly to what ws done in case (B1). Then, it is enough to establish (3.9) on the interval $[\bar{t}, t_1]$ where z remains positive. The interval $[\bar{t}, t_1]$ is the union of $I' = [\bar{t}, t_1')$ and $[t_1', t_1]$. It is easy to see that the procedure that led to (3.9) on I/I' above (the inequalities (3.15, 3.16) and (3.17) together with Lems. 3.12 and 5.1) will also prove (3.9) on $[t_1', t_1]$. Therefore, the last part of the proof of Proposition 3.7 consists of getting (3.9) on I'.

We can suppose that $y(\bar{t}) \geq 8C$ since $t_1 - \bar{t}$ can be taken larger than $\frac{10C}{\sigma_m}$. Indeed, if $y(\bar{t}) \leq 8C$, we reproduce the argument involving the set E, defined in (3.18), and Lemma 3.11 (with $[\bar{t}, t_1]$ now instead of I') and obtain (3.9).

Until the end of the proof of Proposition 3.7, we assume that $y(\bar{t}) \geq 8C$.

We have that $z(\bar{t}) = 0$ and $z(t_1') = y(t_1') + C$. Let

$$T'' \stackrel{\text{def}}{=} t_1' - \bar{t} \text{ and } \bar{y} \stackrel{\text{def}}{=} \frac{\int_{\bar{t}}^{t_1'} y}{T''}.$$

Since

$$-K \leq \dot{y} \leq K \text{ and } T''C \leq T''\bar{y} = \int_{\bar{t}}^{t_1'} y = C + y(\bar{t}),$$

we get $T' \leq 1 + \frac{y(\bar{t})}{C}$ and $y(\bar{t}) - KT'' \leq \bar{y} \leq y(\bar{t}) + KT''$. Then, we get

$$y(\bar{t})\left(1 - \frac{2K}{C}\right) \leq \bar{y} \leq y(\bar{t})\left(1 + \frac{2K}{C}\right).$$

We finally obtain the following estimate for T'':

$$\frac{6}{7} \leq 1 - \frac{2K}{C} \leq \frac{1 + \frac{C}{y(\bar{t})}}{1 + \frac{2K}{C}} \leq T'' \leq \frac{1 + \frac{C}{y(\bar{t})}}{1 - \frac{2K}{C}} \leq \frac{9}{8}\left(1 + \frac{4K}{C}\right). \tag{3.20}$$

Define $t_0'' \in [\bar{t}, t_1']$ such that $z(t_0'') = y(t_0'')$. We have

$$t_1' - t_0'' \leq \frac{\Delta(z-y)\big|_{t_0''}^{t_1'}}{y(\bar{t}) - KT''} \leq \frac{1}{7}.$$

Then, $T_1 \stackrel{\text{def}}{=} t_0'' - \bar{t} \geq 1 - \frac{2K}{C} - \frac{1}{7} \geq \frac{5}{7}$. Using (3.2), we have

$$\Delta V\big|_{\bar{t}}^{t_1'} + \int_{\bar{t}}^{t_1'} |\sigma(z+u)|^p \leq J + J' - \int_{\bar{t}}^{t_1'} \sigma(z)\sigma(z+u), \tag{3.21}$$

where

$$J \stackrel{\text{def}}{=} -2\int_{\bar{t}}^{t_1'} y\Big(\sigma(z+u) - \sigma(z)\Big) \text{ and } J' \stackrel{\text{def}}{=} \int_{\bar{t}}^{t_0''} y\Big(\sigma(z-y) - \sigma(z)\Big).$$

We first have that

$$-\int_{\bar{t}}^{t_1'} \sigma(z)\sigma(z+u) \le 2K^2,$$

and

$$J' \le -\int_{\bar{t}}^{t_0''} y\sigma(z) \le -\frac{\sigma(1)}{2}T_1 y(\bar{t}).$$

Suppose that $|J| \le |J'|$. Then (3.9) follows readily. If $|J| > |J'|$, by taking into account the next equation,

$$|J| \le 3(T'' - |E| - |F|)y(\bar{t})\delta_C + 5Ky(\bar{t})|E|,$$

with $E = \left\{t \in [\bar{t}, t_1'],\ |u| > \frac{z}{3}\ \text{a.e.}\right\}$ and $F = \{t \in [\bar{t}, t_1'],\ |z| \le 1\}$, we obtain, as in Lemma 3.11,

$$|E| \ge \text{const. } T''.$$

This implies that (3.9) holds on $[\bar{t}, t_1']$. Then, case (B) is finished, which ends the proof of Proposition 3.7. $\quad\square$

The second step of (3) of Theorem 2.5 in the case where $p = q < \infty$ consists of providing an appropriate sequence of inputs $u_n \in L^p$ satisfying the conditions of Lemma 2.3. This is done in the Appendix.

We now prove (1) and (2) of Theorem 2.5 in the case where $p = q < \infty$. We first establish the next proposition:

Proposition 3.13. *Let $p \in [1, \infty)$, $u \in L^p$ and $x_u = (z, y)$ be the trajectory of $(SI_2)_{k_L}$ corresponding to u and starting at 0. Then, there exists const. > 0 such that for $\|u\|_p \ge$ const. we have*

$$\|z\|_p \quad \le \quad \text{const. } \|u\|_p^2, \tag{3.22}$$

$$\|y\|_p \quad \le \quad \text{const. } \|u\|_p^{\frac{p+2}{p+1}}, \tag{3.23}$$

$$\|y\|_\infty^2 \quad \le \quad \text{const. } \|z\|_\infty \le \text{const. } \|u\|_p^{\frac{2p}{p+1}}. \tag{3.24}$$

Before starting the proof of the above proposition, we give three preliminary lemmas. The proofs of the last two are given in the Appendix.

Lemma 3.14. *Let $p \in [1, \infty)$. If $\gamma_p > 0$ is the L^p-gain of the linear system given by $\dot{x} = Ax + bu$ with*

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } b = (1\ 1)^T,$$

then, for every $u \in L^p$ with $x_u = (z, y)$ the trajectory of $(SI_2)_{k_L}$ corresponding to u and starting at 0, we have

$$\|x_u\|_p \le \gamma_p \|\bar{u}\|_p, \tag{3.25}$$

where $\bar{u} = z - \sigma(z + u)$.

Proof of Lemma 3.14. This is simply a consequence of the rewriting of (SI_2) as

$$\dot{x}_u = Ax_u + b\bar{u}.$$

Note that, for all $t \in \mathbb{R}^+$,

$$|\bar{u}(t)| \le |u(t)| + K(z(t) + u(t))\sigma(z(t) + u(t)) \le \text{const. } |u(t)| + Kz(t)\sigma\Big(z(t) + u(t)\Big).$$

$\square$

Lemma 3.15. *Let* $p \in [1, \infty)$, $u \in L^p$ *and* $x_u = (z, y)$ *be the trajectory of* $(SI_2)_{k_L}$ *corresponding to* u *and starting at* 0. *Then, we have*

$$\|y^2\|_p \le \text{const.}\, \|z\sigma(z + u)\|_p. \tag{3.26}$$

Let $u \in L^p$ and $x_u = (z, y)$ be the trajectory of $(SI_2)_{k_L}$ corresponding to u and starting at 0. For $R > 0$, consider the set E_R given by

$$E_R = \{t \in \mathbb{R}^+, \ V(t) > R\}\cdot$$

Lemma 3.16. *With the notations above, we either have* $\|V\|_p \le \text{const.}_R\, \|u\|_p$ *or*

$$\|V\|_p^p \le \text{const.}_R \int_{E_R} V^p. \tag{3.27}$$

With the previous lemma at hand, we can start the argument of Proposition 3.13.

For the rest of this proof, we choose $R \overset{\text{def}}{=} C^2$. For $T > 0$, we will "slice" $[0, T] \cup E_R$ into a finite number of disjoint measurable sets L_k and M_k defined below and prove, for each $N = L_k \cup M_k$ the estimates

$$\int_N |z|^p \ \le \ \text{const.} \left(\int_N |u|^p\right)^2, \tag{3.28}$$

$$\int_N |y|^p \ \le \ \text{const.} \left(\int_N |u|^p\right)^{\frac{p+2}{p+1}}, \tag{3.29}$$

with of course const. independent of k and T.

Let $V_m \overset{\text{def}}{=} V_{\infty,[0,T]}$ and $k_m \overset{\text{def}}{=} \left[\frac{\ln \frac{V_m}{R}}{\ln 2}\right]$. For $k = 0, \cdots, k_m$, we define L_k and M_k as

$$L_k \overset{\text{def}}{=} \left\{t \in [0, T] \cup E_R, \ \ \frac{V_m}{2^{k+1}} < V(t) < \frac{V_m}{2^k}\right\}, \tag{3.30}$$

and

$$M_k \overset{\text{def}}{=} \left\{t \in [0, T] \cup E_R, \ \ V(t) = \frac{V_m}{2^k}\right\}. \tag{3.31}$$

It is easy to see that estimates (3.22) and (3.23) of Proposition 3.13 are consequences of Lemma 3.16 together with (3.28, 3.29) and the following inequality: for every $\alpha \ge 1$, integer $n \ge 1$ and positive real numbers $a_1, \cdots, a_n$, we have

$$\sum_1^n a_i^\alpha \le \left(\sum_1^n a_i\right)^\alpha.$$

We first establish (3.22) and (3.23) for the L_k's. In the sequel, we fix $k \in \{0, \cdots, k_m\}$ and use L and $\bar{V}$ to denote L_k and $\frac{V_m}{2^k}$ respectively. Since V is continuous, L is the countable union of disjoint intervals $I_j = (t^j, t'^j)$, $j \in \Omega$, such that for every $t \in (t^j, t'^j)$ we have

$$\frac{\bar{V}}{2} < V(t) < \bar{V},$$

and each I_j falls into one of the three following cases:

(c1) $V(t^j) = V(t'^j)$ $\left(= \frac{\bar{V}}{2} \text{ or } \bar{V}\right)$;

(c2) $V(t^j) = \frac{\bar{V}}{2}$ and $V(t'^j) = \bar{V}$;

(c3) $V(t^j) = \bar{V}$ and $V(t'^j) = \frac{\bar{V}}{2}$.

For $l = 1, 2, 3$ and $j \in \Omega$, we use Ω_l to denote the set of indices $j \in \Omega$ such that I_j verifies (cl). Let $T^{(j)} \stackrel{\text{def}}{=} t'^j - t^j$.

Remark 3.17. The sets Ω_2 and Ω_3 are nonempty. In addition $\left(V^{\frac{1}{2}}\right)^{\cdot}$, the time derivative of $V^{\frac{1}{2}}$ along x_u, is bounded by $5K$. Then, we have that $\Omega_2^{\#}$ and $\Omega_3^{\#}$ are finite. In addition, a simple continuity argument together with Remark 3.2 shows that each interval I_j, $j \in \Omega_3$, has a closest interval $I_{l(j)}$, $l(j) \in \Omega_2$, on its left and

$$\Omega_3^{\#} = \Omega_2^{\#} \leq \text{const.} \frac{T}{\bar{V}^{\frac{1}{2}}} \cdot \tag{3.32}$$

Moreover, for every $j \in \Omega$, we have on I_j that $|\dot{V}| \leq 8\sigma_M \bar{V}^{\frac{1}{2}}$ and

$$\int_{I_j} |z|^p \leq \text{const.} \ T^{(j)} \bar{V}^p \quad \text{and} \quad \int_{I_j} |y|^p \leq \text{const.} \ T^{(j)} \bar{V}^{\frac{p}{2}}. \tag{3.33}$$

We fix $j \in \Omega$ and we will get an estimate of $\Delta V \big|_{I_j}$. We first assume that z has at least two zeros on I_j. Since $|\dot{z}| \geq \text{const.} \ \bar{V}^{\frac{1}{2}}$ at a zero of z and, thanks to Remark 3.2, z has a finite number N_j of isolated zeros on I_j, with $N_j \leq \text{const.} \frac{T_j}{\bar{V}^{\frac{1}{2}}}$. Let $t_0 < \cdots < t_{N_j}$ be the sequence of zeros of z on I_j. Set

$$T_i \stackrel{\text{def}}{=} t_{i+1} - t_i \quad \text{and} \quad \Delta V_i \stackrel{\text{def}}{=} \Delta V \big|_{t_i}^{t_{i+1}}.$$

Notice that $T_i \geq \text{const.} \ \bar{V}^{\frac{1}{2}}$. For $i = 0, \cdots, N_j$, we define the following sets:

$$E_i = \left\{ t \in [t_i, t_{i+1}], \ |u| > \frac{|z|}{3} \text{ a.e.} \right\}; \tag{3.34}$$

$$E_i' = \left\{ t \in [t_i, t_{i+1}], \ |u| \leq \frac{|z|}{3} \text{ a.e.} \right\}; \tag{3.35}$$

$$F_i = \left\{ t \in [t_i, t_{i+1}], \ |y| \leq |z| - C \right\}; \tag{3.36}$$

$$F_i' = \left\{ t \in F_i, \ |u| > \frac{|y|}{2}, \ uy < 0 \text{ a.e.} \right\} \tag{3.37}$$

$$G_i = \left\{ t \in F_i', \ |u| \geq \rho \bar{V}^{\frac{1}{2}} \text{ a.e.} \right\} \cdot \tag{3.38}$$

Set $\Theta = \{ i \in \{0, \cdots, N_j\}, \ |E_i| > \rho \}$. Note that

$$\Theta^{\#} \leq \frac{\sum_0^{N_j} |E_I|}{\rho}.$$

Moreover, for $i \in \{0, \cdots, N_j - 1\}$, we have $y(t_i) y(t_{i+1}) < 0$.

The next lemma is the key estimate for getting Proposition 3.13:

Lemma 3.18. *With the above notations, we have*

$$\Delta V_i + \frac{\sigma_m^2}{2} T_i + \frac{\sigma_m^2}{8} \bar{V}^{\frac{1}{2}} \leq \text{const. } \bar{V}^{\frac{1}{2}} \Big(|E_i| + |G_i| \Big). \tag{3.39}$$

In order to relate the left hand side of equation (3.39) and $\int |u|^p$, we use the set E_i together with the next estimate.

Lemma 3.19. *With the above notations, we have, for $i \in \Theta$,*

$$\int_{E_i} |z|^p \geq \text{const. } |E_i| \bar{V}^{\frac{p}{2}}. \tag{3.40}$$

We now want to establish that, for every $i \in \Omega$,

$$\int_{t_i}^{t_{i+1}} |u|^p \geq \text{const. } \bar{V}^{\frac{p-1}{2}} \left(\Delta V_i + \frac{\sigma_m^2}{2} T_i + \frac{\sigma_m^2}{16} \bar{V}^{\frac{1}{2}} \right). \tag{3.41}$$

Thanks to Lemma 3.18, (3.41) is immediate if $|E_i| \leq \rho$, since we can rewrite (3.39) as

$$\Delta V_i + \frac{\sigma_m^2}{2} T_i + \frac{\sigma_m^2}{16} \bar{V}^{\frac{1}{2}} \leq \text{const. } \bar{V}^{\frac{1}{2}} |G_i|.$$

For the remaining case, $i \in \Theta$ and (3.41) follows from Lemma 3.19.

By adding up (3.41) for $i = 0, \cdots, N_j$, we get

$$\int_{t_0}^{t_{N_j}} |u|^p \geq \text{const. } \bar{V}^{\frac{p-1}{2}} \left(\Delta V \Big|_{t_0}^{t_{N_j}} + \frac{\sigma_m^2}{2}(t_{N_j} - t_0) + N_j \frac{\sigma_m^2}{16} \bar{V}^{\frac{1}{2}} \right).$$

In addition, on each of the two intervals $[t^j, t_0]$ and $[t_{N_j}, t'_j]$, we have

$$\Delta V + \frac{\sigma_m^2}{4} \Delta t \leq \text{const. } \bar{V}^{\frac{1}{2}} \Big(|E^1| + |G^1| \Big),$$

where ΔV and Δt are the variations of V and t respectively, E^1 and G^1 correspond to E_i and G_i defined in (3.34) and (3.38) for $[t^j, t_0]$ and $[t_{N_j}, t'_j]$. We hence obtain, for every $j \in \Omega$,

$$\bar{V}^{\frac{p-1}{2}} \left(\Delta V \Big|_{I_j} + \frac{\sigma_m^2}{4} T^{(j)} + N_j \frac{\sigma_m^2}{8} \bar{V}^{\frac{1}{2}} \right) \leq \text{const. } \bar{V}^{\frac{p-1}{2}} \Big(|E^{(j)}| + |G^{(j)}| \Big) \leq \int_{I_j} |u|^p, \tag{3.42}$$

where

$$E^{(j)} = \left\{ t \in I_j, \ |u| > \frac{|z|}{3} \text{ a.e.} \right\},$$
$$G^{(j)} = \left\{ t \in I_j, \ |u| > \sup\left(\rho \bar{V}^{\frac{1}{2}}, \frac{|y|}{2} \right), \ uy < 0, \ |y| \leq |z| - C \text{ a.e.} \right\}.$$

Note that (3.42) remains valid in the case where z has at most one zero on I_j.

For $j \in \Omega_1$, equation (3.42) implies that

$$\int_{I_j} |u|^p \geq \text{const. } T^{(j)} \bar{V}^{\frac{p-1}{2}}.\tag{3.43}$$

For $j \in \Omega_2$, equation (3.42) implies that

$$\int_{I_j} |u|^p \geq \text{const. } \left(T^{(j)} + \bar{V}\right) \bar{V}^{\frac{p-1}{2}}.\tag{3.44}$$

As for $j \in \Omega_3$, two possibilities occur whether $T^{(j)} > \frac{4\bar{V}}{\sigma_m^2}$ (case $(c31)$) or not (case $(c32)$). Thanks to Remark 3.17, if $j \in \Omega_3$ verifies case $(c32)$, consider $l(j) \in \Omega_2$. Then,

$$\int_{I_j} |u|^p + 2\int_{I_{l(j)}} |u|^p \geq \text{const. } \left(\frac{\sigma_m^2}{4}\left(T^{(j)} + 2T^{(j')}\right) + \Delta V\Big|_{I_j} + 2\Delta V\Big|_{I_{l(j)}}\right) \bar{V}^{\frac{p-1}{2}}.$$

Since $\Delta V\Big|_{I_j} + 2\Delta V\Big|_{I_{l(j)}} = \frac{\bar{V}}{2}$, we have

$$\int_{I_j \cup I_{l(j)}} |u|^p \geq \text{const. } \left(T^{(j)} + T^{(j')} + \frac{\bar{V}}{4}\right) \bar{V}^{\frac{p-1}{2}}.$$

If $j \in \Omega_3$ verifies case $(c31)$, then

$$\frac{\sigma_m^2}{8}T^{(j)} \leq \left(\Delta V\Big|_{I_j} + \frac{\sigma_m^2}{4}T^{(j)}\right),$$

which means that this case is equivalent to case $(c1)$. From now on, Ω_1' is the set of indices j for which case $(c1)$ or $(c32)$ holds and Ω_2' is the set of indices j for which case $(c2)$ or $(c31)$ holds.

Set

$$T_{1,\bar{V}} \stackrel{\text{def}}{=} \sum_{j \in \Omega_1'} T^{(j)} \quad \text{and} \quad T_{2,\bar{V}} \stackrel{\text{def}}{=} \sum_{j \in \Omega_2'} \left(T^{(j)} + \frac{\bar{V}}{4}\right).$$

Note that

$$T_{2,\bar{V}} \geq \frac{\bar{V}}{4} \quad \text{and} \quad T_{1,\bar{V}} + T_{2,\bar{V}} \geq \text{const. } (|L| + \bar{V}).\tag{3.45}$$

Then, we have

$$\int_L |u|^p \geq \text{const. } \bar{V}^{\frac{p-1}{2}}(T_{1,\bar{V}} + T_{2,\bar{V}}),\tag{3.46}$$

$$\int_L |z|^p \leq \text{const. } \bar{V}^p(T_{1,\bar{V}} + T_{2,\bar{V}}),\tag{3.47}$$

and

$$\int_L |y|^p \leq \text{const. } \bar{V}^{\frac{p}{2}}(T_{1,\bar{V}} + T_{2,\bar{V}}).\tag{3.48}$$

Using (3.45–3.48), we get

$$\left(\int_L |u|^p \right)^2 \geq \text{const. } \bar{V}^{p-1} (T_{1,\bar{V}} + T_{2,\bar{V}})^2, \tag{3.49}$$

and

$$\left(\int_L |u|^p \right)^{\frac{p+2}{p+1}} \geq \text{const. } \bar{V}^{\frac{(p-1)(p+2)}{2(p+1)}} (T_{1,\bar{V}} + T_{2,\bar{V}})^{\frac{p+2}{p+1}}. \tag{3.50}$$

It remains to establish (3.49) and (3.50) for M_k (instead of L_k in the aforementioned equations). Similarly to the way in which case $(A2)$ was handled, it is enough to do it for $M_{k,\varepsilon}$ given by

$$M_{k,\varepsilon} = \{t \in [0,T] \cup E_R, \ \ \bar{V} - \varepsilon < V(t) < \bar{V} + \varepsilon\},$$

with $\varepsilon > 0$ small enough. For k and ε fixed, we reproduce the argument given above with L equal now to $M_{k,\varepsilon}$ until getting equation (3.50). The first relevant modification occurs in equation (3.44) where the right-hand side becomes const. $\left(T^{(j)} + 2\varepsilon \right) \bar{V}^{\frac{p-1}{2}}$. The cases (c31) and (c32) are now defined whether $T^{(j)} > \text{const. } \varepsilon$ or not. Then, ones gets (3.45, 3.49) and (3.50) with $T_{2,\bar{V}} \geq \text{const. } \varepsilon$. By taking limits as ε tends to zero, we get (3.49) and (3.50) for M_k with $T_{1,\bar{V}} + T_{2,\bar{V}} \geq |M_k|$. Adding up (3.49) for L_k and (3.49) for M_k, we obtain

$$\left(\int_{L_k \cup M_k} |u|^p \right)^2 \geq \text{const. } \bar{V}^{p-1} T_k^2,$$

with $T_k \geq \sup \left(|L_k \cup M_k|, \text{const. } \bar{V} \right)$. Then

$$\left(\int_{L_k \cup M_k} |u|^p \right)^2 \geq \text{const. } \bar{V}^p T_k \left(\frac{T_k}{\bar{V}} \right) \geq \text{const. } \int_{L_k \cup M_k} |z|^p. \tag{3.51}$$

Similarly, adding up (3.50) for L_k and (3.50) for M_k, we obtain

$$\left(\int_{L_k \cup M_k} |u|^p \right)^{\frac{p+2}{p+1}} \geq \text{const. } \bar{V}^{\frac{(p-1)(p+2)}{2(p+1)}} T_k^{\frac{p+2}{p+1}}.$$

It follows that

$$\left(\int_{L_k \cup M_k} |u|^p \right)^{\frac{p+2}{p+1}} \geq \text{const. } \bar{V}^{\frac{p}{2}} T_k \left(\frac{T_k}{\bar{V}} \right)^{\frac{1}{p+1}} \geq \text{const. } \int_{L_k \cup M_k} |y|^p. \tag{3.52}$$

Therefore the proof of (3.22) and (3.23) is finished. As for (3.24), it is obtained by considering (3.46) on $L_0 \cup M_0$,

$$\int_{L_0 \cup M_0} |u|^p \geq \text{const. } V_m^{\frac{p-1}{2}} T_0 \geq \text{const. } V_m^{\frac{p+1}{2}}.$$

Then, the proof of Proposition 3.13 is completed. $\qquad\square$

Proposition 3.20. *Let $p \le q \in [1, \infty)$, $u \in L^p$ and $x_u = (z, y)$ be the trajectory of $(SI_2)_{k_L}$ corresponding to u and starting at 0. Then, there exists const. > 0 such that for $\|u\|_p \ge$ const. we have*

$$
\begin{align}
\|z\|_q &\le \text{ const. } \|u\|_p^{r_1(p,q)}, \tag{3.53}\\
\|y\|_q &\le \text{ const. } \|u\|_p^{r_2(p,q)}, \tag{3.54}\\
\|\dot{y}\|_q &\le \text{ const. } \|u\|_p^{\frac{p}{q}}, \tag{3.55}
\end{align}
$$

where $r_1(p,q) = \frac{2p(q+1)}{q(p+1)}$ and $r_2(p,q) = \frac{p(q+2)}{q(p+1)}$.

Proof of Proposition 3.20. By interpolating q between p and ∞, and if $\xi = z, y$ or $\dot{x}$, we have

$$
\|\xi\|_q \le \|\xi\|_\infty^{\frac{q-p}{q}} \|\xi\|_p^{\frac{p}{q}}.
$$

Using Propositions 3.13 and 3.7, the conclusions of the above proposition are reached. $\quad\square$

In order to finish the proof of Theorem 2.5, we have to show that the several shapes obtained in Propositions 3.7 and 3.13 are in fact the profiles announced in Theorem 2.5. For that purpose, it remains to exhibit sequences of inputs verifying the hypotheses of Lemma 2.3. This is done in the Appendix, where we construct two such sequences (u_n^1) and (u_n^2). It turns out that (u_n^1) works to get (1) and (2) of Theorem 2.5 and (u_n^2) to get (3) of Theorem 2.5.

3.2. **Proof of Theorem 2.6**

First, recall that $(SI_2)_{k_L}$ is controllable. In addition, we have the following proposition:

Proposition 3.21. *For $p \in [1, \infty]$, there exists C_p such that for every $(z, y) \in \mathbb{R}^2$ and $\|(z, y)\| \ge C_p$ there exists $u_{(z,y)} \in L^p$ such that*

$$
\text{const. } \|u_{(z,y)}\|_p \le V(z, y)^{\frac{p+1}{2p}} \le \text{const. } \|u_{(z,y)}\|_p. \tag{3.56}
$$

Proof of Proposition 3.21. Let $(z, y) \in \mathbb{R}^2$ and $\|(z, y)\| \ge C_p$. It is enough to show that we can reach $(0, \bar{y})$ with $\bar{y} = V(z, y)^{\frac{1}{2}}$. This can easily be deduced from the construction of the sequence (u_n^1) in the Appendix. $\square$

Therefore, any trajectory of $(PSI_2)_{k_L}$ starting at $(z, y) \in \mathbb{R}^2$ and corresponding to $u \in L^p$ and $v \in L^{p_1}$ (with $p_1 \in [1, \infty]$), can be seen as the trajectory of $(PSI_2)_{k_L}$ starting at 0 and corresponding to $\tilde{u} \in L^p$ and $v \in L^{p_1}$ (with $p_1 \in [1, \infty]$), where $\tilde{u}$ is the concatenation of $u_{(z,y)}$ defined in Proposition 3.21 and u.

For the rest of the section, we will only consider trajectories of $(PSI_2)_{k_L}$ starting at 0 and use the estimates given below to get Theorem 2.6: for $p \in [1, \infty]$, for every $(z, y) \in \mathbb{R}^2$ and $\|(z, y)\| \ge C_p$, for every $u \in L^p$, we have

$$
\text{const. } \left(V(z, y)^{\frac{p+1}{2p}} + \|u\|_p \right) \le \|\tilde{u}\|_p \le \text{const. } \left(V(z, y)^{\frac{p+1}{2p}} + \|u\|_p \right). \tag{3.57}
$$

Remark 3.22. The previous paragraph indicates how to get (ISS)-type estimates. By taking into account (3.57), then, for $p \in [1, \infty]$, there exist $C_0, C_1, C_2 > 0$ such that, for $(z, y) \in \mathbb{R}^2$ and $u \in L^p$, if x is the trajectory of $(PSI_2)_{k_L}$ starting at (z, y) and corresponding to $u \in L^p$, the following inequality holds for $t \ge 0$,

$$
\begin{align}
\|x(t)\| &\le \text{ const. } V(x(t)) \le \text{const. } \Big(\sup(\|x(0)\|, \|x(0)\|^2) e^{-C_2 t} (1 - [\text{sgn}(\sup(\|x(0)\|, \|x(0)\|^2) - C_0)]^+) \\
&\quad + (\sup(\|x(0)\|, \|x(0)\|^2) - C_1 t)[\text{sgn}(\sup(\|x(0)\|, \|x(0)\|^2) - C_0)]^+ \Big) + \|u\|_{p,[0,t]}^{\frac{2p}{p+1}}. \tag{3.58}
\end{align}
$$

Let $p \in [1, \infty]$ and $u \in L^p$, $p_1 \in [1, \infty)$, with $p_1 \le p$ and $v = (v_1, v_2)) \in L^{p_1}$. Set $\varepsilon_0 \stackrel{\text{def}}{=} \frac{\sigma_m}{4\gamma_\infty}$, where γ_∞ is the L^∞-gain of the linear control system $(L2)$ defined by

$$(L2) \quad \dot{x} = A_1 x + bu,$$

with $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

Let $x = (z, y)$ be the trajectory of $(PSI_2)_{k_L}$ starting at 0 and corresponding to u and v. Let $x_1 = (z_1, y_1)$ be the trajectory of $(L2)$ starting at 0 and corresponding to v. Then, $(Z, Y) = (z - z_1, y - y_1)$ is the trajectory of $(PSI_2)_{k_L}$ starting at 0 and corresponding to $\bar{u} = z_1 + u$ and $\bar{v} = (0, z_1 + y_1)$. We suppose that $\|v\|_\infty \le \varepsilon_0$. Then, $\bar{v}$ is a locally absolutely continuous (l.a.c. for short) input such that

$$\bar{v} \in L^{p_1}, \quad \|\bar{v}\|_\infty \le \frac{\sigma_m}{2}. \tag{3.59}$$

Therefore, we have reduced the proof of Theorem 2.6 to the case where the trajectory of $(PSI_2)_{k_L}$ starts at 0 and corresponds to an input v of the type $(0, \bar{v})$.

We first prove part (3) of Theorem 2.6 and suppose hence that $p_1 > 2$.

For every $\varepsilon > 0$, we construct a curve x which is l.a.c., unbounded and can be seen as the trajectory of $(PSI_2)_{k_L}$ starting at 0 and corresponding to some continuous and compactly supported input u and to some $\bar{v} \in L^{p_1}$ such that $\|\bar{v}\|_\infty \le \varepsilon$. We will suppose that σ is equal to the standard saturation function σ_0. Let

$$W_0 > \left(\frac{5}{\varepsilon}\right)^{\frac{2p_1}{p_1 - 1}}.$$

There exists a continuous input u steering 0 to $(0, W_0^{\frac{1}{2}})$ in time $T > 0$ along $(SI_2)_{k_L}$. Then, x is defined on $[0, T]$ as the resulting trajectory. For $t > T$, we take $u = 0$ and $\bar{v}$ as follows: let $(t_k)_{k \ge 0}$ be the sequence of zeros of z on $[T, \infty)$. For $k \ge 0$, define $W_k \stackrel{\text{def}}{=} W(z(t_k), y(t_k))$ and

$$c_k = \frac{5}{W_k^{\frac{p_1 - 1}{2p_1}}}.$$

Then, for $k \ge 0$ and $t \in (t_k, t_{k+1}]$, we define the input $\bar{v}$ as

$$\bar{v}(t) \stackrel{\text{def}}{=} c_k \frac{y}{(1 + y^2)^{\frac{p_1 + 1}{2p_1}}}.$$

The derivative of W along x, on $(t_k, t_{k+1}]$, is equal to

$$\dot{W} = -\sigma(z)^2 + y\bar{v} = -\sigma(z)^2 + c_k \frac{y^2}{(1 + y^2)^{\frac{p_1 + 1}{2p_1}}}.$$

If $|y| \ge 1$, we have

$$c_k \frac{y^2}{(1 + y^2)^{\frac{p_1 + 1}{2p_1}}} = \frac{5}{(1 + \frac{1}{y^2})^{\frac{p_1 + 1}{2p_1}}} \left(\frac{y^2}{W_k}\right)^{\frac{p_1 - 1}{2p_1}}.$$

Then $\dot{W}$ is bounded and it is larger than $\frac{1}{2}$ as soon as $y^2 \ge \frac{1}{20}W$, i.e. when $|y| \ge \frac{\sqrt{|z|}}{4}$. This situation occurs during $\frac{3}{4}(t_{k+1} - t_k)$, as a careful examination of the dynamics of (z, y) shows. Then, we prove inductively that

the above sequences are well-defined by using the next equation: for $k \geq 0$, we have

$$
\begin{aligned}
v(t) &\leq \varepsilon, \\
W_k^{\frac{1}{2}} &\leq t_{k+1} - t_k \leq 2W_k^{\frac{1}{2}}, \\
2W_k^{\frac{1}{2}} &\leq W_{k+1} - W_k \leq 6W_k^{\frac{1}{2}}.
\end{aligned}
$$

We also obtain that, for $k \geq 0$,

$$
\text{const. } k^2 + W_0 \leq W_k \leq \text{const. } k^2 + W_0.
$$

The above equations imply that, for $k \geq 1$, we have

$$
\int_{t_k}^{t_{k+1}} |\bar{v}|^{p_1} \leq \text{const. } \frac{\ln k}{k^{p_1-1}}.
$$

Since $\sum_{k \geq 1} \frac{\ln k}{k^{p_1-1}}$ is finite, we have $\|\bar{v}\|_{p_1}$ is finite and we are done. It is easy to adapt the above construction to an arbitrary iS-function.

From now on, we only consider trajectories of $(PSI_2)_{k_L}$ starting at 0 and corresponding to $u, \bar{v}$, with $u \in L^p$, $\bar{v} = (0, v))$ with v l.a.c. and satisfying (3.59). Moreover, we assume that $p_1 \leq 2$.

The key estimates to get Proposition 3.4 ($p = \infty$) and Proposition 3.13 (p finite) respectively are equation (3.6) and equation (3.18). Considering $(PSI_2)_{k_L}$, the two previous equations are still valid with, however, the addition on their respective right-hand side of the term $\int_I |yv|$, where I is the ad hoc interval. Indeed, the important point is that all the estimates on z, $\dot{z}$, y and $\dot{y}$ that were made (in particular the lower bounds), are not essentially modified if $\|v\|_{L^\infty}$ is small enough with respect to σ_m (and thus to K, C).

In the sequel, we will only treat the case where p is finite since the argument when $p = \infty$ follows the same lines.

In the present situation, equation (3.42) becomes

$$
\Delta V \Big|_{I_j} + \text{const. } \left(T^{(j)} + \bar{V}^{\frac{1}{2}}\right) \leq \text{const. } \bar{V}^{\frac{1}{2}} \left(|E^{(j)}| + |G^{(j)}|\right) + \text{const. } \int_{I^{(j)}} |yv|. \tag{3.60}
$$

Setting $T_{\bar{V}} = T_{1,\bar{V}} + T_{2,\bar{V}} \geq \text{const. } \bar{V}$, equation (3.46) becomes

$$
\bar{V}^{\frac{p-1}{2}} T_{\bar{V}} \leq \text{const. } \left(\int_L |u|^p + \bar{V}^{\frac{p}{2}} \int_L |v|\right). \tag{3.61}
$$

We have to consider two sub-cases.

First, we suppose that $p_1 < 2$. By applying Hölder's inequality in (3.61), we get

$$
\bar{V}^{\frac{p+1}{2}} \leq \text{const. } \bar{V}^{\frac{p-1}{2}} T_{\bar{V}} \leq \text{const. } \left(\int_L |u|^p + \bar{V}^{\frac{p}{2}} T_{\bar{V}}^{1-\frac{1}{p_1}} \|v\|_{p_1}\right). \tag{3.62}
$$

Therefore, we obtain

$$
\begin{aligned}
T_{\bar{V}} &\leq \text{const. } \left(\int_L |u|^p + \left(\int_L |v|^{p_1}\right)^{\frac{2}{2-p_1}}\right), \\
\bar{V}^{\frac{p+1}{2}} &\leq \text{const. } \left(\int_L |u|^p + \left(\int_L |v|^{p_1}\right)^{\frac{2}{2-p_1}}\right).
\end{aligned}
$$

Recall that

$$\int_L |z|^p \leq \text{const. } \bar{V}^p T_{\bar{V}},$$

$$\int_L |y|^p \leq \text{const. } \bar{V}^{\frac{p}{2}} T_{\bar{V}},$$

$$\int_L |\sigma(z+u)|^p \leq \text{const. } T_{\bar{V}}.$$

Next, we only prove (2.3). If $\int_L |u|^p \geq \bar{V}^{\frac{p}{2}} T_{\bar{V}}^{1-\frac{1}{p_1}} \|v\|_{p_1}$, then we get (3.22). Otherwise, we get

$$\int_L |z|^p \leq \text{const. } \bar{V}^{p+\frac{p_1}{2}} \|v\|_{p_1}^{p_1} \leq \|v\|_{p_1}^{p_1 + (p+\frac{p_1}{2})p_1}$$

and we conclude.

The second sub-case occurs for $p_1 = 2$. This case is more delicate than the previous one since (3.60) does not provide directly any interesting estimate. We will rewrite it in two different ways: first, we easily get that

$$\Delta V\big|_{I_j} + \text{const. } \left(T^{(j)} + \bar{V}^{\frac{1}{2}}\right) \leq \text{const. } \int_{I_j} \frac{|u|^p}{\bar{V}^{\frac{p-1}{2}}} + \int_{I_j} |yv|, \tag{3.63}$$

and second, if we divide (3.60) by $\bar{V}$, we can write it as

$$\int_{I_j} \frac{\dot{V}}{V} + \text{const. } \int_{I_j} \frac{dt}{V} \leq \text{const. } \int_{I_j} \frac{|u|^p}{\bar{V}^{\frac{p+1}{2}}} + \text{const. } \int_{I_j} \frac{|v|}{V^{\frac{1}{2}}}. \tag{3.64}$$

This leads to rewrite (3.61) as

$$T_{\bar{V}} \leq \text{const. } \int_L \frac{|u|^p}{\bar{V}^{\frac{p-1}{2}}} + \int_L |yv|, \tag{3.65}$$

or

$$1 + \int_L \frac{dt}{V} \leq \text{const. } \int_L \frac{|u|^p}{\bar{V}^{\frac{p+1}{2}}} + \text{const. } \int_L \frac{|v|}{V^{\frac{1}{2}}}. \tag{3.66}$$

Let $J^0 = \int_L \frac{|v|}{V^{\frac{1}{2}}}$. By applying the Cauchy–Scharwz inequality for J^0 and then the Young's inequality for $x \mapsto x^2$, we have that, for every $\mu > 0$,

$$J^0 \leq \text{const. } \left(\int_L \frac{dt}{V}\right)^{\frac{1}{2}} \left(\int_L |v|^2\right)^{\frac{1}{2}} \leq \frac{\text{const.}}{\mu} \int_L \frac{dt}{V} + \text{const. } \mu \int_L |v|^2. \tag{3.67}$$

Then, equations (3.66) and (3.67) imply that

$$1 + \int_L \frac{dt}{V} \leq \text{const. } \int_L \frac{|u|^p}{\bar{V}^{\frac{p+1}{2}}} + \text{const. } \int_L |v|^2. \tag{3.68}$$

We next divide the set of integers $\{0, \cdots, k_m\}$ in two disjoint subsets $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $k \in \mathcal{K}_1$, if

$$\int_{L_k} |u|^p \geq \bar{V}^{\frac{p-1}{2}} \int_{L_k} |yv|,$$

and $k \in \mathcal{K}_2$, if the opposite inequality holds. Let $k_2 = \mathcal{K}_2^{\#}$ and $k_1 = \inf_{\mathcal{K}_1} k$. Then $k_2 \geq k_1$. By applying (3.62) to k_1, we have

$$V_{k_1} \leq \text{const. } \|u\|^{\frac{2p}{p+1}},$$

and by summing up (3.66) for $k \in \mathcal{K}_2$, we obtain

$$k_2 \leq \text{const. } \|v\|_2^2.$$

Since $V_m = 2^{k_1} V_{k_1} \leq 2^{k_2} V_{k_1}$, we get

$$V_m \leq \text{const. } \|u\|^{\frac{2p}{p+1}} e^{\text{const.} \|v\|_2^2},$$

and (2.9) follows. The other inequalities of Theorem 2.6 can be deduced as done before. $\qquad\square$

4. The feedback k_F

From now on, σ is an iS-function and F is an $\mathcal{F}$-function. Note that K is odd, C^1 and increasing. In addition, we have the following obvious estimates:

Lemma 4.1. *The function $y \mapsto \frac{y}{K(y)}$ is even, increasing on $\mathbb{R}^+$ and for every $y \in \mathbb{R}$,*

$$\text{const.} \inf(|y|, 1) \leq |K(y)| \leq \text{const.} \inf(|y|, |y|^{\frac{1}{2}}, |y|^{\frac{1}{r}}) \ \text{and} \ K'(y) \geq \frac{1}{1 + |K(y)|}.$$

Proof of Lemma 4.1. This easily follows from (2.11). $\qquad\square$

As a consequence of Lemma 4.1, we have that, for every $\lambda \in (0, 1)$ and $y \geq 0$,

$$\frac{K(\lambda y)}{K(y)} \geq \lambda. \tag{4.1}$$

Up to a time reparameterization and a linear change of variable in (SI_2), we may assume that $|\sigma(t)| \geq 2$ for $|t| \geq 1$. We use the C^1 change of variable given by

$$z \ \overset{\text{def}}{=} \ x_1 + \bar{F}(x_2), \tag{4.2}$$
$$y \ \overset{\text{def}}{=} \ \bar{F}(x_2), \tag{4.3}$$

to rewrite $(SI_2)_{k_F}$ as

$$\begin{cases} \dot{z} & = & K(y) - \dfrac{\sigma(z + u)}{K'(y)}, \\ \dot{y} & = & -\dfrac{\sigma(z + u)}{K'(y)}. \end{cases}$$

For $y \in \mathbb{R}$, we define the odd function G by

$$G(y) \overset{\text{def}}{=} y - \text{sgn}\,(y)K(y)^2 - \frac{1}{4}K(y), \tag{4.4}$$

and, for $y \in \mathbb{R}$, we have

$$\frac{2}{3}|y|\frac{1 + |y|}{1 + 3|y|} \leq |G(y)| \leq |y|.$$

We define the Lyapunov function V_F by

$$V_F(z, y) = \frac{1}{2}z^2 - zG(y) + \int_0^y G(s)\mathrm{d}s. \tag{4.5}$$

There exist const. > 0 such that for every $z, y \in \mathbb{R}^2$

$$\text{const. } \left(z^2 + y^2\right) \leq V_F(z, y) \leq \text{const. } \left(z^2 + y^2\right),$$

which implies (2.13). Moreover, V_F is a strict Lyapunov function for $(SI_2)_{k_F}$. It follows from the dissipation inequality given below:

Lemma 4.2. *If we use $\dot{V}_F$ to denote the time derivative of V_F along the trajectories of $(SI_2)_{k_F}$, we have*

$$\dot{V}_F + \frac{1 + |K(y)|}{2}V_F^{\frac{1}{2}} \inf\left(V_F^{\frac{1}{2}}, 1\right) + \leq 4(1 + |K(y)|)|z|\sigma(3|u|)M(z, y, u), \tag{4.6}$$

where

$$M(z, y, u) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } |u| > \frac{|z|}{2} > \inf\left(V_F^{\frac{1}{2}}, 1\right)\frac{V_F^{\frac{1}{2}}}{12}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Lemma 4.2. We compute $\dot{V}_F$ and obtain

$$\dot{V}_F = zK(y) - z\sigma(z + u)\left(2|K(y)| + \frac{1}{4}\right) - K(y)G(y).$$

By considering the sub-cases defined whether $\frac{|z|}{2} < |u|$ or not and $V_F^{\frac{1}{2}} > 1$ or not, we conclude. $\qquad\square$

As a consequence, we can see that V_F is an (ISS)-Lyapunov function.

4.1. **Proof of Theorem 2.8**

We first assume that $p = \infty$ and $\|u\|_\infty \geq C$. Then, by Lemma 4.2, we have

$$\dot{V}_F \leq -(1 + |K(y)|)\inf\left(1, V_F^{\frac{1}{2}}\right)\left(V_F^{\frac{1}{2}} - 24K\|u\|_\infty\right).$$

Therefore $\left\|V_F^{\frac{1}{2}}\right\|_\infty \leq 24K\|u\|_\infty$.

For the rest of the pqper, we assume that $p \in (1, \infty)$. Let us show a result similar to Remark 3.2:

Lemma 4.3. *The trajectories of $(SI_2)_{k_F}$ corresponding to inputs $u \in L^p$, with p finite, converge to 0 at infinity.*

Proof of Lemma 4.3. For $X \geq 0$, let $R(X) \stackrel{\text{def}}{=} L_p(X^{\frac{1}{2}})^p$, where L_p was defined in (2.15). We use R and $\dot{R}$ to denote the Lyapunov function $R\left(V_F(z, y)\right)$ and its time derivative along trajectories of $(SI_2)_{k_F}$. Multiplying (4.6) by $\frac{V_F^{\frac{p-1}{2}}}{K(V_F^{\frac{1}{2}})}$, we get

$$\dot{R} + \frac{1 + |K(y)|}{2\sup\left(K\left(V_F^{\frac{1}{2}}\right), 1\right)}V_F^{\frac{p}{2}} \leq 4\frac{1 + |K(y)|}{K\left(V_F^{\frac{1}{2}}\right)}|z|V_F^{\frac{p-1}{2}}\sigma(3|u|)M(z, y, u). \tag{4.7}$$

Then, we have

$$\dot{R} + V_F^{\frac{p}{2}} \quad \leq \quad 8V_F^{\frac{p-1}{2}} \sigma(3|u|)M(z,y,u), \text{ if } V_F \leq 1, \tag{4.8}$$

$$\dot{R} + \frac{1+|K(y)|}{2K\left(V_F^{\frac{1}{2}}\right)} V_F^{\frac{p}{2}} \quad \leq \quad 4\frac{1+|K(y)|}{K\left(V_F^{\frac{1}{2}}\right)} |z| V_F^{\frac{p-1}{2}} \sigma(3|u|)M(z,y,u) \text{ if } V_F > 1. \tag{4.9}$$

We deduce that

$$\dot{R} \leq \text{const. } |u|^p,$$

which implies that $L_p \left(\left\|V_F^{\frac{1}{2}}\right\|_\infty\right) \leq \text{const. } \|u\|_p$. Moreover, since x_u is bounded, $\dot{x}_u$ is also bounded and by (4.8) and (4.9), we also have

$$\text{const.}_{\|u\|_p} \int_0^\infty V_F^{\frac{p}{2}} \leq \int_0^\infty \frac{V_F^{\frac{p}{2}}}{\sup\left(K\left(V_F^{\frac{1}{2}}\right), 1\right)} \leq \text{const. } \|u\|_p^p.$$

Therefore, by Barbălat's lemma, we get that x_u converges to 0 at infinity.   □

Next, we establish a similar result to Lemma 3.15:

Lemma 4.4. *Let* $p \in [1, \infty)$, $u \in L^p$ *and* $x_u = (z, y)$ *be the trajectory of* $(SI_2)_{k_F}$ *corresponding to* u *and starting at* 0. *Then, we have*

$$\|K(y)^2\|_p \leq \text{const. } \|z\sigma(z+u)\|_p, \tag{4.10}$$

with equality if $p = 1$.

Proof of Lemma 4.4.   Consider the function

$$Q_F(z,y) \overset{\text{def}}{=} -zK(y)\left(K(y)^2\right)^{p-1} + \int_0^y K(s)\left(K(y)^2\right)^{p-1} \mathrm{d}s.$$

Let $\dot{Q}_F$ be the time derivative of Q_F along x_u. We have

$$\dot{Q}_F = -|K(y)|^{2p} + (2p-1)\left(K(y)^2\right)^{p-1} z\sigma(z+u). \tag{4.11}$$

Similarly to Lemma 3.15, we integrate (4.11) between 0 and T for every $T > 0$, then we apply Hölder's inequality (for $p > 1$), and we conclude by letting T goes to ∞ and by taking into account Lemma 4.3.   □

As a corollary, we prove Theorem 2.8 for $p = 1$. According to Lemma 4.4, we have

$$\|z\sigma(z+u)\|_1 = \|K(y)^2\|_1 \leq \text{const. } \|y\|_1.$$

Moreover, dividing (4.6) by $V_F^{\frac{1}{2}}$ implies that

$$\left(L_1\left(V_F^{\frac{1}{2}}\right)\right)^{\cdot} + V_F^{\frac{1}{2}} \frac{1+|K(y)|}{2K\left(V_F^{\frac{1}{2}}\right)} \inf\left(V_F^{\frac{1}{2}}, 1\right) \leq 4\frac{1+|K(y)|}{K\left(V_F^{\frac{1}{2}}\right)} |z|\sigma(3|u|)M(z,y,u), \tag{4.12}$$

where L_1 was defined in (2.15). Since $y \mapsto \frac{y}{K(y)}$ is increasing for $y \geq 0$, we have

$$\frac{V_F^{\frac{1}{2}}}{K(V_F^{\frac{1}{2}})} \geq \text{const. } \frac{|y|}{|K(y)|}.$$

Then, we deduce from equation (4.12) that

$$\left(L_1\left(V_F^{\frac{1}{2}}\right)\right)^{\cdot} + \frac{|y|}{2} \leq 12K|u|.$$

After integration, we obtain $\|y\|_1 \leq$ const. $\|u\|_1$. A trajectory of $(SI_2)_{k_F}$ starting at 0 and corresponding to the input u is also the trajectory of the linear system given by $\dot{x} = A_1 x + b_1 w$ with

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad b_1 = (0\ 1)^T,$$

and with $w = \left(z + u - \sigma(z+u)\right) - u - y + K(y)$. By (ii) of Definition 1, we get $|w| \leq$ const. $(|z\sigma(z+u)| + |u| + |y|)$. Since A_1 is Hurwitz, we conclude.

From now on, we assume that $R_m \overset{\text{def}}{=} \|R\|_\infty$, $\left\|V_F^{\frac{1}{2}}\right\|_\infty$, $\|u\|_p \geq C^2$. In order to prove Theorem 2.8, we look for establishing the inequality

$$\int_0^\infty V_F^{\frac{p}{2}} \leq \text{const.} \int_0^\infty V_F^{\frac{p-1}{2}}|u|, \tag{4.13}$$

which provides the result together with Hölder's inequality.

Set $k_m \overset{\text{def}}{=} \left[\frac{\ln \frac{R_m}{C}}{\ln 2}\right]$. We consider

$$\begin{aligned} S_0 &= \left\{t \geq 0,\ R\Big(z(t), y(t)\Big) > C\right\}, \\ S_1 &= \left\{t \geq 0,\ R\Big(z(t), y(t)\Big) \leq C\right\}. \end{aligned}$$

As done before, we approximate S_1 by a decreasing sequence of open sets $(S_{1,\varepsilon})$ such that $|S_{1,\varepsilon}/S_1|$ tends to zero as ε tends to. On each interval I' of $S_{1,\varepsilon}$, we look for

$$\Delta R\Big|_{I'} + \text{const.} \int_{I'} V_F^{\frac{p}{2}} \leq \text{const.} \int_{I'} V_F^{\frac{p-1}{2}}|u|. \tag{4.14}$$

This last equation is a direct consequence of the fact that (4.7) can be simply written on S_1 as

$$\dot{R} + \text{const.}\ V_1^{\frac{p}{2}} \leq \text{const.}\ V_1^{\frac{p-1}{2}}|u|.$$

Similarly to the proof of Proposition 3.13, we "slice" up S_0 according to (3.30) with the obvious modifications. On each interval I of L_k or $M_{k,\varepsilon}$ falling into one of the (ci) cases, we look for obtaining the inequality

$$\Delta R\Big|_I + \text{const.} \int_I V_F^{\frac{p}{2}} \leq \text{const.} \int_I V_F^{\frac{p-1}{2}}|u|. \tag{4.15}$$

Equation (4.13) is finally obtained by summing up (4.15) on S_0 and (4.14) on S_1.

Therefore, it remains to show (4.15) to finish the proof of Theorem 2.8. Let $I = (t_0, t_1)$ be an interval of some L_k or $M_{k,\varepsilon}$ such that $\Delta R\Big|_I = \mu\frac{\bar{R}}{2}$ where $\mu = \pm 1, 0$ and $\bar{R} = \frac{R_m}{2^k}$. Let

$$T \overset{\text{def}}{=} t_1 - t_0 \text{ and } \bar{V} \overset{\text{def}}{=} R^{-1}(\bar{R}).$$

We easily get that $R^{-1}\left(\frac{\bar{R}}{2}\right) = \lambda\bar{V}$ with

$$\left(\frac{1}{2}\right)^{\frac{1}{p}} \le \lambda \le \left(\frac{1}{2}\right)^{\frac{1}{p+1}}.$$

Therefore $\Delta V_F\big|_I = (1-\lambda)\bar{V}$ and we have, for every $t \in I$, $\lambda\bar{V} \le V_F(t) \le \bar{V}$. In addition, we get

$$\frac{1}{\sqrt{2}}T\bar{V}^{\frac{p}{2}} \le \int_I V_F^{\frac{p}{2}} \le T\bar{V}^{\frac{p}{2}}.$$

Consider the following subsets of I:

$$\begin{aligned}
E_1 &= \left\{t \in I, \ |y| \ge \rho V_F^{\frac{1}{2}}\right\}, \\
E_2 &= \left\{t \in I, \ |y| < \rho V_F^{\frac{1}{2}}\right\}, \\
E &= \left\{t \in E_2, \ |u| > \frac{|z|}{2} \text{ a.e.}\right\}.
\end{aligned}$$

We first assume that $|E_1| \ge \rho T$. Then, for $t \in E_1$, we have

$$\frac{1 + |K(y)|}{2K\left(V_F^{\frac{1}{2}}\right)} \ge \frac{\rho}{2},$$

which implies that

$$\int_I \frac{1 + |K(y)|}{2K\left(V_F^{\frac{1}{2}}\right)} V_F^{\frac{p}{2}} \ge \text{const. } \rho \int_{E_1} V_F^{\frac{p}{2}} \ge \text{const. } \rho T\bar{V}^{\frac{p}{2}},$$

and we get (4.15) by integration of (4.9).

We next assume that $|E_1| < \rho T$. Then, $|z| > (1-\rho)V_F^{\frac{1}{2}}$ on E_2. Let $k_0 \stackrel{\text{def}}{=} \sup_{E_2}|K(y)|$. We have $k_0 \le K\left(\rho V_F^{\frac{1}{2}}\right)$ and the following lemma:

Lemma 4.5. *With the above notations, let*

$$U \stackrel{\text{def}}{=} \left\{t \in E_2, \ |K(y)| \ge \frac{k_0}{2}\right\}.$$

Then, either $E_2 = I$ or $|U| + |E| \ge \text{const. } T$.

Proof of Lemma 4.5. If E_1 is not empty, then $k_0 = K\left(\rho V_F^{\frac{1}{2}}\right)$. Since E_2 is open, it is the countable union of open intervals. For every interval $J = (t, t')$ of E_2, let $U_J \stackrel{\text{def}}{=} U \cap J$ and $E_J \stackrel{\text{def}}{=} E \cap J$. We have $|K(y(t))|, |K(y(t'))| \le k_0$ with equality at t or at t'. Similarly to (5.8), we get

$$\left|\sigma(C)(t'-t) - \text{const. } |E_J|\right| \le 2k_0. \tag{4.16}$$

Since the time derivative of $K(y)$ is bounded (by K_σ), we have $k_0 \le \text{const. } |U_J|$. Therefore, we obtain

$$|U_J| + |E_J| \ge \text{const. } (t' - t).$$

Adding up, we conclude.   □

According to the previous lemma, three possibilities may occur: E_1 is empty, $|E| \geq$ const. T or $|U| \geq$ const. T. Suppose first, that $|E| \geq$ const. T. Then, for a.e. $t \in E$, we have

$$|u| > \frac{|z|}{2} > \text{const. } V_F^{\frac{1}{2}}.$$

We deduce that

$$\int_I V_F^{\frac{p-1}{2}} |u| \geq \text{const. } \int_I V_F^{\frac{p}{2}}.$$

This implies (4.15), except in the case where $\Delta R\big|_I = \frac{\bar{R}}{2}$. But we also have

$$\bar{R} \leq \text{const.} \frac{V_F^{\frac{p+1}{2}}}{K(V_F^{\frac{1}{2}})} \text{ and } \bar{V} \leq \text{const. } \Delta V_F\big|_I \leq \text{const. } TK(\bar{V}^{\frac{1}{2}})\bar{V}^{\frac{1}{2}}.$$

Then, $\Delta R\big|_I \leq \text{const. } \int_I V_F^{\frac{p-1}{2}} |u|$ and we conclude this case.

Suppose now that $|U| \geq$ const. T. We may also assume that $k_0 = K\left(\rho V_F^{\frac{1}{2}}\right)$. Then

$$\int_U \frac{1 + |K(y)|}{2K\left(V_F^{\frac{1}{2}}\right)} V_F^{\frac{p}{2}} \geq \text{const. } \frac{K\left(\rho \bar{V}^{\frac{1}{2}}\right)}{K\left(\bar{V}^{\frac{1}{2}}\right)} (1 - \lambda) T\bar{V}^{\frac{p}{2}} \geq T\bar{V}^{\frac{p}{2}}.$$

We easily conclude here.

Finally, suppose that E_1 is empty. Then, z has a constant sign on I, let say $z > (1 - \rho)V_F^{\frac{1}{2}}$ on I. Equation (4.16) becomes

$$\left| \sigma(C)T - \text{const. } |E| \right| \leq 2k_0 \leq \text{const. } K\left(\rho \bar{V}^{\frac{1}{2}}\right). \tag{4.17}$$

If $|\Delta R\big|_I| = \frac{\bar{R}}{2}$, then $|\Delta V_F\big|_I| \geq$ const. $\bar{V}$. Thanks to (4.6), we deduce that $|\Delta V_F\big|_I| \leq$ const. $k_0 \bar{V}^{\frac{1}{2}}$, which implies that

$$T \geq \text{const. } \frac{\bar{V}^{\frac{1}{2}}}{K\left(\rho \bar{V}^{\frac{1}{2}}\right)},$$

with the last const. independent of ρ. Since $K\left(\rho \bar{V}^{\frac{1}{2}}\right)^2 \leq$ const. $\rho \bar{V}^{\frac{1}{2}}$, we obtain, from (4.17), that $|E| \geq$ const. T. That situation was already treated. Therefore, the proof of Theorem 2.8 is finished.

4.2. Proof of Theorem 2.9

Using the change of variables given by (4.3), we rewrite $(PSI_2)_{k_F}$ as

$$\begin{cases} \dot{z} &= K(y) + \dfrac{-\sigma(z + u) + v_2}{K'(y)} + v_1, \\ \dot{y} &= \dfrac{-\sigma(z + u) + v_2}{K'(y)}. \end{cases}$$

We choose $\varepsilon_0 = \frac{\sigma(1)}{4}$. The time derivative of V_F along trajectories of $(PSI_2)_{k_F}$ becomes

$$\dot{V}_F + \frac{1+|K(y)|}{2} V_F^{\frac{1}{2}} \inf\left(V_F^{\frac{1}{2}}, 1\right) \le 8(1+|K(y)|)|z|\sigma\Big(\text{const. }(|u|+|v_1|+|v_2|)\Big) M(z,y,u) + \text{const. }|v_1||y|N(z,y),$$

where

$$N(z,y) \overset{\text{def}}{=} \left\{ \begin{array}{ll} 1, & \text{if } V_F < 1, \\ 0, & \text{otherwise.} \end{array} \right.$$

From the previous equation, we can follow exactly the lines of the proof of Theorem 2.8 only changing $|u|$ by $|u|+|v_1|+|v_2|$ and checking at each step that the extra term $|v_1||y|N(z,y)$ does not alter the computations. The term $L_p(V_F(\bar{x})^{1/2})$, appearing in (2.14), comes from the integration of (4.7).

5. Appendix

We first notice that (SI_2) is controllable. We construct the required sequences assuming that σ is the standard saturation function σ_0. The difference with an arbitrary iS-function simply consists of having a different set of const. below.

5.1. Construction of $\left(u^{(1)}\right)$

We construct a l.a.c. curve $c : [0,\infty) \to \mathbb{R}^2$ such that c is a trajectory of $(SI_2)_{k_L}$ corresponding to an piecewise C^1 input u. Then, each input of the sequence $\left(u_n^{(1)}\right)$ will be the concatenation of u_n, the restriction of u to an finite time interval $[0, T_n]$, and the zero input for $t > T_n$. There exists a continuous input $\bar{u}$ steering 0 to $\left(0, V_0^{\frac{1}{2}}\right)$ with $V_0^{\frac{1}{2}} \ge C$ in time $T_0 > 0$ along $(SI_2)_{k_L}$. Then, c is defined on $[0, T_0]$ as the resulting trajectory. We take $u = C - z$ until $z = y + 2$. In time T', we reach the point $x' = (V', V' - 2)$, with

$$T' = V_0^{\frac{1}{2}} + \sqrt{V_0 - 4} \text{ and } V' = V_0^{\frac{1}{2}} - 2 + \sqrt{V_0 - 4}.$$

From x', we take u to be 0, until $z = 0$. In time T'', we are, hence, at some point $(0, -V'')$, with $T'', V'' = 2V' + o\left(\frac{1}{V'}\right)$. By symmetry with respect to $(0,0)$, we repeat the procedure from $(0, -V'')$ and we end up at $(0, V_1)$ for a time T_1. We clearly have

$$T_1 - T_0, V_1 - V_0 = 4V_0^{\frac{1}{2}} + o\left(\frac{1}{V_0^{\frac{1}{2}}}\right).$$

The above construction is then repeated. We, thus, define two sequences $(V_k)_{k\ge 0}$ and $(T_k)_{k\ge 0}$ such that we have

$$\text{const. } V_k^{\frac{1}{2}} \le T_{k+1} - T_k \le \text{const. } V_k^{\frac{1}{2}}, \quad \text{const. } V_k^{\frac{1}{2}} \le V_{k+1} - V_k \le \text{const. } V_k^{\frac{1}{2}}.$$

We also obtain that, for $k \ge 0$,

$$\text{const. } k^2 + V_0 \le V_k \le \text{const. } k^2 + V_0.$$

It is easy then to get that, for $1 \le p \le q$ and k large,

$$\text{const. } k^{2q+1} \le \int_{T_k}^{T_{k+1}} |z|^q \le \text{const. } k^{2q+1}, \quad \text{const. } k^{q+1} \le \int_{T_k}^{T_{k+1}} |y|^q \le \text{const. } k^{q+1},$$

$$\text{const. } k^p \le \int_{T_k}^{T_{k+1}} |u|^p \le \text{const. } k^p.$$

For $n \geq 0$, the input $u_n^{(1)}$ is defined as u on $[0, T_n]$ and 0 afterwards. If we use $\left(z_n^{(1)}, y_n^{(1)}\right)$ to denote the resulting trajectory, then we get, for $1 \leq p \leq q \leq \infty$,

$$\text{const. } n^{\frac{2(q+1)}{q}} \quad \leq \quad \|z_n^{(1)}\|_q \leq \text{const. } n^{\frac{2(q+1)}{q}}, \tag{5.1}$$

$$\text{const. } n^{\frac{q+2}{q}} \quad \leq \quad \|y_n^{(1)}\|_q \leq \text{const. } n^{\frac{q+2}{q}}, \tag{5.2}$$

$$\text{const. } n^{\frac{p+1}{p}} \quad \leq \quad \|u_n^{(1)}\|_p \leq \text{const. } n^{\frac{p+1}{p}}. \tag{5.3}$$

5.2. Construction of $\left(u^{(2)}\right)$

We already know that the shape found in Proposition 3.7 is the profile (3) in Theorem 2.5, thanks to Remark 2.4. But, even though the L^p-norms of the inputs $\bar{u}_n$ involved to obtain the preceding result can be chosen increasing to ∞, their L^∞-norms are, of course, very small. For $n \geq 0$, the input $u_n^{(2)}$ is simply obtained as the concatenation of an input v_n and an input $\bar{u}_{n_k}$ given as follows: v_n is defined on some $[0, T_{v_n}]$ with large L^∞-norm and such that the resulting trajectory reaches the origin at time T_{v_n}; $\bar{u}_{n_k}$ is chosen such that its L^p-norm is bigger than the L^p-norm of v_n. Therefore, the sequence $\left(u_n^{(2)}\right)$ completes the proof that, the shape found in Proposition 3.7, is the profile (3) in Theorem 2.5 and the $u_n^{(2)}$ have arbitrarily large L^∞-norm.

5.3. Proof of Proposition 3.3

We use $y_{\max}$ and $z_{\max}$ to denote $\|y\|_{\infty,[0,T]}$ and $\|z\|_{\infty,[0,T]}$ respectively.

We start with the argument for (a). We pick $[t_0, t_1] \in [0, T]$ such that we have, for $t \in [t_0, t_1]$,

$$|y(t_0)| = \frac{y_{\max}}{2} \leq |y(t)| \leq |y(t_1)| = y_{\max}.$$

Recall that $\frac{y_{\max}}{2}$ is in the range of y since the initial state is zero. Without loss of generality, we consider the case $y(t) \geq 0$ on $[t_0, t_1]$. Let $L \overset{\text{def}}{=} t_1 - t_0$. Then,

$$\frac{y_{\max}}{2} = \Delta y \Big|_{t_0}^{t_1} = -\int_{t_0}^{t_1} \sigma(z+u) \leq KL,$$

and

$$2z_{\max} \geq \Delta z \Big|_{t_0}^{t_1} = \int_{t_0}^{t_1} y + \Delta y \Big|_{t_0}^{t_1} \geq \frac{y_{\max}}{2}(L+1).$$

The two previous equations imply that $z_{\max} \geq \frac{y_{\max}^2}{8K}$.

As for item (b), we pick $[t_0, t_1] \in [0, T]$ such that we have, for $t \in [t_0, t_1]$,

$$W(t_0) = \frac{W_{\max}}{2} \leq W(t) \leq W(t_1) = W_{\max},$$

where W is given by (3.3) and $W_{\max} = \|W\|_{\infty,[0,T]}$. Note that $\sigma_m \leq \frac{W_{\max}}{z_{\max}} \leq$ const. We may assume that, for every $t \in [t_0, t_1]$, $|y(t)| < \sqrt{\frac{W_{\max}}{2}}$. This clearly implies that, for every $t \in [t_0, t_1]$,

$$|z(t)| \geq \frac{W_{\max}}{4\sigma_M},$$

in which case z has a constant sign on $[t_0, t_1]$. Assume now that $W_{\max} < C$. Then (3.5) follows immediately since $\|u\|_p \geq C$. For the rest of the argument, $W_{\max} \geq C$. Consider the two following subsets of $[t_0, t_1]$ given by

$$E_1 = \left\{ t \in [t_0, t_1], \ |u(t)| < \frac{|z(t)|}{3} \ \text{a.e.} \right\}, \tag{5.4}$$

$$E_2 = \left\{ t \in [t_0, t_1], \ |u(t)| \geq \frac{|z(t)|}{3} \ \text{a.e.} \right\}. \tag{5.5}$$

We derive W along the trajectory of $(SI_2)_{k_L}$ corresponding to u and get

$$\dot{W} = -y\Delta\sigma\Big|_z^{z+u} - \sigma(z)\sigma(z+u) = -\dot{z}\Delta\sigma\Big|_z^{z+u} - \sigma^2(z+u). \tag{5.6}$$

Note that $|\dot{z}| \leq \sqrt{\frac{W_{\max}}{2}} + K$ and $\left|\Delta\sigma\Big|_z^{z+u}\right| \leq \delta_C$ on E_1. We integrate (5.6) between t_0 and t_1 and since $\int_{t_0}^{t_1} \dot{W} = \int_{E_1} \dot{W} + \int_{E_2} \dot{W}$, we get

$$\sigma_m^2(L - |E_2|) + \frac{W_{\max}}{2} \leq (L - |E_2|)\delta_C\left(\sqrt{\frac{W_{\max}}{2}} + K\right) + \left(\sqrt{\frac{W_{\max}}{2}} + K\right)2K|E_2|.$$

The above equation can be written as

$$\sigma_m^2 L + \frac{W_{\max}}{2} \leq L\delta_C\sqrt{2W_{\max}} + 3K|E_2|\left(\sqrt{\frac{W_{\max}}{2}} + K\right). \tag{5.7}$$

On the other hand, we have

$$\Delta y\Big|_{t_0}^{t_1} = -\int_{t_0}^{t_1} \sigma(z) + \int_{t_0}^{t_1} \Delta\sigma\Big|_z^{z+u}, \tag{5.8}$$

which implies that

$$\sigma_m L \leq \int_{t_0}^{t_1} \sigma(z) \leq -\Delta y\Big|_{t_0}^{t_1} + \int_{E_1} \Delta\sigma\Big|_z^{z+u} + \int_{E_2} \Delta\sigma\Big|_z^{z+u}.$$

Using the definition of E_1 and E_2, we get

$$\sigma_m L \leq 2y_{\max} + (L - |E_2|)\delta_C + 2K|E_2|. \tag{5.9}$$

We thus obtain

$$L \leq \frac{4}{\sigma_m}\sqrt{\frac{W_{\max}}{2}} + \frac{6K}{\sigma_m}|E_2|. \tag{5.10}$$

Equations (5.7) and (5.10) give $\sqrt{W_{\max}} \leq \text{const.} \, |E_2|$ which, in turn, implies that for p finite

$$\|u\|_{p,[0,T]}^p \geq \int_{E_2} |u|^p \geq W_{\max}^{p+\frac{1}{2}},$$

and $|\{t \in [t_0, t_1], \ |u(t)| \geq \text{const.} \, W_{\max} \ \text{a.e.}\}| > \text{const.} \, L$. The proof of Proposition 3.3 is finished.

5.4. **Proof of Lemma 3.5**

If $|z(t)| = M + C$ for $t \in [t_0, t_1]$, then $|\dot{z}| \geq |y| - K \geq \frac{\sqrt{\lambda}}{4} M$. Thanks to Remark 3.2, the zeros of $|z| - (M + C)$ are isolated. Then Lemma 3.5 simply says, unless trivial cases, there is an odd number of zeros of $|z| - (M + C)$ such that the variation of W between the smallest and the largest zeros is larger than $\frac{\lambda}{16} M^2$. We do so by examining several cases.

Assume that $|z(t_0)| > M + C$ and $|z(t_1)| > M + C$. In addition, suppose that

$$\text{for all } t \in [t_0, t_1], \ |z(t)| \geq M + C. \tag{5.11}$$

Then, we have $|\dot{y}| \geq \frac{3}{4} \sigma_m$. It implies that

$$\frac{3}{4} \sigma_m (t_1 - t_0) \leq |\Delta \sigma \big|_{t_0}^{t_1} y| \leq 2 y_{\max}.$$

We deduce that $L = t_1 - t_0 \leq \frac{8 y_{\max}}{3 \sigma_m}$. By integrating (5.6) and using (3.6), we get

$$\frac{\lambda}{2} M^2 = \Delta W \big|_{t_0}^{t_1} \leq \int_{t_0}^{t_1} (|y| + K) |\Delta \sigma \big|_{|z|}^{|z| + M}| - \int_{t_0}^{t_1} \sigma^2 (|z| - M).$$

By using (3.6), we have

$$\int_{t_0}^{t_1} (|y| + K) |\Delta \sigma \big|_{|z|}^{|z| + M}| \leq 2 \delta_C L y_{\max}.$$

It implies that

$$\frac{\lambda}{2} M^2 \leq \delta_C \text{const.} \ \lambda M^2 - \left(\frac{3}{4} \sigma_m \right)^2 L,$$

which is impossible. Then (5.11) cannot hold. Therefore, let $t^1 \in [t_0, t_1]$ such that $|z(t^1)| = M + C$ and $|z| \geq M + C$ on $[t_0, t_2]$. Then $\Delta W \big|_{t^1}^{t_1} \geq \frac{\lambda}{4} M^2$. By reproducing the previous argument on $[t^1, t_1]$, we can find $t^2 \in [t^1, t_1]$ such that $|z(t^2)| = M + C$ and $\Delta W \big|_{t^1}^{t^2} \geq \frac{\lambda}{8} M^2$. Again, unless λ is bounded, we can get a third zero of $|z| - (M + C)$ with the required properties. Going on in the same fashion, we prove the conclusions of Lemma 3.5.

Let us study now the case where $|z(t_0)| < M + C$ and $|z(t_1)| < M + C$. Suppose that

$$\text{for all } t \in [t_0, t_1], \ |z(t)| \leq M + C. \tag{5.12}$$

Then, for all $t \in [t_0, t_1]$, we have $|y(t)| \geq \frac{\sqrt{\lambda}}{4} M$. Hence,

$$2(M + C) \geq |\Delta z \big|_{t_0}^{t_1}| \geq L \frac{\sqrt{\lambda}}{8} M.$$

On the other hand, we also have $\frac{\lambda}{2} M^2 = \Delta W \big|_{t_0}^{t_1} \leq \text{const.} \ \sqrt{\lambda} M L$. We finally get that M is bounded by a constant independent of u, which is impossible. Therefore (5.12) cannot hold and there exists $t_0' \in [t_0, t_1]$ such that $|z(t_0')| > M + C$. Then, if t_2 is the first time after t_0 such that $|z(t_2)| = M + C$ and t_3 is the first time before t_1 such that $|z(t_3)| = M + C$, we have that $\Delta W \big|_{t_2}^{t_3} \geq \frac{\lambda}{8} M^2$. Again, unless we are done, we can get a third zero for $|z| - (M + C)$ with the required properties.

All the other cases reduce to those studied above and Lemma 3.5 is thus proved.

5.5. **Proof of Lemma 3.6**

On I_{2l-1}, $|\dot{y}| \geq \frac{3}{4}\sigma_m$ and $|y(t^j)| \geq$ const. M. Therefore there exist $t^i < t^f \in I_{2l-1}$ such that $|y| > \sigma_M$ on $[t^{2l-1}, t^i[$ and $]t^f, t^{2l-1}]$ and $|y| \leq \sigma_M$ on $[t^i, t^f]$. We get $|\dot{z}| > 0$ on $[t^{2l-1}, t^i[$ and $]t^f, t^{2l}]$. We may assume that $z > 0$ on I_{2l-1}. Let

$$z_m \stackrel{\text{def}}{=} \min(z(t^{2l-1}), z(t^i) \text{ and } z'_m = \max\left(z\left(t^{2l-1}\right), z\left(t^i\right)\right).$$

Then

$$
\begin{aligned}
\Delta W\Big|_{t^{2l-1}}^{t^i} &= \int_{t^{2l-1}}^{t^i} -\Big(\dot{z}\Delta\sigma\Big|_z^{z+u} + \sigma^2(z+u)\Big)\mathrm{d}t = -\int_{z(t^{2l-1})}^{z(t^i)} \Delta\sigma\Big|_z^{z+u}\mathrm{d}z - \int_{t^{2l-1}}^{t^i} \sigma^2(z+u)\mathrm{d}t \\
&\leq \int_{z_m}^{z'_m} |\Delta\sigma|_z^{z+u}|\mathrm{d}z - \int_{t^{2l-1}}^{t^i} \sigma^2(z-M)\mathrm{d}t.
\end{aligned}
$$

Since σ is increasing and globally Lipschitz, it has a non negative derivative function σ' locally essentially bounded. Define

$$J \stackrel{\text{def}}{=} \int_{z_m}^{z'_m} |\Delta\sigma|_z^{z+u}|\mathrm{d}z. \tag{5.13}$$

Then, we have

$$J \leq \int_{z_m}^{z'_m} \int_{z-M}^{z+M} \sigma'(s)\mathrm{d}s\,\mathrm{d}z = \iint_{\substack{z_m \leq z \leq z'_m \\ z-M \leq s \leq z+M}} \sigma'(s)\mathrm{d}s\,\mathrm{d}z \leq \iint_{\substack{C \leq z \leq z'_m + M \\ z-M \leq s \leq z+M}} \sigma'(s)\mathrm{d}s\,\mathrm{d}z.$$

By Fubini's theorem, we can permute the order of integration in J and we obtain

$$J \leq \iint_{\substack{s-M \leq z \leq s+M \\ C \leq s \leq \text{const.}\,\lambda M^2}} \sigma'(s)\mathrm{d}s\,\mathrm{d}z \leq \int_C^{\text{const.}\,\lambda M^2} \sigma'(s) \int_{\max(z_m,s-M)}^{\min(z'_m,s+M)} \mathrm{d}z\,\mathrm{d}s \leq 2M\Delta\sigma\Big|_C^{\text{const.}\,\lambda M^2} \leq 2M\delta_C.$$

Then, we get

$$\Delta W\Big|_{t^{2l-1}}^{t^i} \leq 2M\delta_C - \frac{\sigma_m^2}{2}(t^i - t^{2l-1}).$$

In addition, we have

$$\sigma_M(t^i - t^{2l-1}) \geq |\Delta y\Big|_{t^{2l-1}}^{t^i}| \geq \text{const.}\,\sqrt{\lambda}M,$$

in which case

$$\Delta W\Big|_{t^{2l-1}}^{t^i} \leq 2M\delta_C - \text{const.}\,\sqrt{\lambda}M \leq -\text{const.}\,\sqrt{\lambda}M.$$

Moreover, since $\sigma_M(t^f - t^i) \geq |\Delta y\Big|_{t^i}^{t^f}| = 2\sigma_M$, we have

$$\Delta W\Big|_{t^i}^{t^f} \leq \sigma_M\delta_C - \left(\frac{3\sigma_m}{4}\right)^2 (t^f - t^i) \leq 0.$$

Then, we obtain $\Delta W_{2l-1} \leq -\text{const.}\,\sqrt{\lambda}M$. On I_{2l}, $|y| \geq \text{const.}\,\sqrt{\lambda}M$, which implies that

$$\Delta W_{2l} \leq \text{const.}\,\sqrt{\lambda}M(t^{2l+1} - t^{2l}) \leq \text{const.}\,\left|\int_{I_{2l}} y\right| \leq \text{const.}\,|\Delta z\Big|_{t^{2l}}^{t^{2l+1}}| \leq \text{const.}\,M.$$

Adding the appropriate inequalities, we get the conclusion of Lemma 3.6.

5.6. **Proof of Lemma 3.12**

We start with $(J2)$. We assume that $z \geq 0$. If $u + y < 0$, then $|u| > y$, in which case we have

$$|2\dot{z}\Delta\sigma\Big|_{z-\theta y}^{z+u} + \sigma(z-y)\Delta\sigma\Big|_{z-y}^{z+u}| \leq \text{const.} \, |u|^p. \tag{5.14}$$

If $u + y \geq 0$, then $\Delta\sigma\Big|_{z-y}^{z+u} \geq 0$, since σ is increasing. We have to distinguish two subcases, whether $u + \theta y \geq 0$ or not. If $u + \theta y \geq 0$, then $\Delta\sigma\Big|_{z-\theta y}^{z+u} \geq 0$. Then,

$$-\left(2\dot{z}\Delta\sigma\Big|_{z-\theta y}^{z+u} + \sigma(z-y)\Delta\sigma\Big|_{z-y}^{z+u}\right) \leq 0,$$

and $(J2)$ is immediate. If $u + \theta y < 0$, then $|u| > \theta y$ and we get equation (5.14).

We turn to the case where $z \leq 0$. If $u + \theta y < 0$, then $|u| > \theta y$, and equation (5.14) follows readily. Assume, then, that $u + \theta y \geq 0$. We also have that $u + y \geq (1-\theta)y \geq \frac{C}{2}$. We distinguish two subcases, whether $z + u \geq -C$ or not. If $z + u \geq -C$, then $u > |z| - C \geq y$ and equation (5.14) follows. If $z + u < -C$, then $|\Delta\sigma\Big|_{z-y}^{z+u}| \leq \delta_C$ and

$$-\left(2\dot{z}\Delta\sigma\Big|_{z-\theta y}^{z+u} + \sigma(z-y)\Delta\sigma\Big|_{z-y}^{z+u}\right) \leq \text{const.} \, \delta_C,$$

and we get $(J2)$.

For $(J3)$, we may simply assume that $z \geq 0$. We have

$$J_3 \leq -\frac{\sigma_m^2}{4}|I/I'| + K^2|E'|,$$

where $E' = \left\{t \in I/I', \; |u(t)| \geq \frac{z(t)}{3} \text{ a.e.}\right\}$. Since $z \geq 1$ on I/I', we get $(J3)$.

It remains to show $(J1)$. We simply assume that $z \geq 0$. Set $z_1' = z(t_1')$, $z_1 = z(t_1)$. Since $\dot{z} \geq C$ on I/I', we can take z as a new parameter instead of the time t for functions defined on I/I'. In particular, we can consider y as a function of z on $[z_0, z_1]$. In addition, we rewrite (3.15) as

$$\begin{aligned} J_1 &= -\int_{z_1'}^{z_1} \Delta\sigma\Big|_z^{z-\theta y} dz - \int_{z_1'}^{z_1} \Delta\sigma\Big|_{z-y}^{z-\theta y} dz, \\ &= \iint_{\substack{z_1' \leq z \leq z_1 \\ z-\theta y(z) \leq s \leq z}} \sigma'(s) ds\, dz - \iint_{\substack{z_1' \leq z \leq z_1 \\ z-y(z) \leq s \leq z-\theta y(z)}} \sigma'(s) ds\, dz. \end{aligned} \tag{5.15}$$

Note that $z - \theta y$ and $z - y$ can be reparameterized as functions of $z \in [z_0 - y(z_0), z_1]$ and they verify

$$\frac{1}{2} \leq \frac{d(z-\theta y)}{dz}, \; \frac{d(z-y)}{dz} \leq 2.$$

Therefore, we can define, for $s \in [z_0 - y(z_0), z_1]$, two functions $Z(s)$ and $\bar{Z}(s)$ as follows

$$\begin{aligned} Z(s) &= z_0 & \text{for} & \quad s \in [z_0 - y(z_0), z_0], \\ Z(s) - \theta y(Z(s)) &= s & \text{for} & \quad s \in [z_0, z_1 - \theta y(z_1)], \\ Z(s) &= z_1 & \text{for} & \quad s \in [z_1 - \theta y(z_1), z_1] \end{aligned}$$

and

$$\begin{aligned} \bar{Z}(s) - y(\bar{Z}(s)) &= s & \text{for} & \quad s \in [z_0 - y(z_0), z_1 - y(z_1)], \\ \bar{Z}(s) &= z_1 & \text{for} & \quad s \in [z_1 - y(z_1), z_1]. \end{aligned}$$

By Fubini's theorem, we can permute the order of integration in (5.15) and we obtain

$$J_1 = J_1' + J_2' + J_3' + J_4',$$

where

$$
\begin{aligned}
J_1' &= \int_{z_0}^{z_1 - y(z_1)} \Big(2Z(s) - s - \bar{Z}(s)\Big)\sigma'(s)\mathrm{d}s,\\
J_2' &= \int_{z_0 - y(z_0)}^{z_0 - \theta y(z_0)} \Big(Z(s) - \bar{Z}(s)\Big)\sigma'(s)\mathrm{d}s,\\
J_3' &= \int_{z_0 - \theta y(z_0)}^{z_0} \Big(2Z(s) - z_0 - \bar{Z}(s)\Big)\sigma'(s)\mathrm{d}s,\\
J_4' &= \int_{z_1 - y(z_1)}^{z_1} \Big(Z(s) - s\Big)\sigma'(s)\mathrm{d}s.
\end{aligned}
$$

Since $Z(s) \le \bar{Z}(s)$ and $Z(s) \le s$ on $s \in [z_0 - y(z_0), z_0]$, we have $J_2' \le 0$ and $J_3' \le 0$. In addition, we have

$$J_4' = \int_{z_1 - y(z_1)}^{z_1} \Big(\sigma(z) - \sigma(z - \theta y)\Big)\mathrm{d}z \le 2\sup(y(t_1), C)\delta_C, \qquad (5.16)$$

since $z > z - \theta y \ge 2C$ for $s \in [z_1 - y(z_1), z_1]$. As for J_1', we can write it as follows

$$J_1' = \int_{z_0}^{z_1 - y(z_1)} \Big(2\theta y(Z(s)) - y(\bar{Z}(s))\Big)\sigma'(s)\mathrm{d}s.$$

We get $J_1' \le 0$ as a consequence of the next lemma:

Lemma 5.1. *For $s \in [z_0, z_1 - y(z_1)]$, we have $2\theta y(Z(s)) - y(\bar{Z}(s)) \le 0$.*

Proof of Lemma 5.1. We have

$$2\theta y(Z(s)) - y(\bar{Z}(s)) = 2\theta \Delta y\Big|_{Z(s)}^{\bar{Z}(s)} - (1 - 2\theta)y(\bar{Z}(s)) \le -(1 - 2\theta)C + 2\theta \Delta y\Big|_{Z(s)}^{\bar{Z}(s)}.$$

On the other hand, we have

$$\Big|\Delta y^2\Big|_{Z(s)}^{\bar{Z}(s)}\Big| \le 2\Big|\int_{Z(s)}^{\bar{Z}(s)} \frac{y\mathrm{d}y}{\mathrm{d}z}\mathrm{d}z\Big| \le 2\int_{Z(s)}^{\bar{Z}(s)} \frac{|y\sigma(z+u)|}{y - \sigma(z+u)}\mathrm{d}z \le \frac{2K}{1 - \frac{K}{C}}\Big|\Delta z\Big|_{Z(s)}^{\bar{Z}(s)}\Big| \le \frac{2CK}{C - K}y(\bar{Z}(s)).$$

We thus obtain

$$\Big|\Delta y\Big|_{Z(s)}^{\bar{Z}(s)}\Big| \le \frac{2CK}{C - K}\frac{y(\bar{Z}(s))}{y(Z(s)) + y(\bar{Z}(s))} \le \frac{2CK}{C - K}.$$

Then, $2\theta y(Z(s)) - y(\bar{Z}(s)) \le -(1 - 2\theta)C + \frac{2CK}{C-K} = 0$.

5.7. **Proof of Lemma 3.15**

Consider the function

$$Q(z, y) \stackrel{\mathrm{def}}{=} -zy|y|^{2p-2} + \frac{|y|^{2p}}{p}.$$

If we use $\dot{Q}$ to denote the time derivative of Q along x_u, we get

$$\dot{Q} = -|y|^{2p} + (2p - 1)|y|^{2p-2}z\sigma(z + u).$$

For every $T > 0$, we integrate the above inequality on $[0, T]$ and get

$$Q(T) + \int_0^T |y|^{2p} \leq \text{const.} \int_0^T (|y|^{2p})^{\frac{p-1}{p}} |z\sigma(z + u)|.$$

By applying Hölder's inequality for $p > 1$, we get

$$Q(T) + \int_0^T |y|^{2p} \leq \text{const.} \left(\int_0^T |y|^{2p} \right)^{\frac{p-1}{p}} \|z\sigma(z + u)\|_p.$$

By letting T tend to infinity and thanks to Remark 3.8, we get the result.

5.8. **Proof of Lemma 3.16**

The two previous lemmas clearly imply that

$$\|V\|_p \leq \text{const.} \left(\|V\sigma(z + u)\|_p + \|u\|_p \right).$$

Therefore, we have, by using Proposition 3.7,

$$\int_0^\infty V^p \leq \text{const.} \left(\int_0^\infty V^p |\sigma(z + u)|^p + \|u\|_p^p \right) \leq \text{const.} \left(\int_{E_R} V^p |\sigma(z + u)|^p + \int_{\mathbb{R}^+/E_R} V^p |\sigma(z + u) + \|u\|_p^p|^p \right)$$

$$\leq \text{const.} \left(\int_{E_R} V^p |\sigma(z + u)|^p + \text{const.} \, R^p \|\sigma(z + u)\|_p^p + \|u\|_p^p \right) \leq \text{const.} \int_{E_R} V^p + \text{const.} \, R^p \|u\|_p^p.$$

From the last equation, we conclude.

5.9. **Proof of Lemma 3.18**

The set F_i is an interval $[t', t'']$ such that $\frac{1}{2} \leq t' - t_i \leq \frac{3}{2}$ and $\frac{1}{2} \leq t_{i+1} - t'' \leq \frac{3}{2}$. Set $S_i = [t_i, t_{i+1}]/F_i$. We have $1 \leq |S_i| \leq 3$ and $|y| \geq \frac{\bar{V}^{\frac{1}{2}}}{2}$ on S_i. Then,

$$\int_{S_i} \dot{V} = \int_{S_i \cap E_i} \dot{V} + \int_{S_i \cap E_i'} \dot{V} \leq \text{const.} \, |S_i \cap E_i| \bar{V}^{\frac{1}{2}} - |S_i \cap E_i'| \frac{\sigma_m^2 + \bar{V}^{\frac{1}{2}}}{2}$$

$$\leq -\frac{1}{4} \bar{V}^{\frac{1}{2}} + \text{const.} \, \bar{V}^{\frac{1}{2}} |S_i \cap E_i|.$$

On F_i, using the estimates obtained in Lemma 3.12 of case $(B1)$, we get

$$\int_{F_i} \dot{V} \leq -\frac{\sigma_m^2}{4} \bar{V}^{\frac{1}{2}} + \text{const.} \, |F_i \cap E_i| + 2 \int_{F_i'} |\dot{z}| |\Delta\sigma|_{z+u}^{z-\theta y}|.$$

We also have

$$\int_{F_i'/G_i} |\dot{z}| |\Delta\sigma|_{z+u}^{z-\theta y}| \leq \int_{F_i'} |\dot{z}| |\Delta\sigma|_{z-\rho\bar{V}^{\frac{1}{2}}}^{z} \leq \int_{F_i} |\dot{z}| |\Delta\sigma|_{z-\rho\bar{V}^{\frac{1}{2}}}^{z}.$$

We need the next lemma to complete the proof of Lemma 3.18:

Lemma 5.2. *With the previous notations, we have*

$$\int_{F_i} |\dot{z}| |\Delta\sigma|_{z-\rho\bar{V}^{\frac{1}{2}}}^{z} \leq \text{const.} \, \delta_{\bar{V}^{\frac{1}{2}}} \left(|E_i| + \bar{V}^{\frac{1}{2}} \right).$$

Proof of Lemma 5.2. Define $L_i = [t', \bar{t}'] \cup [\bar{t}'', t'']$ and $M_i = (\bar{t}', \bar{t}'')$ in such a way that $|y| > \sigma_M + 1$ on L_i and $|y(\bar{t}')| = |y(\bar{t}'')| = \sigma_M + 1$. Assume first that

$$|M_i \cap E_i| < \frac{\sigma_m}{4K}|L_i|. \tag{5.17}$$

Let us show that $|M_i|$ is bounded, in which case $|y|$ remains also bounded. For that purpose, consider the following equation

$$\Delta y\Big|_{\bar{t}'}^{\bar{t}''} \leq -\int_{M_i} \sigma(z) + \int_{M_i} |\Delta\sigma\Big|_{z+u}^{z}|.$$

It implies that

$$\sigma_m|M_i| \leq 2(\sigma_M + 1) + \delta_C|M_i| + 2K|M_i \cap E_i|.$$

Then, by using (5.17), we at once get that $|M_i|$ is bounded. Therefore, we have

$$\int_{M_i} |\dot{z}|\Delta\sigma\Big|_{z-\rho\bar{V}^{\frac{1}{2}}}^{z} \leq \text{const. } \delta_{\bar{V}^{\frac{1}{2}}}.$$

Assume now that $|M_i \cap E_i| \geq \frac{\sigma_m}{4K}|L_i|$. Then, $|M_i| \leq \text{const. } |E_i|$, in which case we obtain

$$\int_{M_i} |\dot{z}|\Delta\sigma\Big|_{z-\rho\bar{V}^{\frac{1}{2}}}^{z} \leq \text{const. } \delta_{\bar{V}^{\frac{1}{2}}}|E_i|.$$

Finally, we have

$$\int_{L_i} |\dot{z}|\Delta\sigma\Big|_{z-\rho\bar{V}^{\frac{1}{2}}}^{z} \leq 3\int_{\bar{t}'}^{\bar{t}''} \Delta\sigma\Big|_{z-\rho\bar{V}^{\frac{1}{2}}}^{z} \leq \iint_{\substack{z(\bar{t}')\leq z\leq z(\bar{t}'') \\ z-\rho\bar{V}^{\frac{1}{2}}\leq s\leq z}} \sigma'(s)\mathrm{d}s\,\mathrm{d}z \leq \rho\bar{V}^{\frac{1}{2}}\Delta\sigma\Big|_{z(\bar{t}')-\rho\bar{V}^{\frac{1}{2}}}^{z(\bar{t}'')} \leq \text{const. } \delta_{\bar{V}^{\frac{1}{2}}}\bar{V}^{\frac{1}{2}}. \quad \square$$

We also have $\int_{G_i} |\dot{z}||\Delta\sigma\Big|_{z+u}^{z-\theta y}| \leq \text{const. } |G_i|\bar{V}^{\frac{1}{2}}$. Adding up all the inequalities, we get (3.39).

5.10. **Proof of Lemma 3.19**

Define $U_i = \left\{t \in E_i, \ |z| \geq \frac{\bar{V}^{\frac{1}{2}}}{2\sigma_M}\right\}$. Then, we clearly have

$$\int_{U_i} |z|^p \geq \text{const. } |U_i|\bar{V}^{\frac{p}{2}}.$$

We may assume that $|U_i| < \rho|E_i|$, otherwise we are done. Moreover, since E_i/U_i is included in S_i, $|E_i|$ is bounded. Recall that $|E_i/U_i| \geq \frac{\rho}{2}$ and $|y| \geq \bar{V}^{\frac{1}{2}}$ on E_i/U_i. In addition, we have

$$|z(t)| \geq \bar{V}^{\frac{1}{2}}|t - t_s|, \tag{5.18}$$

where $s = i$ or $i + 1$. The set E_i/U_i contains a subset E_i'' of measure larger than $\frac{\rho}{4}$ and disjoint from $[t_i, t_i + \frac{\rho}{8}] \cup [t_{i+1} - \frac{\rho}{8}, t_{i+1}]$. We deduce from (5.18) that

$$\int_{E_i''} |z|^p \geq \text{const. } \int_{\frac{\rho}{8}}^{\frac{\rho}{4}} \left(\frac{\bar{V}}{2}s\right)^{\frac{p}{2}} \mathrm{d}s \geq \text{const. } \bar{V}^{\frac{p}{2}} \geq \text{const. } |E_i|\bar{V}^{\frac{p}{2}}.$$

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