ON THE CONTROLLABILITY OF THE BURGER EQUATION

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ABSTRACT. We present here a return method to describe some attainable sets on an interval of the classical Burger equation by means of the variation of the domain.

1. STATEMENT OF THE MAIN RESULT

We are here interested in the following problem of controllability: Let T_c be an arbitrary real number, X a given normed space of real functions of the real variable defined on $[1, 2]$, z_1, z_0 elements of X. Does there exist a weak (entropic) solution of:

$$u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$$
$$u \in L^\infty\left((0, +\infty), X\right), \tag{1.1}$$
$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{1.2}$$
$$u(0, .)_{/[1,2]} = z_0, \tag{1.3}$$
$$u(T_c, .)_{/[1,2]} = z_1? \tag{1.4}$$

If this holds for all $(z_0, z_1) \in X \times X$ one says that equation (1.2) is controllable in X. Moreover if one replaces condition (1.4) by

$$||u(T_c, .)_{/[1,2]} - z_1||_X < \varepsilon, \tag{1.5}$$

where $\varepsilon > 0$ is a priori given, and if this holds for any $(z_0, z_1) \in X \times X$ and any $\varepsilon > 0$, one says that there is approximate controllability in X. According to the settlement of the controllability problem, we take here $X = BV([1, 2])$ equipped with the L^1 norm. If $A \subset \mathbb{R}$, $BV(A)$ denotes the space of mesurable functions f such that

$$\int_A |f| dx + \int |Df| < +\infty,$$

here Df is the derivative of f in the sense of measures.

The following conditions (ES) are the conditions necessarily satisfied by spatial values of weak entropic solutions of the Cauchy problem (1.2) with initial value in the class of functions with bounded variation: we say that a

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function f satisfies (ES), and is said to be of entropic shape, if

$$f \in BV$$
$$\forall x,\, f(x-) := \lim_{t \to x, t < x} f(t) \geq f(x+) := \lim_{t \to x, t > x} f(t), \qquad (1.6)$$
$$(1.6) \text{ is strict for at most a sequence } (x_\nu)$$

The main proposed result in this paper is the following:

THEOREM 1.1. *Let z_0 be given in $BV([1,2]) \cap L^\infty([1,2])$ and $\varepsilon > 0$ and $T > 2$. Let z_1 be an element of $BV([1,2])$ satisfying the necessary conditions (ES) and, such that*

(i): $\forall x_0 \in (1,2)$, if $z_1(x_0-) > 0$ then $z_1(x) > \dfrac{x-1}{T}$ $\forall 1 \leq x < x_0$, if $z_1(x_0+) < 0$, there holds $z_1(x) < \dfrac{x-2}{T}$ $\forall x_0 < x \leq 2$,

(ii): There exists at most one $x_1 \in (1,2)$ such that $z_1 > 0$ on $[1,x_1)$ and $z_1 < 0$ on $(x_1,2]$: moreover for any $x \in (1,2)$ being a point of continuity of z_1 such that $z_1(x) \neq 0$ one has

$$D^+ z_1(x) \leq \frac{1}{T},$$

then there exists a time $T_c \geq T$ (T_c is the time of approximate controllability) such that the equation (1.2) has at least a weak solution such that (1.3) and (1.4) hold if (1.4) is replaced by the following: there exists an interval of length 2ε outside of which $u(T_c, .)$ and z_1 are equal and moreover

$$\int_1^2 |u(T_c, x) - z_1(x)| dx \leq C(u_0, u_1)\varepsilon.$$

This last statement is a little bit more than L^1 approximate controllability. Here $D^+ z_1(x)$ means

$$\sup_{h > 0} \frac{z_1(x+h) - z_1(x)}{h}.$$

To our knowledge, the problem of the controllability of the equation (1.2) has been initially studied in [1]. The major difference between [1] and the present work relies on the fact that the problem considered in [1] concerns only the solution on a half-plane (see [13], or [10]), and only for vanishing initial data. Nevertheless the method that we present here is of quite different nature, though, of course, it relies on the characteristic method. It has been inspired by the finite dimensional case in the theory of control.
Let us stress that *(i)* is given in [1]. Condition *(ii)* has also an equivalent in [1], which is

$$D^+ z_1(x) \leq \frac{z_1}{x-1}$$

written in our situation, but (ii) requires more.

Let us also mention that for the Burger equation with addition of a diffusion term, a negative answer to a problem of partial boundary controllability was given by A. Fursikov & O.Yu. Imanuvilov, see [8] and (it seems previously) by J.I. Diaz, see [5].

It is well known, see [14] or [15], that the space of bounded variation functions is a good regularity space for the Cauchy problem (1.2) & (1.3) in the class of entropic solution. One also knows that $\forall t \in (0, +\infty)$ $u(t, .)$

has compact support (provided that the initial value has compact support) and is continuous except on a countable set of points being the boundary of a dense union of open intervals. At each discontinuity x then the entropic shape of the solution (1.6) gives that

$$u(t, x-) > u(t, x+).$$

According to these remarks, the conditions (ES) are necessary for z_1 to be attainable. In particular z_1 is continuous on the countable union of open intervals. The standard ideas on the Cauchy problem (1.2) & (1.3) can be found in [11] and in [12]. We also recall, see [18], that standard rules of differentiation can be used in the space BV provided that one replaces the functions by some regularization. In particular, if $f \in BV$ and if one defines $\tilde{f}(x) = \frac{1}{2}(f(x+) + f(x-))$ one has the formula $(f^2)_x = 2\tilde{f}f_x$.

2. Proof of the result

A natural idea to show the controllability (at least locally) is to deal with the linearized operator around the null solution. This method, a classical one for finite dimensional systems, gives also many results for partial differential equations, even if it is harder to proceed in that case, as it has been shown recently by A. Fursikov and O.Yu. Imanuvilov, see [8] and L. Rosier [16]. This trick, astutely rethought, gives also global results, when the linearized operators around any trajectories are controllable, as it was proven by C. Fabre & J-P. Puel & E. Zuazua in [6] & [7].
Unfortunately, for the equation (1.2), the linearized operator around 0 is

$$z_t = 0 \text{ on } [1, 2],$$

which is not even approximately controllable.

In order to obtain our result we have been interested in the so called "return method" introduced by J-M. Coron in [2] to show a result of stabilization of a finite dimensional system. He used it afterwards in [3] and [4] to study the controllability of incompressible fluids in dimension 2. The idea is to consider the controllability of the linearized operator, not around the zero solution, but around trajectories vanishing at the times 0 and T: if this linearized operator is controllable, one may hope the nonlinear system is locally controllable.

Of course such an entropic trajectory of (1.2) does not exist except if it is 0 on $[0, T] \times [1, 2]$, and the linearized operator around the zero solution is not even approximately controllable, since any solution of the linearized equation is of course a constant of time.

Moreover, if one rescales such a solution (or more exactly an approximation), the time to attain the zero value is scaled too, thus one cannot expect to apply correctly the implicit function theorem, the essential reason being that for asymptotic large time a small difference between two solutions becomes very large, due to the term uu_x in (1.2).

However it is possible to provide solutions allowing to perform some kind of linearization as will be described later. Let us mention that the method used here relies more on a variation of the domain than on a standard linearization and our proof can also be related to the "extension method" introduced by D.L. Russell, see [17]. The control in our case is the initial value and can be realized by means of boundary values when it has some sense for (1.2).

In order to show Theorem 1.1, we first prove:

THEOREM 2.1. *If $z_0 = 0$, $T > 2$ be given, there exists a solution to (1.2), (1.3), (1.4), (2.2) provided that z_1 satisfies (ES) and conditions (i) and (ii) of Theorem 1.1. Moreover we can choose $T_c = T$.*

The proofs of Theorem 1.1 and Theorem 2.1 are organized as follows: In 2.1 we give the necessary condition for z_1 to be attainable, then in 2.2, we describe the sufficient conditions that lead to the conclusion of those theorems.

2.1. NECESSARY CONDITIONS FOR CONTROLLABILITY

We will now assume that $T > 2$. Since the problem we are concerned with in proposition 2.3, requires that the initial value of u must be 0 on $[1, 2]$ whatever is z_1, we see that it is necessary to impose

$$\forall y \in [1, 2], z_1(y) \geq \frac{y - 1}{T} \, (\text{ or } z_1(y) \leq \frac{y - 2}{T}). \qquad (2.1)$$

We will treat the first case since the second is similar. Now assume that for some $\bar{y} \in [1, 2]$ equality holds, then in order for z_1 to be attained by the Cauchy problem, we have

ASSERTION 2.2. *For $y > \bar{y}$ equality holds in (2.1).*

Proof: If there exists $y > \bar{y}$ such that inequality holds, then the generalized backward characteristics issued from the point $(\bar{y}, T)$ and the point (y, T) would meet on $(0, T)$ contradicting the assumption on the entropic shape of z_1. So equality holds which proves this assertion.

Now by the method of characteristics one immediately sees that such a z_1 can only be attained by imposing a Dirac mass at time $t = 0$ for $x = 1$, since $z_{0/[1,2]} = 0$, and thus it is not reachable by our method. Thus the only possibility for equality is $\bar{y} = 2$ but this also leads to the same conclusion.

So we found one necessary controllability condition for z_1 (in the present case) which is $z_1(x) > \frac{x-1}{T}$. Let us stress that condition (2.1) does not depend on the device introduced here.

2.2. PROOF OF THE SUFFICIENT CONDITIONS FOR CONTROLLABILITY

2.2.1. PROOF OF THEOREM 2.1: CASE OF CONTINUOUS FINAL STATE. We first of all assume stronger conditions.

PROPOSITION 2.3. *Assume moreover that $z_1 > 0$ on $[1, 2]$. Then the result of Theorem 2.1 is true.*

PROOF OF PROPOSITION 2.3

We fix once for all a real nonnegative small number $\frac{1}{2} > \rho > 0$. Let $a_{1,\rho}$ be the unique entropic solution of (1.2) defined by

$$a_{1,\rho}(x,0) = 0, \qquad x \le 0,$$
$$a_{1,\rho}(x,t) = \min\left(\frac{x}{t+\rho}, 1\right), \quad 0 \le x \le 1,$$
$$a_{1,\rho}(x,t) = 0, \qquad x \ge 1.$$

Standard computations using the Rankin-Hugoniot condition show that

$$a_{1,\rho}(x,t) = \frac{x}{t+\rho} \cdot \chi_{x \le \sqrt{(2-\rho).(t+\rho)}}, \ t \ge 2 - 2\rho.$$

Here χ means the characteristic function of the set in subscript.

Assume that z_1 is continuous. We look for solutions of (1.2) of the form (for sake of simplicity we forget the subscript 1 for the moment being in $a_{1,\rho}$).

$$u(x,t) = \alpha a_\rho\left(\frac{x + \tau(x,t)}{\alpha}, t\right), \tag{2.2}$$

where τ is of course to be precised and where we take $\alpha > 0$. This is precisely the variation of the domain. Assume that the standard rule of differentiation of composed functions is valid, then if we put (2.2) into (1.2) it gives

$$\alpha \partial_t a_\rho\left(\frac{x+\tau(x,t)}{\alpha}, t\right) + \partial_x a_\rho\left(\frac{x+\tau(x,t)}{\alpha}, t\right) \partial_t \tau(x,t) +$$
$$\alpha a_\rho\left(\frac{x+\tau(x,t)}{\alpha}, t\right) \partial_x a_\rho\left(\frac{x+\tau(x,t)}{\alpha}, t\right).\left(1 + \partial_x \tau(x,t)\right) = 0.$$

Thus, if

$$\partial_t \tau(x,t) + \alpha a_\rho\left(\frac{x+\tau(x,t)}{\alpha}, t\right) \partial_x \tau(x,t) = 0, \tag{2.3}$$

the function u will be, at least formally, a weak solution of (1.2) in some region of the (x,t) plane. Moreover we require (1.4) to be satisfied in order to obtain the desired final state.

We wish to solve (2.3) by the method of characteristics (see e.g. [12]).

Let $y \in [1,2]$ be given. Define ϕ_y the solution (provided it is defined) of

$$\frac{d}{dt}\phi_y(t) = \alpha a_\rho\left(\frac{\phi_y(t)+\tau(\phi_y(t),t)}{\alpha}, t\right) \quad \forall t \in [0,T]$$
$$\phi_y(T) = y.$$

We will also assume that, for any $t \in [0,T]$, $\frac{\phi_y(t)+\tau(\phi_y(t),t)}{\alpha}$ lies in the domain

$$\left[0, \min\left(t + \rho, \sqrt{(2-\rho)(t+\rho)}\right)\right).$$

This gives, at least formally, the following formula

$$\phi_y(t) + \tau(y,T) = \frac{t+\rho}{T+\rho}(y + \tau(y,T)).$$

In terms of z_1 this formula is

$$\phi_y(t) = (t - T)z_1(y) + y.$$

At this point of the proof, it is necessary, for the sake of completeness, to distinguish which conditions in our results are required for the state z_1 for any kind of method of controllability and which are sufficient to apply our technique.

Since we wish to define the initial data $u(0,.)$ with respect to z_1 we have to require that for two different values of $(y_0, y_1) \in [1, 2]$ there holds $\phi_{y_0}(t) \neq \phi_{y_1}(t)$. Thus for any $t \in (0, T]$ we want to have

$$(t - T)z_1(y_0) + y_0 \neq (t - T)z_1(y_1) + y_1. \qquad (2.4)$$

Assume that $y_0 < y_1$ then we must have $(t - T)(z_1(y_0) - z_1(y_1)) \neq y_1 - y_0$ or else $\frac{z_1(y_0) - z_1(y_1)}{y_0 - y_1} \neq \frac{1}{T-t}$; that implies

$$D^+ z_1(y) = \sup_{h>0} \frac{z_1(y + h) - z_1(y)}{h} \leq \frac{1}{T}$$

what we then will assume.

Suppose that equality holds for a point $y \in [1, 2]$ such that there exists a sequence (h_n), $h_n > 0$, for which

$$\frac{z_1(y + h_n) - z_1(y)}{h_n} > \frac{1}{T}.$$

Then the backward characteristics issuing from the points (y, T) and $(y + h_n, T)$ will meet before time $t = 0$, thus contradicts the entropic shape of z_1. Thus the necessary condition is $D^+ z_1 \leq \frac{1}{T}$.

Let us stress that this condition is not necessary for z_1 to be attainable by a general scheme, and relies only on the method introduced here.

Now we have to check that for any $y \in [1, 2]$ one has

$$(t + \rho)z_1(y) = \phi_y(t) + \tau(\phi_y(t), t) \in \left[0, \alpha \min(t + \rho, \sqrt{(2 - \rho)(t + \rho)}) \right), \qquad (2.5)$$

which is equivalent to

$$\forall y \in [1, 2], z_1(y) \in [0, \alpha \frac{\sqrt{2 - \rho}}{\sqrt{T + \rho}}].$$

But this is trivially possible provided one chooses α sufficiently large.

Now we have obtained the conditions announced for z_1 to be, formally for the moment being, attainable.

CONSTRUCTION OF THE SOLUTION.

Now assume that the condition of Theorem 2.1 are fulfilled. It is obvious that

$$\Psi : (t, y) \to (t, \phi_y(t) = y + (t - T)z_1(y))$$

is a continuous function and that our assumption on uniqueness (2.4) shows that, if $t > 0$ is fixed, it is a one-to-one mapping onto its image. Thus the converse map is also continuous and one-to-one (for t fixed). This means that if t is fixed then y depends continuously on $\phi_y(t)$. It is also clear that $\Psi([0, T] \times [1, 2])$ is the convex set whose boundary is the union of the four intervals

$$t \to 1 + (t - T)z_1(1), t \in [0, T],$$
$$t \to 2 + (t - T)z_1(2), t \in [0, T],$$
$$[1 - Tz_1(1), 2 - Tz_1(2)],$$
$$[1, 2].$$

We thus are able to define τ in this region, that is to say u, by means of the preceding formulas. This gives also an initial value.

Let us also remark that the initial value may not be continuous, this being the case if z_1 is a line of slope $\frac{1}{T}$. However, it is of no importance if one uses the following trick.
On the interval $[2 - Tz_1(2), 2 - \frac{T}{T+\rho}2]$ we define $u(0, .)$ to be the line joining the points $(2 - Tz_1(2), z_1(2))$ and $(2 - \frac{T}{T+\rho}2, \frac{2}{T+\rho})$. On the interval $(-\infty, 1 - Tz_1(1))$ we define $u(0, .)$ to be 0. Now, for the points of $[2 - Tz_1(2), 2 - \frac{T}{T+\rho}2]$ which are attained by more than one characteristics, we impose to the initial value to have left and right limit at that point. This is possible, because that case occurs when there exists $1 \le y_0 < y_1 \le 2$ such that $y_0 - Tz_1(y_0) = y_1 - Tz_1(y_1)$. Now the condition on the derivative of z_1 and the uniqueness of the curve $t \to \phi_y(t)$ on $(0, T]$ shows that in fact for all $y \in [y_0, y_1]$ we have $y - Tz_1(y) = y_0 - Tz_1(y_0)$. Then if $y_0 = 1$ and $y_1 = 2$ then $u(0, .)$ will be 0 on the left of $y_0 - Tz_1(y_0)$ and on the right will be the affine function just as above. If not, let us assume that y_0 is the infimum value of y such that $y - Tz_1(y) = y_1 - Tz_1(y_1)$ and we redefine y_1 by the supremum with this property. Then we define

$$u(0, y_0 - Tz_1(y_0)-) = z_1(y_0)$$

and

$$u(0, y_0 - Tz_1(y_0)+) = z_1(y_1).$$

Thus we have defined the initial value, for the Cauchy problem. Moreover this method gives also u by means of τ, following the above manner. We have to check that it is in fact a weak solution to the problem and that it satisfies both final and initial conditions.

We have

$$\tau(\phi_y(t), t) = \tau((t - T)z_1(y) + y, t) = \tau(y, T).$$

Put $x = \phi_y(t)$, we get the formula for $t \in (0, T]$ $\tau(x, t) = \tau(\xi_x^{-1}(t), T)$ with $\xi_x(t) = y$ if and only if $x = \phi_y(t)$. Now it is well known that functions with bounded variations are almost everywhere differentiable. Since $\phi_y(t) = y + (t - T)z_1(y)$ and that $y \to \phi_y(t)$ is one-to-one onto its image, and because of the assumption on $D^+ z_1$ we immediately see that τ is almost everywhere differentiable with respect to x and also with respect to t, thus it is a weak solution. It is by construction clear that u coincides with z_1 at times T.

Now let us show that for almost every $x \in \mathbb{R}$ one has $\lim_{t \to 0} u(x, t) = z_0$ except for a set of t of measure 0. Let us make a partition of $\mathbb{R}$ in the following way. Let

$$I_1 = \{x \in \mathbb{R}, \exists! y \in [1, 2], x = y - Tz_1(y)\},$$

$$I_2 = \{x \in \mathbb{R}, \exists(y_1, y_2), y_1 \ne y_2, x = y_1 - Tz_1(y_1) = y_2 - Tz_1(y_2)\}.$$

We have

ASSERTION 2.4. I_1 *is a dense union set of open intervals.* I_2 *is at most countable.*

Indeed, for any $x \in I_2$ there exists a unique maximal interval non reduced to a point, $[y_{1,x}, y_{2,x}]$ such that for any $y \in [y_{1,x}, y_{2,x}]$ there holds $x = y - Tz_1(y)$. Thus since for two different such x the corresponding intervals are distinct, I_2 is at most countable. It is then clear that I_1 is of the desired form.

Now let $x \in I_1$. Assume that there exists a decreasing sequence of nonnegative numbers (t_n) such that $t_n \to 0$ and such that $(u(t_n, x))$ does not tend to $u(0, x)$. Then we can determine a sequence (y_n) and y in $[1, 2]$ such that $u(t_n, x) = u(T, y_n)$ and $u(0, x) = u(T, y)$, but it is also clear that we have $y_n \to y$ (eventually up to a subsequence) so that we have a contradiction.

It is also clear that $u(t, x) \to u(T, x) = z_1(x)$ as $t \to T$. Thus we have a weak solution of the Burger equation satisfying the desired properties.

Let us remark that the initial data is of bounded variation. Indeed there is a dense union of open intervals on which $u(0, .)$ is of bounded variation. Let (x_n) denote the sequence of the points of I_2. For each x_n we can determine two elements of $[1, 2]$, say y_n^1 and y_n^2 such that z_1 is affine on the interval $[y_n^1, y_n^2]$ with slope $\frac{1}{T}$, this interval being as large as possible. Now we have $|u(0, x_n-) - u(0, x_n+)| = |z_1(y_n^1) - z_1(y_n^2)|$ and thus we see that

$$\sum_{n \in \mathbb{N}} |u(0, x_n-) - u(0, x_n+)| < +\infty,$$

proving that $u(0, .)$ is of bounded variation. Thus we see that in the case of a continuous final state, we can obtain a solution to our control problem. This concludes the proof of Proposition 2.3. $\square$

2.2.2. CASE OF MORE GENERAL POSITIVE FINAL STATES. Now we consider the case when z_1 is no longer continuous. We have.

PROPOSITION 2.5. *Assume that z_1 satisfies the conditions of theorem 2.1 and is $BV([1, 2])$ then there exists u satisfying the control problem (1.2), (1.3) et (1.4), with $z_{0/[1,2]} = 0$.*

Proof: Let us denote $I_n = (\alpha_n, \beta_n)$ an open interval on which z_1 is continuous. By means of the trick used in the proof of the previous proposition, we immediately see that there exists an interval (a_n, b_n) with $a_n = \alpha_n - Tz_1(\alpha_n+)$, $b_n = \beta_n - Tz_1(\beta_n-)$ such that we might define an initial data on (a_n, b_n) extended with the same preceding manner producing the desired value z_1 on the interval I_n. But here we need to patch all such initial datas defined on such intervals. We use the fact that z_1 is of entropic shape that is if y is a discontinuity point of z_1 then there holds

$$z_1(y-) > z_1(y+).$$

So if we have a singularity y, we determine an interval

$$[y - Tz_1(y-), y - Tz_1(y+)]$$

on which we impose $u(0, .)$ to be an affine function joining the point $(y - Tz_1(y-), z_1(y-))$ to the point $(y - Tz_1(y+), z_1(y+))$. Now outside the interval $[1 - Tz_1(1+), 2 - Tz_1(2-)]$ we define $u(0, .)$ just as in the case when z_1 is continuous.

Finally the same technique as before shows that u is a weak solution to the (1.2) and satisfies both conditions (1.2), (1.3) and (1.4).$\square$

2.2.3. CASE OF NONNEGATIVE FINAL STATES. In this subsection we give the way when z_1 is equal to 0 for $x > x_0$, $x_0 \in (1, 2)$ and satisfies also the other conditions.

This is the object of:

PROPOSITION 2.6. *Assume that there exists $x_0 \in (1, 2)$ such that $\forall x > x_0$ we have $z_1(x) = 0$ then if z_1 satisfies the assumptions of Theorem 2.1, there exists u satisfying (1.2) and (1.3) and (1.5).*

Proof: For $p \in (1, +\infty)$, $\delta \in (0, \frac{1}{2})$ and $\rho \in (0, \frac{1}{2})$, let $b_{\delta,\rho,p}$ be as follows: $b_{\delta,\rho,p}$ is the unique entropic solution of (1.2), with the following initial data

$$b_{\delta,\rho,p}(0, x) = \max\left(0, \tfrac{x}{\rho}\right), \quad x \leq \rho, \tag{2.6}$$

$$b_{\delta,\rho,p}(0, x) = 1, \qquad \rho \leq x \leq 1 - \delta, \tag{2.7}$$

$$b_{\delta,\rho,p}(0, x) = -p, \qquad 1 - \delta < x \leq 1, \tag{2.8}$$

$$b_{\delta,\rho,p}(x, 0) = 0, \qquad \text{else.} \tag{2.9}$$

The classical theory of entropic solutions shows that there exist three curves, say,

$$t \to x_1(t), \, x_2(t), \, x_3(t),$$

such that for a short time $t \leq t_1$ one has (the subscripts are temporarily omitted in $b_{\delta,\rho,p}$)

$$x_1(t) = t + \rho,$$
$$x_2(t) = \tfrac{1-p}{2}t + 1 - \delta,$$
$$x_3(t) = -pt + 1,$$
$$b(x, t) = \tfrac{x}{t+\rho}, \, 0 \leq x \leq x_1(t), t \leq t_1,$$
$$b(x, t) = 1, \, x_1(t) \leq x \leq x_2(t), t \leq t_1,$$
$$b(x, t) = -p, \, x_2(t) \leq x \leq x_3(t),$$
$$b(x, t) = \tfrac{x-1}{t}, \, x_3(t) \leq x \leq 1.$$

In fact this holds as long as $x_1(t) < x_2(t) < x_3(t)$.

One has $x_1(t) = x_2(t) \iff t = \dfrac{2(1 - \delta - \rho)}{p + 1}$ and $x_2(t) = x_3(t) \iff t = \dfrac{2\delta}{p + 1}$. Let us take $\delta < \dfrac{1 - \rho}{2}$, this yields $t_1 = 2\frac{\delta}{p+1}$.

For $t > t_1$, one takes $x_2(t) = x_3(t)$, and $x_2(t)$ is computed for $t > t_1$ with the conservation of the mass

$$\int_0^{t+\rho} \frac{x}{t + \rho}dx + \int_{t+\rho}^{x_2(t)} 1dx + \int_{x_2(t)}^1 \frac{x - 1}{t}dx = 1 - \delta(p + 1) - \frac{\rho}{2}.$$

Thus

$$x_2(t) = t + 1 - \sqrt{2\delta t(p + 1)}$$

as long as $x_2(t) > x_1(t) = t + \rho$. This holds for $t < \dfrac{(1-\rho)^2}{2\delta(p+1)} = t_2$. Let us point out that $t_2 > t_1$ since

$$(1-\rho) > 2\delta.$$

For $t > t_2$, one settles $x_2(t) = x_1(t)$ and x_1 is given by

$$\int_0^{x_1(t)} \frac{x}{t+\rho}dx + \int_{x_1(t)}^1 \frac{x-1}{t}dx = 1 - \delta(p+1) - \frac{\rho}{2},$$

i.e.

$$x_1(t) = \frac{t + \rho - \sqrt{t(t+\rho)}\sqrt{(1-\rho)^2 + 2\delta\rho(p+1)}}{\rho}.$$

Of course this is true as long as $x_1(t) \leq 1$. For $x_1(t) \geq 1$ one has, using again the conservation of mass,

$$\int_0^{x_1(t)} \frac{x}{t+\rho}dx = 1 - \delta(p+1) - \frac{\rho}{2},$$

i.e.

$$x_1(t) = \sqrt{2(t+\rho)(1 - \delta(p+1) - \frac{\rho}{2})}.$$

Let us check, for sake of completeness, that the equality

$$\frac{t + \rho - \sqrt{t(t+\rho)}\sqrt{(1-\rho)^2 + 2\delta\rho(p+1)}}{\rho} = \sqrt{2(t+\rho)(1 - \delta(p+1) - \frac{\rho}{2})}$$

is possible. This is in fact true for

$$t_3 := t = \frac{(1-\rho)^2 + 2\delta\rho(p+1)}{2 - \rho - 2\delta(p+1)}.$$

If one fixes T, so that $x_1(T) = x_0$, we need

$$(1 - \delta(p+1) - \frac{\rho}{2}) = \frac{x_0^2}{2(T+\rho)}, \text{ or else } \delta(p+1) = 1 - \frac{\rho}{2} - \frac{x_0^2}{2(T+\rho)}.$$

This will be possible if $T > t_3$.

Now the same method than in the proof of Proposition 2.3 where one replaces $a_{1,\rho}$ by $b_{\rho,\delta,p}$ and choosing α such that

$$0 < z_1(x) < \alpha\frac{\sqrt{2(1 - \delta(p+1) - \frac{\rho}{2})}}{\sqrt{T+\rho}},$$

gives z_1 satisfying $z_1(x) > \frac{x-1}{T}$, if $x \leq x_0$ and $z_1(x) = 0$ if $x > x_0$.□

2.2.4. GENERAL FINAL STATE UNDER THE ASSUMPTIONS OF THEOREM 2.1.
Now we examine the case when z_1 can take nonpositive value.

The statement of the main result says that in this case there exists $x_0 \in (1, 2]$ such that, either $x < x_0$ and $z_1(x) > \frac{x-1}{T}$, either $x > x_0$ and $z_1(x) < \frac{x-2}{T}$, moreover equality possibly holds when $x = x_0$.

We have then:

PROPOSITION 2.7 (Pf. of Theorem 2.1 completed). *With the above assumption, one can find u such that the problem (1.2) (1.3) (1.4) has a solution with vanishing initial data on $[1, 2]$.*

Proof: For $x \leq 1$, let us define $b_{\delta,\rho,p}(0,x)$ as before and for $x > 2$ $b_{\delta',\rho',p'}(0,x) = -b_{\delta',\rho,p'}(0, \frac{3}{2} - x)$. Here p' and δ' are chosen so that x_0 is replaced by $3 - x_0$. Now we are able by the preceding methods to construct an initial data with vanishing value on $[1,2]$ coinciding with z_1 on $[1,2]$ (in fact at least, almost everywhere). □

2.3. CASE OF INITIAL DATA PRESCRIBED ON $[1,2]$

Now we are directly involved in the proof of Theorem 1.1.
We thus assume that $u(0,.)$ restricted to $[1,2]$ has to be z_0. We fix real nonnegative numbers, υ, ζ and denote $a_{2,\upsilon,\zeta}$ the following data

$$a_{2,\upsilon,\zeta}(x) = \zeta, \text{ if } 1 - \upsilon < x < 1.$$

We then consider u^1 the solution of (1.2) with initial value

$$u^1(0,x) = a_{2,\upsilon,\zeta}(x) \text{ if } x \in (1 - \upsilon, 1)$$
$$u^1(0,x) = z_0(x) \text{ if } x \in [1,2]$$
$$u^1(0,x) = 0 \text{ else.}$$

We will also denote $a_{2,\upsilon,\zeta}$ the solution in the entropic class of (1.2) coinciding with its definition at $t = 0$. We then have:

ASSERTION 2.8. *There exists a time T_1 and ζ such that the solution of (1.2) u^1 is $x \to \frac{x-1+\upsilon}{t}$ on $[1,2]$ for $t > T_1$.*

Proof: We assume a priori that $\zeta > |z_0|_\infty$. Let $\mathfrak{a} = ||z_{0/[1,2]}||_\infty$. It is then clear that u^1 is singular at least along a curve, which is Lipschitz denoted $t \to \xi(t)$ and satisfying

$$\xi(0) = 1$$

and

$$\frac{d}{dt}\xi(t) \geq \frac{1}{2}\left(a_{2,\upsilon,\zeta}(t, \xi(t)+) - \mathfrak{a}\right), \text{ for a.e } t.$$

Let us remark that this comes from the conservation of the mass of the entropic solutions of (1.2) and can also be interpreted by the characteristic method when the solutions are replaced by their regularization in the BV space.

It will be clear that the claim will be true if one can show that for large enough ζ the curve $t \to \xi(t)$ intersects the curve $t \to \mathfrak{a}t + 2$ in the region $x > 1$.

For a short time one has

$$\text{for a.e. } t > 0, \frac{d}{dt}\xi(t) \geq \frac{1}{2}(\zeta - \mathfrak{a})t.$$

Thus, for some short time,

$$\text{for a.e. } t > 0 \, \xi(t) \geq 1 + \frac{1}{2}(p - \mathfrak{a})t.$$

This is available at least up until the following equality holds

$$\zeta t + 1 - \upsilon = 1 + \frac{1}{2}(\zeta - \mathfrak{a})t.$$

This gives $t = 2\frac{v}{\zeta+\mathfrak{a}}$ for which $\xi\left(2\frac{v}{\zeta+\mathfrak{a}}\right) \geq 1 + \mu\left(\frac{\zeta-\mathfrak{a}}{\zeta+\mathfrak{a}}\right)$. We thus can only insure afterwards that $\frac{d}{dt}\xi(t) \geq \frac{1}{2}\left(\frac{\xi+v-1}{t} - \mathfrak{a}\right)$, which gives $\xi(t) \geq C\sqrt{t} - v + 1 - \frac{\mathfrak{a}}{2}t$, with

$$C\sqrt{2\frac{v}{\zeta+\mathfrak{a}}} - v + 1 - \frac{\mathfrak{a}v}{\zeta+\mathfrak{a}} \geq 1 + v\frac{\zeta-\mathfrak{a}}{\zeta+\mathfrak{a}}.$$

We thus obtain

$$C \geq \sqrt{\frac{\zeta+\mathfrak{a}}{2v}}\left(v\frac{2\zeta+\mathfrak{a}}{\zeta+\mathfrak{a}}\right).$$

Let us point out that as $p \to +\infty$ one has $\xi(t) \to +\infty$. Thus if $\xi(t) > \mathfrak{a}t + 2$, for $t > \frac{2v}{\zeta+\mathfrak{a}}$ we are done.

Let us point out (it will become clearer in the following) that one has to take ζ as small as possible to obtain an optimal time of control. Let us also point out that this trick becomes unnecessary in the following if one has $z_0 \geq 0$ on $[1,2]$. Now we choose $\eta > 0$ and assume than T_1 in the preceding claim is chosen so that $|u^1|_{L^\infty([1,2])}(t) < \eta$ if $t > T_1$. This always is possible because of the asymptotic properties of solutions of (1.2) with finite mass.

We are now ready to give the final proof to theorem 1.1.
We first of all determine T_1 as before. We modify $u^1(T_1,.)$ by imposing that it is 0 outside the interval $[1-v, 2+v]$. For the sake of simplicity let us assume that z_1 satisfy the condition of Theorem 2.1. We may modify $b_{\rho,\delta,p}$ in the following way. Let $\mathbb{T}$ be the increasing affine transformation of $[\delta, 1]$ onto $[\delta, 1-v]$. Then we put $b^{\#}_{\delta,\rho,p}(T_1, x) = b_{\delta,\rho,p}(0, \mathbb{T}^{-1}(x))$ and choose p as in the proof of theorem 2.1, such that the singularity line issuing from $t = T_1$ and $x = 1 - v$ is exactly some x_0 at the time $T_1 + T$. We then determine $u(T_1,.)$ outside $[1-v, 2]$ such that at the time T the solution u coinciding with $u(T_1,.)$ at the time T_1 is z_1 on $(1, x_0)$ at the time $T + T_1$. This is in fact possible by increasing shortly the value α determined in the proof of theorem 2.1.

Now we glue $u(T_1,.)$ with $u^1(T_1,.)$ and consider the solution $u^{\#}$ of (1.2) with this initial value at the time $t = T_1$.

We have:

ASSERTION 2.9. *The solution $u^{\#}$ coincides with z_1 on $(1, x_0)$ at the time $T + T_1$ Moreover one has $|u^{\#}(T + T_1, .) - z_1(.)|_{L^1((1,2))} < \eta(1 + T|z_1|_{L^\infty})$.*

The proof is immediate. Indeed one has only to see that the singularity line issuing from the point $(T_1, 1 - v)$ differs from the one if the initial value on $[1-v, 2]$ was zero at the time $T + T_1$ of not more than ηT. Since the final state on this interval is not larger than $|z_1|_{L^\infty}$ we have the claim. Then this proves theorem 1.1, in the case when z_1 is nonnegative, by taking v small enough.

Now we sketch the case when z_1 can possibly take negative values.
As in the preceding case one glues $u^1(t_1,.)$ with the initial value giving z_1 outside $[1-v, 2+v]$. Following the trajectory of the singularity line issuing from the point shows that it also differs from the one with vanishing initial value at the time $T + T_1$ of ηT we thus have such an equality as in the preceding claim. Here, however we can not insure the coincidence of

the final state with z_1 on $(x_0, 2)$ but only on $(x_0 + \eta T, 2)$ thus we require also that η satisfies $\eta(T) < 2 - x_0$. We have then the desired result. This concludes the proof of the main result and the paper.

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