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GENERICITY OF OBSERVABILITY OF CONTROL-AFFINE SYSTEMS

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Abstract. For smooth or real-analytic single-input, control-affine, nonlinear systems, with at least two ouputs, observability for any input of a given class is generic. This class can be either the class of inputs bounded with their derivatives up to a certain order, or the class of polynomial inputs with bounded degree.

1. Introduction

In this paper we deal with genericity of observability of control-affine systems of the form:

$$\Sigma = \begin{cases} \dot{x} = f(x) + u_0 g(x) & x \in X, \quad u_0 \in \mathbb{R} \\ y = h(x) & y \in \mathbb{R}^p, \end{cases}$$

where the state space X is compact and d-dimensional.

Such a single-input p-outputs system is said to be observable for an input $t \mapsto u_0(t)$ if any two initial states are distinguished by the outputs (corresponding to a given input) on any time interval, i.e. if:

$$\forall x_1 \neq x_2 \in X, \quad \forall T > 0, \quad \exists \tau \in [0, T[\text{ s.t.}$$
$$h[x_1(\tau)] \neq h[x_2(\tau)],$$

where $x_i(t)$ is the solution of the dynamical system for the input $t \mapsto u_0(t)$ and the initial condition $x_i(0) = x_i$.

This definition of observability is not the standard one that was proposed by Hermann and Krener in [10]. But our definition is justified by the genericity results at least when there are more outputs than inputs. It is used in [7],[13] and [16].

In fact, we deal with the stronger property of differential observability: *a C^k-input being given, the system Σ is said to be differentially observable for this input if, for an integer $k' \leq k$, the mapping which associates to the state the outputs together with their k' first derivatives is an embedding.* This definition permits the use of transversality tools, that are involved in order to prove the following results:

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Theorem 3.2. Fix $\mathbf{p} \geq \mathbf{2}$*. The set of systems with* p *outputs which are differentially observable for any* C^{2d}*-input is residual in the set of all* C^r*-systems, for* $r \in \mathbb{N} \cup \{\infty, \omega\}$ *large enough.*

Theorem 3.3. A bound $B > 0$ *being given, the set of systems that are differentially observable for any* C^{2d}*-input* $t \mapsto u_0(t)$ *such that* $\left|\frac{d^i u_0}{dt^i}(0)\right| \leq B$ *is open and dense in the set of all* C^r*-systems, for* $r \in \mathbb{N} \cup \{\infty, \omega\}$ *large enough, and for* $\mathbf{p} \geq \mathbf{2}$*.*

Analogous statements are proved in [7], [8] for general nonlinear systems. Notice, however, that since the interior of the subset of control-affine systems is empty in the set of all non-linear systems, our results cannot be deduced from those of [7], [8].

In the statement of the above theorems, one can notice the requirement $p \geq 2$. This is in fact the general requirement that the number of ouputs is strictly larger than the number of inputs, which is also made in [7], [8]. If the number of ouputs is equal to the number of inputs the theorems are false as it is shown in [6].

In section 5 we prove the following result: *Theorem 5.1. The set of systems that are differentially observable for any polynomial input whose degree is smaller than a given bound contains an open and dense subset of the set of all* C^r*-systems, for* $r \in \mathbb{N} \cup \{\infty, \omega\}$ *large enough, and for* $\mathbf{p} \geq \mathbf{2}$*.*

The proof of this theorem is based on a method of compactification of the inputs, that was used for the first time by the second author in the case of general non-linear systems (see [14], [15]).

One of the main interests of differential observability for all inputs whose derivatives belong to a bounded set is the possibility of constructing non-linear observers. We do not give this construction here, because it follows the same line as the construction made for general non linear-systems and can be found in [6] or [7]. The computations are particularly detailed in [16] where it is moreover shown that differential observability is a sufficient but not necessary condition for the possibility of the constructing of nonlinear observers.

To finish we want to recall another result proved in the uppermentionned paper [7], and that is immediatly available for our particular systems:

Theorem 5.1 of [7]. For an analytic system Σ, the following properties are equivalent:

- Σ is observable for any C^ω-input
- Σ is observable for any L^∞-input.

For the sake of clarity, we have chosen to deal with single-input systems only ; but our results can be rather easily extended to multi-input control-affine systems assuming that the number of ouputs is strictly larger than the number of inputs.

For systems whose state manifold is not compact, similar results are certainly true in the Whitney topology. Nevertheless, the observer design problem is well posed only if the state space is compact or when restricting to a compact subset (see for instance [16]). For this reason we consider the compact case only.

2. Definitions and notations

Let X be a d-dimensional, compact, connected C^{∞}-manifold.

Let $h : X \longrightarrow \mathbb{R}^p$ be a C^r-mapping and f, g two C^r-vector-fields on X, where $r \in \mathbb{N} \cup \{\infty\}$. We consider a single-input, p-outputs system $\Sigma = (h, f, g)$ on X defined by:

$$\Sigma = (h, f, g) = \begin{cases} \dot{x} = f(x) + u_0 g(x) & x \in X, \quad u_0 \in \mathbb{R} \\ y = h(x) & y \in \mathbb{R}^p. \end{cases} \tag{2.1}$$

The set $C^r(X, \mathbb{R}^p)$ (resp. the set $\Gamma^r(TX)$ of C^r-vector-fields on X) will be denoted by H^r (resp. Γ^r) and endowed with the C^r-topology. S^r will stand for the set $H^r \times (\Gamma^r)^2$ of C^r-systems. For $r < \infty$, H^r, $\Gamma^r$and S^r have the natural structure of Banach spaces (see [1]).

In what follows, the integer r will be supposed to be large enough in order to perform all considered operations.

The k^{th} dynamical extension of the vector-field $f + u_0 g$ is the vector-field F^k on $X \times \mathbb{R}^k$ controled by u_k and defined by:

$$\begin{aligned} &F^k(x, u, u_k) = f(x) + u_0 g(x) + \textstyle\sum_{i=0}^{k-1} u_{i+1} \frac{\partial}{\partial u_i}, \\ &\text{where} \qquad u = (u_0, u_1, ..., u_{k-1}). \end{aligned}$$

A system $\Sigma = (h, f, g) \in S^r$ being given and for $1 \leq k \leq r+1$ we define the mapping $R_{k,\Sigma}$ by:

$$\begin{aligned} R_{k,\Sigma} &: X \times \mathbb{R}^k \longrightarrow \mathbb{R}^{kp} \\ (x, u) &\mapsto \Big(h(x), L_{F^k} h(x, u), ..., L_{F^k}^{k-1} h(x, u)\Big). \end{aligned}$$

Remark 2.1.

1. $R_{k,\Sigma}$ depends on $u = (u_0, u_1, ..., u_{k-1})$ but not on u_k.

2. Let $t \longmapsto u_0(t)$ be an input which verifies $\dfrac{d^i u_0}{dt^i}(0) = u_i$ for $i = 0 \cdots k-1$. It is clear that

$$R_{k,\Sigma}(x, u) = \left(h(\phi(t)), \frac{d}{dt}[h(\phi(t))], ..., \frac{d^{k-1}}{dt^{k-1}}[h(\phi(t))]\right)_{t=0},$$

where $t \longmapsto \phi(t)$ is the solution of $\dot{x} = f(x) + u_0(t) g(x)$ passing by the initial condition $\phi(0) = x$.

We will also use the suspension mapping $SR_{k,\Sigma}$, which is merely:

$$SR_{k,\Sigma}(x, u) = (R_{k,\Sigma}(x, u), u) \in \mathbb{R}^{kp} \times \mathbb{R}^k$$

Since X is a compact manifold, $SR_{k,\Sigma}$ is always a proper mapping. We are interested in systems Σ for which $SR_{k,\Sigma}$ is an embedding, at least when restricted to $X \times U$ where U is a subset of $\mathbb{R}^k$, for the following reasons:

1. If the restriction of $SR_{k,\Sigma}$ to $X \times U$, is one-to-one, then, according to remark 2.1(2), Σ is observable for any input whose $(k-1)$ first derivatives belong to U.
2. The fact that $SR_{k,\Sigma}$ is additionally an embedding is a sufficient condition to construct observers.

3. Transversality results

3.1. Statement of the results

The following theorems holds only for $\mathbf{p} \geq \mathbf{2}$, i.e. if *the number of ouputs is strictly larger than the number of inputs.*

Theorem 3.1. *The set of systems Σ such that $SR_{2d,\Sigma}$ is an immersion contains an open and dense subset of S^r (for $r \in \mathbb{N} \cup \{\infty\}$, large enough).*

Theorem 3.2. *The set of systems Σ such that $SR_{2d+1,\Sigma}$ is an embedding is residual hence dense in S^r (for $r \in \mathbb{N} \cup \{\infty\}$, large enough).*

Theorem 3.3. *Let $B > 0$ and $I_B = [-B, B]$. The set of systems Σ such that the restriction of $SR_{2d+1,\Sigma}$ to $X \times (I_B)^{2d+1}$ is an embedding is open and dense in S^r (for $r \in \mathbb{N} \cup \{\infty\}$, large enough).*

3.2. Some remarks

1. We recall the following definition: a subbundle B of a vector bundle E is called partially semi-algebraic (PSA) if its typical fiber is a semi-algebraic subset of the typical fiber of E (see [4],[1],[17],[11]). Standard properties of PSA subbundles are used in the following proofs.
2. We denote by $\oplus$ the fiber product over X. Thus the bundle of k-jets of systems is the bundle:
$$J^k S = J^k H \oplus J^k \Gamma \oplus J^k \Gamma.$$
3. The mapping $R_{k,\Sigma}$ (resp. $T_X R_{k,\Sigma}$) depends only on the $(k-1)$-jet (resp. the k-jet) of Σ, therefore it induces a mapping on $J^k S \times \mathbb{R}^k$ that is also denoted by $R_{k,\Sigma}$ (resp. $T_X R_{k,\Sigma}$).
4. If a point (x, u_0) is a singularity of the system Σ, i.e., if $f(x) + u_0 g(x) = 0$, we will sometimes denote by (C, A) the linearized system $(dh(x), T_X(f(x) + u_0 g(x)))$.

3.3. Proof of the theorem 3.1

Let $\hat{B}(k)$ denote the subset of $J^k S \times \mathbb{R}^k$ of all $\left(j_x^k \Sigma, u\right)$ such that *rank* $[T_X R_{k,\Sigma}(x, u)] < d$. We consider the following partition of $\hat{B}(k)$:

$$\hat{B}(k) = \hat{B}_1(k) \cup \hat{B}_2(k) \cup \hat{B}_3(k),$$

where:

- $\left(j_x^k \Sigma, u\right) \in \hat{B}_1(k) \Longleftrightarrow f(x) + u_0 g(x) \neq 0$
- $\left(j_x^k \Sigma, u\right) \in \hat{B}_2(k) \Longleftrightarrow \begin{cases} f(x) + u_0 g(x) = 0 \\ \text{and the linearized system} \\ (dh(x), T_X(f(x) + u_0 g(x))) \text{ is not observable.} \end{cases}$
- $\left(j_x^k \Sigma, u\right) \in \hat{B}_3(k) \Longleftrightarrow \begin{cases} f(x) + u_0 g(x) = 0 \\ \text{and the linearized system} \\ (dh(x), T_X(f(x) + u_0 g(x))) \text{ is observable.} \end{cases}$

$\hat{B}(k)$, $\hat{B}_1(k)$, $\hat{B}_2(k)$, $\hat{B}_3(k)$ and their respective projections $B(k)$, $B_1(k)$, $B_2(k)$, $B_3(k)$ into $J^k S$ are PSA.

3.3.1. $\mathrm{CODIM}(B_1(k)) = k(p-1)+1-d$. Let

$$O_1 = \{\left(j_x^k\Sigma, u\right) \in J^kS \times \mathbb{R}^k, \quad f(x) + u_0 g(x) \neq 0\}.$$

Notice that O_1 is an open subset of $J^kS \times \mathbb{R}^k$. Let μ_1 be the mapping:

$$\begin{aligned} \mu_1 : \qquad O_1 &\longrightarrow (T^*X)^{\oplus kp} \\ \left(j_x^k\Sigma, u\right) &\longmapsto T_X R_{k,\Sigma}(x,u). \end{aligned}$$

The mapping μ_1 is a submersion because, by Lemma 6.2, the linear mapping $j_x^k h \mapsto T_X R_{k,\Sigma}(x,u)$ is surjective as soon as $f(x) + u_0 g(x) \neq 0$. Notice that $\hat{B}_1(k)$ is the pullback by μ_1 of the set of elements of $(T^*X)^{\oplus kp}$ whose rank is less than d. By the product of coranks theorem (see [9]), the codimension of this set is $(d-(d-1))(kp-(d-1)) = kp+1-d$. Thus the codimension of $\hat{B}_1(k)$ in $J^kS \times \mathbb{R}^k$ is also $kp+1-d$, and the codimension of $B_1(k)$ in J^kS is $kp+1-d-k = k(p-1)+1-d$.

3.3.2. $\mathrm{CODIM}\left(B_2(k), J^kS\right) = d+p-1$. Let $x \in X$ and let J_x^kS be the (typical) fiber of J^kS over x. Let V be the subset of $J_x^kS \times \mathbb{R}^k$ defined by $\left(j_x^k\Sigma, u\right) \in V \iff f(x) + u_0 g(x) = 0$. The set V has codimension d in $J_x^kS \times \mathbb{R}^k$. Let μ_2 be the mapping:

$$\begin{aligned} \mu_2 : \qquad V &\longrightarrow (T_x^*X)^p \times End(T_xX) \\ \left(j_x^k\Sigma, u\right) &\longmapsto (dh(x), T_X(f(x) + u_0 g(x))). \end{aligned}$$

We see that μ_2 is clearly surjective and the typical fiber of $\hat{B}_2(k)$ is the pullback by μ_2 of the set of unobservable couples (C, A) of $(T_x^*X)^p \times End(T_xX)$, whose codimension, according to lemma 6.1, is p. Therefore $\mathrm{codim}\left(\hat{B}_2(k), J^kS \times \mathbb{R}^k\right) = d+p$. But $\hat{B}_2(k)$ does not depend on the variables $u_1, ..., u_{k-1}$ and $\mathrm{codim}\left(B_2(k), J^kS\right) = d+p-1$.

3.3.3. $\mathrm{CODIM}\left(B_3(k), J^kS\right) \geq \min\{k(p-1)+1; 2d\}$. First let us define by O_3 the subset of J^kS of elements that verify $(f(x), g(x)) \neq (0,0)$. The set O_3 is clearly an open, dense and PSA subbundle of J^kS. Let $\hat{B}'_3(k) = \hat{B}_3(k) \cap \left(O_3 \times \mathbb{R}^k\right)$ and $B'_3(k) = B_3(k) \cap O_3$. Observe that $\left(j_x^k\Sigma, u\right) \in \hat{B}'_3(k)$ implies $g(x) \neq 0$.

For $1 \leq \rho \leq d-1$ let us define $\hat{B}_6(k,\rho)$ by :

$$\hat{B}_6(k,\rho) = \left\{\left(j_x^k\Sigma, u\right) \in \hat{B}'_3(k); \quad u_1 = u_2 = \cdots = u_{\rho-1} = 0 \text{ and } u_\rho \neq 0\right\}$$

We may notice that $u_1 = u_2 = \cdots = u_{d-1} = 0$ implies $\left(j_x^k\Sigma, u\right) \notin \hat{B}'_3(k)$ because of the observability of the linearized system, and conclude that:

$$\hat{B}'_3(k) = \bigcup_{1 \leq \rho \leq d-1} \hat{B}_6(k,\rho).$$

For $1 \leq \rho \leq d-1$, both $\hat{B}_6(k,\rho)$ and its projection $B_6(k,\rho)$ into O_3 are PSA.

Let $G_\rho \subset (O_3 \oplus TX) \times \mathbb{R}^k$ be the set of elements $\left(j_x^k\Sigma, \Lambda, u\right)$ such that:

1. $f(x) + u_0 g(x) = 0$
2. $u_1 = u_2 = \cdots = u_{\rho-1} = 0$
3. the vectors $A^i\Lambda$, $0 \leq i \leq \rho$ are linearly independant.

G_ρ is a subbundle and a submanifold of $(O_3 \oplus TX) \times \mathbb{R}^k$; its codimension equals $d + \rho - 1$. Let μ_3 be the mapping:

$$\begin{aligned} \mu_3 : \qquad & G_\rho \longrightarrow \mathbb{R}^{kp} \\ & \left(j_x^k \Sigma, \Lambda, u\right) \longmapsto T_X R_{k,\Sigma}(x,u).\Lambda. \end{aligned}$$

By Lemma 6.3 of the appendix, μ_3 is a submersion and $\operatorname{codim}\left(\mu_3^{-1}(0), G_\rho\right) = kp$. Therefore, G_ρ being conical w.r.t. Λ, the codimension of the projection of $\mu_3^{-1}(0)$ in $O_3 \times \mathbb{R}^k$ is $kp + (d + \rho - 1) - (d-1) = kp + \rho$.

Let $\left(j_x^k \Sigma, u\right) \in \hat{B}_6(k,\rho)$. There exists $\Lambda \in T_x X$, $\Lambda \neq 0$, such that $T_X R_{k,\Sigma}(x,u).\Lambda = 0$. By lemma 6.3 of the appendix, $CA^i\Lambda = 0$ for $i = 0, ..., \rho - 1$. The pair (C, A) is observable and this implies that the vectors $A^i\Lambda,\ 0 \leq i \leq \rho$ are linearly independant. As a consequence $\left(j_x^k\Sigma, u\right)$ belongs to the projection of $\mu_3^{-1}(0)$ into $O_3 \times \mathbb{R}^k$. Hence $\operatorname{codim}\left(\hat{B}_6(k,\rho), O_3 \times \mathbb{R}^k\right) \geq kp + \rho$ and $\operatorname{codim}(B_6(k,\rho), O_3) \geq kp + \rho - k = k(p-1) + \rho$. Therefore

$$\operatorname{Codim}\left(B_3'(k), J^k S\right) \geq \min_{1 \leq \rho \leq d-1} \{k(p-1) + \rho\} = k(p-1) + 1.$$

Now $B_3(k) \setminus B_3'(k)$ is included in $J^k S \setminus O_3$. The codimension of $B_3(k) \setminus B_3'(k)$ in $J^k S$ is therefore larger or equal to $2d$ and

$$\operatorname{Codim}\left(B_3(k), J^k S\right) \geq \min\{k(p-1) + 1; 2d\}.$$

3.3.4. End of the proof. It is now possible to compute the codimension of $B(k)$. We have

$$\begin{aligned} \operatorname{Codim}\left(B(k), J^k S\right) &\geq \min_{i=1,2,3} \operatorname{Codim}\left(B_i(k), J^k S\right) \\ &\geq \min\{k(p-1) + 1 - d, d + p - 1, k(p-1) + 1, 2d\}. \end{aligned}$$

For $p \geq 2$ and $k \geq 2d$, we have $\operatorname{Codim}\left(B(k), J^k S\right) \geq d+1$. The set $B(k)$ is PSA and the same is true for the closure $\overline{B(k)}$ of $B(k)$ in $J^k S$. Therefore $\overline{B(k)}$ is a finite union of submanifolds of $J^k S$ whose codimension is larger or equal to $d+1$. For $r \geq 2d+1$, the evaluation mapping: $(\Sigma, x) \mapsto j_x^k\Sigma$ is a C^1-mapping. Then by Abraham's theorems on transversality (see [1]), the set of systems Σ such that $j^k\Sigma$ avoids $\overline{B(k)}$ is open and dense in S^r. The fact that $j^k\Sigma$ avoids $B(k)$ implies obviously that $SR_{k,\Sigma}$ is an immersion. This proves Theorem 3.1 for $2d + 1 \leq r < \infty$. Since the C^∞-topology is the inductive limit of the C^r-topologies, the result holds for $r = \infty$.

3.4. Proof of the theorem 3.2

Let $\widetilde{X}$ stand for $X \times X \setminus \Delta X$ where ΔX is the diagonal of $X \times X$, and let $\widetilde{J^k S}$ be the restriction of $\left(J^k S\right)^{\oplus 2}$ to $\widetilde{X}$.

Let $\widehat{B}_4(k)$ be the subset of $\widetilde{J^k S} \times \mathbb{R}$ whose elements $\left(j_{x_1}^k \Sigma, j_{x_2}^k \Sigma, u_0\right)$ verify:

- $f(x_1) + u_0 g(x_1) = f(x_2) + u_0 g(x_2) = 0$
- $h(x_1) = h(x_2)$.

The codimension of $\widehat{B}_4(k)$ in $\widetilde{J^k S} \times \mathbb{R}$ equals $2d + p$. Since $\widehat{B}_4(k)$ is PSA, the codimension in $\widetilde{J^k S}$ of its projection $B_4(k)$ is $2d + p - 1$.

Let $\widehat{B}_5(k)$ be the PSA subbundle of $\widetilde{J^kS} \times \mathbb{R}^k$ whose elements $\left(j^k_{x_1}\Sigma, j^k_{x_2}\Sigma, u\right)$ verify:

- $f(x_1) + u_0 g(x_1) \neq 0$ or $f(x_2) + u_0 g(x_2) \neq 0$
- $R_{k,\Sigma}(x_1, u) = R_{k,\Sigma}(x_2, u)$.

Let O_5 be the open subset of $\widetilde{J^kS} \times \mathbb{R}^k$ consisting of elements $\left(j^k_{x_1}\Sigma, j^k_{x_2}\Sigma, u\right)$ such that $f(x_1) + u_0 g(x_1) \neq 0$ or $f(x_2) + u_0 g(x_2) \neq 0$. By Lemma 6.2 of the appendix the mapping:

$$\mu_5 : \begin{array}{c} O_5 \longrightarrow \mathbb{R}^{kp} \\ \left(j^k_{x_1}\Sigma, j^k_{x_2}\Sigma, u\right) \longmapsto R_{k,\Sigma}(x_2, u) - R_{k,\Sigma}(x_1, u) \end{array}$$

is a submersion. Clearly $\widehat{B}_5(k)$ is equal to $\mu_5^{-1}(0)$, and thus its codimension in O_5, and also in $\widetilde{J^kS} \times \mathbb{R}^k$, is kp. Finally, the codimension in $\widetilde{J^kS}$ of the projection $B_5(k)$ of $\widehat{B}_5(k)$ is $kp - k = k(p-1)$.

Let $B'(k) = B_4(k) \cup B_5(k)$. The codimension of $B'(k)$ is larger or equal to $2d+1$ as soon as $p \geq 2$ and $k \geq 2d+1$. By Abraham's theorem on density [1], the set of systems Σ such that $\left(j^k_{x_1}\Sigma, j^k_{x_2}\Sigma\right)$ avoids $B'(k)$ for any $(x_1, x_2) \in \widetilde{X}$ is residual (hence dense) in S^r for $r \geq 2d+2$. Thus Theorem 3.2 is proved for $2d+2 \leq r < \infty$. Since the C^∞-topology is the inductive limit of the C^r-topologies, the result holds for $r = \infty$.

To finish, let K be a compact subset of $\widetilde{X} = X \times X \setminus \Delta X$. Since $B'(k)$ is PSA, its closure $\overline{B'(k)}$ is again a PSA subbundle of $\widetilde{J^kS}$ of codimension larger or equal to $2d+1$ as soon as $p \geq 2$ and $k \geq 2d+1$. By Abraham's theorem on openness of non-intersection, we can state:

LEMMA 3.4. *The set of systems Σ such that the mapping:*

$$\begin{array}{l} K \times \mathbb{R}^k \longrightarrow \mathbb{R}^{kp} \\ (x_1, x_2, u) \longmapsto R_{k,\Sigma}(x_2, u) - R_{k,\Sigma}(x_1, u) \end{array}$$

does not vanish at any point is open and dense in S^r for $r \geq 2d+2$, $k \geq 2d+1$ and $p \geq 2$.

This lemma will be used in Section 5.

3.5. Proof of the theorem 3.3

The mapping:

$$\begin{array}{l} S^r \longrightarrow C^{r-2d}\left(X \times (I_B)^{2d}; \mathbb{R}^{(2d+1)p}\right) \\ \Sigma \longmapsto SR_{2d+1,\Sigma} \end{array}$$

being continuous, and the set of embeddings being open in $C^{r-2d}\left(X \times (I_B)^{2d}; \mathbb{R}^{(2d+1)p}\right)$ for $p \geq 2$ and $r > 2d$, the set of systems Σ such that the restriction of $SR_{2d+1,\Sigma}$ to $X \times (I_B)^{2d}$ is an embedding is open in S^r (for r large enough); it is dense by Theorem 3.2 and thus Theorem 3.3 is a straightforward consequence of the previous ones.

4. The analytic case

We consider in this section the set S^ω of real-analytic systems of the form (2.1):

$$\Sigma = (h, f, g) = \begin{cases} \dot{x} = f(x) + u_0 g(x) & x \in X, \quad u_0 \in \mathbb{R} \\ y = h(x) & y \in \mathbb{R}^p, \end{cases}$$

i.e., we assume that X, h, f and g are analytic.

For any system $\Sigma_0 = (h_0, f_0, g_0) \in S^\omega$, one can build Banach spaces of analytic functions (resp. analytic vector-fields), (resp. analytic systems) H, F, G and $S = H \times F \times G$, containing respectively h_0, f_0, g_0 and Σ_0 and such that:

1. the norm of H (resp. F, G, S) is stronger that the norm of H^r (resp. F^r, G^r, S^r) for $r \in \mathbb{N}$;
2. for any distinct points x_1 and x_2 in X, and for any integer k, there exists an element in H (resp. F, G, S) whose k-jets at x_1 and x_2 have arbitrary prescribed values.

The proof of the existence of such Banach spaces can be found in [7].

Now, replacing S^r by S, we can exactly repeat the statements and proofs of Theorems 3.1, 3.2 and 3.3. Thus these theorems remain true in the analytic case.

5. Observability for any polynomial input with bounded degree

In this section we examine systems that are observable for any polynomial input whose degree is smaller than an arbitrary but given bound. For the sake of clarity we consider the case $p = 2$. The main result is:

Theorem 5.1. *Let b be a positive integer. The set of systems that are observable for any polynomial input whose degree is smaller or equal to b contains an open and dense subset of S^r (for $r \in \mathbb{N} \cup \{\infty, \omega\}$, large enough).*

This theorem is a straightforward consequence of Theorem 5.2, whose statement needs the following definition according to the input compactification in Section 5.1.

Let $b \geq 2d$ be an integer. We denote by G_b the subset of $\mathbb{R}^K$, where $K = (d-1)(b+1) + 1$, whose elements $(u, \bar{u}) = (u_0, ..., u_b; u_{b+1}, ..., u_{K-1})$ verify:

- either

$$\max_{0 \leq i \leq b} |u_i| \leq 1$$

- or

$$|u_i| \leq [\lambda(u)]^{i+1} \text{ for } \quad b+1 \leq i \leq K-1$$
$$\text{where} \quad \lambda(u) = \max\nolimits_{0 \leq i \leq b} |u_i|^{\frac{1}{i+1}} > 1.$$

We can now state:

Theorem 5.2. *The set of systems Σ such that the restriction of $SR_{K,\Sigma}$ to $X \times G_b$ is an embedding contains an open and dense subset of S^r (for $r \in \mathbb{N} \cup \{\infty, \omega\}$, large enough).*

In order to prove this theorem, we will need two lemmas.

5.1. Inputs compactification

For $\Sigma = (h, f, g) \in S^r$ and $\lambda \in \mathbb{R}_+^* = \{x \in \mathbb{R} \mid x > 0\}$, Σ_λ will stand for the system $(h, \lambda f, g)$. For the same positive real λ:

- Δ_λ is the linear mapping from $\mathbb{R}^k$ into $\mathbb{R}^k$ defined by:

$$\Delta_\lambda u = \Delta_\lambda (u_0, u_1, ..., u_{k-1}) = \left(\lambda u_0, \lambda^2 u_1, ..., \lambda^k u_{k-1}\right)$$

- A_λ is the linear mapping from $\mathbb{R}^{2k}$ into $\mathbb{R}^{2k}$ defined by:

$$A_\lambda (y_1, z_1; y_2, z_2; ...; y_k, z_k) = \left(y_1, z_1; \lambda y_2, \lambda z_2; ...; \lambda^{k-1} y_k, \lambda^{k-1} z_k\right).$$

Lemma 5.3. *For any system $\Sigma \in S^r$, any $\lambda \in \mathbb{R}_+^*$, and any $k \geq 1$ we have*

$$R_{k,\Sigma_\lambda} (., \Delta_\lambda (.)) = A_\lambda R_{k,\Sigma} (., .).$$

Proof. Let $u = (u_0, u_1, ..., u_{k-1}) \in \mathbb{R}^k$ and let $t \longmapsto u(t)$ be a C^{k-1}-input that verifies:

$$\frac{d^i u}{dt^i}(0) = u_i \quad \text{for } i = 0, ..., k-1.$$

Let us denote by $t \longmapsto \phi(t)$ the solution of $\dot{x} = f(x) + u_0 g(x)$ passing by the initial condition $\phi(0) = x_0$ and corresponding to the input $t \longmapsto u(t)$. The trajectory ϕ verifies:

$$\frac{d}{dt}[\phi(\lambda t)] = \lambda [f \circ \phi(\lambda t) + u(\lambda t) g \circ \phi(\lambda t)].$$

Therefore $t \longmapsto \phi(\lambda t)$ is the solution of $\dot{x} = \lambda f(x) + u_0 g(x)$ for the initial condition x_0 and the input $t \longmapsto \lambda u(\lambda t)$, whose derivatives verify:

$$\left.\frac{d^i}{dt^i}[\lambda u(\lambda t)]\right|_{t=0} = \lambda^{i+1} u_i \quad \text{for } i = 0, ..., k-1.$$

Since:

$$\frac{d^i}{dt^i}[h \circ \phi(\lambda t)] \mid_{t=0} = \lambda^i \frac{d^i}{dt^i}(h \circ \phi)(0) \quad \text{for } i = 0, ..., k-1$$

and according to the remark 2.1(2), we have:

$$\begin{aligned} R_{k,\Sigma_\lambda}(x_0, \Delta_\lambda u) &= \left(h \circ \phi(\lambda t), \frac{d}{dt}[h \circ \phi(\lambda t)], ..., \frac{d^{k-1}}{dt^{k-1}}[h \circ \phi(\lambda t)]\right)_{t=0} \\ &= A_\lambda \left(h \circ \phi(t), \frac{d}{dt}[h \circ \phi(t)], ..., \frac{d^{k-1}}{dt^{k-1}}[h \circ \phi(t)]\right)_{t=0} \\ &= A_\lambda R_{k,\Sigma}(x_0, u). \end{aligned}$$

□

5.2. A transversality lemma

Lemma 5.4. *The set O of systems $\Sigma = (h, f, g)$, such that the mapping from X into $\mathbb{R}^{2d}$*

$$x \longmapsto \left(h, L_g h, ..., L_g^{d-1} h\right)(x)$$

is an immersion, is open and dense in S^r for $r \geq d$.

This lemma is a standard application of the transversality theorems to the observability theory of uncontrolled systems. Its proof can be found in [5] or in [13].

5.3. Proof of theorem 5.2

Proof. Let $\Re$ be the set of systems Σ that verify:

1. Σ belongs to the open set O defined in lemma 5.4.
2. $SR_{2d+1,\Sigma}$ is an embedding.
3. Σ belongs to the interior of the set of systems Σ' such that $SR_{2d,\Sigma'}$ is an immersion.

The set $\Re$ is residual in S^r. Let $\Sigma = (h, f, g) \in \Re$. There exists a neighbourhood of Σ in S^r whose elements Σ' are such that $SR_{2d,\Sigma'}$ is a proper immersion. In order to prove theorem 5.2, we have to prove the existence of a neighbourhood of Σ in S^r whose elements Σ' are such that the $SR_{2d,\Sigma'}$ are one-to-one.

Let $(\Sigma^n)_{n\geq 1}$ be a sequence of systems that converges to Σ in S^r. We have only to show that the restriction of SR_{K,Σ^n} to $X \times G_b$ is one-to-one for n large enough. Let us assume that for $n \geq 1$ one can find:

$$\begin{array}{l} x_1^n \neq x_2^n \in X \\ (u^n, \bar{u}^n) = \left(u_0^n, ..., u_b^n; u_{b+1}^n, ..., u_{K-1}^n\right) \in G_b, \end{array}$$

such that $R_{K,\Sigma^n}\left(x_1^n, (u^n, \bar{u}^n)\right) = R_{K,\Sigma^n}\left(x_2^n, (u^n, \bar{u}^n)\right)$.

X being compact, we can extract a subsequence such that:

$$\begin{array}{l} x_1^n \underset{n\to+\infty}{\longrightarrow} x_1 \\ x_2^n \underset{n\to+\infty}{\longrightarrow} x_2 \end{array}$$

According to lemma 3.4, the equality $x_1 = x_2$ holds: (x_1^n, x_2^n) converges to (x_1, x_2) in $X \times X$; the pair (x_1, x_2) cannot belong to any compact subset of $X \times X \setminus \Delta X$. Hence $(x_1, x_2) \in \Delta X$.

Let us denote by x the point $x = x_1 = x_2$ and choose a local chart centered at x. We can assume that x_1^n and x_2^n belong to this chart for all $n \geq 1$. Now we can define:

$$\Lambda_n = \frac{1}{\|x_2^n - x_1^n\|}\left(x_2^n - x_1^n\right), \qquad n \geq 1,$$

where $\|\cdot\|$ stands for the Euclidian norm in $\mathbb{R}^d$; clearly, Λ_n belongs to the sphere S^{d-1}.

Furthermore, theorem 3.3 implies:

$$\|u^n\|_\infty = Max\left\{|u_o^n|, ..., |u_b^n|\right\} \underset{n\to+\infty}{\longrightarrow} +\infty.$$

In order to obtain convergences, we use the compactification of Section 5.1:

$$\begin{array}{c} \lambda_n = \lambda\left(u^n\right) = \underset{0\leq i\leq b}{Max} |u_i^n|^{\frac{1}{i+1}} \\ v_i^n = \dfrac{u_i^n}{(\lambda_n)^{i+1}} \text{ for } i = 0, ..., K-1 \\ (v^n, \bar{v}^n) = \left(v_o^n, ..., v_b^n; v_{b+1}^n, ..., v_{K-1}^n\right) \\ = \Delta_{\lambda_n^{-1}}\left(u^n, \bar{u}^n\right). \end{array}$$

Taking another subsequence, we have the following convergences:

$$\begin{array}{l} \lambda_n \underset{n\to+\infty}{\longrightarrow} +\infty \\ x_1^n, x_2^n \underset{n\to+\infty}{\longrightarrow} (x,x) \in \Delta X \\ \Lambda_n \underset{n\to+\infty}{\longrightarrow} \Lambda \in S^{d-1} \\ (v^n,\bar{v}^n) \underset{n\to+\infty}{\longrightarrow} (v,\bar{v}) \\ \text{where } \|v\|_\infty = 1 \text{ and } \|\bar{v}\|_\infty \le 1 \end{array}$$

Consider now:

$$\delta^n = \frac{1}{\|x_2^n - x_1^n\|}\left[R_{K,\Sigma^n_{\lambda_n^{-1}}}\left(x_2^n,(v^n,\bar{v}^n)\right) - R_{K,\Sigma^n_{\lambda_n^{-1}}}\left(x_1^n,(v^n,\bar{v}^n)\right)\right].$$

According to Lemma 5.3:

$$\delta^n = \frac{1}{\|x_2^n - x_1^n\|}A_{\lambda_n^{-1}}\left[R_{K,\Sigma^n}\left(x_2^n,(u^n,\bar{u}^n)\right) - R_{K,\Sigma^n}\left(x_1^n,(u^n,\bar{u}^n)\right)\right].$$

By assumption $\delta^n = 0$, and hence the same is true for its limit that we denote by:

$$\lim_{n\to+\infty}\delta^n = (L_0, L_1, ..., L_{K-1}) \in \mathbb{R}^{2K}.$$

Let us denote by δ_k^n the k^{th} double coordinate of δ^n. Clearly δ_k^n is a polynomial in the $k-1$ first components of $(v^n,\bar{v}^n)$ and its coefficients are derivatives of h w.r.t. $\lambda_n^{-1}f_n$ and g_n. Since $\lambda_n^{-1} \underset{n\to+\infty}{\longrightarrow} 0$, all the monomials where derivation w.r.t. $\lambda_n^{-1}f_n$ arises tend to 0 when n tends to infinity. Therefore:

$$\begin{array}{l} L_0 = h^{(0)} \\ L_1 = v_0 h^{(1)} \\ L_2 = v_0^2 h^{(2)} + v_1 h^{(1)} \\ \cdots \\ L_k = P_{k,k}h^{(k)} + \cdots + P_{k,1}h^{(1)} \\ \cdots \end{array}$$

where

$$h^{(l)} = dL_g^l h\,(x)\cdot\Lambda, \qquad 0 \le l \le K-1.$$

L_k belongs to $\mathbb{R}^2$ and $P_{k,l}$ is a polynomial in $v_0, v_1, ..., v_{k-1}$.

Moreover, for $0 \le k \le K-1$, we have that $P_{k,l}$ is a sum of monomials of the form:

$$\zeta v_0^{p_0} v_1^{p_1} \cdots v_{k-1}^{p_{k-1}},$$

where $\zeta \in \mathbb{N}^* = \mathbb{N}\backslash\{0\}$ and:

$$(*)\left\{\begin{array}{l} p_0 + p_1 + \cdots + p_{k-1} = l \\ p_0 + 2p_1 + \cdots + kp_{k-1} = k. \end{array}\right.$$

We prove the property $(*)$ by induction. Clearly $L_0 = h^{(0)}$ and $L_1 = v_0 h^{(1)}$. Let us assume the property is true for L_k. Clearly L_{k+1} is obtained by derivation of L_k w.r.t. $v_0 g$ and $v_{i+1}\dfrac{\partial}{\partial v_i}$ for $0 \le i \le k-1$. Let $m =$

$\zeta v_0^{p_0} v_1^{p_1} \cdots v_{k-1}^{p_{k-1}}$ be a monomial of $P_{k,l}$ where $0 \leq l \leq k-1$. By assumption we have:

$$\begin{cases} p_0 + p_1 + \cdots + p_{k-1} = l \\ p_0 + 2p_1 + \cdots + kp_{k-1} = k. \end{cases}$$

- If $mh^{(l)}$ is derivated w.r.t. $v_0 g$ we obtain:

$$L_{v_0 g}\left(mh^{(l)}\right) = v_0 m h^{(l+1)},$$

$v_0 m = \zeta v_0^{p_0+1} v_1^{p_1} v_2^{p_2} \cdots v_{k-1}^{p_{k-1}}$ is a monomial of $P_{k+1,l+1}$ that verifies:

$$\begin{cases} (p_0+1) + p_1 + \cdots + p_{k-1} = l+1 \\ (p_0+1) + 2p_1 + \cdots + kp_{k-1} = k+1. \end{cases}$$

- if $mh^{(l)}$ is derivated w.r.t. $v_{i+1}\dfrac{\partial}{\partial v_i}$, $0 \leq i \leq k-1$, we obtain (if $p_i \geq 1$):

$$v_{i+1}\frac{\partial}{\partial v_i}\left(mh^{(l)}\right) = p_i \zeta v_0^{p_0} \cdots v_i^{p_i-1} v_{i+1}^{p_{i+1}+1} \cdots v_{k-1}^{p_{k-1}} h^{(l)}.$$

The monomial $p_i \zeta v_0^{p_0} \cdots v_i^{p_i-1} v_{i+1}^{p_{i+1}+1} \cdots v_{k-1}^{p_{k-1}}$ of $P_{k+1,l}$ verifies:

$$\begin{cases} p_0 + p_1 + \cdots + (p_i - 1) + (p_{i+1}+1) + \cdots + p_{k-1} = l \\ p_0 + 2p_1 + \cdots + (i+1)(p_i - 1) + (i+2)(p_{i+1}+1) + \cdots + kp_{k-1} \\ = k - (i+1) + (i+2) = k+1. \end{cases}$$

Thus the property $(*)$ is true for L_{k+1}.

A consequence of this property is the existence of an integer j, $1 \leq j \leq \dfrac{k}{l} - 1$, such that $p_j \neq 0$. Otherwise we would have:

$$\sum_{0 \leq i \leq k-1} (i+1) p_i > \frac{k}{l} l = k.$$

Let s, $0 \leq s \leq d-1$, be the smallest integer for which $h^{(s)} = dL_g^s h(x) \cdot \Lambda \neq 0$. Such an s does exist by lemma 5.4. If $s = 0$, then $L_0 = h^{(0)} \neq 0$. Otherwise, let q, $0 \leq q \leq b$, be the smallest integer for which $v_q \neq 0$. Such a q does exist since $\|v\|_\infty = 1$. We have:

$$s(q+1) \leq (d-1) \times (b+1) \leq K - 1.$$

We will show that:

$$L_{s(q+1)} = \sum_{l=1}^{s(q+1)} P_{s(q+1),l} h^{(l)} \neq 0.$$

- if $l < s$, $h^{(l)} = 0$ by assumption, hence $P_{s(q+1),l} h^{(l)} = 0$.
- if $l > s$ let us consider a monomial m of $P_{s(q+1),l}$:

$$m = \zeta v_0^{p_0} v_1^{p_1} \cdots v_{s(q+1)-1}^{p_{s(q+1)-1}}.$$

  There exists $j \leq \dfrac{s(q+1)}{l} - 1 < q$ for which $p_j \neq 0$. But $j < q$ emplies $v_j = 0$. Hence $m = 0$ and $P_{s(q+1),l} = 0$.
- It remains the case $l = s$.
  - one of the monomials of $L_{s(q+1),s}$ is $\zeta v_q^s h^{(s)}$, $\zeta \neq 0$. It can easily be obtained by suitable derivations of $L_0 = h^{(0)}$.

– let $m = \zeta v_0^{p_0} v_1^{p_1} \cdots v_{s(q+1)-1}^{p_{s(q+1)-1}}$ be another monomial of $L_{s(q+1),s}$. There exists $j \neq q$ for which $p_j \neq 0$. Moreover:

$$p_0 + p_1 + \cdots + p_q + \cdots + p_{s(q+1)-1} = s$$
$$p_0 + 2p_1 + \cdots + (q+1)p_q + \cdots + s(q+1)p_{s(q+1)-1} = s(q+1).$$

Hence:

$$\frac{1}{q+1}p_0 + \frac{2}{q+1}p_1 + \cdots + p_q + \cdots + sp_{s(q+1)-1} = s.$$

If $p_0 = p_1 = \cdots = p_{q-1} = 0$, then on the one hand:

$$\frac{q+2}{q+1}p_{q+1} + \cdots + sp_{s(q+1)-1} = p_{q+1} + \cdots + p_{s(q+1)-1}$$

and on the other hand there exists j, $q+1 \leq j \leq s(q+1)-1$, such that $p_j \neq 0$. This is impossible, hence one of the p_j's, $0 \leq j \leq q-1$, is positive. Therefore $m = 0$, since $v_j = 0$.

Finally, we have obtained:

$$L_s(q+1) = \zeta v_q^s h^{(s)} \neq 0$$

This is in contradiction with the assumptions $L_k = 0$, $0 \leq k \leq K-1$, and the theorem is proved. □

6. Appendix

We state here three lemmas that are proved in [7]. They are used in the proofs of our transversality results.

Lemma 6.1. *Let d and p be two positive integers and let L be the set of linear mappings $L = L\left(\mathbb{R}^d, \mathbb{R}^p\right) \times L\left(\mathbb{R}^d, \mathbb{R}^p\right)$. An element (C, A) of L is said to be observable if $\bigcap_{i \geq 0} KerCA^i = \{0\}$ or, equivalently, if the rank of $\left(C, CA, CA^2, ..., CA^{d-1}\right)$ is d.*

The set of unobservable elements of L is an algebraic subset of L of codimension p.

In the statement of the two next lemmas the notations are those of the third section.

Lemma 6.2. *Let $\left(j_{x_0}^k f, j_{x_0}^k g, u\right) \in \left(J^k\Gamma\right)^{\oplus 2} \times \mathbb{R}^k$ such that $f(x_0) + u_0 g(x_0) \neq 0$. Then*

1. *the mapping*

$$\begin{aligned} J_{x_0}^k H &\longrightarrow \mathbb{R}^{kp} \\ j_{x_0}^k h &\longmapsto R_{k,\Sigma}(x_0, u) \end{aligned}$$

is linear and surjective.

2. *the mapping*

$$\begin{aligned} J_{x_0}^k H &\longrightarrow \left(T_{x_0}^* X\right)^{kp} \\ j_{x_0}^k h &\longmapsto T_X R_{k,\Sigma}(x_0, u) \end{aligned}$$

is linear and surjective.

Lemma 6.3. *Let $\left(j_{x_0}^k f, j_{x_0}^k g, \Lambda, u\right) \in \left(\left(J^k\Gamma\right)^{\oplus 2} \oplus TX\right) \times \mathbb{R}^k$ and $\rho \geq 1$ such that:*

- $f(x_0) + u_0 g(x_0) = 0$;
- $f(x_0) \neq 0$ *or* $g(x_0) \neq 0$ *(hence* $g(x_0) \neq 0$*);*
- $u_1 = u_2 = \cdots = u_{\rho-1} = 0$ *but* $u_\rho \neq 0$;

  the vectors $A^i\Lambda$, $0 \leq i \leq \rho$, *are linearly independant (where* $A = T_X(f(x_0) + u_0 g(x_0))$.

Then:

1. *the mapping*

$$\begin{aligned} J^k_{x_0} H &\longrightarrow \mathbb{R}^{kp} \\ j^k_{x_0} h &\longmapsto T_X R_{k,\Sigma}(x_0, u).\Lambda \end{aligned}$$

   is linear and surjective.

2. *for* $0 \leq i < \rho$

$$d_X L^i_{F^k} h(x_0, u).\Lambda = \frac{1}{i!} C A^i \Lambda$$

   where $C = dh(x_0)$.

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References

[1] R. Abraham, J. Robbin: *Tranversal mappings and flows*, W.A. Benjamin, Inc., 1967.

[2] D. Aeyels: Generic observability of differentiable systems, *SIAM J. Contr. Opt.*, **19**, 1981, 595–603.

[3] D. Aeyels: Global observability of Morse-Smale systems, *J. Diff. Equations*, **45**, 1982, 1–15.

[4] J. Bochnak, M. Coste, M-F. Roy: *Géométrie algébrique réelle*, Ergebnisse der Mathematik, Springer-Verlag, 1987.

[5] J.-P. Gauthier, H. Hammouri, I. Kupka: Observers for non-linear systems, *IEEE CDC Conference*, december 1991, 1483–1489.

[6] J.-P. Gauthier, I. Kupka: Observability and observers for nonlinear systems, *SIAM J. Contr. Opt.*, **32**, 4, 1994, 975–994.

[7] J.-P. Gauthier, I. Kupka: Observability for systems with more outputs than inputs and asymptotic observers, *Math. Z.* **223**, 1996, 47–78.

[8] J.-P. Gauthier, I. Kupka: Genericity of observability and the existence of nonlinear observers, *Banach Center publications*, **32**, Warszawa 1995, 227–244.

[9] M. Golubitsky, V. Guillemin: *Stable mappings and their singularities*, Graduate texts in mathematics, Springer-Verlag, 1973.

[10] R. Hermann, J. Krener: Nonlinear controlability and observability, *IEEE Trans. Automat. Contr.*, AC-**22**, 1977, 728–740.

[11] H. Hironaka: Subanalytic sets, in *Number theory, Algebraic geometry and Commutative algebra*, in honor of Y. Akizuki, Kinokuniya, Tokyo, 1973, 453–493.

[12] M.W. Hirsch: *Differential topology*, Graduate texts in mathematics, Springer-Verlag, 1976.

[13] Ph. Jouan: Observability of real analytic vector-fields on compact manifolds, *Systems Control Lett.*, **26**, 2, Septembre 1995.

[14] Ph. Jouan: Contribution à l'étude de la généricité de l'observabilité des systèmes contrôlés, submitted to *C. R. Acad. Sci. Paris.*

[15] Ph. Jouan: *Singularités des systèmes non linéaires. Observabilité et observateurs*, Thèse de l'Université de Rouen, 1995.

[16] Ph. Jouan, J.-P. Gauthier: Finite singularities of nonlinear systems. Output stabilization, observability and observers, *Journal of Dynamical and Control Systems*, **2**, 2, 1996, 255–288.

[17] S. Lojasiewicz: Triangulation of semi analytic sets, *Annal. Sc. Nor. Sup. PISA*, 1966, 449–474.
[18] H.J. Sussmann: Single-input observability of continuous-time systems, *Math. Syst. Theory*, **12**, 1979, 371–393.
[19] F. Takens: Detecting strange attractors in turbulence, in *Dynamic Systems and Turbulence*, Warwick, 1980, Springer-Verlag, Berlin, 1981, 366–381.
[20] K. Tchon: On solvability of several affine system problems, *Systems Contr. Lett.*, **4**, 1984, 373–379.