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# ESTIMATIONS OF THE BEST CONSTANT INVOLVING THE $L^2$ NORM IN WENTE'S INEQUALITY AND COMPACT $H$-SURFACES IN EUCLIDEAN SPACE

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Abstract. In the first part of this paper, we study the best constant involving the $L^2$ norm in Wente's inequality. We prove that this best constant is universal for any Riemannian surface with boundary, or respectively, for any Riemannian surface without boundary. The second part concerns the study of critical points of the associate energy functional, whose Euler equation corresponds to $H$-surfaces. We will establish the existence of a non-trivial critical point for a plan domain with small holes.

## 1. Introduction

Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^2$. We denote $V = \{a \in H^1(\Omega), a \neq \text{constant}\}$ and $V_0 = V \cap H^1_0(\Omega)$. Given two functions $a$, $b \in V$, we denote by $\varphi$ the unique solution in $W^{1,1}(\Omega)$ of the Dirichlet problem

$$\begin{cases} -\triangle\varphi &= a_x b_y - a_y b_x, \quad \text{in } \Omega \\ \varphi &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where subscripts denote partial differentiation with respect to coordinates.

By developing a previous work from H. Wente [22], H. Brezis and J.-M. Coron [7] showed the following result:

Theorem 1.1. *The solution $\varphi$ of equation (1.1) is a continuous function on $\bar{\Omega}$ and $\varphi \in H^1(\Omega)$. Moreover there exists a constant $C_0(\Omega)$ which depends only on $\Omega$ such that*

$$\|\varphi\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)} \leq C_0(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)} \tag{1.2}$$

This result is sharp in the sense that since the right hand side of (1.1) is in $L^1(\Omega)$, the classical theory of Calderon-Zygmund does provide estimates for $\varphi$ only in $L^q(\Omega)$ and $W^{1,p}(\Omega)$ for $q < \infty$ and $p < 2$. Note that equation (1.1) appears in many problems arising in physics and geometry, and Theorem 1.1 has many applications.

Later on, F. Bethuel and J.-M. Ghidaglia [5] proved that in fact one can find a constant $C_0(\Omega)$ which does not depend on $\Omega$. We are interested here in the optimal (i.e. smallest) value of this constant such that estimates analogous to (1.2) hold. To be more precise we denote by $C_\infty(\Omega)$ the best

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constant involving the $L^\infty$-norm in the estimations and by $C_2(\Omega)$ for the $L^2$-norm, i.e.

$$C_\infty(\Omega) = \sup_{a,b\in V} \frac{\|\varphi\|_\infty}{\|\nabla a\|_2 \|\nabla b\|_2}, \tag{1.3}$$

$$C_2(\Omega) = \sup_{a,b\in V} \frac{\|\nabla\varphi\|_2^2}{\|\nabla a\|_2^2 \|\nabla b\|_2^2}. \tag{1.4}$$

S. Baraket [3] obtained that $C_\infty(\Omega) = \frac{1}{2\pi}$ for simply connected domain $\Omega$. This result has been recently extended to any domain by P. Topping [21]. Our aim in this paper is to study $C_2(\Omega)$. Thus we consider the following energy functional defined on $V \times V$

$$E(a,b,\Omega) = \frac{\|\nabla\varphi\|_2^2}{\|\nabla a\|_2^2 \|\nabla b\|_2^2}, \tag{1.5}$$

where $a$, $b \in V$, and $\varphi$ is given by (1.1).

In this paper, we will prove the following main results.

Theorem 1.2. *Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$. Then we have*

$$C_2(\Omega) = \frac{3}{16\pi}.$$

*Moreover, the best constant is achieved if and only if $\Omega$ is simply connected.*

Notice that the functional $E(a,b,\Omega)$ is invariant under the action of conformal diffeomorphisms on the domain $\Omega$ (see [15]). As a consequence we deduce that $C_2(\Omega)$ and $C_\infty(\Omega)$ depend only on the conformal type of $\Omega$. Moreover it implies that the functional $E$ makes sense on any Riemann surface (i.e. a surface equipped with a conformal structure) with or without boundary. In section 4, we prove generalizations of Theorem 1.2, namely

Theorem 1.3. *Let $M$ be a Riemann surface with a non empty boundary, then*

$$C_2(M) = \frac{3}{16\pi}$$

*and the maximum in (1.5) is achieved if and only if $M$ is topologically a disc.*

Theorem 1.4. *Let $M$ be a Riemann surface without boundary, then*

$$C_2(M) = \frac{3}{32\pi}$$

*and the maximum in (1.5) is achieved if and only if $M$ is topologically a sphere.*

An interesting observation, due to F. Hélein [15], is that the study of $E$ leads to a solution of the $H$-surface equation $-\triangle u = u_x \times u_y$, satisfied by surfaces of constant mean curvature in $\mathbb{R}^3$ in conformal representation. For this purpose, we will look for critical points of $E$. Note that direct variational approaches on that problem were developed in [7], [17] and [22]. In view of Theorem 1.2, we can not maximise the problem if $\Omega$ is not simply connected. The major obstruction in proving the existence of a maximum comes from the fact that the norms $\|\nabla a\|_{L^2}$ and $\|\nabla b\|_{L^2}$ are not continuous under weak convergence in $L^2$. Indeed, for any smooth bounded domain in plan, concentration phenomena occur in the maximizing sequence as shown in section 7 of this paper. However, making use of a topological method, invented by J.-M. Coron [8], we establish the following result of existence.

THEOREM 1.5. *Let $\Omega$ be the unit disc perfored with small holes. Then $E$ admits a non trivial critical point.*

This paper consists of two parts; sections 2-5 are concerned with the estimations of the best constant involving the $L^2$ norm in Wente's inequality, the remainder is devoted to search of a critical point for $E$: a study of the compactness of minimizing sequences, of the Palais-Smale condition and some existence results through a topological argument.

# PART A. ESTIMATIONS OF THE BEST CONSTANT INVOLVING THE $L^2$ NORM

## 2. OUTLINE

In this part, we will study the energy functional $E$ and estimate the value of $C_2(\Omega)$. Our approach is the following. In section 3, we will look for the Euler-Lagrange equation for critical points of the functional $E(a, b, \Omega)$ on the "manifold" where $\|\nabla a\|_2 = \|\nabla b\|_2 = 1$. After a scaling which uses the Lagrange multiplier, we see that any critical point leads by a canonical way to a solution of the $H$-surface equation, that is, the equation satisfied by a conformal parameterization of a surface when its mean curvature is constant.

In section 4, we will calculate $C_2(\Omega)$ in the case where $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$. With the help of the isoperimetric inequality, we will show that $C_2(\Omega) = \frac{3}{16\pi}$. If $\Omega$ is a disc, it is easy to show also that this constant is achieved. The next question is to know whether the maximum of $E$ is achieved for a multiply connected domain. This is an interesting problem related to surfaces of constant mean curvature. Recall that for a long time, we thought that there does not exist an immersion with constant mean curvature from torus into $\mathbb{R}^3$. In 1984, H. Wente has given a counterexample. In view of Euler equation, the torus of Wente gives rise to a critical point of our functional $E$ on an annulus. Indeed, let $\Psi = (a, b, \varphi)$ be a critical point of $E$ on an annulus, we construct a compact oriented Riemannian surface $M = \Omega \bigcup_{\partial\Omega} \tilde{\Omega}$ by sticking $\Omega$ and a copy of $\Omega$, provided with opposing orientation and define a $C^\infty$ map $\tilde{\Psi}$ from $M$ into $\mathbb{R}^3$ by $\tilde{\Psi} = \Psi$ on $\Omega$ and $\tilde{\Psi} = (a, b, -\varphi)$ on $\tilde{\Omega}$. Would this map be conformal, then its image would be a torus of constant mean curvature. Conversely the torus of Wente corresponds to a critical point of our functional $E$ on some annulus. Unfortunately, this surface can not be obtained by maximizing the energy functional $E$ and Wente tori thus correspond to nonmaximizing critical points of $E$. We will prove this fact in section 5.

At end of this part, we will also generalize all these results on a compact manifold without boundary. An interesting fact is that $C_2(M)$ is also universal and is just half of $C_2(\Omega)$. Furthermore, a maximal critical point on a domain in the plan gives rise to a maximal critical point on a compact manifold, by sticking.

## 3. The Euler-Lagrange Equation

Definition 3.1. A point $(a,b)\in V\times V$ is critical for the energy functional $E$ if it satisfies the following conditions:

(i) $\nabla E(a+t\alpha, b+s\beta,\Omega)|_{(s,t)=(0,0)}=0$, for all $\alpha$, $\beta\in H^1(\Omega)$,

(ii) if $\sigma_t:\Omega\longrightarrow\Omega$ is a family of diffeomorphisms, depending differentiably on $t$, with $\sigma_0=id_\Omega$, then we have

$$\left.\frac{d}{dt}\right|_{t=0} E(a\circ\sigma_t, b\circ\sigma_t,\Omega)=0.$$

We remark that $E$ is invariant under a conformal transformation of $\Omega$ and $E(\lambda a,\mu b,\Omega)=E(a,b,\Omega)$ for all $\lambda$, $\mu\in\mathbb{R}^*$. Hence, without loss of generality, we can assume that $\|\nabla a\|_2=\|\nabla b\|_2=1$.

Theorem 3.2. *Assume that $(a,b)\in V\times V$ is a critical point of $E$ such that $\varphi\neq 0$. Then*

(i) $\int_\Omega \nabla a\nabla b=0$,

(ii) $\frac{\partial a}{\partial n}=\frac{\partial b}{\partial n}=0$ *on $\partial\Omega$ where $n=(n^1,n^2)$ is the normal vector on $\partial\Omega$,*

(iii) *there exists $\lambda\in\mathbb{R}^*$ such that $\Psi=(a_1,b_1,\varphi_1)=(\lambda a,\lambda b,\lambda^2\varphi)$ satisfies:*

$$\begin{cases} -\triangle\varphi_1 &=\{a_1,b_1\},\\ -\triangle a_1 &=\{b_1,\varphi_1\},\\ -\triangle b_1 &=\{\varphi_1,a_1\},\end{cases}\tag{3.1}$$

*where $\{\xi,\eta\}=\xi_x\eta_y-\xi_y\eta_x$,*

(iv) *$\Psi$ is $C^\infty$ on $\overline{\Omega}$,*

(v) *the Hopf differential $\omega=\langle\partial_z\Psi,\partial_z\Psi\rangle$ is holomorphic, i.e.*

$$\partial_{\bar z}\langle\partial_z\Psi,\partial_z\Psi\rangle=0.$$

*Moreover, if we denote $t=-n^2+in^1$ the unit complex number tangent to $\partial\Omega$, we have*

$$Im(\omega t^2)=0 \text{ on } \partial\Omega.$$

(vi) *If $\Omega$ is simply connected, then the Hopf differential vanishes:*

$$\langle\partial_z\Psi,\partial_z\Psi\rangle=0,$$

*where $\partial_z=\frac{1}{2}(\partial_x-i\partial_y)$, which implies that $\Psi$ is conformal,*

(vii) *if $\Omega$ is an annulus, then there exists $c\in\mathbb{R}$ such that*

$$\langle\partial_z\Psi,\partial_z\Psi\rangle=\frac{c}{z^2}.$$

First we prove some technical lemma.

Lemma 3.3. *(see [7] and also [22]). If $\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)$ (resp. $\varphi\in H_0^1(\Omega)$), $a\in H^1(\Omega)\cap L^\infty(\Omega)$ (resp. $a\in H^1(\Omega)$) and $b\in H^1(\Omega)$ (resp. $b\in W^{1,\infty}(\Omega)$), then we have*

$$\int_\Omega\varphi\{a,b\}=\int_\Omega a\{b,\varphi\}.$$

*Proof.* Assuming first that $\varphi$, $a$, $b \in C^2(\bar{\Omega})$, we have

$$\begin{aligned}\int_\Omega \varphi\{a,b\} &= \int_\Omega \varphi(a_x b_y - a_y b_x)\\ &= \int_\Omega \varphi[(ab_y)_x - (ab_x)_y].\end{aligned}$$

Integrating by parts and using the fact $\varphi = 0$ on $\partial\Omega$, we obtain

$$\int_\Omega \varphi\{a,b\} = \int_\Omega a(b_x\varphi_y - b_y\varphi_x) = \int_\Omega a\{b,\varphi\}.$$

Now, we consider $\varphi \in L^\infty(\Omega) \cap H_0^1(\Omega)$, $b \in H^1(\Omega)$ and $a \in L^\infty(\Omega) \cap H^1(\Omega)$. We choose three suitable sequences of smooth functions $\{\varphi_n\}_{n\in\mathbb{N}}$, $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ satisfying the following conditions:

$$\begin{aligned}&\varphi_n \longrightarrow \varphi \text{ in } H^1(\Omega) \text{ and } \varphi_n \longrightarrow \varphi \text{ weakly} \star \text{ in } L^\infty(\Omega),\\ &b_n \longrightarrow b \text{ in } H^1(\Omega),\\ &a_n \longrightarrow a \text{ in } H^1(\Omega) \text{ and } a_n \longrightarrow a \text{ weakly} \star \text{ in } L^\infty(\Omega).\end{aligned}$$

We state that

$$\begin{aligned}\left|\int_\Omega \varphi\{a,b\}\right| &\le \|\varphi\|_{L^\infty}\|\nabla a\|_2\|\nabla b\|_2,\\ \left|\int_\Omega a\{b,\varphi\}\right| &\le \|a\|_{L^\infty}\|\nabla\varphi\|_2\|\nabla b\|_2.\end{aligned}$$

Passing to the limit in the inequality for $a_n$, $b_n$ and $\varphi_n$, this completes the proof. □

LEMMA 3.4. *(see* [5] *and see also* [22]*). Let $\Psi \in H^1(\Omega;\mathbb{R}^3)$ be a solution of equation* (3.1) *in the sense of distributions. Then $\Psi \in C^\infty(\Omega;\mathbb{R}^3)$.*

*Proof.* (of Theorem 3.2). Let $a_t = a + tb$, $b_t = b$. We denote by $\varphi_t$ the unique solution in $H_0^1(\Omega)$ of equation (1.1). Obviously, we have $\varphi_t = \varphi$ for all $t \in \mathbb{R}$ and $\|\nabla a_t\|_2^2 = \|\nabla a\|_2^2 + 2t\int_\Omega \nabla a \nabla b + O(t^2)$. Then (i) follows from the definition of a critical point.
Given $a_t = a + t\alpha$, $b_t = b$ with $\alpha \in C^\infty(\bar{\Omega})$. We denote $\psi$ the unique solution in $H_0^1(\Omega)$ of equation (1.1) with $a = \alpha$, that is,

$$\begin{cases} -\triangle\psi &= \{\alpha, b\}, \quad \text{in } \Omega\\ \quad\ \psi &= 0, \qquad\quad \text{on } \partial\Omega.\end{cases}$$

It is clear that

$$\int_\Omega |\nabla\varphi_t|^2 = \int_\Omega |\nabla\varphi|^2 + 2t\int_\Omega \nabla\varphi\cdot\nabla\psi + O(t^2).$$

By Lemma 3.3,

$$\int_\Omega \varphi\{\alpha,b\} = \int_\Omega \alpha\{b,\varphi\}.$$

Hence, we obtain

$$\begin{aligned}\int_\Omega |\nabla\varphi_t|^2 &= \int_\Omega |\nabla\varphi|^2 + 2t\int_\Omega \varphi(-\triangle\psi) + O(t^2)\\ &= \int_\Omega |\nabla\varphi|^2 + 2t\int_\Omega \varphi\{\alpha,b\} + O(t^2)\\ &= \int_\Omega |\nabla\varphi|^2 + 2t\int_\Omega \alpha\{b,\varphi\} + O(t^2).\end{aligned}$$

On the other hand,

$$\int_\Omega |\nabla a_t|^2 = \int_\Omega |\nabla a|^2 + 2t\int_\Omega \nabla a\cdot\nabla\alpha + O(t^2).$$

Thus, we have

$$E(a_t,b_t,\Omega) = \frac{\|\nabla\varphi\|_2^2 + 2t\displaystyle\int_\Omega \alpha\{b,\varphi\} + O(t^2)}{\left(\|\nabla a\|_2^2 + 2t\displaystyle\int_\Omega \nabla a\cdot\nabla\alpha\right)\|\nabla b\|_2^2 + O(t^2)}.$$

With the definition of critical point, we conclude that

$$\int_\Omega \alpha\{b,\varphi\} = \|\nabla\varphi\|_2^2\int_\Omega \nabla a\cdot\nabla\alpha,\ \forall\alpha\in C^\infty(\bar\Omega).$$

Performing analogous deformations for $b$, we obtain

$$\int_\Omega \beta\{\varphi,a\} = \|\nabla\varphi\|_2^2\int_\Omega \nabla b\cdot\nabla\beta,\ \text{for any } \beta\in C^\infty(\bar\Omega).$$

In particular, if we set $\alpha$, $\beta\in C_0^\infty(\Omega)$, we deduce that

$$\begin{cases} -\triangle a = \dfrac{1}{\|\nabla\varphi\|_2^2}\{b,\varphi\}, \\ -\triangle b = \dfrac{1}{\|\nabla\varphi\|_2^2}\{\varphi,a\}. \end{cases} \tag{3.2}$$

In order to establish the property (ii), we put $\alpha$, $\beta\in C^\infty(\bar\Omega)$. Setting $\lambda = 1/\|\nabla\varphi\|_2$, the property (iii) is demonstrated.
In view of Lemma 3.4, $\Psi$ is $C^\infty$ on $\Omega$. To prove the regularity of $u$ up to the boundary, fix $x\in\partial\Omega$. So there exists a conformal map $I$ from $B(x,r)\cap\Omega$ onto $B_+ = B\cap\{x>0\}$, where $B$ is a unit disc. Without loss of generality, we can assume that $\Psi$ is defined on $B_+$. We define the extensions of $\Psi$ on $B$ as follows:

$$\widetilde{\varphi_1}(x,y) = \begin{cases} \varphi_1(x,y), & \text{if } x\geq 0,\\ -\varphi_1(-x,y), & \text{if } x\leq 0,\end{cases}$$

$$\widetilde{a_1}(x,y) = \begin{cases} a_1(x,y), & \text{if } x\geq 0,\\ a_1(-x,y), & \text{if } x\leq 0,\end{cases}$$

and

$$\widetilde{b_1}(x,y) = \begin{cases} b_1(x,y), & \text{if } x\geq 0,\\ b_1(-x,y), & \text{if } x\leq 0.\end{cases}$$

Clearly, $\tilde\Psi$ is in $H^1(B,\mathbb{R}^3)$. We will prove that $\tilde\Psi$ is also a solution of equation (3.1). Thus, by Lemma 3.4, we conclude that $\Psi$ is $C^\infty$ on $\overline{\Omega}$. Set

$\psi \in C_0^\infty(B)$. From the properties (ii) and (iii), we have

$$\begin{aligned}
\int_B \nabla\widetilde{a}_1 \cdot \nabla\psi &= \int_{B_+} \nabla a_1 \cdot \nabla\psi + \int_{B_-} \nabla\widetilde{a}_1 \cdot \nabla\psi \\
&= \int_{B_+} \nabla a_1 \cdot \nabla\psi + \int_{B_+} \nabla a_1(x,y) \cdot \nabla(\psi(-x,y)) \\
&= \int_{B_+} \{b_1, \varphi_1\}\psi + \int_{B_+} \{\widetilde{b}_1, \widetilde{\varphi}_1\}(x,y)\psi(-x,y) \\
&= \int_{B_+} \{b_1, \varphi_1\}\psi + \int_{B_-} \{\widetilde{b}_1, \widetilde{\varphi}_1\}(-x,y)\psi(x,y) \\
&= \int_B \{\widetilde{b}_1, \widetilde{\varphi}_1\}\psi.
\end{aligned}$$

i.e. $-\triangle\widetilde{a}_1 = \{\widetilde{b}_1, \widetilde{\varphi}_1\}$.
With the same arguments, we deduce that

$$-\triangle\widetilde{b}_1 = \{\widetilde{\varphi}_1, \widetilde{a}_1\}.$$

On the other hand, we have

$$\begin{aligned}
\int_B \nabla\widetilde{\varphi}_1 \cdot \nabla\psi &= \int_{B_+} \nabla\varphi_1 \cdot \nabla\psi + \int_{B_-} \nabla\widetilde{\varphi}_1 \cdot \nabla\psi \\
&= \int_{B_+} \nabla\varphi_1 \cdot \nabla\psi - \int_{B_+} \nabla\varphi_1(x,y) \cdot \nabla(\psi(-x,y)) \\
&= -\int_{B_+} \triangle\varphi_1\psi + \int_{B_+} \triangle\varphi_1(x,y)\psi(-x,y) \\
&= \int_{B_+} \{a_1, b_1\}\psi - \int_{B_+} \{a_1, b_1\}(x,y)\psi(-x,y) \\
&= \int_B \{\widetilde{a}_1, \widetilde{b}_1\}\psi,
\end{aligned}$$

that is, $-\triangle\widetilde{\varphi}_1 = \{\widetilde{a}_1, \widetilde{b}_1\}$.
To prove the property (v), set $a_t = a \circ \sigma_t$, $b_t = b \circ \sigma_t$ where $\sigma_t$ is a family of smooth diffeomorphisms of $\Omega$. Suppose that $\left.\dfrac{d\sigma_t}{dt}\right|_{t=0} = (X^1, X^2)$. Clearly, $(X^1, X^2) \cdot n = 0$ on $\partial\Omega$ where $n$ is the normal vector on $\partial\Omega$. Moreover, we have

$$\begin{aligned}
\int_\Omega (-\triangle\varphi_t)\varphi &= \int_\Omega (\{a,b\} \circ \sigma_t)\det(\nabla\sigma_t)\varphi \\
&= \int_\Omega \{a,b\}(\varphi \circ \sigma_{-t}) \\
&= -\int_\Omega \triangle\varphi(\varphi \circ \sigma_{-t}) \\
&= \int_\Omega \nabla\varphi \cdot \nabla(\varphi \circ \sigma_{-t}),
\end{aligned}$$

i.e. $\int_\Omega \nabla\varphi_t\cdot\nabla\varphi = \int_\Omega \nabla\varphi\cdot\nabla(\varphi\circ\sigma_{-t})$.
However, from Theorem 1.1, we get

$$-\frac{1}{2}\int_\Omega |\nabla\varphi|^2 + \int_\Omega \nabla\varphi_t\cdot\nabla\varphi = \frac{1}{2}\int_\Omega |\nabla\varphi_t|^2 + O(t^2),$$
$$-\frac{1}{2}\int_\Omega |\nabla\varphi|^2 + \int_\Omega \nabla\varphi\cdot\nabla(\varphi\circ\sigma_{-t}) = \frac{1}{2}\int_\Omega |\nabla(\varphi\circ\sigma_{-t})|^2 + O(t^2).$$

Thus, we get

$$\int_\Omega |\nabla(\varphi\circ\sigma_{-t})|^2 = \int_\Omega |\nabla\varphi_t|^2 + O(t^2).$$

This means that

$$E(a_t, b_t, \Omega) = \frac{\|\nabla(\varphi\circ\sigma_{-t})\|_2^2}{\|\nabla a_t\|_2^2\|\nabla b_t\|_2^2} + O(t^2).$$

On the other hand, it is easy to get the following relations:

$$\begin{aligned}
\left.\frac{d(\|\nabla a_t\|_2^2)}{dt}\right|_{t=0} &= 2\int_\Omega [((\partial_x a)^2 - (\partial_y a)^2)(\partial_x X^1 - \partial_y X^2) \\
&\qquad + 2\partial_x a\partial_y a(\partial_y X^1 + \partial_x X^2)] \\
\left.\frac{d(\|\nabla b_t\|_2^2)}{dt}\right|_{t=0} &= 2\int_\Omega [((\partial_x b)^2 - (\partial_y b)^2)(\partial_x X^1 - \partial_y X^2) \\
&\qquad + 2\partial_x b\partial_y b(\partial_y X^1 + \partial_x X^2)] \\
\left.\frac{d(\|\nabla(\varphi\circ\sigma_{-t})\|_2^2)}{dt}\right|_{t=0} &= -2\int_\Omega [((\partial_x \varphi)^2 - (\partial_y \varphi)^2)(\partial_x X^1 - \partial_y X^2) \\
&\qquad + 2\partial_x \varphi\partial_y \varphi(\partial_y X^1 + \partial_x X^2)].
\end{aligned}$$

Thus, we get the equality

$$\begin{aligned}
\int_\Omega \Big[(\partial_x\varphi)^2 - (\partial_y\varphi)^2 + \|\nabla\varphi\|_2^2((\partial_x a)^2 - (\partial_y a)^2 + (\partial_x b)^2 - (\partial_y b)^2)\Big] \\
\times (\partial_x X^1 - \partial_y X^2) \\
+ 2[\partial_x\varphi\partial_y\varphi + \|\nabla\varphi\|_2^2(\partial_x a\partial_y a + \partial_x b\partial_y b)](\partial_y X^1 + \partial_x X^2) = 0
\end{aligned}$$

i.e.

$$\int_\Omega \Big[(|\partial_x\Psi|^2 - |\partial_y\Psi|^2)(\partial_x X^1 - \partial_y X^2) + 2 < \partial_x\Psi, \partial_y\Psi > (\partial_y X^1 + \partial_x X^2)\Big] = 0. \tag{3.3}$$

A convenient way to rewrite this equation is to set $\omega = |\partial_x\Psi|^2 - |\partial_y\Psi|^2 - 2i < \partial_x\Psi, \partial_y\Psi >$, and we obtain

$$\mathrm{Re}\int_\Omega \omega\partial_{\bar z}(X^1 + iX^2)dxdy = 0,$$

where $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$. In particular, if we put $X^1 + iX^2 \in C_0^\infty(\Omega)$, we deduce that

$$\partial_{\bar z}\omega = 0,$$

i.e. $\omega$ is holomorphic.
Now, if we use $(X^1, X^2)$ such that $(X^1, X^2) = f(-n^2, n^1)$ on $\partial\Omega$, where $f$ is an arbitrary continuous real-valued function on $\partial\Omega$, we obtain

$$\begin{aligned} 0 &= \text{Re} \int_\Omega \partial_{\bar z}(\omega(X^1 + iX^2))dxdy \\ &= \text{-Im} \int_{\partial\Omega} \frac{\omega}{2} f t^2 ds, \end{aligned}$$

thus $\text{Im}(\omega t^2) = 0$ on $\partial\Omega$. The property (v) is proved.

If $\Omega$ is a disc or an annulus, from (v), we obtain $\text{Im}(\omega z^2) = 0$ on $\partial\Omega$. From the principle of maximum, we have $\text{Im}(\omega z^2) = 0$ on $\Omega$ since $\text{Im}(\omega z^2)$ is harmonic. So we deduce that there exists $c \in \mathbb{R}$ such that $\omega z^2 = c$. In the case where $\Omega$ is a disc, we have moreover

$$\lim_{z \to 0} \omega z^2 = 0.$$

So we conclude the properties (vi) and (vii). □

Remark 3.5. If $(a, b) \in V_0 \times V_0$ is a critical point of $E$ in $V_0 \times V_0$, then all the conclusions of Theorem except (ii) are also right.

Remark 3.6. We know that every plane domain of one connectivity can be mapped conformally onto some annulus (see Ahlfors [1]). Thus, we obtain a characterization of Hopf's differential $\omega$. But for a multiply connected domain $\Omega$, the characterization of $\omega$ is less simple.

## 4. Isoperimetric inequality

In the following $\Omega$ denotes a smooth simply connected domain. For simplicity, we suppose that $\Omega$ is a disc, that is, $\Omega = B = \{(x, y)/r < 1\}$. We check easily that a stereographic representation of the upper hemi-sphere

$$(a, b, \varphi) = \left(\frac{4x}{1 + r^2}, \frac{4y}{1 + r^2}, \frac{2(1 - r^2)}{1 + r^2}\right)$$

verifies all the properties of Theorem 3.2, i.e. is a critical point of $E$. It is just a maximum of $E$. More precisely, we have the following result.

Theorem 4.1. *Let $\Omega = B$, then*

(i) $\sup_{a,b \in V} E(a, b, \Omega) = \dfrac{3}{16\pi}$ *and the map* $\left(\dfrac{x}{1 + r^2}, \dfrac{y}{1 + r^2}\right)$ *achieves the best constant,*

(ii) $\sup_{a,b \in V_0} E(a, b, \Omega) = \dfrac{3}{32\pi}$ *and the best constant is not achieved in $V_0 \times V_0$.*

First, we will introduce the following notations. Given $\Psi,\ \Theta \in H^1(\Omega;\mathbb{R}^n)$, we define

$$\begin{aligned}
\langle \Psi, \Theta\rangle_D &= \int_\Omega \langle \Psi_x, \Theta_x\rangle + \langle \Psi_y, \Theta_y\rangle = \int_\Omega \langle \nabla\Psi, \nabla\Theta\rangle \\
|\Psi|_D^2 &= \langle \Psi, \Psi\rangle_D \\
V(\Psi) &= \frac{1}{3}\int_\Omega (\Psi \cdot \Psi_x \times \Psi_y),\ \text{if } \Psi \in C^0(\bar{\Omega};\mathbb{R}^3) \\
L(\Psi) &= \int_\Omega \sqrt{\{\varphi_1, \varphi_2\}^2 + \{\varphi_2, \varphi_3\}^2 + \{\varphi_3, \varphi_1\}^2}, \\
&\quad \text{where } \Psi = (\varphi_1, \varphi_2, \varphi_3) \\
(a, b, \varphi)_V &= L(\Theta),\ \text{where } \Theta = (a, b, \varphi).
\end{aligned}$$

In the proof, we will make use of the following lemmas.

LEMMA 4.2. *(see [22]) Let $\Psi,\ \Theta \in C^0(\bar{\Omega};\mathbb{R}^3) \cap H^1(\Omega;\mathbb{R}^3)$ be two mappings such that $\Psi|_{\partial\Omega}$ and $\Theta|_{\partial\Omega}$ describe the same oriented Jordan curve $\gamma$; then*

$$|V(\Psi) - V(\Theta)|^2 \le \frac{[L(\Psi) + L(\Theta)]^3}{36\pi}. \tag{4.1}$$

In fact, this Lemma is equivalent to the isoperimetric inequality.

LEMMA 4.3. *(see [23]). Let $\Psi \in C^0(\bar{\Omega};\mathbb{R}^3) \cap C^2(\Omega;\mathbb{R}^3) \cap H_0^1(\Omega;\mathbb{R}^3)$ be a solution of equation* (3.1)*; then $\Psi \equiv 0$.*

*Proof.* (of Theorem 4.1). Let $a,\ b \in C^\infty(\bar{\Omega};\mathbb{R}^3)$ and $\varphi$ be the corresponding solution of (1.1). By Lemma 3.3, we get

$$(a, b, \varphi)_V = \int_\Omega a\{b, \varphi\} = \int_\Omega b\{\varphi, a\} = \int_\Omega \varphi\{a, b\}.$$

Now the two vector functions

$$\Psi = \left(\frac{a}{|a|_D}, \frac{b}{|b|_D}, \frac{\varphi}{|\varphi|_D}\right) \text{ and } \Theta = \left(\frac{a}{|a|_D}, \frac{b}{|b|_D}, \frac{-\varphi}{|\varphi|_D}\right).$$

have the same boundary values. Noting that

$$V(\Psi) = -V(\Theta) \text{ and } L(\Psi) = L(\Theta) \le \frac{1}{2}|\Psi|_D^2 = \frac{3}{2},$$

from Lemma 4.2, we obtain that

$$|V(\Psi)|^2 \le \frac{3}{16\pi}.$$

Consequently,

$$\|\nabla\varphi\|_2^2 = \int_\Omega (-\triangle\varphi)\varphi = \int_\Omega \varphi\{a, b\} = (a, b, \varphi)_V \le \sqrt{\frac{3}{16\pi}}|a|_D|b|_D|\varphi|_D,$$

that is, $E(a, b, \Omega) \le \frac{3}{16\pi}$. Then the density of $C^\infty(\bar{\Omega})$ into $H^1(\Omega)$ implies that

$$\sup_{a,b\in V} E(a, b, \Omega) \le \frac{3}{16\pi}.$$

On the other hand, it is easy to check that

$$E\left(\frac{x}{1 + r^2}, \frac{y}{1 + r^2}, \Omega\right) = \frac{3}{16\pi}.$$

Hence, we deduce the property (i). Similarly, putting $\Theta = 0$, we get

$$E(a,b,\Omega) \leq \frac{3}{32\pi}, \text{ for all } a,b \in H_0^1(\Omega).$$

We set $a_{\varepsilon,1} = \dfrac{\varepsilon x}{\varepsilon^2 + r^2}$, $b_{\varepsilon,1} = \dfrac{\varepsilon y}{\varepsilon^2 + r^2}$, $a_{\varepsilon,2} = \dfrac{\varepsilon x}{1+\varepsilon^2}$ and $b_{\varepsilon,2} = \dfrac{\varepsilon y}{1+\varepsilon^2}$.
We claim that

$$\begin{aligned} \|\nabla a_{\varepsilon,1}\|_2^2 &= \|\nabla b_{\varepsilon,1}\|_2^2 &= \pi \int_0^{\frac{1}{\varepsilon^2}} \frac{(1+r^2)dr}{(1+r)^4}, \\ \|\nabla a_{\varepsilon,2}\|_2^2 &= \|\nabla b_{\varepsilon,2}\|_2^2 &= \frac{\varepsilon^2 \pi}{(1+\varepsilon^2)^2}, \end{aligned}$$

where $r = \sqrt{x^2+y^2}$. Set $a_\varepsilon = a_{\varepsilon,1} - a_{\varepsilon,2}$ and $b_\varepsilon = b_{\varepsilon,1} - b_{\varepsilon,2}$. We denote by $\varphi_\varepsilon$ the unique solution of equation (1.1). Then $\varphi_\varepsilon$ can be written as follows:

$$\varphi_\varepsilon = \frac{\varepsilon^2 - r^2}{8(\varepsilon^2 + r^2)} - \frac{\varepsilon^2 - 1}{8(\varepsilon^2+1)} + \psi_\varepsilon,$$

where $\psi_\varepsilon$ is the unique solution of the following equation

$$\begin{cases} -\triangle \psi_\varepsilon &= -\{a_{\varepsilon,1}, b_{\varepsilon,2}\} - \{a_{\varepsilon,2}, b_{\varepsilon,1}\} + \{a_{\varepsilon,2}, b_{\varepsilon,2}\}, \quad \text{in } \Omega \\ \psi_\varepsilon &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Using Theorem 1.1, we have $\|\nabla \psi_\varepsilon\|_2^2 = O(\varepsilon^2)$. Hence,

$$\|\nabla \varphi_\varepsilon\|_2^2 = \frac{\pi}{16} \int_0^{\frac{1}{\varepsilon^2}} \frac{r dr}{(1+r)^4} + O(\varepsilon).$$

It is easy to see that

$$E(a_\varepsilon, b_\varepsilon, \Omega) \longrightarrow \frac{3}{32\pi} \text{ as } \varepsilon \longrightarrow 0.$$

Finally, we obtain

$$\sup_{a,b \in V_0} E(a,b,\Omega) = \frac{3}{32\pi}.$$

Now we suppose that the best constant is achieved in the point $(a,b) \in V_0 \times V_0$. By Theorem 3.2, there exists $\lambda \in \mathbb{R}^*$ such that $(\lambda a, \lambda b, \lambda^2 \varphi)$ satisfies equation (3.1). From Lemma 4.2 and Theorem 1.1, $(\lambda a, \lambda b, \lambda^2\varphi) \in C^0(\bar{\Omega}; \mathbb{R}^3) \cap C^\infty(\Omega; \mathbb{R}^3)$. And, applying Lemma 4.3, we obtain $(\lambda a, \lambda b, \lambda^2 \varphi) = 0$. Thus, this contradiction completes the proof. □

REMARK 4.4. Because of the isoperimetric inequality, we always have

$$\sup_{a,b \in V} E(a,b,\Omega) \leq \frac{3}{16\pi} \quad \text{and} \quad \sup_{a,b\in V_0} E(a,b,\Omega) \leq \frac{3}{32\pi}$$

for any multiply connected domain $\Omega$ in $\mathbb{R}^2$. Moreover in the light of [9], this theorem implies that the embedding of Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ into $H^{-1}(\mathbb{R}^2)$ is not compact. Indeed, let $(a_n, b_n) \in V_0 \times V_0$ be a maximizing sequence of $E$ in $V_0 \times V_0$. Clearly, $\{a_n, b_n\}$ is bounded in $\mathcal{H}^1(\mathbb{R}^2)$, but it does not converge strongly in $H^{-1}(\mathbb{R}^2)$.

In the following, we consider a multiply connected domain $\Omega$. We set $m(\Omega) = \sup_{a,b\in V_0} E(a,b,\Omega)$. The analogue of Theorem 4.1 is following result.

THEOREM 4.5. *Let $\Omega$, $\Omega_1$ be two smooth bounded domains such that $\Omega \subseteq \Omega_1$. Then $m(\Omega) \leq m(\Omega_1)$. Moreover, we have*

$$m(\Omega) = \frac{3}{32\pi}. \tag{4.3}$$

*Furthermore, the best constant is not achieved in $V_0 \times V_0$.*

*Proof.* Let $a$, $b$ be two functions in $H_0^1(\Omega)$. We define a embedding of $H_0^1(\Omega)$ into $H_0^1(\Omega_1)$ as follows, to any $\alpha \in H_0^1(\Omega)$, we associated $\bar{\alpha} \in H_0^1(\Omega_1)$ such that

$$\begin{cases} \bar{\alpha}(x,y) = \alpha(x,y), & \text{if } (x,y) \in \Omega \\ \bar{\alpha}(x,y) = 0, & \text{if } (x,y) \notin \Omega. \end{cases}$$

We define a energy functional $E_1$ on $H_0^1(\Omega_1)$ by following:

$$E_1(\beta) = \frac{1}{2}\int_{\Omega_1} |\nabla\beta|^2 - \int_{\Omega_1} \{\bar{a}, \bar{b}\}\beta,$$

where $\beta \in H_0^1(\Omega_1)$. We denote $\varphi_1$ the unique solution of equation (1.1) in $H_0^1(\Omega_1)$, i.e.

$$\begin{cases} -\triangle\varphi_1 &= \{\bar{a}, \bar{b}\}, & \text{in } \Omega_1 \\ \varphi_1 &= 0, & \text{on } \partial\Omega_1. \end{cases} \tag{4.4}$$

Recall that $\varphi_1$ is the unique minimal point of functional $E_1$. Thus, we get $E_1(\bar{\varphi}) \geq E_1(\varphi_1)$ where $\varphi$ is the unique solution of equation (1.1) in $H_0^1(\Omega)$. Therefore, we obtain that

$$\begin{aligned} E_1(\bar{\varphi}) &= \frac{1}{2}\int_{\Omega} |\nabla\varphi|^2 - \int_{\Omega} \{a, b\}\varphi \\ &= \frac{1}{2}\int_{\Omega} |\nabla\varphi|^2 - \int_{\Omega} (-\triangle\varphi)\varphi \\ &= -\frac{1}{2}\int_{\Omega} |\nabla\varphi|^2. \end{aligned}$$

Similarly, $E_1(\varphi_1) = -\frac{1}{2}\int_{\Omega_1} |\nabla\varphi_1|^2$.

Consequently, we deduce that

$$\|\nabla\varphi\|^2_{L^2(\Omega)} = \|\nabla\bar{\varphi}\|^2_{L^2(\Omega_1)} \leq \|\nabla\varphi_1\|^2_{L^2(\Omega_1)}.$$

But, stating that $\|\nabla\bar{a}\|^2_{L^2(\Omega_1)} = \|\nabla a\|^2_{L^2(\Omega)}$ and $\|\nabla\bar{b}\|^2_{L^2(\Omega_1)} = \|\nabla b\|^2_{L^2(\Omega)}$, we conclude that

$$E(a, b, \Omega) \leq E(\bar{a}, \bar{b}, \Omega_1),$$

that is, $m(\Omega) \leq m(\Omega_1)$. Now we choose $B(z_0, r_0) = \{z \in \mathbb{C} \mid |z - z_0| < r_0\}$ and $B(z_1, r_1) = \{z \in \mathbb{C} \mid |z - z_1| < r_1\}$ such that $B(z_0, r_0) \subseteq \Omega \subseteq B(z_1, r_1)$. Thus, we obtain

$$\frac{3}{32\pi} = m(B(z_0, r_0)) \leq m(\Omega) \leq m(B(z_1, r_1)) = \frac{3}{32\pi}.$$

Hence, (4.3) follows.

We suppose that the best constant is achieved in the point $(a, b) \in V_0 \times V_0$. It is clear that

$$\frac{3}{32\pi} = E(a, b, \Omega) \leq E(\bar{a}, \bar{b}, B(z_1, r_1)) \leq \frac{3}{32\pi}.$$

Then in the point $(\bar{a}, \bar{b})$, the best constant is achieved in $V_0(B(z_1, r_1)) \times V_0(B(z_1, r_1))$. By Theorem 4.1, we obtain a contradiction. Thus, the theorem is proved. □

Now we write $\Omega = B - \bigcup_{i=1}^{n} \overline{\Omega}_i$ where $\overline{\Omega}_i \subset B$ for $i = 1$ to $n$ is simply connected. We will show the following result.

THEOREM 4.6. *Under the above notations, we have*

$$C_2(\Omega) = \frac{3}{16\pi}. \tag{4.5}$$

*Proof.* Set $a = \dfrac{x}{r^2+1}$ and $b = \dfrac{y}{r^2+1}$. Then, it is clear that the unique solution in $H_0^1(B)$ of (1.1) is

$$\varphi = \frac{1-r^2}{8(r^2+1)}.$$

Choosing a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $0 < t_n < 1$ and $t_n \longrightarrow 1$ as $n \longrightarrow \infty$. We define the maps $T_n$ by:

$$T_n(z) = \frac{z - t_n}{1 - t_n z}$$

which are conformal transformations from $B$ to $B$. Denote $a_n = a \circ T_n$ and $b_n = b \circ T_n$, clearly,

$$E(a_n, b_n, B) = \frac{3}{16\pi}$$

and the unique solution of (1.1) for $a_n$ and $b_n$ is $\varphi_n = \varphi \circ T_n$. Clearly, $a_n - ⨍_\Omega a_n$, $b_n - ⨍_\Omega b_n$ and $\varphi_n$ tend to 0 weakly in $H^1$. Let $\bigcup_{i=1}^{n} \overline{\Omega}_i \subset B(0, r)$. Choosing $\xi \in C^\infty(\mathbb{R}^2)$ such that $0 \le \xi \le 1$, $supp(\xi) \subset \mathbb{R}^2 \setminus B(0, r)$ and $\xi = 1$ on $\mathbb{R}^2 \setminus B(0, r')$ with $r < r' < 1$. Setting $\tilde{a}_n = \xi a_n$ and $\tilde{b}_n = \xi b_n$ and $\tilde{\varphi}_n$ the unique solution of (1.1) for $a = \tilde{a}_n$ and $b = \tilde{b}_n$ in $H_0^1(\Omega)$. Therefore, it is easy to obtain (see Lemma 7.5 below)

$$\lim_{n\to\infty} \|\nabla(\tilde{\varphi}_n - \xi^2 \varphi_n)\|_{L^2(\Omega)} = 0,$$

since $\varphi_n \longrightarrow 0$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. A simple computation leads to

$$\lim_{n\to\infty} \frac{\|\nabla(\xi^2\varphi_n)\|_2^2}{\|\nabla\tilde{a}_n\|_2^2 \|\nabla\tilde{b}_n\|_2^2} = \frac{3}{16\pi}.$$

Thus, we deduce that

$$\lim_{n\to\infty} E(\tilde{a}_n, \tilde{b}_n, \Omega) = \frac{3}{16\pi}.$$

On the other hand,

$$C_2(\Omega) \le \frac{3}{16\pi}.$$

Hence, (4.5) is proved. □

## 5. Generalization on manifolds

Recall first some definitions and notations (see [2]). Let $(M, g)$ be a smooth two dimensional Riemannian manifold without boundary. Let $\{x^i\}$ $(i = 1, 2)$ be a local coordinate system. We can write $g$ as following:

$$g = g_{ij} dx^i \otimes dx^j,$$

Where $g^{ij}$ are the components of inverse matrix of the metric matrix $(g_{ij})$.

Assume that $M$ is oriented and $A$ an atlas compatible with orientations. In the coordinate system $\{x^i\}$ corresponding to $(\Omega, \varphi) \in A$, define the differential 2-form by

$$dV = \eta = \sqrt{|g|} dx^1 \wedge dx^2, \tag{5.1}$$

where $|g|$ is the determinant of the metric matrix $(g_{ij})$. $\eta$ is called oriented volume element, denoted by $dV$. In the following, we will use a local isothermal coordinate. Let $\alpha \in \wedge^p(M)$. We associate to $\alpha$, a $(2-p)$-form $*\alpha$, called the adjoint of $\alpha$, defined as follows:

$$*1 = \eta, *dx^1 = dx^2, *dx^2 = -dx^1, *\eta = 1. \tag{5.2}$$

Now, we define $\delta\alpha$ by

$$\delta\alpha = (-1)^p *^{-1} d * \alpha, \text{ where } p = deg(\alpha). \tag{5.3}$$

Then, the Laplacian operator $\Delta$ is defined by

$$\Delta_g = d\delta + \delta d. \tag{5.4}$$

Assume that $p = 0$, clearly in a chart, we have

$$\Delta_g = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right). \tag{5.5}$$

Moreover, let $M$ be compact, we define the global scalar product $\langle \alpha, \beta \rangle$ of two $p$-forms $\alpha$ and $\beta$, as follows:

$$\langle \alpha, \beta \rangle = \int_M (\alpha, \beta) \eta.$$

Now we consider the vector space of smooth functions. We denote $H = \{\varphi \in C^\infty(M, \mathbb{R}), \|\varphi\|_{H^1} < \infty\}$ where

$$\|\varphi\|_{H^1} = \int_M (g^{ij} (d\varphi)_i (d\varphi)_j + \varphi^2) \eta = \int_M (\nabla^i \varphi \nabla_i \varphi + \varphi^2) \eta.$$

The Sobolev space $H^1(M)$ is completion of $H$ with respect to the norm $\|\|_{H^1}$. In fact, $H^1$ is independent on the metric $g$. Then we have the Sobolev embedding theorem and the Kondrakov theorem, that is,

Lemma 5.1. *For any $p < \infty$, the embedding $H^1(M) \hookrightarrow L^p(M)$ is compact.*

On the manifold $M$, we consider the Dirichlet problem, that is, to solve the following linear elliptic equations

$$\begin{cases} \triangle_g \varphi = f, \\ \displaystyle\int_M \varphi dV = 0, \end{cases} \tag{5.6}$$

where $f \in L^2(M)$.

It is well known that there exists a unique weak solution $\varphi \in H^1$ of (5.6) if and only if $\int_M f = 0$. Moreover, if $f \in C^{r+\alpha}$ then $\varphi \in C^{2+r+\alpha}$ ($r \geq 0$ an integer and $1 > \alpha > 0$).

We denote $H_0^1(M) = \{a \in H^1(M),\ \int_M a dV = 0\}$ and define $\{a, b\}_g$ as follows:

$$\{a, b\}_g = *(da \wedge db), \tag{5.7}$$

where $a$, $b \in H_0^1(M)$. Thus, if in the chart $U$ $(M, g)$ is conformal to the Euclidian metric, under corresponding local coordinate system, we can write

$$\{a, b\}_g = \frac{1}{\sqrt{|g|}}(a_{x^1} b_{x^2} - b_{x^1} a_{x^2}) \quad \text{and} \quad \triangle_g = -\frac{1}{\sqrt{|g|}}\left(\frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2}\right).$$

We consider the following equation:

$$\begin{cases} \triangle_g \varphi = \{a, b\}_g, \\ \displaystyle\int_M \varphi dV = 0. \end{cases} \tag{5.8}$$

We will generalize Wente's inequality on the manifold $M$. Our result is the following.

THEOREM 5.2. *There exists a unique solution $\varphi \in H_0^1$ of (5.8). Furthermore, the solution is continuous on $M$ and there exists a constant $C_0(M)$ which depends on $M$ such that*

$$\|\varphi\|_\infty + \|\nabla\varphi\|_2 \leq C_0(M)\|\nabla a\|_2\|\nabla b\|_2, \tag{5.9}$$

*where* $\|\nabla a\|_2^2 = \int_M g^{ij}(da)_i(da)_j dV = \int_M \nabla^i a \nabla_i a dV$ *for any* $a \in H^1(M)$.

In the proof, we will use Green's function. First, we give some properties of Green's function on manifolds.

LEMMA 5.3. *Under the above notations, there exists $G(P, Q)$ a Green's function of the Laplacian which has the following properties:*

(i) *for all functions $\varphi \in C^2$*

$$\varphi(P) = V^{-1}\int_M \varphi(Q)dV(Q) + \int_M G(P, Q)\Delta_g\varphi(Q)dV(Q), \tag{5.10}$$

*where $V$ is the volume of the manifold $M$,*

(ii) *$G(P, Q)$ is $C^\infty$ on $M \times M$ minus the diagonal (for $P \neq Q$),*

(iii) *there exists a constant $K$ such that*

$$\begin{cases} |G(P, Q)| & < K(1 + |\log r|), \\ |\nabla_Q G(P, Q)| & < Kr^{-1}, \\ |\nabla_Q^2 G(P, Q)| & < Kr^{-2}, \end{cases} \tag{5.11}$$

*where $r = d(P, Q)$,*

(iv) *there exists a constant $B$ such that $G(P, Q) \geq B$. Since the Green's function is defined up to a constant, we can thus choose the Green's function so that its integral equals to zero,*

(v) *$G(P, Q) = G(Q, P)$.*

*Proof.* (of Theorem 5.2). Set $a,\, b \in C^\infty(M)$. First, by Stokes' Formula, we see that

$$\int_M \{a,b\}_g dV = \int_M da \wedge db = \int_M d(a \wedge db) = 0.$$

Thus there exists the unique $C^\infty$ solution of (5.8). On the other hand, there exists $r_0 > 0$ such that for any $P \in M$ the set $B(P, 2r_0) = \{Q \in M,\ d(P,Q) < 2r_0\}$ is included in a local chart where $g$ is conformal to the Euclidian metric and corresponding coordinate system is $\{x^i\}(i = 1,2)$. First, we assume that there exists $P_1 \in M$ such that $supp(a) \subset B(P_1, \frac{r_0}{4})$. We divide $M$ into two parts, that is, $M = M_1 \cup M_2$ where $M_1 = \{Q \in M,\ d(P_1,Q) \leq \frac{r_0}{2}\}$ and $M_2 = \{Q \in M,\ d(P_1,Q) \geq \frac{r_0}{2}\}$.
*Case 1: $P \in M_2$.* Hence, applying Lemma 5.3, we conclude that

$$\begin{aligned} |\varphi(P)| &= \left| \int_M G(P,Q)\Delta_g \varphi(Q) dV(Q) \right| \\ &= \left| \int_{M \setminus B(P,\frac{r_0}{4})} G(P,Q)\Delta_g \varphi(Q) dV(Q) \right| \\ &= \left| \int_{M \setminus B(P,\frac{r_0}{4})} G(P,Q) da \wedge db \right| \\ &\leq CK\left(1 + \log\left|\frac{r_0}{4}\right|\right) \|\nabla a\|_2 \|\nabla b\|_2. \end{aligned} \tag{5.12}$$

*Case 2: $P \in M_1$.* We consider the solution $\varphi_1$ of the following equation:

$$\begin{cases} \triangle_g \varphi_1 = \{a,b\}_g, \text{ on } B(P,r_0), \\ \varphi_1 = 0, \text{ on } \partial B(P,r_0). \end{cases} \tag{5.13}$$

So $\triangle_g(\varphi - \varphi_1) = 0$ on $B(P,r_0)$. Using the maximum principle, we obtain

$$\|\varphi - \varphi_1\|_{L^\infty(B(P,r_0))} \leq \|\varphi - \varphi_1\|_{L^\infty(\partial B(P,r_0))} = \|\varphi\|_{L^\infty(\partial B(P,r_0))}. \tag{5.14}$$

However, by Theorem 1.1 and using the conformal chart $\{x^i\}(i = 1,2)$, we have

$$\|\varphi_1\|_{L^\infty(B(P,r_0))} \leq C\|\nabla a\|_{L^2(B(P,r_0))} \|\nabla b\|_{L^2(B(P,r_0))} = C\|\nabla a\|_2 \|\nabla b\|_2. \tag{5.15}$$

Combining (5.12), (5.14) and (5.15), we get

$$\|\varphi\|_{L^\infty(B(P,r_0))} \leq C\|\nabla a\|_2 \|\nabla b\|_2.$$

*In general case:* using partition of unity, we deduce

$$\|\varphi\|_\infty \leq C\|\nabla a\|_2 \|\nabla b\|_2.$$

Finally,

$$\begin{aligned} \|\nabla\varphi\|_2^2 &= \int_M \varphi \Delta_g \varphi = \int_M \varphi\{a,b\}_g = \int_M \varphi da \wedge db \\ &\leq C\|\varphi\|_\infty \|\nabla a\|_2 \|\nabla b\|_2. \end{aligned}$$

Thus, by density, the conclusion follows. □

Now, we consider the energy functional $E(a,b,M)$ and $C_2(M)$ defined as the same way as before, i.e.,

$$E(a,b,M)=\frac{\|\nabla\varphi\|_2^2}{\|\nabla a\|_2^2\|\nabla b\|_2^2},$$

and

$$C_2(M)=\sup_{a,b\in H_0^1} E(a,b,M).$$

First, we will give the Euler equation for critical point.

DEFINITION 5.4. A point $(a,b)\in H_0^1\times H_0^1$ is critical for the energy functional $E$ if it satisfies the following condition:

$\nabla E(a+t\alpha,b+s\beta,M)|_{(s,t)=(0,0)}=0$, for all $\alpha$, $\beta\in H_0^1(M)$.

Clearly, $E$ is also invariant under a conformal transformation of $M$ and $E(\lambda a,\mu b,M)=E(a,b,M)$ for all $\lambda$, $\mu\in\mathbb{R}^*$, so without loss of generality, we can assume that $\|\nabla a\|_2=\|\nabla b\|_2=1$.

THEOREM 5.5. *Assume that $(a,b)\in H_0^1\times H_0^1$ is a critical point of $E$ such that $\varphi\neq 0$; then there exists $\lambda\in\mathbb{R}^*$ such that*

(i) $\displaystyle\int_M(\nabla a,\nabla b)=0$,

(ii) *denote $\Psi=(a_1,b_1,\varphi_1)=(\lambda a,\lambda b,\lambda^2\varphi)$. Then we have*

$$\left\{\begin{array}{ll}\triangle_g\varphi_1 & =\{a_1,b_1\}_g,\\ \triangle_g a_1 & =\{b_1,\varphi_1\}_g,\\ \triangle_g b_1 & =\{\varphi_1,a_1\}_g,\end{array}\right. \tag{5.16}$$

(iii) *if $M$ is a surface homeomorphic to $S^2$, then $\Psi$ is conformal.*

We need some similar technical lemmas as Lemmas 3.3 to 4.3.

LEMMA 5.6. *If $\varphi\in H^1(M)\cap L^\infty(M)$, $a\in H^1(M)\cap L^\infty(M)$ and $b\in H^1(M)$, then we have*

$$\int_M\varphi\{a,b\}_g=\int_M a\{b,\varphi\}_g. \tag{5.17}$$

*Proof.* Setting $\varphi$, $a$, $b\in C^2(M)$, we have

$$\begin{aligned}\int_M\varphi\{a,b\}_g &=\int_M\varphi da\wedge db\\ &=\int_M d(a\varphi)\wedge db-\int_M ad\varphi\wedge db\\ &=\int_M d(a\varphi b)+\int_M a\{b,\varphi\}_g\\ &=\int_M a\{b,\varphi\}_g \text{ (Stokes' Formula).}\end{aligned}$$

However, we see that

$$\begin{aligned}\left|\int_M\varphi\{a,b\}_g\eta\right| &\le\|\varphi\|_\infty\|\nabla a\|_2\|\nabla b\|_2,\\ \left|\int_M a\{b,\varphi\}_g\eta\right| &\le\|a\|_\infty\|\nabla\varphi\|_2\|\nabla b\|_2.\end{aligned}$$

By approximation, (5.17) follows. ☐

LEMMA 5.7. *(see [16]). Let $\Sigma$ be a surface homeomorphic to $S^2$ with a metric tensor given in the local coordinates by bounded measurable functions satisfying*

$$g_{11}g_{22} - g_{12}^2 \geq \lambda > 0 \text{ almost everywhere.}$$

*Then there is a homeomorphism $h : S^2 \longrightarrow \Sigma$ satisfying the conformality relations*

$$\begin{cases} g_{ij}\dfrac{\partial h^i}{\partial x}\dfrac{\partial h^j}{\partial x} & = g_{ij}\dfrac{\partial h^i}{\partial y}\dfrac{\partial h^j}{\partial y}, \\ g_{ij}\dfrac{\partial h^i}{\partial x}\dfrac{\partial h^j}{\partial y} & = 0, \end{cases} \tag{5.18}$$

*almost everywhere.*
*If $(g_{ij}) \in C^\alpha$, then $h$ is a diffeomorphism of class $C^{1,\alpha}$, satisfying (5.18) everywhere. If $\Sigma$ is of class $C^{k,\alpha}$, $C^\infty$ or $C^\omega$, then so is $h$.*

*Proof.* (of Theorem 5.5). We only need to prove the property (iii). The proof of other assertions is the same that for Theorem 3.2. Thanks to Lemma 5.7, for simplicity, we can assume that $M$ is $S^2$. We use the coordinates of stereographic projection, that is,

$$\begin{array}{rlcl} P: & \mathbb{R}^2 & \longrightarrow & S^2 - (0,0,-1) \\ & (x,y) & \longmapsto & \left(\dfrac{2x}{1+r^2}, \dfrac{2y}{1+r^2}, \dfrac{1-r^2}{1+r^2}\right). \end{array}$$

With these coordinates, we have

$$-\Delta\Psi = \Psi_x \wedge \Psi_y.$$

Hence, we define the Hopf's differential $\omega$ by $\omega = |\Psi_x|^2 - |\Psi_y|^2 - 2i\langle\Psi_x, \Psi_y\rangle$. Clearly, a simple computation leads

$$\partial_{\bar{z}}\omega = 0.$$

So, $\omega$ is holomorphic on $\mathbb{R}^2$. On the other hand,

$$\omega(x,y) = \frac{-4}{(x+iy)^4}\langle(\partial_{\bar{z}'})\Psi(x',y'), (\partial_{\bar{z}'})\Psi(x',y')\rangle,$$

where $(x',y') = (\frac{x}{r^2}, \frac{y}{r^2})$ and $z' = x' + iy'$. Therefore,

$$\lim_{|z|\to\infty} \omega(z) = 0.$$

Thus, the conclusion follows. □

REMARK 5.8. By Lemma 3.4, $\Psi$ is $C^\infty$.

Actually, we calculate $C_2(M)$. In fact, we show that $C_2(M)$ is independent on the compact manifold $M$.

THEOREM 5.9. *Let $M$ be a compact oriented Riemannian surface; then*

$$C_2(M) = \frac{3}{32\pi}. \tag{5.19}$$

*Proof.* Denote $\Psi = \left(\frac{a}{\|\nabla a\|_2}, \frac{b}{\|\nabla b\|_2}, \frac{\varphi}{\|\nabla \varphi\|_2}\right) = (a_1, b_1, \varphi_1)$. Thus, the area of surface $\Psi(M)$ is:

$$\begin{aligned} A(\Psi) &= \sqrt{\int_M \{a_1, b_1\}_g^2 + \{b_1, \varphi_1\}_g^2 + \{\varphi_1, a_1\}_g^2 dV} \\ &\leq \frac{1}{2}\int_M (d\Psi, d\Psi) dV = \frac{3}{2}. \end{aligned}$$

On the other hand, the oriented volume bounded by $\Psi(M)$ is

$$\begin{aligned} V(\Psi) &= \frac{1}{3}\int_M (\varphi_1\{a_1, b_1\}_g + a_1\{b_1, \varphi_1\}_g + b_1\{\varphi_1, a_1\}_g) dV \\ &= \int_M \varphi_1\{a_1, b_1\}_g dV \\ &= \frac{1}{\|\nabla a\|_2 \|\nabla b\|_2 \|\nabla \varphi\|_2}\int_M \varphi da \wedge db = \frac{\|\nabla \varphi\|_2}{\|\nabla a\|_2 \|\nabla b\|_2\|}. \end{aligned}$$

In view of the isoperimetric inequality, we have

$$|V(\Psi)|^2 \leq |A(\Psi)|^3.$$

Hence,

$$E(a, b, M) \leq \frac{3}{32\pi}.$$

Now, fix $Q \in M$. Choose a local chart $U$ of $Q$ which is conformal to an open subset $W$ of $\mathbb{R}^2$. Denote by $\{x_i\}(i = 1, 2)$ the corresponding coordinates. Choose a function $\xi \in C_0^\infty(W)$. Set

$$a_\varepsilon = \xi(x_1, x_2)\frac{\varepsilon x_1}{\varepsilon^2 + r^2} \quad \text{and} \quad b_\varepsilon = \xi(x_1, x_2)\frac{\varepsilon x_1}{\varepsilon^2 + r^2}$$

where $r^2 = x_1^2 + x_2^2$. It is easy to check that

$$\lim_{\varepsilon \to 0} E(a_\varepsilon, b_\varepsilon, M) = \frac{3}{32\pi}.$$

Hence, the theorem is proved. □

THEOREM 5.10. *If $M$ is not homeomorphic to $S^2$, the maximum is not achieved.*

First, we need a result of Hartman and Wintner.

LEMMA 5.11. *(see [14]). Let $L = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{2} b_i(x)\frac{\partial}{\partial x_i} + c(x)$, where $b_i$ and $c$ be continuous functions in $B$. Let $u$ be a solution of class $C^2$ for the equation*

$$L(u) = 0, \text{ in } B,$$

*or, more generally, let $u$ be a function of class $C^1$ satisfying*

$$\int_J \frac{\partial u}{\partial x_2} dx_1 - \int_J \frac{\partial u}{\partial x_1} dx_2 = \int_E \left(\sum_{i=1}^{2} b_i \frac{\partial u}{\partial x_i} + cu\right) dx_1 dx_2,$$

*for every domain $E$ bounded by a piecewise smooth $C^1$ Jordan curve $J$, contained in $B$. Then if $u$ satisfies*

$$u(x) = o(| x |^n), \text{ for some } n \in \mathbb{N}, \tag{5.20}$$

*it must satisfy that*

$$\lim_{x \to 0} \frac{\partial_z u}{z^n}(x) \text{ exists,} \tag{5.21}$$

*where $z = x_1 + ix_2$ and $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$. Moreover, if $u \not\equiv 0$, then $\exists n' \in \mathbb{N}$, such that*

$$\limsup_{x \to 0} \frac{| u(x) |}{| x |^{n'}} > 0. \tag{5.22}$$

*Proof.* (of Theorem 5.10). Suppose that $\Psi = (a, b, \varphi)$ is a maximum of $E$ with $\|\nabla a\|_2 = \|\nabla b\|_2 = \|\nabla \varphi\|_2 = \sqrt{32\pi/3}$. From the proof of Theorem 5.9, $\Psi$ is a conformal map and $\Psi(M)$ is a sphere $S_2^2$ with radius equal to 2. By the property of degree, we deduce that

$$\deg(\Psi) = \frac{\int_M \Psi^* \Omega}{\int_{S_2^2} \Omega},$$

where $\Omega = \frac{1}{2}(x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2)$ is the area element on the sphere $S_2^2$. A simple calculation leads to $\deg(\Psi) = 1$ if we choose a suitable orientation on the sphere. On the other hand, if $(x_0, y_0)$ is a branch-point, using Lemma 5.11, we obtain that there exists $n \in \mathbb{N}^*$ and $c \in \mathbb{C}^3 - \{0\}$ such that

$$\partial_z \Psi = c(z - z_0)^n + o((z - z_0)^n),$$

where $z_0 = x_0 + iy_0$. Thus, the branch-points are isolated. By the condition of conformality and using the stereographic coordinates, we conclude that $\det(\partial\Psi/\partial x) \geq 0$ and $\Psi$ is a harmonic map. Moreover, $\Psi$ is holomorphic. We claim that $\Psi$ has no branch-points. Otherwise, there exists $c \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$ such that

$$\tilde{\Psi}(z) = c(z - z_0)^{n+1} + o((z - z_0)^{n+1}),$$

where $\tilde{\Psi}$ is the stereographic coordinates on the sphere and $z_0$ is a branch-point. This contradicts the fact that the degree of $\Psi$ is equal to 1. Hence, we deduce that $\Psi$ is a covering map since $M$ is compact. And since the degree of $\Psi$ is one, it is a diffeomorphism. This is a contradiction. □

Corollary 5.12. *If $\Omega$ is a multiply connected domain in $\mathbb{R}^2$, then $C_2(\Omega)$ can not be achieved.*

*Proof.* Suppose that $\Psi = (a, b, \varphi)$ is a maximum of $E$. In view of Theorem 3.2, $\Psi$ is $C^\infty$. We construct a compact oriented Riemannian surface $M = \Omega \bigcup_{\partial\Omega} \tilde{\Omega}$ by sticking $\Omega$ and a copy of $\Omega$, provided with opposing orientation. We define a $C^\infty$ map $\tilde{\Psi}$ on $M$ by

$$\tilde{\Psi} = \Psi \text{ on } \Omega \text{ and } \tilde{\Psi} = (a, b, -\varphi) \text{ on } \tilde{\Omega}.$$

Thus, $\tilde{\Psi}$ is a maximum of $E$ on $M$. The result follows from the previous theorem. □

# Part B. Compact $H$-surfaces in Euclidean space

## 6. Precise statement of the problem and setting of the results

In this part, we consider the following equation

$$-\triangle u = u_x \wedge u_y, \text{ in } \Omega, \tag{6.1}$$

where $u \in C^2(\Omega; \mathbb{R}^3)$. The equation (6.1) is satisfied by surfaces of mean curvature $\frac{1}{2}$ in $\mathbb{R}^3$ in conformal representation. Thus we will call (6.1) the incomplete $H$-system. Moreover, it is of variational type. The classical energy functional associated with this equation is

$$E_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dxdy - \frac{1}{3}\int_\Omega u \cdot u_x \wedge u_y dxdy.$$

As before, we study a new variational approach of (6.1) proposed by Hélein in [15]. In fact, in view of Theorem 1.1, we can consider the new energy functional $E(a, b, \Omega)$, or equivalently,

$$F(a,b,\Omega) = \frac{\|\nabla a\|^2_{L^2(\Omega)} + \|\nabla b\|^2_{L^2(\Omega)}}{2\|\nabla\varphi\|_{L^2(\Omega)}}, \text{ defined for } a, b \in H^1(\Omega),$$

or,

$$F_1(a,b,\Omega) = \frac{1}{2}(\|\nabla a\|^2_{L^2(\Omega)} + \|\nabla b\|^2_{L^2(\Omega)}), \text{ defined for all } a, b \in M,$$

where $M = \{(a,b) \in H^1 \times H^1, \|\nabla\varphi\|_{L^2} = 1\}$. Recall that, by Theorem 3.2, we can recognize in (6.1) the Euler-Lagrange equation associated to the critical points of these functionals, through the substitution $u = (\lambda a, \lambda b, \lambda^2\varphi)$ for $\lambda = \|\nabla b\|_2$. Moreover, we have

$$\varphi = \frac{\partial a}{\partial n} = \frac{\partial b}{\partial n} = 0 \text{ on } \partial\Omega, \tag{6.2}$$

where $n = (n_1, n_2)$ is the normal vector on $\partial\Omega$. The conditions on boundary allow us to construct a solution of (6.1) from a compact oriented Riemannian surface into $\mathbb{R}^3$ by sticking two copies of $\Omega$. Thus, if $\Omega$ is an annulus, we may expect to find again Wente's torus, which is an immersion of a torus into $\mathbb{R}^3$ with a constant mean curvature. For this purpose, we will look for critical points of $F$ on an annulus.

Our first task is to study a minimizing sequence for $F$. In part one, we saw that the minimum of $F$ is a universal constant for any bounded and smooth domain. Here we will deal with a minimizing sequence for the energy functional $F_1$ and we will show that we can not minimize the energy functional $F$ on a multiply connected domain. Our first result provides a complete description of a minimizing sequence.

Theorem 6.1. *If $\Omega$ is simply connected, there exists some $(a, b, \varphi)$ which is solution of (6.1) such that*

$$F(a,b,\Omega) = G(\Omega) = \inf_{a,b\in M} F(a,b,\Omega).$$

*Moreover, if* $(a_n, b_n, \varphi_n)$ *is a minimizing sequence for* $F$ *with* $(a_n, b_n) \in M$ *and* $\int_\Omega a_n = \int_\Omega b_n = 0$, *then* $(a_n, b_n, \varphi_n)$ *up to conformal transformations is relatively compact in* $H^1$. *If* $\Omega$ *is multiply connected, then there exists* $x_0 \in \partial\Omega$ *such that*

$$(a_n, b_n, \varphi_n) \longrightarrow (G(\Omega), G(\Omega), 1)\delta_{x_0} \ \ in\ \mathcal{D}'(\mathbb{R}^2),$$

*where* $\delta_{x_0}$ *is the Dirac-mass of mass 1 concentrated at* $x_0$.

REMARK 6.2. Clearly,

$$C_2(\Omega) = \frac{1}{G(\Omega)^2}.$$

Thus, we see that concentration phenomena occur for a minimizing sequence. In some way, our problem is similar to the problem of the best constant of Sobolev embedding for the limiting case. For a multiply connected domain, we can not produce a solution of (6.1) by minimizing this energy. So we must study the compactness properties of $F$ at higher energy levels as well. The next result is to analyze the behavior of a Palais-Smale sequence. It can be viewed as an extension of P.-L. Lions' concentration compactness method for minimizing problems. A similar phenomenon had been observed by M. Struwe [19] in the context of Sobolev embedding for the limiting case. Our proof is inspired by the method of concentration compactness.

THEOREM 6.3. $F_1$ *satisfies the Palais-Smale condition for all* $C \in (G(\Omega), \sqrt{2}G(\Omega))$.

The value $\sqrt{2}G(\Omega)$ is optimal in the following sense. Let $\Omega = D = \{(x,y); x^2 + y^2 < 1\}$ be the unit disc. Let $u = (a, b, \varphi)$ be a solution of (6.1) satisfying the boundary conditions (6.2). After an extension by symmetry has been performed, we are led to a finite energy solution of (6.1) on all of $\mathbb{R}^2$. In view of H. Brezis and J.-M. Coron's result, we deduce that there exists $k \in \mathbb{N}^*$ such that $F(a, b, \Omega) = \sqrt{k}G(\Omega)$. Now let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$ converging to 1 as $n$ tends to infinity. After the Möbius transformations $\sigma_n(z) = \frac{z-t_n}{1-t_n z}$ with $z = x + iy$, we obtain a sequence $(a_n, b_n) = (a \circ \sigma_n, b \circ \sigma_n)$ in $H^1 \times H^1$. Obviously, $(a_n, b_n)$ is a Palais-Smale sequence. But it is not compact in $H^1 \times H^1$. It proves that Palais-Smale condition fails at the energy values $\gamma = \sqrt{k}G(\Omega)$. Now, with the help of Theorem 6.3, we can prove our main result in this part.

THEOREM 6.4. *Let* $\Omega = D \setminus \bigcup_{i=1}^n B(x_i, r_i)$ *be a multiply connected domain in* $\mathbb{R}^2$. *Assume that the set of points* $\{x_i\}$ *is fixed. Then, there exists* $\varepsilon > 0$ *such that if* $r_i < \varepsilon$ *for all* $i = 1, ..., n$ *and there exists a solution of (6.1) satisfying the boundary conditions (6.2).*

A similar conclusion for Sobolev embedding has been obtained by J.-M. Coron [8]. Here we will use the same strategy. For $t \geq G(\Omega)$ denote by $E_M^t = \{(a,b) \in M / F_1(a,b) \leq t\}$ the level set of $F_1$. In fact, the topology of $E_M^\gamma$ is equivalent to $\partial\Omega$ when $\gamma$ is near $G(\Omega)$. We will argue by contradiction. We will construct a topological disc $\Delta$ in $E_M^{\sqrt{2}G(\Omega)}$ whose boundary is a non contractible circle $\partial\Delta$ in $E_M^{G(\Omega)}$. And if the system (6.1), (6.2) does not admit a solution in $E_M^{\sqrt{2}G(\Omega)}$, then it implies that there exists a contraction $h$ of $\Delta$ onto $\partial\Delta$, which is a contradiction.

This part is organized as follows. In the section 7, we prove Theorem 6.1. In the section 8, we establish Theorem 6.3. In the section 9, we show Theorem 6.4. In the last section, we describe some additional properties for a solution of equation (6.1) and (6.2).

## 7. Study of a minimizing sequence

Now we consider the minimum of energy functional $F$. Let $(a_n, b_n, \varphi_n)$ be a minimizing sequence, that is, $(a_n, b_n, \varphi_n)$ satisfying the equation (1.1) and

$$F(a_n, b_n, \Omega) = G(\Omega) + o(1).$$

Without loss of generality, we can assume that

$$(a_n, b_n) \in M \text{ and } \int_\Omega a_n = \int_\Omega b_n = 0.$$

After extracting a subsequence, we may assume that

$$\begin{aligned} a_n &\longrightarrow \alpha \text{ weakly in } H^1 \text{ and strongly in } L^2, \\ b_n &\longrightarrow \beta \text{ weakly in } H^1 \text{ and strongly in } L^2, \\ \varphi_n &\longrightarrow \psi \text{ weakly in } H^1 \text{ and strongly in } L^2. \end{aligned}$$

We will show the following result.

Theorem 7.1. *Under the above assumptions, we have the alternative:*

(i) *if $\psi = 0$, then $\alpha = \beta = 0$,*
*or*
(ii) *if $\psi \neq 0$, then $(\alpha, \beta, \psi)$ is a minimum of energy $F$. Moreover, the following holds:*

$$\begin{aligned} a_n &\longrightarrow \alpha \textit{ strongly in } H^1, \\ b_n &\longrightarrow \beta \textit{ strongly in } H^1, \\ \varphi_n &\longrightarrow \psi \textit{ strongly in } H^1. \end{aligned}$$

First, we recall a technical lemma.

Lemma 7.2. *(see [22] and also [7]). We assume that $\varphi_n$ is a bounded sequence in $H_0^1 \cap L^\infty$. Let $a_n \longrightarrow 0$ weakly in $H^1$ and strongly in $L^2$. Then for every $b \in H^1$, we have*

$$\lim_{n\to\infty} \int \varphi_n\{a_n, b\} = 0. \tag{7.1}$$

*Proof.* We state that

$$\left| \int \varphi\{a, b\} \right| \le \|\varphi\|_\infty \|\nabla a\|_2 \|\nabla b\|_2 \text{ for all } \varphi \in H_0^1 \cap L^\infty, a \in H^1, b \in H^1.$$

Given $\varepsilon > 0$, we fix $\bar{b} \in C^\infty(\bar{\Omega})$ such that $\|b - \bar{b}\|_{H^1} < \varepsilon$. Thus we obtain

$$\left| \int \varphi_n\{a_n, b\} - \int \varphi_n\{a_n, \bar{b}\} \right| \le C\varepsilon.$$

On the other hand, in view of Lemma 3.3, we have

$$\left| \int \varphi_n\{a_n, \bar{b}\} \right| = \left| \int a_n\{\varphi_n, \bar{b}\} \right| \le \|\nabla \varphi_n\|_2 \|a_n\|_2 \|\bar{b}\|_{C^1},$$

that is,

$$\lim_{n\to\infty}\int \varphi_n\{a_n,\bar{b}\}=0.$$

Therefore, we obtain

$$\limsup_{n\to\infty}\left|\int \varphi_n\{a_n,b\}\right|\le C\varepsilon, \text{ for any } \varepsilon>0,$$

which implies that

$$\lim_{n\to\infty}\int \varphi_n\{a_n,b\}=0.$$

□

COROLLARY 7.3. *Under the above notations, if $\varphi\in H_0^1$, we have*

$$\lim_{n\to\infty}\int \varphi\{a_n,b_n\}=\int\varphi\{\alpha,\beta\}.$$

*Proof.* (of Theorem 7.1). By the corollary, $(\alpha,\beta,\psi)$ is also a solution of equation (1.1). Set $\alpha_n=a_n-\alpha$, $\beta_n=b_n-\beta$ and $\psi_n=\varphi_n-\psi$ so that

$$\begin{aligned}\alpha_n &\longrightarrow 0 \text{ weakly in } H^1 \text{ and strongly in } L^2,\\ \beta_n &\longrightarrow 0 \text{ weakly in } H^1 \text{ and strongly in } L^2,\\ \psi_n &\longrightarrow 0 \text{ weakly in } H^1 \text{ and strongly in } L^2.\end{aligned}$$

Denote by $\psi_{n,1}$ (resp. $\psi_{n,2}$) the unique solution of equation (1.1) for $a=\alpha_n$ and $b=\beta$ (resp. $a=\alpha$ and $b=\beta_n$). So $\gamma_n=\psi_n-\psi_{n,1}-\psi_{n,2}$ is the unique solution of equation (1.1) for $a=\alpha_n$ and $b=\beta_n$. Applying Lemma 7.2, we deduce that

$$\lim_{n\to\infty}\int|\nabla\psi_{n,1}|^2=\lim_{n\to\infty}\int(-\Delta\psi_{n,1})\psi_{n,1}=\lim_{n\to\infty}\int\psi_{n,1}\{\alpha_n,\beta\}=0.$$

Similarly, we get

$$\lim_{n\to\infty}\int|\nabla\psi_{n,2}|^2=0.$$

Clearly,

$$\begin{aligned}\|\nabla a_n\|_2^2&=\|\nabla\alpha_n\|_2^2+\|\nabla\alpha\|_2^2+o(1),\\ \|\nabla b_n\|_2^2&=\|\nabla\beta_n\|_2^2+\|\nabla\beta\|_2^2+o(1),\\ \|\nabla\varphi_n\|_2^2&=\|\nabla\psi_n\|_2^2+\|\nabla\psi\|_2^2+o(1).\end{aligned}$$

Therefore, we deduce that

$$1=\|\nabla\varphi_n\|_2^2=\|\nabla\psi_n\|_2^2+\|\nabla\psi\|_2^2+o(1)=\|\nabla\gamma_n\|_2^2+\|\nabla\psi\|_2^2+o(1),$$

which implies

$$\|\nabla a_n\|_2^2+\|\nabla b_n\|_2^2\ge 2G(\Omega)(\|\nabla\psi\|_2+\|\nabla\gamma_n\|_2).$$

Now passing to the limit as $n\longrightarrow\infty$, we obtain

$$G(\Omega)\ge G(\Omega)(\|\nabla\psi\|_2+\sqrt{1-\|\nabla\psi\|_2^2}).$$

That is, $\|\nabla\psi\|_2=0$ or $\|\nabla\psi\|_2=1$. In the first case, we infer that $\alpha=\beta=0$. The second case implies that

$$\lim_{n\to\infty}\|\nabla(\varphi_n-\psi)\|_2=0.$$

Moreover, we have

$$\|\nabla\alpha\|_2^2 + \|\nabla\beta\|_2^2 \leq \liminf_{n\to\infty}(\|\nabla a_n\|_2^2 + \|\nabla b_n\|_2^2) \leq 2G(\Omega).$$

Hence, we achieve the proof. □

The proof of Theorem 6.1 is divided into several steps. First, we need only study the case

$$a_n \rightharpoonup 0, b_n \rightharpoonup 0 \text{ and } \varphi_n \rightharpoonup 0 \text{ in } H^1.$$

*Step 1.* In this step, assume that $\Omega$ is the unit disc. Clearly, we have $\|\varphi_n\|_\infty \geq \|\nabla\varphi_n\|_2 = 1$. By the continuity of $\varphi_n$ on $\bar{\Omega}$, there exists a point $z_n \in \Omega$ such that

$$|\varphi_n(z_n)| = \|\varphi_n\|_\infty.$$

Then, after a homographic transformation $\dfrac{z - z_n}{1 - \bar{z_n} z}$, we may assume that

$$|\varphi_n(0)| = \|\varphi_n\|_\infty.$$

LEMMA 7.4. *For any $1 > \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\limsup_{n\to\infty} \int_{B(0,\varepsilon)} |\nabla a_n|^2 + |\nabla b_n|^2 \geq \delta(\varepsilon), \tag{7.2}$$

*where $B(0,\varepsilon) = \{(x,y), x^2 + y^2 < \varepsilon^2\}$.*

*Proof.* Suppose that there exists $\varepsilon_0 > 0$ such that

$$\lim_{n\to\infty} \int_{B(0,\varepsilon_0)} |\nabla a_n|^2 + |\nabla b_n|^2 = 0.$$

Denote by $\psi_n$ the unique solution of equation (1.1) in $H_0^1(B(0,\varepsilon_0))$, i.e.,

$$\begin{cases} -\triangle\psi_n &= \{a_n, b_n\}, \quad \text{in } B(0,\varepsilon_0) \\ \quad\psi &= 0, \qquad\qquad \text{on } \partial B(0,\varepsilon_0). \end{cases}$$

So $\varphi_n - \psi_n$ is harmonic in $B(0,\varepsilon_0)$. Applying the mean value property, we deduce that

$$⨍_{B(0,\varepsilon_0)} (\varphi_n - \psi_n) = \varphi_n(0) - \psi_n(0).$$

Obviously, from (1.2), we get

$$\lim_{n\to\infty} \int_{B(0,\varepsilon_0)} \psi_n = 0,$$

and

$$\lim_{n\to\infty} \psi_n(0) = 0.$$

On the other hand, but by the fact that $\varphi_n \to 0$ in $L^2(\Omega)$, we deduce

$$\lim_{n\to\infty} \int_{B(0,\varepsilon_0)} \varphi_n = 0.$$

Consequently,

$$\lim_{n\to\infty} \varphi_n(0) = 0.$$

This contradiction completes our proof. □

*Step 2.* Denote by $\mathcal{M}(\Omega)$ the space of non-negative measures on $\Omega$ with finite mass. Set $\mu_n = \frac{1}{2}(|\nabla a_n|^2 + |\nabla b_n|^2)dx$ and $\nu_n = |\nabla\varphi_n|^2 dx$. We consider the extensions of $\mu_n$ and $\nu_n$ to all of $\mathbb{R}^2$ by valuing 0 in $\mathbb{R}^2 \setminus \Omega$. Then $\{\mu_n\}$ and $\{\nu_n\}$ are bounded in $\mathcal{M}(\mathbb{R}^2)$. Modulo a subsequence, we may assume that $\mu_n \longrightarrow \mu$, $\nu_n \longrightarrow \nu$ weakly in the sense of measures where $\mu$ and $\nu$ are bounded non-negative measures on $\mathbb{R}^2$.

Lemma 7.5. *Under the above notations, then we have that there exists a point $x_0 \in \bar{\Omega}$ such that*

$$\nu = \delta_{x_0} \text{ and } \mu = G(\Omega)\delta_{x_0}. \tag{7.3}$$

*Proof.* Clearly, $\mu(\mathbb{R}^2 \setminus \bar{\Omega}) = \nu(\mathbb{R}^2 \setminus \bar{\Omega}) = 0$. Choose $\xi \in C^\infty(\mathbb{R}^2)$. Denote by $\psi_n$ the unique solution of equation (1.1) for $a = \xi a_n$ and $b = \xi b_n$, that is

$$\begin{cases} -\triangle\psi_n &= \{\xi a_n, \xi b_n\}, \quad \text{in } \Omega \\ \quad\psi &= 0, \qquad\qquad\quad \text{on } \partial\Omega. \end{cases}$$

Thus,

$$\xi a_n \rightharpoonup 0 \text{ and } \xi b_n \rightharpoonup 0 \text{ in } H^1.$$

From (1.2) and by Lemma 7.2, we obtain

$$\psi_n \longrightarrow 0 \text{ weakly in } H^1 \text{ and strongly in } L^2.$$

Since

$$\begin{aligned} &\int_\Omega |\nabla(\psi_n - \xi^2\varphi_n)|^2 \\ &= \int_\Omega (-\triangle(\psi_n - \xi^2\varphi_n))(\psi_n - \xi^2\varphi_n) \\ &= \int_\Omega (\{\xi a_n, \xi b_n\} - \xi^2\{a_n, b_n\} + 2\nabla(\xi^2)\nabla\varphi_n + (\triangle\xi^2)\varphi_n)(\psi_n - \xi^2\varphi_n) \\ &= \int_\Omega (b_n\{a_n, \xi\} + a_n\{\xi, b_n\} + 2\nabla(\xi^2)\nabla\varphi_n + (\triangle\xi^2)\varphi_n)(\psi_n - \xi^2\varphi_n) \\ &\le C[(\|b_n\|_2 + \|a_n\|_2)(\|\nabla b_n\|_2 + \|\nabla a_n\|_2)\|\xi\|_{C^1}\|\psi_n - \xi^2\varphi_n\|_\infty \\ &\quad + \|\xi\|_{C^2}^2\|\varphi_n\|_2\|\psi_n - \xi^2\varphi_n\|_2 + \|\xi\|_{C^1}^2\|\nabla\varphi_n\|_2\|\psi_n - \xi^2\varphi_n\|_2], \end{aligned}$$

and $\varphi_n$, $\psi_n$, $a_n$ and $b_n$ tend to 0 strongly in $L^2$, we deduce that

$$\lim_{n\to\infty} \|\nabla(\psi_n - \xi^2\varphi_n)\|_2 = 0.$$

Hence, we obtain

$$G(\Omega)\|\nabla(\xi^2\varphi_n)\|_2 + o(1) \le \frac{1}{2}(\|\nabla(\xi a_n)\|_2^2 + \|\nabla(\xi b_n)\|_2^2),$$

i.e.

$$\begin{aligned} &G(\Omega)\sqrt{\int (\xi^4|\nabla\varphi_n|^2 + 2\nabla\xi^2\nabla\varphi_n + \varphi_n^2|\nabla\xi^2|^2)} + o(1) \\ &\le \frac{1}{2}(\int \xi^2(|\nabla b_n|^2 + |\nabla a_n|^2) + 2\nabla\xi(\nabla a_n + \nabla b_n) + |\nabla\xi|^2(a_n^2 + b_n^2)). \end{aligned}$$

Passing to the limit as $n \longrightarrow \infty$, there holds

$$G(\Omega)\sqrt{\int \xi^4 d\nu} \le \int \xi^2 d\mu, \ \forall\xi \in C_0^\infty(\mathbb{R}^2). \tag{7.4}$$

By approximation, therefore,

$$G(\Omega)\sqrt{\nu(E)} \le \mu(E) \; (E \subset \mathbb{R}^2, E \text{ Borel}). \tag{7.5}$$

Let $\tilde{\Omega}$ be a open domain containing $\bar{\Omega}$. Clearly, we have

$$\nu(\tilde{\Omega}) \le \liminf_{n\to\infty} \nu_n(\tilde{\Omega}) = 1.$$

On the other hand, we obtain

$$\nu(\bar{\Omega}) \ge \limsup_{n\to\infty} \nu_n(\bar{\Omega}) = 1.$$

Hence, $\nu(\bar{\Omega}) = 1$. With the same argument, we deduce that $\mu(\bar{\Omega}) = G(\Omega)$. Now, let $A$ be a Borel set contained in $\bar{\Omega}$. It follows from (7.5) that

$$G(\Omega)\sqrt{\nu(A)} \le \mu(A) \text{ and } G(\Omega)\sqrt{\nu(\bar{\Omega}\setminus A)} \le \mu(\bar{\Omega}\setminus A).$$

Or, $\nu(\bar{\Omega}) = 1$ and $\mu(\bar{\Omega}) = G(\Omega)$. Therefore, we deduce that

$$\nu(A) = \mu(A) = 0 \text{ or } \nu(\bar{\Omega}\setminus A) = \mu(\bar{\Omega}\setminus A) = 0.$$

Then we conclude the result. □

*Proof.* (of Theorem 6.1 completed). Suppose first that $\Omega$ is a disc. Applying Lemma 7.4, we deduce that

$$\mu(\bar{B}(0,r)) \ge \limsup \mu_n(\bar{B}(0,r)) \ge \delta(r) > 0.$$

Using Lemma 7.5, we conclude that

$$\mu = \delta_0 \text{ and } \nu = G(\Omega)\delta_0.$$

Choose $\xi \in C_0^\infty(\mathbb{R}^2)$ such that $0 \le \xi \le 1$ and $\xi|_{B(0,r)} = 1$ with $r < 1$. Setting $\bar{a}_n = \xi a_n$ and $\bar{b}_n = \xi b_n$, denote by $\bar{\varphi}_n$ the unique solution of (1.1) for $a = \bar{a}_n$ and $b = \bar{b}_n$. Therefore, going back to (7.3), we obtain

$$\begin{aligned}\|\nabla \bar{a}_n\|_2 &= \|\nabla a_n\|_2 + o(1),\\ \|\nabla \bar{b}_n\|_2 &= \|\nabla b_n\|_2 + o(1),\\ \|\nabla \bar{\varphi}_n\|_2 &= \|\nabla \varphi_n\|_2 + o(1).\end{aligned}$$

This implies that $(\bar{a}_n, \bar{b}_n, \bar{\varphi}_n)$ is also a minimizing sequence. Or, $\bar{a}_n, \bar{b}_n \in H_0^1$, we infer

$$F(\bar{a}_n, \bar{b}_n, \Omega) \ge \inf_{a,b\in V\cap H_0^1} F(a,b,\Omega) \ge \sqrt{2}G(\Omega) > G(\Omega).$$

This contradiction completes the proof of the first part. Now, let $\Omega$ be multiply connected. We know that we can not minimize the energy $F$. Therefore, with the same arguments as above, we establish the result. □

REMARK 7.6. For any compact Riemann surface without boundary, we have the same result that in Theorem 7.1 and Lemmas 7.4 to 7.5.

## 8. Proof of Theorem 6.3

Consider the energy functional $F_1$ on $M$ and the energy level sets

$$E_M^\gamma = \{(a,b) \in M;\ F_1(a,b) \le \gamma\}.$$

A simple calculation leads to

$$\begin{aligned} DF_1(a,b)(\alpha,\beta) &= \int_\Omega \nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta \\ &- \frac{1}{2}\Big(\int_\Omega |\nabla a|^2 + |\nabla b|^2\Big)\Big(\int_\Omega \varphi\{\alpha, b\} + \varphi\{a, \beta\}\Big), \quad (8.1) \end{aligned}$$

for all $\alpha$, $\beta \in H^1(\Omega)$. First, we introduce a result which is essential in our proof of Theorem 6.3.

Lemma 8.1. *(see [7]). Let $\omega \in L^2_{loc}(\mathbb{R}^2;\mathbb{R}^3)$ be such that*

$$\triangle \omega = 2\omega_x \wedge \omega_y, \text{ on } \mathbb{R}^2, \tag{8.2}$$

*and*

$$\int_{\mathbb{R}^2} |\nabla \omega|^2 < \infty. \tag{8.3}$$

*Then $\omega$ has precisely the form*

$$\omega = \pi\left(\frac{P(z)}{Q(z)}\right) + C, \tag{8.4}$$

*where $\pi : \mathbb{C} \longrightarrow S^2$ denotes stereographic projection, $P$, $Q$ are polynomials and $C$ is a constant. In addition,*

$$\int_{\mathbb{R}^2} |\nabla \omega|^2 = 8\pi Max\{degP, degQ\}.$$

Let $\{(a_n, b_n)\}_{n\in\mathbb{N}} \subset M$ be a Palais-Smale sequence such that

$$F_1(a_n, b_n) \to C \in (G(\Omega), \sqrt{2}G(\Omega)),\ DF_1(a_n,b_n) \to 0, \text{ as } n \to \infty. \tag{8.5}$$

By the boundedness of $(a_n, b_n, \varphi_n)$ where $\varphi_n$ is a solution of (1.1) for $a = a_n$ and $b = b_n$, there exists $a$, $b$, $\varphi \in H^1(\Omega)$ such that, modulo a subsequence,

$$a_n \rightharpoonup a, b_n \rightharpoonup b, \varphi_n \rightharpoonup \varphi, \text{ in } H^1(\Omega).$$

Applying the Rellich's theorem, we have also

$$a_n \to a, b_n \to b, \varphi_n \to \varphi, \text{ in } L^2(\Omega).$$

Fix $\alpha$, $\beta \in C^\infty(\bar\Omega)$. From (8.1) and (8.5), it follows that

$$\begin{aligned} DF_1(a_n,b_n)(\alpha,\beta) &= \int_\Omega \nabla a_n \cdot \nabla \alpha + \nabla b_n \cdot \nabla \beta \\ &\quad - \frac{1}{2}\Big(\int_\Omega \nabla {a_n}^2 + \nabla {b_n}^2\Big)\Big(\int_\Omega \alpha\{b_n, \varphi_n\} + \beta\{\varphi_n, a_n\}\Big) \\ &= o(1). \end{aligned}$$

Lemma 7.2 implies

$$\int \nabla a \cdot \nabla \alpha + \int \nabla \beta \cdot \nabla b - C\int \alpha\{b,\varphi\} - C\int \beta\{\varphi, a\} = 0,$$

that is,

$$\left\{\begin{array}{ll} -\triangle a = C\{b,\varphi\}, & \text{in } \Omega \\ -\triangle b = C\{\varphi, a\}, & \text{in } \Omega \\ \dfrac{\partial a}{\partial n} = \dfrac{\partial b}{\partial n} = 0, & \text{on } \partial\Omega, \end{array}\right. \tag{8.6}$$

On the other hand, $(a, b, \varphi)$ satisfies (1.1). Thus,

$$C = \frac{-\displaystyle\int a\triangle a}{\displaystyle\int a\{b,\varphi\}} = \frac{\|\nabla a\|_2^2}{-\displaystyle\int \varphi\triangle\varphi} = \frac{\|\nabla a\|_2^2}{\|\nabla\varphi\|_2^2}. \tag{8.7}$$

Similarly,

$$C = \frac{\|\nabla b\|_2^2}{\|\nabla\varphi\|_2^2}. \tag{8.8}$$

Set $\tilde{a}_n = a_n - a$, $\tilde{b}_n = b_n - b$ and $\tilde{\varphi}_n = \varphi_n - \varphi$. Denote by $\psi_n$ the solution of (1.1) for $a = \tilde{a}_n$ and $b = \tilde{b}_n$. Similarly to the proof of Theorem 7.1, we deduce that

$$\|\nabla(\tilde{\varphi}_n - \psi_n)\|_{L^2} = o(1).$$

Set $\mu_n = \frac{1}{2}(|\nabla\tilde{a}_n|^2 + |\nabla\tilde{b}_n|^2)dx$ and $\nu_n = |\nabla\psi_n|^2 dx$. Then $\{\mu_n\}$ and $\{\nu_n\}$ are bounded in $M(\mathbb{R}^2)$. Modulo a subsequence, we may assume that $\mu_n \to \mu$, $\nu_n \to \nu$ weakly in the sense of measures where $\mu$ and $\nu$ are bounded non-negative measures on $\mathbb{R}^2$. It is clear that $\mu(\mathbb{R}^2 \setminus \bar{\Omega}) = \nu(\mathbb{R}^2 \setminus \bar{\Omega}) = 0$. Fix $\xi \in C_0^\infty(\mathbb{R}^2)$. Recall (8.5), we have

$$DF_1(a_n, b_n)(\xi\tilde{a}_n, \xi\tilde{b}_n) = o(1),$$

which implies

$$\begin{aligned} \int \nabla a_n \cdot \nabla(\xi\tilde{a}_n) - C\int \varphi_n\{\xi\tilde{a}_n, b_n\} = o(1), \\ \int \nabla b_n \cdot \nabla(\xi\tilde{b}_n) - C\int \varphi_n\{a_n, \xi\tilde{b}_n\} = o(1). \end{aligned} \tag{8.9}$$

Using the equation (8.6), we get the following equalities

$$\begin{aligned} \int \nabla a \cdot \nabla(\xi\tilde{a}_n) = C\int \varphi\{\xi\tilde{a}_n, b\}, \\ \int \nabla b \cdot \nabla(\xi\tilde{b}_n) = C\int \varphi\{a, \xi\tilde{b}_n\}. \end{aligned} \tag{8.10}$$

Combining (8.9) and (8.10), we deduce that

$$\begin{aligned} \int \nabla\tilde{a}_n \cdot \nabla(\xi\tilde{a}_n) - C\int \tilde{\varphi}_n\{\xi\tilde{a}_n, \tilde{b}_n\} + \tilde{\varphi}_n\{\xi\tilde{a}_n, b\} + \varphi\{\xi\tilde{a}_n, \tilde{b}_n\} = o(1), \\ \int \nabla\tilde{b}_n \cdot \nabla(\xi\tilde{b}_n) - C\int \tilde{\varphi}_n\{\tilde{a}_n, \xi\tilde{b}_n\} + \tilde{\varphi}_n\{a, \xi\tilde{b}_n\} + \varphi\{\tilde{a}_n, \xi\tilde{b}_n\} = o(1). \end{aligned}$$

Applying Lemma 7.2, we obtain

$$\begin{aligned} \int \xi|\nabla\tilde{a}_n|^2 - C\int \tilde{\varphi}_n\{\xi\tilde{a}_n, \tilde{b}_n\} &= o(1), \\ \int \xi|\nabla\tilde{b}_n|^2 - C\int \tilde{\varphi}_n\{\tilde{a}_n, \xi\tilde{b}_n\} &= o(1). \end{aligned}$$

Consequently,

$$\frac{1}{2}\int \xi(|\nabla \tilde{a}_n|^2 + |\nabla \tilde{b}_n|^2) = C\int \xi|\nabla \psi_n|^2 + o(1),$$

since

$$\begin{aligned}\int \tilde{\varphi}_n\{\xi\tilde{a}_n, \tilde{b}_n\} &= \int \xi\tilde{\varphi}_n\{\tilde{a}_n, \tilde{b}_n\} + o(1) = -\int \xi\tilde{\varphi}_n \triangle\psi_n + o(1)\\ &= \int \xi\nabla\tilde{\varphi}_n\cdot\nabla\psi_n + o(1) = \int \xi|\nabla\psi_n|^2 + o(1),\end{aligned}$$

and

$$\int \tilde{\varphi}_n\{\tilde{a}_n, \xi\tilde{b}_n\} = \int \xi|\nabla\psi_n|^2 + o(1).$$

Thus, we conclude that

$$\mu = C\nu. \tag{8.11}$$

With the same arguments that in the proof of Lemma 7.5 and Theorem 6.1, we deduce that

$$\begin{aligned}\sqrt{\nu(E)} &\le \frac{1}{G(\Omega)}\mu(E)\ (E\subset\Omega, E \text{ borel }),\\ \sqrt{\nu(E)} &\le \frac{1}{\sqrt{2}G(\Omega)}\mu(E)\ (E\subset\subset\Omega, E \text{ borel }).\end{aligned} \tag{8.12}$$

Thus, it follows from (8.11) and (8.12) that if $\nu(E)\neq 0$, then

$$\nu(E) > \frac{1}{2}\,(E\subset\Omega) \text{ and } \nu(E) > 1\,(E\subset\subset\Omega).$$

Hence, there exists $x_0\in\partial\Omega$ and $\lambda > \frac{1}{2}$ such that

$$\nu = \lambda\delta_{x_0},$$

since $\nu(\bar{\Omega})\le 1$. On the other hand, from (8.7) and (8.8), we have

$$\|\nabla\varphi\|_2 = \frac{(\|\nabla a\|_2^2 + \|\nabla b\|_2^2)}{2C\|\nabla\varphi\|_2} > \frac{1}{\sqrt{2}}, \text{ if } \|\nabla\varphi\|_2\neq 0$$

Or,

$$1 = \|\nabla\varphi_n\|_2^2 = \|\nabla\psi_n\|_2^2 + \|\nabla\varphi\|_2^2 + o(1).$$

This leads to

$$\varphi\equiv 0 \text{ or } \nu\equiv 0.$$

For the case $\nu\equiv 0$, in view of (8.11), we have $\mu\equiv 0$. Therefore, $(a_n, b_n)$ is compact in $M$.

For the case $\varphi\equiv 0$, then from (8.7) and (8.8), we have $a = b = 0$. So we have $\nu(\bar{\Omega}) = 1$. From (8.11), it follows that $\nu = \delta_{x_0}$ and $\mu = C\delta_{x_0}$ for some $x_0\in\partial\Omega$. Notice that our problem is invariant under the conformal mapping. Without loss of generality, we can assume $\Omega = D\setminus\bigcup_{i=1}^n\bar{\Omega}_i$ where $\Omega_i$ is a simply connected domain verifying $\bar{\Omega}_i\subset D$ and suppose that $x_0\in\partial D$. Choose a function $\xi\in C_0^\infty(\mathbb{R}^2\setminus\bigcup_{i=1}^n\bar{\Omega}_i)$ such that $\xi|_{\partial D}\equiv 1$. Let $(\bar{a}_n, \bar{b}_n) = (\frac{\xi a_n}{e_n}, \frac{\xi b_n}{e_n})$, where $e_n$ is a constant such that $(\bar{a}_n, \bar{b}_n)\in M$. So

$(\bar{a}_n, \bar{b}_n)$ can be extended to $D$. Obviously, modulo a subsequence, we can assume that

$$\begin{array}{ll} \bar{a}_n \to 0 \text{ and } \bar{b}_n \to 0, & \text{strongly in } L^2(D) \text{ and weakly in } H^1(D), \\ \frac{1}{2}(|\nabla \bar{a}_n|^2 + |\nabla \bar{b}_n|^2)dx \to C\delta_{x_0} & \text{in } \mathcal{M}(\mathbb{R}^2). \end{array}$$

Moreover, it is easy to check that $(\bar{a}_n, \bar{b}_n)$ is a Palais-Smale sequence for $\Omega = D$. Now, we can choose a sequence of Möbius transformations $\{\sigma_n\}_{n\in\mathbb{N}}$ such that

$$\int_{B(0,\frac{1}{2})} |\nabla \bar{a}_n \circ \sigma_n|^2 + |\nabla \bar{b}_n \circ \sigma_n|^2 \geq \varepsilon_0, \text{ for some } \varepsilon_0 > 0,$$

since we can use the same arguments as in the proof of Lemma 7.4. We repeat the above procedure so that $(\bar{a}_n \circ \sigma_n, \bar{b}_n \circ \sigma_n)$ is compact in $M$ for $\Omega = D$. Let $\bar{\psi}_n$ be a solution of (1.1) for $a = \bar{a}_n \circ \sigma_n$ and $b = \bar{b}_n \circ \sigma_n$ with $\Omega = D$. Assume that

$$(\bar{a}_n \circ \sigma_n, \bar{b}_n \circ \sigma_n, \bar{\psi}_n) \rightharpoonup (a, b, \varphi), \text{ in } H^1(D).$$

Thus, $u = (\sqrt{C}a, \sqrt{C}b, C\varphi)$ is a solution of (6.1). We consider the following extension of $u$ to all of $\mathbb{R}^2$

$$\tilde{u}(z) = \begin{cases} u(z) & \text{in } D, \\ \left(\sqrt{C}a(\frac{z}{|z|^2}), \sqrt{C}b(\frac{z}{|z|^2}), C\varphi(\frac{z}{|z|^2})\right) & \text{in } \mathbb{R}^2 \setminus D. \end{cases}$$

Hence, $\tilde{u}$ is a solution of (6.1). By Lemma 8.1, we obtain

$$\|\nabla u\|^2_{L^2(D)} = 16\pi k, \text{ for some } k \in \mathbb{N}^*.$$

Observing that $\|\sqrt{C}\nabla a\|_2^2 = \|\sqrt{C}\nabla b\|_2^2 = \|C\nabla\varphi\|_2^2$ and $\|\nabla\varphi\|_2 = 1$, we deduce:

$$C = \sqrt{\frac{16\pi k}{3}} = \sqrt{k}G(\Omega) \notin (G(\Omega), \sqrt{2}G(\Omega)).$$

Therefore, this contradiction completes the proof of Theorem 6.3.

## 9. Proof of Theorem 6.4

In this section, arguing by contradiction, we assume that $F_1$ does not admit a critical value in $(G(\Omega), \sqrt{2}G(\Omega))$ on $M$. For simplicity, we consider annular domains. Let $\Omega = D \setminus B(0, r)$. We divide the proof into several steps.

*Step 1.* First we show a technical lemma.

Lemma 9.1. *$M$ is a complex $C^2$ Finsler manifold.*

*Proof.* Let us consider a map $I$

$$\begin{array}{rlcl} I: & H^1(\Omega) \times H^1(\Omega) & \longrightarrow & \mathbb{R} \\ & (a, b) & \longmapsto & \displaystyle\int_\Omega |\nabla\varphi|^2 dx, \end{array}$$

where $\varphi$ is a solution of (1.1). Clearly, $I$ is a smooth analytical multilinear map and the differential of $I$ at $(a, b)$ is

$$DI(a,b)(\alpha,\beta) = \int_\Omega \varphi\{\alpha, b\} + \int_\Omega \varphi\{a, \beta\}, \text{ for all } \alpha, \beta \in H^1(\Omega).$$

Note that $M = \{(a,b) \in H^1(\Omega) \times H^1(\Omega), I(a,b) = 1\}$ and $DI \neq 0$ on $M$. Hence, we conclude the result. □

*Step 2.* We show that, for sufficiently small $\mu$, $E_M^{G+\mu}$ has the same topology as $\partial\Omega$ (where $G$ denotes $G(\Omega)$). For this purpose, we introduce a map $C$ from $H^1(\Omega) \times H^1(\Omega)$ into $\mathbb{R}^2$,

$$\begin{array}{rccl} C: & H^1(\Omega) \times H^1(\Omega) & \longrightarrow & \mathbb{R}^2 \\ & (a,b) & \longmapsto & \dfrac{1}{2G}\displaystyle\int_\Omega x \cdot (|\nabla a|^2 + |\nabla b|^2)dx \in \mathbb{R}^2. \end{array}$$

It is easy to prove that $C$ is continuous. We have the following result.

Lemma 9.2. $\forall \delta > 0$, $\exists \mu > 0$ *such that*

$$\forall (a,b) \in E_M^{G+\mu}, \ \ dist(C(a,b), \partial\Omega) < \delta. \tag{9.1}$$

*Proof.* Argue by contradiction. Suppose that (9.1) is not right. Then, there exists a sequence $(a_n, b_n)$ in $M$ such that

$$\text{dist}(C(a,b), \partial\Omega) \geq \delta, \ \ \text{for some } \delta > 0,$$

and

$$F_1(a_n, b_n) \longrightarrow G(\Omega).$$

By Theorem 6.1, there exists $x_0 \in \partial\Omega$ such that

$$C(a_n, b_n) = \frac{1}{2G}\int_\Omega x \cdot (|\nabla a_n|^2 + |\nabla b_n|^2)dx \longrightarrow x_0.$$

This contradiction terminates our proof. □

The main result of this step is the following.

Lemma 9.3. *There exists* $\varepsilon_0 > 0$ *such that* $\forall \mu < \varepsilon_0$, $E_M^{G+\mu}$ *and* $\partial\Omega$ *are of the same homotopy type.*

*Proof.* Set $W_\delta = \{x \in \mathbb{R}^2, \ \text{dist}(x, \partial\Omega) < \delta\}$. Choose a small $\delta > 0$ such that we can define the nearest point projection $P : W_\delta \longrightarrow \partial\Omega$, i.e.,

$$\text{dist}(x, \partial\Omega) = |P(x) - x|.$$

Clearly, $P$ is a continuous map. In view of Lemma 9.2, we construct a continuous map $\pi$ for all small $\mu > 0$

$$\begin{array}{rccl} \pi: & E_M^{G+\mu} & \longrightarrow & \partial\Omega \\ & (a,b) & \longmapsto & \pi(a,b) = P(C(a,b)). \end{array}$$

Let

$$(a,b) = \left(\frac{2x^1}{1+r^2}, \frac{2x^2}{1+r^2}\right) \quad \text{and} \quad \sigma_{x,t}(z) = \frac{z + tz_0}{1 + t\bar{z}_0 z}$$

where $r = \sqrt{(x^1)^2 + (x^2)^2}$, $t \in [0,1)$ and $z_0 = x_0^1 + ix_0^2$. Denote $\sigma(z) = \frac{rz}{|z|^2}$. Now, we define another continuous map $\tau$ from $\partial\Omega$ to $M$ such that

$$\tau(x) = \begin{cases} e(a \circ \sigma_{x,t}, b \circ \sigma_{x,t})|_\Omega & \text{if } x \in \partial B(0,1), \\ e(a \circ \sigma_{\frac{x}{|x|},t} \circ \sigma, b \circ \sigma_{\frac{x}{|x|},t} \circ \sigma)|_\Omega & \text{if } x \in \partial B(0,r), \end{cases}$$

where $t \in [0,1]$ and $e \in \mathbb{R}$ are well chosen such that $\tau(x) \in E_M^{G+\mu}$. Using Theorem 6.1, we deduce that $\tau \circ \pi$ and $Id_{E_M^{G+\mu}}$ are homotopic and that $\pi \circ \tau$ and $Id_{\partial\Omega}$ are homotopic. Thus, Lemma 9.3 is proved. □

*Step 3.* We prove the existence of an embedded two-disc $\Delta$ in $E_M^{\sqrt{2}G}$ whose boundary is in $E_M^G$. Consider the unit circle in $\mathbb{R}^2$

$$S^1 = \{x \in \mathbb{R}^2, |x| = 1\}.$$

Let

$$(a, b, \varphi) = \left( \frac{2\sqrt{3}x^1}{\sqrt{\pi}(1+r^2)}, \frac{2\sqrt{3}x^2}{\sqrt{\pi}(1+r^2)}, \frac{\sqrt{3}(1-r^2)}{2\sqrt{\pi}(1+r^2)} \right).$$

Note that $(a, b, \varphi)$ is a minimizer of $F_1$ for $\Omega = D$. For $x \in S^1$, $0 \le t < 1$ let $\sigma_{x,t}(z)$ be defined as above. Set $a_{x,t} = a \circ \sigma_{x,t}|_\Omega$, $b_{x,t} = b \circ \sigma_{x,t}|_\Omega$ and $\varphi_{x,t} = \varphi \circ \sigma_{x,t}|_\Omega$. We see that $(a_{x,t}, b_{x,t}, \varphi_{x,t})$ "concentrates" at $x$ as $t \to 1$. Moreover, letting $t \to 0$, we have

$$(a_{x,t}, b_{x,t}, \varphi_{x,t}) \longrightarrow (a, b, \varphi), \text{ in } H^1.$$

The set $\Delta \equiv \{(a_{x,t}, b_{x,t})/x \in S^1, t \in [0, 1[\}$ is a disc embedded in $E_M^\gamma$ with $G < \gamma < \sqrt{2}G$, as a consequence of the following lemma.

LEMMA 9.4. *Let $\psi_{x,t}$ be a solution of*

$$\begin{cases} -\triangle \psi_{x,t} &= \{a_{x,t}, b_{x,t}\}, \quad \textit{in } \Omega = D \setminus B(0,r), \\ \psi_{x,t} &= 0, \quad \textit{on } \partial\Omega. \end{cases} \tag{9.2}$$

*Then, for any $\varepsilon > 0$, there exists $\eta > 0$ independent of $t$ and $x$ such that for any $r < \eta$*

$$\|\nabla(\psi_{x,t} - \varphi_{x,t})\|^2_{L^2(\Omega)} < \varepsilon.$$

*Proof.* First, we see that

$$-\triangle \varphi_{x,t} = \{a_{x,t}, b_{x,t}\} \text{ in } D.$$

We will decompose $\varphi_{x,t}$ into its harmonic $\theta_{x,t}$ and non-harmonic $\psi_{x,t}$ components

$$\varphi_{x,t} = \theta_{x,t} + \psi_{x,t},$$

where

$$\begin{cases} -\triangle \theta_{x,t} &= 0, \quad \text{in } \Omega, \\ \theta_{x,t} &= \varphi_{x,t}, \quad \text{on } \partial\Omega. \end{cases} \tag{9.3}$$

Hence, for any $\varepsilon > 0$, there exists $\eta > 0$ such that for any $r < \eta$

$$\|\nabla \varphi_{x,t}\|_{L^2(B(0,r))} < \varepsilon,$$

since

$$\lim_{r \to 0} \operatorname{diam}(\sigma_{x,t}(B(0,r))) = 0.$$

Set $\tilde{\theta}_{x,t} = \theta_{x,t}(r^2 z/|z|^2)$. Thus, $\tilde{\theta}_{x,t}$ is harmonic in $\Omega = B(0,r) \setminus B(0,r^2)$. Choose $\xi \in C_0^\infty(\mathbb{R}^2 \setminus B(0,r^2))$ such that $\xi|_{B(0,r)\setminus B(0,r/2)} \equiv 1$ and $|\nabla \xi| < \frac{4}{r}$. So, we have

$$\begin{aligned} \|\nabla(\xi(\varphi_{x,t} - \bar{\varphi}))\|_{L^2(B(0,r))} &\le \|\nabla \varphi_{x,t}\|_{L^2(B(0,r))} + \frac{4}{r}\|\varphi_{x,t} - \bar{\varphi}\|_{L^2(B(0,r))} \\ &\le C\|\nabla \varphi_{x,t}\|_{L^2(B(0,r))}, \end{aligned}$$

where $\bar{\varphi} = \fint_{B(0,r)} \varphi_{x,t}$. Consequently,

$$\begin{aligned} \|\nabla\theta_{x,t}\|_{L^2(B(0,1)\setminus B(0,r))} &= \|\nabla\tilde{\theta}_{x,t}\|_{L^2(B(0,r)\setminus B(0,r^2))} \\ &\leq \|\nabla(\xi(\varphi_{x,t}-\bar{\varphi}))\|_{L^2(B(0,r))} \\ &\leq C\|\nabla\varphi_{x,t}\|_{L^2(B(0,r))} \leq C\varepsilon. \end{aligned}$$

Hence, we get the result. □

Hence, we deduce that, for $r < \eta$, $\Delta$ is embedded in $E_M^\gamma$, for $\gamma < G + \varepsilon$.
*Step 4.* Conclusion. By the deformation lemma, for any $\gamma \in (G, \sqrt{2}G)$ there exists a continuous flow $\Psi : E_M^\gamma \times [0,1] \to E_M^\gamma$ such that

$$\begin{aligned} &\Psi(u,0) = u, && \text{for all } u \in E_M^\gamma, \\ &\Psi(\cdot,1) \in E_M^{G+\mu}, && \\ &\Psi(u,t) = u, && \text{for all } u \in E_M^{G+\mu}, \end{aligned}$$

where $\gamma > G+\mu$. Thus, by Step 3, we can define the map $h : S^1\times[0,1] \to \partial\Omega$, given by

$$h(x,t) = \pi(\Psi((a_{x,t}, b_{x,t}), 1)),$$

then it is continuous and satisfies

$$\begin{aligned} &h(x,0) = \pi(\Psi((a,b),1)) =: x_0 \in \partial\Omega && \text{for all } x \in S^1, \\ &h(x,1) = x && \text{for all } x \in S^1. \end{aligned}$$

Hence, $h$ is a contraction of $S^1$ in $\Omega$. This contradicts our assumptions. Thus, Theorem 6.4 is proved.

## 10. Some extensions

In this section, we study the properties of a solution of the incomplete $H$-system, i.e., a solution of equation (6.1). We remark first that a conformal covering map of a sphere is such a solution. But these solutions are not interesting, from a geometric point of view. Hence one difficulty for our approach to the construction of $H$-tori is that there are holomorphic maps from a torus $T$ into a sphere of arbitrary degrees $\geq 2$. However, we expect that we may find a non-trivial solution for $H$-system. And, we will give here a criterion.

Let $(N,g)$ be a compact orientable smooth Riemannian surface without boundary. Given $a,b \in H^1(N,\mathbb{R})$, we define

$$\{a,b\} = *(da \wedge db),$$

where $*$ is the Hodge operator associated to $g$. We consider the following equation, called $H$-system,

$$\begin{cases} \triangle_g\varphi = \{a,b\} & \text{on } N, \\ \triangle_g a = \{b,\varphi\} & \text{on } N, \\ \triangle_g b = \{\varphi,a\} & \text{on } N, \end{cases} \tag{10.1}$$

where $u = (a,b,\varphi) \in C^\infty(N,\mathbb{R}^3)$ and $\triangle_g$ is Laplacian operator associated to $g$. This equation is of variational type associated to a energy functional arising from the generalized Wente's inequality on a manifold as above (see also [15]). In isothermal charts, it follows from (10.1)

$$-\triangle u = u_x \wedge u_y \text{ on } N. \tag{10.2}$$

If $u$ maps $N$ into a sphere, then $u$ is also a harmonic map. J. Eells and J.C. Wood have shown the following useful result for a harmonic map.

LEMMA 10.1. *Let $X$ and $Y$ be closed orientable smooth surfaces and $\varphi : X \to Y$ be a smooth map. If $\varphi$ is a harmonic map relative to Riemannian metrics $g$ and $h$, and if*

$$e(X) + |d_\varphi e(Y)| > 0, \tag{10.3}$$

*then $\varphi$ is holomorphic or anti-holomorphic relative to the complex structures determinated by $g$ and $h$.*

Here $e(X) = 2 - 2p$ and $e(Y) = 2 - 2q$ denote Euler characteristics, and $d_\varphi$ is the degree of $\varphi$. With the help of this result, we have the following:

THEOREM 10.2. *Let $N$ be a Riemannian surface with a genus $p = 0$ or $1$. Assume that $u$ is a solution of (10.2) and $u$ maps $N$ into a sphere. Then, there exists $k \in \mathbb{N}$ such that*

$$\|\nabla a\|_2 = \|\nabla b\|_2 = \|\nabla \varphi\|_2 = \sqrt{\frac{32k\pi}{3}}. \tag{10.4}$$

*Proof.* Suppose $u : N \to S$, where $S$ is a sphere. Note first that $u$ is a harmonic map. It is clear that

$$\|\nabla a\|_2^2 = \int_N a db \wedge d\varphi = \int_N d(ab) \wedge d\varphi - \int_N b da \wedge d\varphi = \int_N b d\varphi \wedge da = \|\nabla b\|_2^2.$$

Similarly, we obtain

$$\|\nabla a\|_2^2 = \|\nabla \varphi\|_2^2.$$

*Case 1: $N$ is simply connected.* Clearly, it follows from Lemma 8.1 and the fact that $N$ is conformal to $S^2$.

*Case 2: the genus of $N$ is equal to 1.* Assume $u$ is not a constant map. We claim that $\deg(u) \neq 0$. Indeed, assuming that $u(N) \subset \partial B(0,r)$ and by the properties of degree, we have

$$\deg(u) = \frac{\displaystyle\int_N \Psi^* \Omega}{\displaystyle\int_{\partial B(0,r)} \Omega},$$

where $\Omega = \frac{1}{r}(x^1 dx^2 \wedge dx^3 - x^2 dx^1 \wedge dx^3 + x^3 dx^1 \wedge dx^2)$ is the element of volume on the sphere $\partial B(0,r)$. Hence, from equation (10.1), it follows

$$\begin{aligned} 4\pi r^3 \deg(u) &= \int_N a db \wedge d\varphi - b da \wedge d\varphi + \varphi da \wedge db \\ &= \|\nabla a\|_2^2 + \|\nabla b\|_2^2 + \|\nabla \varphi\|_2^2 = 3\|\nabla a\|_2^2. \end{aligned} \tag{10.5}$$

Now, applying Lemma 10.1 and using $e(M) = 0$ and $e(\partial B(0,r)) = 2$, we deduce that $u$ is a conformal map. Suppose that $z_0$ is a branch-point. Thanks of Hartman's and Wintner's result (see [14] and [16]), there exist $n \in \mathbb{N}^*$ and $c \in \mathbb{C}^3 \setminus \{0\}$ such that

$$\partial_z u = c(z - z_0)^n + o((z - z_0)^n),$$

where $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. This implies that the branch-point is isolated. Recalling (10.1), we conclude that $u(N)$ is a sphere with radius equal to 2. Therefore, by using (10.5), (10.4) is proved. Moreover, we have $k = \deg(u)$. □

With the same method, we have the following general result.

THEOREM 10.3. *Under the above assumptions and supposing that $N$ is a Riemannian surface with a genus $p > 1$, then we have that either*

(i) $\|\nabla a\|_2 = \|\nabla b\|_2 = \|\nabla \varphi\|_2 = \sqrt{\frac{32deg(u)\pi}{3}}$, *for* $deg(u) \geq p$,
*or*
(ii) $\|\nabla a\|_2 = \|\nabla b\|_2 = \|\nabla \varphi\|_2 \geq \sqrt{\frac{32deg(u)\pi}{3}}$, *for* $deg(u) < p$.

*Proof.* It is easy to check that

$$(|\nabla u|^2)^2 = |\mathrm{Hopf}(u)|^2 + (2|u_x \wedge u_y|)^2,$$

where $\mathrm{Hopf}(u) = |u_x|^2 - |u_y|^2 + 2i\langle u_x, u_y\rangle$. From (10.1), it follows that

$$|\mathrm{Hopf}(u)|^2 = (r^2 - 4)|u_x \wedge u_y|^2,$$

since $u$ is harmonic, i.e.,

$$\triangle(\frac{u}{r}) = -\frac{u}{r}\left|\nabla(\frac{u}{r})\right|^2.$$

Thus,

$$r \geq 2.$$

Using (10.5), we terminate the proof. □

COROLLARY 10.4. *Let $u$ be a solution of (10.1) on a torus obtained by Theorem 6.4. Then, $u$ is not a covering map of a sphere.*

Now, return to equation (6.1). Let $\Omega$ be an annulus. We know that for each solution of (6.1) satisfying the boundary condition (6.2), there exists $c \in \mathbb{R}$ such that

$$\langle \partial_z u, \partial_z u\rangle = \frac{c}{z^2}. \tag{10.6}$$

Here we study the branch-points. Set $P = \{(x,y) \in \Omega,\ \mathrm{rank}(\nabla u(x,y)) \leq 1\}$. So we have the following result.

THEOREM 10.5. *Under the above assumptions we have*

$$\mathcal{H}^1(P) < \infty,$$

*where $\mathcal{H}^1$ designates the 1-dimensional Hausdorff measure.*

*Proof.* Set $H = xu_x + yu_y$ and $J = yu_x - xu_y$. Hence, we obtain

$$\begin{aligned}\langle H, J\rangle &= xy(\langle u_y, u_y\rangle - \langle u_x, u_x\rangle) + (x^2 - y^2)\langle u_x, u_y\rangle \\ &= -2\mathrm{Im}(z^2\langle \partial_z u, \partial_z u\rangle).\end{aligned}$$

It follows from (10.6) that $H$ and $J$ are orthogonal.

*Case 1: $c = 0$.* Thus, $u$ is a conformal map. By Hartman's and Wintner's result on real-valued vector functions, we conclude that a branch-point is isolated.

*Case 2: $c \neq 0$.* By definition, we have

$$H_x = u_x + xu_{xx} + yu_{xy},\ H_y = u_y + xu_{xy} + yu_{yy}.$$

Assume that $x_0 \neq 0$ and $H(x_0, y_0) = 0$. Hence, using (10.1), we deduce

$$\begin{aligned} -\triangle H &= -2\triangle u - x(\triangle u)_x - y(\triangle u)_y \\ &= 2u_x \wedge u_y + x(u_x \wedge u_y)_x + y(u_x \wedge u_y)_y \\ &= (xu_x \wedge u_y)_x + (yu_x \wedge u_y)_y \\ &= (H \wedge u_y)_x + (\frac{y}{x} H \wedge u_y)_y. \end{aligned}$$

Therefore, by Hartman's and Wintner's result, there exist $n \in \mathbb{N}^*$ and $c \in \mathbb{C}^3 \setminus \{0\}$ such that

$$\lim_{z \to 0} H_z z^{-n} = c,$$

which implies that there exists some neighborhood $V$ of $(x_0, y_0)$ such that

$$\mathcal{H}^1(V \cap \{(x,y), H(x,y) = 0\}) < \infty.$$

Now let $y_0 \neq 0$ and $J(x_0, y_0) = 0$. With the same arguments, there exists some neighborhood $V'$ of $(x_0, y_0)$ such that

$$\mathcal{H}^1(V' \cap \{(x,y), J(x,y) = 0\}) < \infty.$$

Hence, we prove Theorem 10.5. □

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