REGULARIZATION OF LINEAR LEAST SQUARES PROBLEMS BY TOTAL BOUNDED VARIATION

G. CHAVENT AND K. KUNISCH

Abstract. We consider the problem

(P) Minimize $\frac{1}{2}\,|\,Tu - z\,|_Y^2 + \frac{\alpha}{2}\,|\,u\,|_{L^2}^2 + \beta \int_\Omega |\nabla u\,|$ over $u \in K \cap X$,

where $\alpha \geq 0$, $\beta > 0$, K is a closed convex subset of $L^2(\Omega)$, and the last additive term denotes the BV-seminorm of u, T is a linear operator from $L^2 \cap BV$ into the observation space Y. We formulate necessary optimality conditions for (P). Then we show that (P) admits, for given regularization parameters α and β, solutions which depend in a stable manner on the data z. Finally we study the asymptotic behavior when $\alpha = \beta \to 0$. The regularized solutions $\hat{u}_\beta$ of (P) converge to the $L^2 \cap BV$ minimal norm solution of the unregularized problem. The rate of convergence is $\beta^{\frac{1}{2}}$ when the minimum-norm solution $\hat{u}$ is smooth enough.

1. Problem statement and functions of bounded variation

We consider the problem of finding approximate solutions to

$$Tu = z \tag{1.1}$$

where T is a bounded linear operator from $L^2(\Omega)$ to a Hilbert space Y by using regularization techniques based on bounded variation functionals. Here Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitzian boundary and $z \in Y$. The operator T is not assumed to be injective and it may be compact. In this case approximate solutions that are stable with respect to z can be obtained by solving the regularized problem.

$$\min \frac{1}{2}|Tu - z|_Y^2 + \frac{\beta}{2}|u|_{L^2}^2 \tag{1.2}$$

or

$$\min \frac{1}{2}|Tu - z|_Y^2 + \frac{\beta}{2}|\nabla u|_{L_n^2}^2, \tag{1.3}$$

for example. In (1.2) and (1.3) one refers to $\beta > 0$ as the regularization parameter. Regularization by the square of a Hilbertian norm or seminorm as in (1.2) and (1.3) is well-studied. We refer to [2], [7], [8] and the references given there. While (1.2) and (1.3) have attractive analytical properties concerning the stability of their solution with respect to perturbations in the data z and their asymptotic behavior as $\beta \to 0$ they can have serious practical disadvantages. The L^2-regularization term in (1.2) is not satisfactory

to dampen undesired oscillations in the solution u which result from highly noisy data and/or compactness of T. Regularization with the square of the gradient as in (1.3) reduces these oscillations but it overregularizes the solution in the neighborhood of edges and corners. This is well-known in the context of image enhancement, see for example [11] and the references given there, and one can quickly convince oneself of this fact by a short MATLAB-code even with $T = $ identity. Due to these shortcomings with (1.2) and (1.3) regularization based on total bounded variation functionals has recently been investigated in several publications [3], [11], [13], for example. In this case the regularized problems are given by (variations of)

$$\min \frac{1}{2}|Tu - z|_Y^2 + \beta \int_\Omega |\nabla u|, \qquad (1.4)$$

where $\int_\Omega |\nabla u|$ denotes the bounded variation seminorm, which will be described at the end of this section. For the moment we may think of the term $\int_\Omega |\nabla u|$ as the $W^{1,1}(\Omega)$ seminorm. In the context of image enhancement (1.4) has proved to be extremely effective. Comparing (1.4) to (1.2) or (1.3) we immediately recognize that while (1.2) and (1.3) are quadratic, (1.4) is not. In fact, (1.4) is highly nonlinear and this results in interesting problems both analytically and numerically. In [3] basic facts concerning existence of solutions to (1.4), the optimality system, approximation of (1.4) by finite dimensional problems and algorithms that are specifically adapted for the nonlinear structure of problem (1.4) were presented. In the present analysis we focus on the problem of stability of the solutions to (1.4) with respect to perturbations in the data z and on the asymptotic behavior as $\beta \to 0^+$.

Let us now specify the problem to be investigated:

$$\min \frac{1}{2}|Tu - z|_Y^2 + \frac{\alpha}{2}|u|_{L^2}^2 + \beta \int_\Omega |\nabla u| \text{ over } u \in K \cap X, \qquad (P)$$

where $\alpha \geq 0, \beta > 0, K$ is a closed, convex subset of $L^2(\Omega), X = L^2(\Omega) \cap BV(\Omega)$, and $BV(\Omega)$ denotes the space of bounded variation functions to be defined below. The space X endowed with the norm $|u|_X = |u|_{L^2} + |u|_{BV}$ is a Banach space. We shall see that due to the assumption that Ω is bounded X and $BV(\Omega)$ are equivalent if $n \leq 2$. The use of the quadratic regularization term serves two purposes. First, for $\alpha > 0$ it provides a coercive term for the subspace of constant functions which are in the kernel of the ∇-operator (and can be in the kernel of T) and secondly it gives the possibility to distinguish the structure of stability results for quadratic regularization terms from that of the nonquadratic BV-term.

Let us now summarize results from the theory of functions of bounded variation that are relevant to our work. We refer to [3], [6], [12] for details. As mentioned above Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitzian boundary. A function $u \in L^1(\Omega)$ is said to be of bounded variation if

$$\sup \left\{ \int_\Omega u(x) \text{ div } v(x)dx : v \in C_0^\infty(\Omega)^n, |v(x)|_\infty \leq 1, x \in \Omega \right\} < \infty. \quad (1.5)$$

Here $|\cdot|_\infty$ denotes the l^∞-norm on $\mathbb{R}^n$. The sup in (1.5) will be denoted by $\int_\Omega |\nabla u|$. The space of all functions in $L^1(\Omega)$ with bounded variation is

abbreviated by $BV(\Omega)$. Endowed with

$$|u|_{BV} = |u|_{L^1} + \int_\Omega |\nabla u|, \qquad (1.6)$$

$BV(\Omega)$ is a Banach space. Equivalently $BV(\Omega)$ can be introduced in terms of measures. Let $M(\Omega)$ denote the space of real Borel measures on Ω which is a Banach space when endowed with the norm

$$|\mu|(\Omega) = \int_\Omega |\mu|, \ \mu \in M(\Omega),$$

$|\mu| = \mu^+ + \mu^-$ being the total variation measure associated to μ and $|\mu|(\Omega)$ is the total variation of μ in Ω, [10]. It is known that $M(\Omega)$ is the dual space of $C_0(\Omega)$, the space of continuous function vanishing on the boundary of Ω and

$$\int_\Omega |\mu| = \sup \left\{ \int_\Omega v(x) d\mu(x) : v \in C_0(\Omega), |v|_{C(\Omega)} \le 1 \right\}.$$

Thus $BV(\Omega)$ can equivalently be expressed as the space of functions $u \in L^1(\Omega)$ for which $\partial_{x_i} u \in M(\Omega)$, for every $1 \le i \le n$, with the partial derivatives ∂_{x_i} understood in the distributional sense. If the vector valued measure space $M^n(\Omega)$ is endowed with the l^1-norm, in the sense that

$$\int_\Omega | \ \vec{\mu} \ | = \sum_{i=1}^n \int_\Omega | \ \vec{\mu}_i \ |, \ \vec{\mu} \in M^n(\Omega),$$

then

$$|u|_{BV} = |u|_{L^1} + \int_\Omega |\nabla u| = |u|_{L^1} + \sum_{i=1}^n \int_\Omega |\partial_{x_i} u|, \text{ for } u \in BV(\Omega).$$

If instead of the l^∞-norm on $\mathbb{R}^n$ in (1.5) one uses the l^p-norm, $1 \le p < \infty$, then this induces an equivalent norm on $BV(\Omega)$.

PROPOSITION 1.1.

a) *If $\{u_j\}_{j=1}^\infty \subset BV(\Omega)$ and $\lim u_j = u$ in $L^1(\Omega)$ then*

$$\int_\Omega |\nabla u| \le \liminf \int_\Omega |\nabla u_j|.$$

b) *For every $u \in BV(\Omega) \cap L^r(\Omega), r \in [1, \infty)$ there exists $\{u_j\}_{j=1}^\infty \subset C^\infty(\Omega)$ such that*

$$\lim u_j = u \text{ in } L^r(\Omega) \text{ and } \lim \int_\Omega |\partial_{x_i} u_j| = \int_\Omega |\partial_{x_i} u|, \ 1 \le i \le n.$$

c) *For every bounded sequence $\{u_j\}_{j=1}^\infty \subset BV(\Omega)$ there exists a subsequence $\{u_{j_k}\}_{k=1}^\infty$ and $u \in BV(\Omega)$ such that $\lim u_{j_k} = u$ in $L^p(\Omega), p \in [1, \frac{n}{n-1})$, if $n \ge 2$, and $\lim u_{j_k} = u$ in $L^p, p \in [1, \infty)$, if $n = 1$. (Compactness).*

d) *There exists a constant $C = C(n)$ such that*

$$\left(\int_\Omega |u - \tilde{u}|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \le C \int_\Omega |\nabla u|, \text{ for all } u \in BV(\Omega),$$

with $\tilde{u} = \frac{1}{|\Omega|} \int_\Omega u \, dx$. (Sobolev inequality). If $n = 1$, then the $L^{n/n-1}$-norm is understood to be the L^∞-norm.

As a consequence of Proposition 1.1 d) we have the following

COROLLARY 1.2. *If $n = 1$ or 2 then $BV(\Omega)$ and X are equivalent Banach spaces.*

Sketch of proof of Proposition 1.1. The assertions of Proposition 1.1 are small modification of results given in [6]. The semicontinuity of the bounded variation functional asserted in a) is proved in ([6], Theorem 1.9). As for b) the proof of Theorem 1.17 in [6] can be generalized from $r = 1$ to arbitrary $r \in [1, \infty)$. An application of Rellich's Theorem, see ([6], Theorem 1.19) implies c). Finally d) is well-known for $u \in C^\infty(\Omega)$ [9]. For arbitrary $u \in BV(\Omega)$ there exists as a consequence of b) a sequence $u_j \in C^\infty(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$ and $\int_\Omega |\nabla u_j| \to \int_\Omega |\nabla u|$. Applying d) to the sequence $\{u_j\}$ we obtain that $\{u_j\}_{j=1}^\infty$ is bounded in $L^{\frac{n}{n-1}}(\Omega)$ and hence u_j converges weakly to u in $L^{\frac{n}{n-1}}(\Omega)$ and $\lim \tilde{u}_j = \tilde{u}$. It follows that

$$|u - \tilde{u}|_{L^{\frac{n}{n-1}}} \leq \underline{\lim} |u_j - \tilde{u}_j| \leq \lim C \int_\Omega |\nabla u_j| = C \int_\Omega |\nabla u|.$$

The subsequent sections are organized as follows. In Section 2 sufficient conditions for existence of a unique solution to (P) and the optimality system are given. Section 3 is dedicated to the stability of the solutions with respect to the data z. The asymptotic behavior of the solutions to (P) as $\alpha \to 0, \beta \to 0$ is analyzed in Section 4.

2. EXISTENCE AND OPTIMALITY CONDITIONS

We shall use the following hypothesis.
One of the following properties holds:

  (i) $\alpha > 0$,
  (ii) $n = 1$ or $n = 2$, and const $\notin \ker T$,   (H)
  (iii) K is bounded in $L^2(\Omega)$.

THEOREM 2.1. *If (H) holds, then there exists a solution to (P). If moreover $\alpha > 0$ or T is injective then the solution is unique.*

Proof. Let $\{u_j\}_{j=1}^\infty$ be a minimizing sequence. If (H) (i) or (iii) hold, then $\{u_j\}$ is bounded in $L^2(\Omega)$. In the case of (H) (ii) boundedness of $\{u_j\}$ in $L^2(\Omega)$ follows from Proposition 1.1 d) and the fact that $\beta > 0$. Thus in either case (H) and the form of the cost functional imply that $\{u_j\}$ is bounded in X. By Proposition 1.1 c) there exists a subsequence of $\{u_j\}$ denoted by the same symbol and $\bar{u} \in X$, such that $\{u_j\}$ converges to u strongly in $L^1(\Omega)$ and weakly in $L^2(\Omega)$. It follows that $\bar{u} \in K$. Moreover, T maps weakly convergent sequence in $L^2(\Omega)$ into weakly convergent sequences in Y. By Proposition 1.1 a) we find

$$\frac{1}{2} |T\bar{u} - z|^2 + \frac{\alpha}{2} |\bar{u}|^2 + \beta \int_\Omega |\nabla \bar{u}|$$

$$\leq \liminf \left\{ \frac{1}{2} |Tu_j - z|^2 + \frac{\alpha}{2} |u_j|_{L^2}^2 + \beta \int_\Omega |\nabla u_j| \right\}$$

$$\leq \inf \left\{ \frac{1}{2} |Tu - z|^2 + \frac{\alpha}{2} |u|_{L^2}^2 + \beta \int_\Omega |\nabla u| : u \in K \right\}$$

and thus $\bar{u}$ is a solution to (P). Concerning uniqueness we note that in the case $\alpha > 0$ the cost functional of (P) is strictly convex and hence $\bar{u}$ is unique. In the case of injectivity of T the cost functional of (P) is again strictly convex and uniqueness of $\bar{u}$ follows. $\qquad\qquad$ $\square$

We now specify the optimality system for (P).

THEOREM 2.2. *Let $\bar{u} \in K \cap X$. Then $\bar{u}$ is a solution of (P) if and only if there exists $\bar{\lambda}$ in the dual space of $M^n(\Omega)$ such that*

$$\left(T^*(T\bar{u} - z) + \alpha\,\bar{u}, u - \bar{u}\right)_{L^2(\Omega)} - \beta\left\langle div\,\bar{\lambda}, u - \bar{u}\right\rangle_{BV^*,BV} \geq 0 \qquad (2.1)$$
$$for\ all\ u \in K \cap X$$

$$\left\langle\bar{\lambda}, \mu - \nabla\bar{u}\right\rangle_{M^{n,*},M^n} + \int_\Omega |\nabla\bar{u}| \leq \int_\Omega |\mu|,\ for\ all\ \mu \in M^n(\Omega). \qquad (2.2)$$

Here T^* denotes the adjoint of $T \in \mathcal{L}(L^2(\Omega), Y)$ and - div stands for the conjugate of $\nabla : BV(\Omega) \to M^n(\Omega)$, i.e.

$$\left\langle\text{- } div\,\bar{\lambda}, u\right\rangle_{BV^*,BV} = \left\langle\bar{\lambda}, \nabla u\right\rangle_{M^{n,*},M^n}\ for\ u \in BV(\Omega).$$

The proof can be obtained with only minor modifications from that of ([3], Theorem 2.2) and it is therefore omitted. Let us note that (2.2) is equivalent to

$$\left\langle\bar{\lambda}, \nabla\bar{u}\right\rangle = \int_\Omega |\nabla\bar{u}| \qquad (2.3)$$

and

$$\left\langle\bar{\lambda}, \mu\right\rangle \leq \int_\Omega |\mu|\ \text{for all}\ \mu \in M^n(\Omega). \qquad (2.4)$$

Alternatively (2.2) can be expressed as $\bar{\lambda} \in \partial\varphi(\nabla\bar{u})$ where $\partial\varphi$ is the subdifferential of $\varphi \colon M^n(\Omega) \to \mathbb{R}$ given by

$$\varphi(\mu) = \sum_{i=1}^n \int_\Omega |\mu_i|.$$

Note that (2.4) implies that the restriction to L^1_n of $\mu \to \left\langle\bar{\lambda}, \mu\right\rangle$ can be identified with an element $\bar{\lambda}$ in $L^\infty_n(\Omega)$ with $|\bar{\lambda}|_{L^\infty} \leq 1$.

The Lagrange multiplier $\bar{\lambda}$ of Theorem 2.2 is not unique. In the following theorem we give a necessary and sufficient optimality system that involves a unique Lagrange multiplier $\bar{q} \in H^1(\Omega)$. Let us recall some function spaces involving the divergence operator [5]:

$$H(\text{div}) = \left\{v \in L^2_n(\Omega) : \text{div}\ v \in L^2(\Omega)\right\},$$

which is a Hilbert space for the norm

$$|v|_{H(\text{div})} = \left(|v|^2_{L^2_n} + |\text{div}\ v|^2_{L^2}\right)^{1/2},$$

and the closed subspace of $H(\text{div})$:

$$H_0(\text{div}) = \overline{\mathcal{D}(\Omega)^n},$$

where $\mathcal{D}(\Omega)^n$ denotes the space of infinitely differentiable vector-valued functions with support in Ω and the closure is taken in the norm of $H(\text{div})$. The trace operator $\tau(v) = v \cdot n\,|\,\Gamma$ can be extended as continuous linear operator from $H(\text{div})$ to $H^{-1/2}(\Gamma)$ ([5], §2). Here n denotes the outer unit normal to

the Lipschitzian boundary Γ of Ω. The space $H_0(\mathrm{div})$ can be characterized as

$$H_0(\mathrm{div}) = \{v \in H(\mathrm{div}) : \tau(v) = 0\}.$$

THEOREM 2.3. *Let $K = X$ and let $\bar{u}$ be a solution to (P). Then there exists a unique $\bar{q} \in H^1(\Omega)$ with $\int_\Omega \bar{q}\,dx = 0$ such that for $\tilde{\lambda} = \nabla\bar{q}$ we have $\tilde{\lambda} \in H_0(\mathrm{div})$, $|\tilde{\lambda}|_{L_n^2} \leq \sqrt{\mathrm{meas}(\Omega)}$,*

$$(T^*T + \alpha)\bar{u} - T^*z = \beta\ \mathrm{div}\ \tilde{\lambda}\ \text{in } L^2(\Omega) \tag{2.5}$$

$$(-\ \mathrm{div}\ \tilde{\lambda}, u - \bar{u})_{L^2(\Omega)} + \int_\Omega |\nabla\bar{u}| \leq \int_\Omega |\nabla u|\ \text{for all } u \in X. \tag{2.6}$$

Conversely, if there exists a pair $(\bar{u}, \tilde{\lambda}) \in X \times H_0(\mathrm{div})$ such that (2.5), (2.6) hold, then $\bar{u}$ is a solution to (P).

REMARK 2.4. From the proof it will follow that $\tilde{\lambda}$ of Theorem 2.3 is obtained from the Lagrange multiplier $\bar{\lambda}$ of Theorem 2.2 by restriction of the functional $\mu \to \langle\bar{\lambda}, \mu\rangle_{M^{n,*},M^n}$ to $L_n^2(\Omega)$ and subsequent projection of $\bar{\lambda}$ onto the orthogonal complement in $L_n^2(\Omega)$ of the divergence-free space. We recall the decomposition of $L_n^2(\Omega)$ as

$$L_n^2(\Omega) = H \oplus H^\perp,$$

where $H = \{v \in H_0(\mathrm{div}) : \mathrm{div}\ v = 0\}$, and $H^\perp$ denotes the orthogonal complement of H in $L_n^2(\Omega)$. Accordingly the restriction to $L_n^2(\Omega)$ of $\bar{\lambda}$ of Theorem 2.2 can be decomposed as $\bar{\lambda}_1 + \bar{\lambda}_2 \in H \oplus H^\perp$ and $\bar{\lambda}_2$ can be identified with the Lagrange multiplier $\tilde{\lambda}$ of Theorem 2.3.

Proof of Theorem 2.3. By Theorem 2.2 there exists $\bar{\lambda} \in M^n(\Omega)^*$, the dual of $M^n(\Omega)$, such that (2.1) and (2.2) hold. Since $K = X$ we find

$$((T^*T + \alpha)\bar{u} - T^*z, v)_{L^2} = -\beta\langle\bar{\lambda}, \nabla v\rangle_{M^{n,*},M^n}\ \text{for all } v \in X. \tag{2.7}$$

Let us consider the restriction of the functional $\mu \to \langle\bar{\lambda}, \mu\rangle_{M^{n,*},M^n}$ to $L_n^2(\Omega)$. By the Riesz representation theorem it can be identified with an element of $L_n^2(\Omega)$ which we denote by $\hat{\lambda}$. Due to (2.4) we find that $|\hat{\lambda}|_{L_n^2} \leq \sqrt{\mathrm{meas}(\Omega)}$. Taking the distributional derivative in

$$((T^*T + \alpha)\bar{u} - T^*z, v)_{L^2} = -\beta(\hat{\lambda}, \nabla v)_{L_n^2}\ \text{for all } v \in H^1(\Omega),$$

it follows that $\mathrm{div}\ \hat{\lambda}$ can be considered as bounded linear functional on $L^2(\Omega)$. Hence $\hat{\lambda}$ can be identified with an element of $H(\mathrm{div})$ and

$$(T^*T + \alpha)\bar{u} - T^*z = \beta\ \mathrm{div}\ \hat{\lambda}\ \text{in } L^2(\Omega). \tag{2.8}$$

Combining (2.7) and (2.8) it follows that

$$\left(\mathrm{div}\ \hat{\lambda}, v\right)_{L^2} = -\left\langle\bar{\lambda}, \nabla v\right\rangle_{M^{n,*},M^n}\ \text{for all } v \in X. \tag{2.9}$$

Let us argue that $\hat{\lambda} \in H_0(\mathrm{div})$. By Green's formula and (2.9) we have

$$\left(\mathrm{div}\ \hat{\lambda}, v\right)_{L^2} + \left(\hat{\lambda}, \nabla v\right)_{L^2} = \int_\Gamma v\,\hat{\lambda}\,n\,ds = 0\ \text{for every } v \in \mathcal{D}(\bar{\Omega}).$$

Hence the trace of $\hat{\lambda}$ satisfies $\tau(\hat{\lambda}) = 0$ and $\hat{\lambda} \in H_0(\mathrm{div})$. Let $H = \{v \in H_0(\mathrm{div}) : \mathrm{div}\, v = 0\}$. Since H is a closed subspace of $L_n^2(\Omega)$, we have

$$L_n^2 = H \oplus H^\perp, \tag{2.10}$$

with $H^\perp$ the orthogonal complement of H in $L_n^2(\Omega)$. It is well-known that $H^\perp = \{\nabla q : q \in H^1(\Omega)\}$ [5]. Let us decompose

$$\hat{\lambda} = \lambda_1 + \lambda_2 \in H \oplus H^\perp,$$

with $\mathrm{div}\, \lambda_1 = 0$ and $\lambda_2 = \nabla \bar{q}$. The element $\bar{q}$ is unique when normalized by $\int_\Omega \bar{q}\, dx = 0$. Then (2.5) follows from (2.8). To verify (2.6) let us observe that by (2.2)

$$\langle \bar{\lambda}, \nabla(u - \bar{u}) \rangle_{M^{n,*}, M^n} + \int_\Omega |\nabla \bar{u}| \leq \int_\Omega |\nabla u| \text{ for all } u \in X$$

and by (2.9)

$$(-\mathrm{div}\, \lambda_2, u - \bar{u})_{L^2} + \int_\Omega |\nabla \bar{u}| \leq \int_\Omega |\nabla u| \text{ for all } u \in X.$$

Finally concerning the L_n^2-norm of λ_2 we have due to the orthogonal decomposition (2.10)

$$|\lambda_2|_{L_n^2} \leq |\bar{\lambda}|_{L_n^2} \leq \sqrt{\mathrm{meas}\,(\Omega)}.$$

This concludes the first part of the proof with $\tilde{\lambda} = \lambda_2$.

Conversely assume that $(\bar{u}, \tilde{\lambda}) \in X \times H_0(\mathrm{div})$ satisfies (2.5), (2.6), and let u be an arbitrary element in X. Let F denote the cost functional in (P). Then by (2.5), (2.6)

$$F(u) - F(\bar{u}) \geq ((T^*T + \alpha)\bar{u}, u - \bar{u})_{L^2} + \beta \int_\Omega |\nabla u| - \beta \int_\Omega |\nabla \bar{u}|$$

$$= \beta(\mathrm{div}\, \tilde{\lambda}, u - \bar{u}) - \beta(\mathrm{div}\, \tilde{\lambda}, u - \bar{u}) \geq 0$$

and $\bar{u}$ is a solution to (P).

COROLLARY 2.5. *Under the assumptions of Theorem 2.3, $\bar{q}$ is the unique variational solution satisfying $\int_\Omega \bar{q}\, dx = 0$ to the Neumann problem*

$$\begin{cases} -\beta\Delta q &= (T^*T + \alpha)\bar{u} + T^*z \text{ in } \Omega \\ \frac{\partial q}{\partial n} &= 0 \text{ on } \Gamma. \end{cases} \tag{2.11}$$

The compatibility condition is clearly satisfied since

$$\int_\Omega ((T^*T + \alpha)\bar{u} - T^*z)\, dx = 0$$

by (2.7) with $v = 1$. If the boundary of Ω is $C^{1,1}$-smooth then $\bar{q} \in H^2(\Omega)$.

COROLLARY 2.6. *Under the assumption of Theorem 2.3*

$$\left(-\mathrm{div}\, \tilde{\lambda}, \bar{u}\right)_{L^2(\Omega)} = \int_\Omega |\nabla \bar{u}| \tag{2.12}$$

and

$$\left(\mathrm{div}\, \tilde{\lambda}, u\right)_{L^2(\Omega)} \leq \int_\Omega |\nabla u| \quad \text{for all} \quad u \in X. \tag{2.13}$$

Equality (2.12) follows from (2.6) with $u = 0$ and $u = 2\bar{u}$. By (2.6) and (2.12) one obtains (2.13).

3. Stability with respect to data

This section is dedicated to the analysis of the stability of the solution to (P) with respect to the data z. It will be convenient to denote (P) by (P_z) and solutions to (P_z) by u_z. Let us first give the stability result that can be attained from a-priori estimates.

THEOREM 3.1. *Assume that (H) holds, let $\{z_n\}$ be a sequence of observations in Y converging to z, and let $\{u_{z_n}\}$ denote a sequence of solutions to (P_{z_n}). Then there exists a subsequence $\{u_{z_{n_k}}\}$ of $\{u_{z_n}\}$ and $u_z \in L^2(\Omega)$ such that*

$$\lim u_{z_{n_k}} = u_z \ \text{weakly in } L^2(\Omega) \ \text{and} \ \lim \int_\Omega |\nabla u_{z_{n_k}}| = \int_\Omega |\nabla u_z|, \quad (3.1)$$

and every such sequence has the property that u_z is a solution to (P). If (H) (i) holds then $\lim u_{z_{n_k}} = u_z$ strongly in $L^2(\Omega)$.

Proof. For simplicity let us denote $u_n = u_{z_n}$. We have for every $u \in K$

$$\frac{1}{2}|Tu_n - z_n|_Y^2 + \frac{\alpha}{2}|u_n|_{L^2}^2 + \beta \int_\Omega |\nabla u_n| \le \frac{1}{2}|Tu - z_n|_Y^2 \quad (3.2)$$
$$+ \frac{\alpha}{2}|u|_{L^2}^2 + \beta \int_\Omega |\nabla u|.$$

If (H) (i) or (iii) hold then it follows directly that $\{u_n\}$ is bounded in X. In the case of (H) (ii) boundedness of $\{u_n\}$ in X follows from Proposition 1.1 d). Thus there exists a subsequence of $\{u_n\}$ denoted, for simplicity by the same symbol and $u_z \in L^2(\Omega)$ such that

$$\lim u_n = u_z \ \text{weakly in } L^2(\Omega) \ \text{and} \ \lim u_n = u_z \text{ in } L^1(\Omega). \quad (3.3)$$

It follows that $u_z \in K$ and

$$\lim Tu_n = Tu_z \ \text{weakly in } Y \ \text{and} \ \int_\Omega |\nabla u_z| \le \underline{\lim} \int_\Omega |\nabla u_n|.$$

We can now pass to the limit in (3.2) to obtain

$$\frac{1}{2}|Tu_z - z|_Y^2 + \frac{\alpha}{2}|u_z|_{L^2}^2 + \int_\Omega |\nabla u_z| \le \frac{1}{2}|Tu - z|_Y^2 + \frac{\alpha}{2}|u|_{L^2}^2 + \beta \int |\nabla u|$$

for every $u \in K$ and hence u_z is a solution to (P_z). Let us show next the second convergence statement in (3.1). We have

$$\frac{1}{2}|Tu_z - z|^2 + \frac{\alpha}{2}|u_z|^2 + \overline{\lim} \ \beta \int_\Omega |\nabla u_n|$$
$$\le \underline{\lim} \frac{1}{2}|Tu_n - z_n|^2 + \underline{\lim} \frac{\alpha}{2}|u_n|^2 + \overline{\lim} \ \beta \int_\Omega |\nabla u_n|$$
$$\le \overline{\lim} \left(\frac{1}{2}|Tu_n - z_n|^2 + \frac{\alpha}{2}|u_n|^2 + \beta \int_\Omega |\nabla u_n|\right)$$
$$\le \overline{\lim} \ \left(\frac{1}{2}|Tu_z - z_n|^2 + \frac{\alpha}{2}|u_z|^2 + \beta \int_\Omega |\nabla u_z|\right)$$
$$= \frac{1}{2}|Tu_z - z|^2 + \frac{\alpha}{2}|u_z|^2 + \beta \int_\Omega |\nabla u_z|,$$

and hence $\overline{\lim} \int_{\Omega} |\nabla u_n| \leq \int_{\Omega} |\nabla u_z|$, which together with $\int_{\Omega} |\nabla u_z|$ $\leq \underline{\lim} \int_{\Omega} |\nabla u_n|$ implies the second statement in (3.1). If (H) (i) holds then one shows similarly that $\lim |u_n|_{L^2} = |u_z|$, which together with weak convergence implies strong convergence in $L^2(\Omega)$ of u_n to u_z. $\qquad \square$

The following lemma will be useful.

LEMMA 3.2. *For every pair of solutions u_z and $u_{\bar{z}}$ with associated Lagrange multipliers λ_z and $\lambda_{\bar{z}}$ we have*

$$\langle \lambda_z - \lambda_{\bar{z}}, \nabla(u_z - u_{\bar{z}}) \rangle_{M^n,*,M^n} \geq 0.$$

Proof. Since $\lambda_z \in \partial \varphi(\nabla u_z)$ and $\lambda_{\bar{z}} \in \partial \varphi(\nabla u_{\bar{z}})$ the Lemma follows from a wellknown property of subdifferentials. $\qquad \square$

THEOREM 3.3. *Let (H) (i) hold. Then for every pair $z, \bar{z} \in Y$*

$$|u_z - u_{\bar{z}}|_{L^2} \leq \frac{1}{\alpha} \| T \|_{\mathcal{L}(L^2,Y)} |z - \bar{z}|_Y.$$

Proof. From (2.1) of Theorem 2.2 we have

$$\langle (T^*T + \alpha)u_z - T^*z - \beta \text{ div } \lambda_z, u_{\bar{z}} - u_z \rangle \geq 0$$
$$\langle (T^*T + \alpha)u_{\bar{z}} - T^*\bar{z} - \beta \text{ div } \lambda_{\bar{z}}, u_z - u_{\bar{z}} \rangle \geq 0.$$

The duality pairing $\langle \cdot, \cdot \rangle$ must be interpreted as in (2.1) and hence

$$\langle (T^*T + \alpha)(u_z - u_{\bar{z}}) - T^*(z - \bar{z}) - \beta \text{ div } (\lambda_z - \lambda_{\bar{z}}), u_{\bar{z}} - u_z \rangle \geq 0.$$

Lemma 3.2 implies that

$$\langle (T^*T + \alpha)(u_z - u_{\bar{z}}) - T^*(z - \bar{z}), u_z - u_{\bar{z}} \rangle \leq \beta \langle \lambda_z - \lambda_{\bar{z}}, \nabla(u_{\bar{z}} - u_z) \rangle \leq 0,$$

and hence

$$\alpha |u_z - u_{\bar{z}}|^2 \leq \| T^* \| |u_z - u_z| |\bar{z} - z|,$$

and the result follows. $\qquad \square$

Let B_r denote the ball with center at the origin and radius r in Y.

THEOREM 3.4. *Let (H) (i) hold and let $r > 0$. Then*

$$\beta | \int_{\Omega} |\nabla u_z| - \int_{\Omega} |\nabla u_{\bar{z}}| | \leq \frac{r}{\alpha} \left(2 + \frac{1}{\alpha} \right) \ max \ (1, \| T \|^4)|z - \bar{z}|,$$

for every $z, \bar{z} \in B_r$.

Proof. From Theorem 3.3 we conclude that

$$|u_z|_{L^2} \leq \frac{r}{\alpha} \| T \| \text{ for every } z \in B_r. \tag{3.4}$$

Utilizing the fact that u_z is a solution to (P_z) we find

$$\beta \left(\int_{\Omega} |\nabla u_z| - \int_{\Omega} |\nabla u_{\bar{z}}| \right) \leq \frac{1}{2} \left[|Tu_{\bar{z}} - z|^2 - |Tu_z - z|^2 + \alpha |u_{\bar{z}}|^2 - \alpha |u_z|^2 \right].$$

Similarly, since $u_{\bar{z}}$ is a solution to $(P_{\bar{z}})$

$$\beta \left(\int_{\Omega} |\nabla u_{\bar{z}}| - \int_{\Omega} |\nabla u_z| \right) \leq \frac{1}{2} \left[|Tu_z - \bar{z}|^2 - |Tu_{\bar{z}} - \bar{z}|^2 + \alpha |u_z|^2 - \alpha |u_{\bar{z}}|^2 \right].$$

Combining these two inequalities implies by Theorem 3.3 and (3.4) that

$$\beta \, | \int_\Omega | \nabla u_z | - \int_\Omega | \nabla u_{\bar z} | \, | \leq \frac{\alpha}{2} \, | \, u_z - u_{\bar z} \, | \, | \, u_z + u_{\bar z} \, |$$

$$+ \frac{1}{2} \, | \, Tu_z - Tu_{\bar z} \, | \, (\, | \, Tu_z + Tu_{\bar z} \, | + 2r)$$

$$\leq \frac{r}{\alpha} \, \| \, T \, \|^2 \, | \, z - \bar z \, | + \| \, T \, \|^3 \, | \, u_z - u_{\bar z} \, | \left(\frac{r}{\alpha} + r \right)$$

$$= \frac{r}{\alpha} \left(2 + \frac{1}{\alpha} \right) \, \| \, T \, \|^4 \, | \, z - \bar z \, |,$$

and the desired estimate follows. $\square$

The last aspect that we address in this section is the stability of the dual variable $\lambda = \lambda_z$ with respect to perturbations in the data z. It will be assumed that (H) (i) is satisfied. This implies existence of a unique solution u_z to (P_z) for every $z \in Y$.

THEOREM 3.5. *Assume that (H) (i) holds and that $K = X$. Let $\{z_n\}$ be a sequence in Y with limit z, and let (u_{z_n}, λ_{z_n}) be the solutions and La-grange multipliers to (P_{z_n}) according to Theorem 2.2. Then u_{z_n} converges strongly in $L^2(\Omega)$ to the solution u_z of (P_z) and $\lim \int_\Omega | \nabla u_{z_n} | = \int_\Omega | \nabla u_z |$. Furthermore there exists a subsequence $\{\lambda_{z_{n_k}}\}$ of $\{\lambda_{z_n}\}$ converging weakly in H (div) and weak* in $L^\infty(\Omega)$ to some $\lambda_z \in L_n^\infty(\Omega) \cap H_0$ (div) and for every such cluster point we have*

$$(T^*T + \alpha)u_z = T^*z + \beta \, \mathrm{div} \, \lambda_z \, \text{ in } \, L^2(\Omega), \tag{3.5}$$

$$- (\mathrm{div} \, \lambda_z, u_z)_{L^2} = \int_\Omega | \nabla u_z |, \, \text{ and} \tag{3.6}$$

$$(\lambda_z, \mu)_{L^\infty, L^1} \leq \int_\Omega | \mu | \, dx, \, \text{ for all } \, \mu \in L_n^1(\Omega). \tag{3.7}$$

Proof. The first part of the claim follows from Theorem 3.1. For simplicity we shall write (u_n, λ_n) in place of (u_{z_n}, λ_{z_n}). The optimality systems for (P_{z_n}) are given by

$$(T^*T + \alpha) \, u_n = T^*z + \beta \, \mathrm{div} \, \lambda_n, \tag{3.8}$$

$$\langle \lambda_n, \nabla u_n \rangle_{M^{n,*}, M^n} = \int_\Omega | \nabla u_n |, \tag{3.9}$$

$$\langle \lambda_n, \mu \rangle_{M^{n,*}, M^n} \leq \int_\Omega | \mu |, \, \text{ for all } \, \mu \in M^n(\Omega). \tag{3.10}$$

Since $L_n^1(\Omega) \subset M^n(\Omega)$, the restrictions of $\mu \to \langle \lambda_n, \mu \rangle_{M^{n,*}, M^n}$ to $L_n^1(\Omega)$ can be identified with elements in $L^\infty(\Omega)$ with the property that $| \lambda_n |_{L^\infty} \leq 1$ for all n, by (3.10). From (3.8) it follows that $\{\lambda_n\}$ is bounded in $H_0(\mathrm{div})$ and thus there exists a subsequence of $\{\lambda_n\}$ denoted by the same symbol and $\lambda_z \in L_n^\infty(\Omega) \cap H_0(\mathrm{div})$ such that $| \lambda_z |_{L^\infty} \leq 1, \lambda_n \to \lambda_z$ weak* in $L_n^\infty(\Omega)$ and $\lambda_n \to \lambda_z$ weakly in $H_0(\mathrm{div})$. Taking the limit in (3.8) implies (3.5) and

(3.7) is equivalent to $|\lambda_z|_{L_n^\infty} \leq 1$. To verify (3.6) one argues as in the proof of Theorem 2.3 — see (2.9) — that (3.9) implies

$$-(\text{div } \lambda_n, u_n)_{L^2} = \int |\nabla u_n|. \tag{3.11}$$

Taking the limit in (3.11) one obtains (3.6).   □

It is simple to pass to the limit in the optimality system (2.5), (2.6) and to obtain the following result.

THEOREM 3.6. *Let the assumptions of Theorem 3.5 hold and let $(u_{z_n}, \tilde{\lambda}_n)$ denote the solutions to (P_{z_n}) with the associated Lagrange multipliers according to Theorem 2.3. Then $\tilde{\lambda}_{z_n}$ converges strongly in $L_n^2(\Omega)$ to $\tilde{\lambda}_z = \nabla q_z$ and weakly in $H_0(div)$ and*

$$(T^*T + \alpha)u_z = T^*z + \beta \ div \ \tilde{\lambda}_z \ in \ L^2(\Omega)$$
$$\left(-div \ \tilde{\lambda}_z, u - u_z\right)_{L^2} + \int_\Omega |\nabla u_z| \leq \int_\Omega |\nabla u| \ for \ all \ u \in X.$$

4. ASYMPTOTIC ANALYSIS

In this section convergence and rate of convergence of the solutions of the regularized problems as the regularization parameter tends to zero is analyzed. We consider the case where $\alpha = \beta$ and for convenience we repeat the problem formulation:

$$\min \frac{1}{2}|Tu - z|_Y^2 + \frac{\beta}{2}|u|_{L^2}^2 + \beta \int_\Omega |\nabla u| \text{ over } u \in K \cap X. \tag{P_β}$$

From Theorem 2.1 it is known that (P_β) admits a unique solution u_β for every $\beta > 0$ and $z \in Y$. The unregularized least squares problem is given by

$$\min \frac{1}{2}|Tu - z|_Y^2 \text{ over } u \in K \cap X. \tag{P_0}$$

Let us henceforth assume that $z \in Y$ is fixed and that z has a projection $\hat{z} \in Y$ on range $T(K \cap X)$. Since K is assumed to be convex this projection is necessarily unique. It then follows that (P_0) admits a solution and we define the set of solutions

$$\mathcal{S} = \{u \in K \cap X : Tu = \hat{z}\}.$$

Let us introduce the notion of minimal solution to (P_0) in $\mathcal{S}$. For that purpose we consider

$$\min \frac{1}{2}|u|_{L^2}^2 + \int_\Omega |\nabla u|, \quad u \in \mathcal{S}. \tag{Q}$$

By Proposition 1.1 it is simple to argue the existence of a solution to (Q). Due to the strict convexity of the L^2-part of the cost functional in (Q) this

solution is unique. It will be denoted by $\hat{u}_z$ and it is called the minimal solution of (P_0).

PROPOSITION 4.1. *There exists Λ_z in the dual of $M^n(\Omega)$ such that*

$$(\hat{u}_z, u - \hat{u}_z)_{L^2} - \langle div\, \Lambda_z, u - \hat{u}_z \rangle_{BV^*,BV} \geq 0 \quad \textit{for all} \quad u \in \mathcal{S},$$

$$\langle \Lambda_z, \mu - \nabla \hat{u}_z \rangle_{M^{n,*},M^n} + \int_\Omega |\nabla \hat{u}_z| \leq \int_\Omega |\mu| \quad \textit{for all} \quad \mu \in M^n(\Omega).$$

Here, as in Section 2, $-div$ is the adjoint to $\nabla : BV(\Omega) \to M^n(\Omega)$.

Let us give a first result on the behavior of the solutions u_β as a function of β.

PROPOSITION 4.2.

(i) $\dfrac{1}{2}\,|\,u_\beta\,|^2_{L^2} + \displaystyle\int_\Omega |\nabla u_\beta| \leq \dfrac{1}{2}\,|\,\hat{u}_z\,|^2_{L^2} + \int_\Omega |\nabla \hat{u}_z|$, *for every $\beta > 0$,*

(ii) $|\,T\hat{u}_z - z\,|_Y \leq |\,T u_\beta - z\,|_Y$, *for every $\beta > 0$,*

(iii) $\displaystyle\lim_{\beta \to 0^+} u_\beta = \hat{u}_z$ *strongly in* $L^p(\Omega), p < \dfrac{n}{n-2}$, $\displaystyle\lim_{\beta \to 0^+} u_\beta = \hat{u}_z$ *weakly in* $L^2(\Omega)$ *and* $\displaystyle\lim_{\beta \to 0^+} T u_\beta = T\hat{u}_z$ *strongly in Y.*

(iv) *If $z \in range\,(T)$, then $|\,T u_\beta - z\,| = o\left(\sqrt{\beta}\right)$.*

Proof. The second assertion follows from the fact that $\hat{u}_z \in \mathcal{S}$. To verify (i) we use (ii) and the fact that

$$\frac{1}{2}\,|\,T u_\beta - z\,|^2 + \frac{\beta}{2}\,|\,u_\beta\,|^2_{L^2} + \beta \int_\Omega |\nabla u_\beta|$$

$$\leq \frac{1}{2}\,|\,T\hat{u}_z - z\,|^2_Y + \frac{\beta}{2}\,|\hat{u}_z|^2_{L^2} + \beta \int_\Omega |\nabla \hat{u}_z|. \qquad (4.1)$$

From (i) it can be deduced that $\{|\,u_\beta\,|_X\}_{\beta > 0}$ is bounded. Hence there exists a subsequence of $\{u_\beta\}_{\beta > 0}$, denoted by the same symbol and $v \in X$ such that $u_\beta \to v$ strongly in $L^p(\Omega), p < \frac{n}{n-1}$, and weakly in $L^2(\Omega)$ as $\beta \to 0$. By Proposition 1.1 and (i)

$$\frac{1}{2}\,|\,v\,|^2_{L^2} + \int_\Omega |\nabla v| \leq \frac{1}{2}\underline{\lim}\,|\,u_\beta\,|^2_{L^2} + \underline{\lim} \int_\Omega |\nabla u_\beta|$$

$$\leq \overline{\lim} \left(\frac{1}{2}\,|\,u_\beta\,|^2_{L^2} + \int_\Omega |\nabla u_\beta| \right) \leq \frac{1}{2}\,|\,\hat{u}_z\,|^2_{L^2} + \int_\Omega |\nabla \hat{u}_z|. \qquad (4.2)$$

Taking the limit with respect to $\beta \to 0^+$ in (4.1) we obtain

$$|\,T v - z\,|_Y \leq |\,T\hat{u}_z - z\,|_Y. \qquad (4.3)$$

Combining (4.2) and (4.3) we find that v is a minimal solution to (P_0). But the minimal solution to (P_0) is necessarily unique and therefore $\hat{u}_z = v$. Since $T u_\beta \to T\hat{u}_z$ weakly in Y and $|\,T u_\beta - z\,| \to |\,T\hat{u}_z - z\,|$ it follows that $T u_\beta \to T\hat{u}_z$ strongly in Y. If $z \in range\,(T|K)$ then $T\hat{u}_z = z$ and (iv) follows from (4.1) and (4.2). $\qquad \square$

We consider now the case where noise is admitted in the data. To obtain convergence and convergence rate of u_β to $\hat{u}_z$, additional assumptions will be used. We consider

$$\min \frac{1}{2}\,|\,T u - z_\delta\,|^2 + \frac{\beta}{2}\,|\,u\,|^2_{L^2} + \beta \int_\Omega |\nabla u| \text{ over } u \in K \cap X \qquad (P_{\beta,\delta})$$

where

$$| z - z_\delta |_Y \le \delta. \tag{4.4}$$

The solution to $(\mathrm{P}_{\beta,\delta})$ will again be denoted by u_β, i.e. we omit the dependence on δ.

To interpret properly the assumptions that will be used below we reconsider the optimality condition for the minimal solution $\hat{u}_z$. For a convex set C in X the tangent cone to C at $u \in C$ is defined by

$$T(C, u) = \left\{ v \in X : \exists\, v_n \in C,\, \lambda_n > 0 \text{ with } \lim_{n \to \infty} \lambda_n (v_n - u) = v \right\},$$

the limit being taken in X, and the negative polar cone $N(C, u) \subset X^*$ at $u \in C$ is given by

$$N(C, u) = T(C, u)^- = \left\{ v \in X^* : \langle v, \bar{u} \rangle_{X^*, X} \le 0 \text{ for all } \bar{u} \in T(C, u) \right\}.$$

Note that $\hat{u}_z - \operatorname{div} \Lambda_z$ defines an element of X^* which we denote by the same symbol. The action of $\hat{u}_z - \operatorname{div} \Lambda_z$ on $\varphi \in X = L^2(\Omega) \times BV(\Omega)$ is given by

$$\langle \hat{u}_z - \operatorname{div} \Lambda_z, \varphi \rangle_{X^*, X} = (\hat{u}_z, \varphi)_{L^2} + \langle \Lambda_z, \nabla \varphi \rangle_{M^{n,*}, M^n}.$$

Referring to Proposition 4.1 we have

$$\operatorname{div} \Lambda_z - \hat{u}_z \in T(\mathcal{S}, \hat{u}_z)^-. \tag{4.5}$$

We shall denote the intersections of $\ker T$ and K with BV by $\ker_{BV} T$ and K_{BV} respectively. It is simple to argue that

$$T(\mathcal{S}, \hat{u}_z)^- \subset T(K_{BV}, \hat{u}_z) \cap \ker_{BV} T. \tag{4.6}$$

DEFINITION 4.3. The minimal solution $\hat{u}_z$ is called qualified if

$$T(\mathcal{S}, \hat{u}_z) = T(K_{BV}, \hat{u}_z) \cap \ker_{BV} T.$$

Clearly $\hat{u}_z$ is qualified if $\ker_{BV} T = \{0\}$ or if $K_{BV} = BV(\Omega)$. If

$$0 \in \operatorname{int}\, (K_{BV} - \ker_{BV} T)$$

then $\hat{u}_z$ is qualified [1]. If

$$K_{BV} = \{ u \in X : f_i(u) \le b_i, i = 1, \ldots, M \},$$

with M fixed, $f_i \in X^*, b_i \in \mathbb{R}$, then one can proceed as in ([4], Prop. 3) to show that every element of K_{BV} is qualified.

PROPOSITION 4.4. *If $\hat{u}_z$ is qualified then*

$$div\, \Lambda_z - \hat{u}_z \in \overline{T(K_{BV}, \hat{u}_z)^- + (\ker_{BV} T)^-}^{X^*},$$

where the bar with index X^* denotes closure in X^*.

Proof. Due to (4.5) and the qualification condition

$$\operatorname{div} \Lambda_z - \hat{u}_z \in (T(K_{BV}, \hat{u}_z) \cap \ker_{BV} T)^- = \overline{T(K_{BV}, \hat{u}_z)^- + (\ker_{BV} T)^-}^{X^*},$$

the last identity being well-known for polar cones ([1], p. 175). $\quad\square$

Let us note that $(\ker_{BV} T)^-$ cannot be replaced by $\overline{Rg(T')}$ since X is not reflexive and hence $\overline{Rg(T')}$ may be a strict subset of $(\ker_{BV} T)^-$. Here $T' : Y^* \to X^*$ denotes the conjugate of $T|X \in \mathcal{L}(X, Y)$. Let us also note that due to the assumption that $T \in \mathcal{L}(L^2(\Omega), Y)$ the elements in the range of T' can be identified with elements of L^2. The condition, which insures strong convergence of u_β to $\hat{u}_z$ is given by (H1)

$$-\hat{u}_z \in \overline{- \operatorname{div} \Lambda_z + T(K_{BV}, \hat{u}_z)^- + Rg\, T^*}^{L_2}. \tag{H1}$$

Here the bar with index L_2 denotes the closure in L^2. Since $Rg\, T^* \subset L^2(\Omega)$, it is assumed implicitly in (H1) that $T(K_{BV}, \hat{u}_z)^- - \operatorname{div} \Lambda_z$ can be identified with a subset of $L^2(\Omega)$. If $T(K_{BV}, \hat{u}_z)^-$ equals $\{0\}$, it is thus required that $\operatorname{div} \Lambda_z$ can be identified with an element of $L^2(\Omega)$. We shall discuss this requirement after Theorem 4.6. Under the additional assumption that

$$-\hat{u}_z \in - \operatorname{div} \Lambda_z + T(K_{BV}, \hat{u}_z)^- + Rg\, T^*, \tag{H2}$$

rate of convergence can be proved.

THEOREM 4.5.

   i) *If (H1) holds and $\beta \to 0^+, \delta^2/\beta(\delta) \to 0$ then the solutions u_β to $(P_{\beta,\delta})$ converge strongly to $\hat{u}_z$ in $L^2(\Omega)$.*

   ii) *If (H2) holds and $\beta \to 0^+, \beta \sim \delta$, then*

$$|u_\beta - \hat{u}_z|_{L^2} = O\left(\sqrt{\beta}\right),$$

   and

$$|Tu_\beta - \hat{z}|_Y = O(\beta).$$

Proof. By the definition of u_β

$$\frac{1}{2}|Tu_\beta - z_\delta|_Y^2 + \frac{\beta}{2}|u_\beta|_{L^2}^2 + \beta \int_\Omega |\nabla u_\beta|$$
$$\leq \frac{1}{2}|T\hat{u}_z - z_\delta|_Y^2 + \frac{\beta}{2}|\hat{u}_z|_{L^2}^2 + \beta \int_\Omega |\nabla \hat{u}_z|,$$

and hence

$$\frac{1}{2}|Tu_\beta - \hat{z}|_Y^2 + \frac{\beta}{2}|u_\beta - \hat{u}_z|^2$$
$$\leq \frac{1}{2}|T\hat{u}_z - z_\delta|^2 + \frac{1}{2}|Tu_\beta - \hat{z}|^2 - \frac{1}{2}|Tu_\beta - z_\delta|^2$$
$$+ \frac{\beta}{2}|\hat{u}_z|^2 + \frac{\beta}{2}|u_\beta - \hat{u}_z|^2 - \frac{\beta}{2}|u_\beta|^2 + \beta \left(\int_\Omega |\nabla \hat{u}_z| - \int_\Omega |\nabla u_\beta|\right).$$

Using $a^2 + b^2 - |a + b|^2 = -2(a, b)$ twice yields

$$\frac{1}{2} |Tu_\beta - \hat{z}|_Y^2 + \frac{\beta}{2} |u_\beta - \hat{u}_z|^2$$

$$\leq (Tu_\beta - \hat{z}, z_\delta - \hat{z})_Y + \beta (\hat{u}_z, \hat{u}_z - u_\beta) + \beta \left(\int_\Omega |\nabla \hat{u}_z| - \int_\Omega |\nabla u_\beta| \right)$$

$$\leq (Tu_\beta - \hat{z}, z_\delta - z) + (Tu_\beta - \hat{z}, z - \hat{z}) + \beta (\hat{u}_z, \hat{u}_z - u_\beta)$$

$$+ \beta \left(\int_\Omega |\nabla \hat{u}_z| - \int_\Omega |\nabla u_\beta| \right) \qquad (4.7)$$

$$\leq (Tu_\beta - \hat{z}, z_\delta - z) + \beta (\hat{u}_z, \hat{u}_z - u_\beta) \quad + \beta \left(\int_\Omega |\nabla \hat{u}_z| - \int_\Omega |\nabla u_\beta| \right).$$

Let $\varepsilon > 0$ be arbitrary and by (H1) choose $w \in Y, \eta \in T(K_{BV}, \hat{u}_z)^-$, $\mu \in L^2(\Omega)$ such that

$$\hat{u}_z = \operatorname{div} \Lambda_z - \eta + T^* w + \mu, \quad |\mu|_{L^2} \leq \varepsilon.$$

The L^2-inner product $(\hat{u}_z, \hat{u}_z - u_\beta)$ can be interpreted as duality pairing $(\hat{u}_z, \hat{u}_z - u_\beta)_{X^*, X}$ and hence

$$\frac{1}{2} |Tu_\beta - \hat{z}|_Y^2 + \frac{\beta}{2} |u_\beta - \hat{u}_z|_{L^2}^2$$

$$\leq (Tu_\beta - \hat{z}, z_\delta - z)_Y + \beta (T^* w, \hat{u}_z - u_\beta)_{X^*, X}$$

$$+ \beta (\Lambda_z, \nabla u_\beta - \nabla \hat{u}_z)_{M^{n,*}, M^n} + \beta (\mu, \hat{u}_z - u_\beta)_{L^2} \qquad (4.8)$$

$$+ \beta (\eta, u_\beta - \hat{u}_z)_{X^*, X} + \beta \left(\int_\Omega |\nabla \hat{u}_z| - \int_\Omega |\nabla u_\beta| \right)$$

$$\leq (\delta + \beta |w|_Y) |Tu_\beta - \hat{z}| + \beta \varepsilon |u_\beta - \hat{u}_z|_{L^2}.$$

The following argument is now standard [4]. The last inequality can be expressed as

$$a^2 + b^2 \leq 2Aa + 2Bb, \qquad (4.9)$$

where $a = |Tu_\beta - \hat{z}|$, $b = \sqrt{\beta} |u_\beta - \hat{u}_z|$, $A = (\delta + \beta |w|_Y)$ and $B = \sqrt{\beta} \varepsilon$.

It is simple to argue that (4.9) implies

$$a \leq 2A + B \text{ and } b \leq A + 2B,$$

and (i) follows. If (H2) holds then one can take $\mu = 0$ and (4.8) implies

$$|Tu_\beta - \hat{z}|_Y^2 + \beta |u_\beta - \hat{u}_z|^2 \leq 2(\delta + \beta |w|_Y) |Tu_\beta - \hat{z}|.$$

If $\delta \sim \beta$ then the asymptotic behavior asserted in (ii) follows. $\qquad \square$

Since the regularization term involves an L^2- as well as a BV-part we aim in the final result of this paper to improve the convergence properties of Theorem 4.5. We cannot expect convergence of $\int_\Omega |\nabla u_\beta - \nabla \hat{u}_z|$. In fact, let us assume that $\Omega = (0, 1)$, and let u_β be Heaviside functions with jump at $\frac{1}{2} + \beta, \beta \in (0, \frac{1}{2})$. Then $u_\beta \in BV(0, 1)$ and $|u_\beta - \bar{u}|_{L^2} \to 0$, $|\int_\Omega |\nabla u_\beta| - \int_\Omega |\nabla \bar{u}|| \to 0$ where $\bar{u}$ is the Heaviside function with jump at $\frac{1}{2}$. On the other hand $\int |\nabla u_\beta - \nabla \bar{u}| = 2$ for all $\beta \in (0, \frac{1}{2})$.

THEOREM 4.6. *In addition to the assumptions of Theorem 4.5 (ii) assume that $\operatorname{div} \Lambda_z \in X^*$ can be identified with an element of $L^2(\Omega)$. Then*

$$|\int_\Omega |\nabla u_\beta| - \int_\Omega |\nabla \hat u_z|\,| \leq K\,|\,u_\beta - u_z\,|_{L^2}$$

for a constant K independent of $\beta \in (0,1]$. In particular

$$|\int_\Omega |\nabla u_\beta| - \int_\Omega |\nabla \hat u_z|\,| = O\left(\sqrt{\beta}\right).$$

Proof. By Proposition 4.1 we have

$$\int_\Omega |\nabla \hat u_z| - \int_\Omega |\nabla u_\beta| \leq (\operatorname{div}\Lambda_z, u_\beta - \hat u_z) \leq |\operatorname{div}\Lambda_z|_{L^2}|u_\beta - \hat u_z| \quad (4.10)$$

and further by (2.2) it follows that

$$\int_\Omega |\nabla u_\beta| - \int_\Omega |\nabla \hat u_z| \leq \frac{1}{\beta}\left((T^*T + \beta)u_\beta, \hat u_z - u_\beta\right) - \frac{1}{\beta}(T^*z_\delta, \hat u_z - u_\beta)$$

$$= (u_\beta, \hat u_z - u_\beta) + \frac{1}{\beta}\left(T^*(Tu_\beta - z_\delta), \hat u_z - u_\beta\right)$$

$$\leq |u_\beta|_{L^2}|\hat u_z - u_\beta|_{L^2} + \frac{1}{\beta}\left(T^*(Tu_\beta - \hat z + (\hat z - z) + z - z_\delta), \hat u_z - u_\beta\right)$$

$$\leq |u_\beta|_{L^2}|\hat u_z - u_\beta|_{L^2} + \frac{1}{\beta}\|T\|_{\mathcal{L}(L^2, Y)}(|Tu_\beta - \hat z| + |z - z_\delta|)|\hat u_z - u_\beta|$$

$$= C|\hat u_z - u_\beta|_{L^2}$$

for some constant C independent of $\beta > 0$. Combined with (4.10) this gives the desired result. $\qquad\square$

REMARK 4.7. We consider several examples illustrating the condition $\operatorname{div} \Lambda_z \in L^2(\Omega)$, where Λ_z satisfies the second condition stated in Proposition 4.1, i.e.

$$\langle \Lambda_z, \mu - \nabla \hat u_z \rangle_{M^{n,*}, M^n} + \int_\Omega |\nabla \hat u_z| \leq \int_\Omega |\mu| \text{ for all } \mu \in M^n(\Omega),$$

which can equivalently be expressed as

$$\langle \Lambda_z, \nabla \hat u_z \rangle_{M^{n,*}, M^n} = \int_\Omega |\nabla \hat u_z|, \quad (4.11)$$

and

$$\langle \Lambda_z, \mu \rangle_{M^{n,*}, M^n} \leq \int_\Omega |\mu| \text{ for all } \mu \in M^n(\Omega). \quad (4.12)$$

While these examples will be one-dimensional with $\Omega = (0,1)$, higher dimensional examples can be obtained by extending the functions by constant values in the remaining directions.

Let us briefly discuss the requirements that are imposed on Λ_z by (4.11), (4.12) and Λ_z, $\operatorname{div} \Lambda_z \in L^2(\Omega)$ in the one-dimensional case. First of all this necessitates that $\Lambda_z \in H^1(\Omega) \subset C(\Omega)$. Satisfying (4.11) amounts, formally, to setting $\Lambda_z = \operatorname{sign} \hat u'_z$ at points where $\hat u_z$ is smooth but nonconstant. Since $\Lambda_z \in C(\Omega)$, condition (4.12) necessitates by the definition of $M(\Omega)$ as the dual of $C(\Omega)$ to choose $\Lambda_z \in C(\Omega)$ such that $|\Lambda_z(x)| \leq 1$ for all $x \in \Omega$.

Summarizing we see that (at least for piecewise smooth $\hat{u}_z$) the existence of Λ_z satisfying the specified requirements depends on the possibility of extending the graph of $\{(x, \operatorname{sgn} \hat{u}'_z(x)) : x \text{ such that } \hat{u}'_z(x) \neq 0\}$ to a graph with domain Ω and $|\hat{u}'_z(x)| \leq 1$ for all $x \in \Omega$. We now give three examples which illustrate the existence of Λ_z - or the lack thereof.

(i) Let

$$\hat{u}_z = \begin{cases} 3 & \text{on} & (0, 1/4) \\ 4 & \text{on} & (1/4, 1/2) \\ 2 & \text{on} & (1/2, 3/4) \\ 1 - x & \text{on} & (3/4, 1), \end{cases}$$

then Λ_z can be chosen as a $C'^\infty(\Omega)$-function with $|\Lambda_z|_C \leq 1$ (so that (4.12) is satisfied) and such that it equals
- 1 in a neighborhood of 1/4
- - 1 in a neighborhood of 1/2
- - 1 in a neighborhood of 3/4
- - 1 on $(3/4, 1)$.

Then Λ_z thus defined satisfies (4.11), (4.12) and div $\Lambda_z \in L^2(\Omega)$.

(ii) If $\hat{u}_z$ is chosen as in (i) but with $\hat{u}_z = x - 1$ on $(3/4, 1)$ instead of $1 - x$, then satisfying (4.11) requires that Λ_z equals
- 1 in a neighborhood of 1/4
- - 1 in a neighborhood of 1/2
- - 1 in a neighborhood of 3/4
- 1 on $(3/4, 1)$,

which can clearly not be satisfied by any function in $C'^\infty(\Omega)$. So in this case there exists no regular Λ_z satisfying (4.11) and (4.12).

(iii) For $\hat{u}_z$ given by

$$\hat{u}_z = \begin{cases} \sin \pi x \alpha & \text{on} & \left(0, \frac{1}{2\alpha}\right) \\ 1 & \text{on} & \left[\frac{1}{2\alpha}, 1 - \frac{1}{2\alpha}\right] \\ \sin \pi \left(\alpha(x - 1) + 1\right) & \text{on} & \left(1 - \frac{1}{2\alpha}, 1\right), \end{cases}$$

with $\alpha \in (1, \infty)$ one chooses $\Lambda_z \in C'^\infty(\Omega), |\Lambda_z|_C \leq 1$ such that

$$\Lambda_z = \begin{cases} 1 & \text{on} & \left(0, \frac{1}{2\alpha}\right) \\ -1 & \text{on} & \left(1 - \frac{1}{2\alpha}, 1\right). \end{cases}$$

Clearly Λ_z satisfies (4.11), (4.12) and div $\Lambda_z \in L^2(\Omega)$. For $\alpha = 1$ no such Λ_z can be found.

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