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LOCAL AND GLOBAL NULL CONTROLLABILITY OF TIME VARYING LINEAR CONTROL SYSTEMS

F. COLONIUS AND R. JOHNSON

Abstract. For linear control systems with coefficients determined by a dynamical system null controllability is discussed. If uniform local null controllability holds, and if the Lyapounov exponents of the homogeneous equation are all non-positive, then the system is globally null controllable for almost all paths of the dynamical system. Even if some Lyapounov exponents are positive, an irreducibility assumption implies that, for a dense set of paths, the system is globally null controllable.

1. Introduction

The purpose of this paper is to study the local and global null controllability of the family of linear control systems

$$\begin{gathered} x' = A(T_t(\omega))x + B(T_t(\omega))u \\ x \in \mathbb{R}^n, \quad u \in U \subset \mathbb{R}^m, \quad U \text{ compact convex} \end{gathered} \tag{1.1}$$

where the coefficients $A(T_t(\omega)) \in \mathbb{R}^{n\times n}$ and $B(T_t(\omega)) \in \mathbb{R}^{n\times m}$ are determined by a dynamical system $T_t : \Omega \to \Omega$ $(t \in \mathbb{R})$ on a compact metric space Ω. We assume that an ergodic measure μ on Ω is given and that the results of interest may only hold on a set $\{\omega\}$ with μ-probability 1. The variables x and u represent, respectively, the state and control of the system. The control function $u(\cdot)$ will be assumed to be a measurable map of $\mathbb{R}$ into U. The linear control system is driven by the dynamical system $\{T_t\}$ which is not influenced by the control system. It can be interpreted in a variety of ways. For example, if the coefficients are almost periodic functions of t, a classical construction in the theory of differential equations yields the system $\{T_t\}$ as the shift on the closure Ω of the set of translates of the coefficient functions in the space of continuous functions.

We devote special attention to control problems where only the amplitudes of time varying perturbations in the coefficients are known:

$$\begin{gathered} x' = A(w(t))x + B(w(t))u \\ x \in \mathbb{R}^n, \quad u \in U \subset \mathbb{R}^m, \quad w \in W \subset \mathbb{R}^k. \end{gathered} \tag{1.2}$$

Here U is a compact convex set containing the origin. In regard to the values of the coefficients A and B, we assume that $\{(A(w), B(w)) \mid w \in W\}$ is bounded in $\mathbb{R}^{n\times n}\times\mathbb{R}^{m\times n}$. Systems of the form (1.2) can be reformulated as

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F. Colonius, Universität Augsburg, Germany. Research supported by the Deutsche Forschungsgemeinschaft.

R. Johnson, Università di Firenze, Italy. Research supported by the Ministero delle Università e delle Ricerche Scientifiche e Tecnologiche.

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systems of type (1.1); see Section 2 below. The quantity w can be interpreted as a background noise which influences a reference control system

$$x' = A(w_0)x + B(w_0)u$$

via the perturbation terms $A(w) - A(w_0)$ and $B(w) - B(w_0)$. We can regard $w(\cdot)$ as a typical path of a stationary ergodic process $\{w_t\}$ with values in W. The stability properties of the homogeneous equations $x' = A(w(t))x$, $w(\cdot) \in W$, have been studied in [3], see also the survey [2]. Here the sets of Lyapounov exponents and the corresponding initial points have been characterized.

Our goal in the present paper is to prove local controllability results which are uniform in the path $w(\cdot)$, then to prove global controllability statements which hold for almost all paths $w(\cdot)$ (see § 2 for the definitions of local and global null controllability). Such results have been proved previously by Johnson and Nerurkar [7, 8] when the stationary ergodic process satisfies a uniform recurrence assumption. The main point here is to relax this assumption. We will instead impose the hypothesis that the hull (see § 2) Ω of the stationary ergodic process $\{w_t\}$ is the topological support of an ergodic measure μ. This hypothesis is very natural in the present context: it leads to no restriction on the class of systems (1.2) which we can study, and avoids the uniform recurrence assumption.

Under this hypothesis we will prove the following results.

(i) Uniform local null controllability holds over Ω if it holds for at least one point in each minimal subset of Ω.

(ii) If uniform local null controllability holds, and if the Lyapounov exponents of the homogeneous equation $x' = A(w(t))x$ are all non-positive, then (1.2) is globally null controllable for almost all paths $w(\cdot)$.

(iii) Even if some Lyapounov exponents are positive, an "irreducibility" assumption implies that, for a dense set of paths $w(\cdot)$, the process (1.2) is globally null controllable.

The paper is organized as follows. In § 2 we repeat some basic definitions, including those of local and global null controllability. We also review the "randomization" procedure by which the stationary ergodic process $\{w(t)\}$ is identified with a topological dynamical system. This construction permits the application to (1.2) of various techniques of topological dynamics. In fact such tools will be applied in § 3, where we study (uniform) local null controllability, and in § 4, where global null controllability is treated.

We wish to note that Baranova [1] has published a proof of our Theorem 4.5 regarding the global null controllability of (1.2) when the Lyapounov exponents of the homogeneous equation are non-positive.

2. Preliminaries

We begin by considering the control process (1.2) for a fixed measurable function $w : \mathbb{R} \to W$, so that the coefficients $A(w(t))$ and $B(w(t))$ are bounded measurable functions of t.

Definition 2.1. (a) A point $x_0 \in \mathbb{R}^n$ can be *steered to zero* in time $T > 0$ by the process (1.2) if there is a measurable control function $u : [0, T] \to U$

such that the solution of the initial value problem

$$\begin{aligned} x' &= A(w(t))x + B(w(t))u \\ x(0) &= x_0 \end{aligned}$$

satisfies $x(T) = 0$.

(b) The process (1.2) is said to be *locally null controllable* if there exists a neighborhood V of zero in $\mathbb{R}^n$ such that each $x_0 \in V$ can be steered to zero in some finite time $T > 0$.

REMARK 2.2. In Definition 2.1 (b) the time T may a priori be a function of $x_0 \in V$: $T = T(x_0)$. However the convexity of U and the fact that $0 \in U$ can be used to show that, if 2.1 (b) holds, then there is a neighborhood V_1 of 0 in $\mathbb{R}^n$ and a fixed time $T_1 > 0$ such that each $x_0 \in V_1$ can be steered to zero in some finite time T_1. See, e.g., [7, Corollary 2.7].

Next we briefly discuss the randomization process; see e.g. [6] for more details. For each path $w(\cdot)$ of the stationary ergodic process, consider the pair

$$\omega = (A(w(\cdot)), B(w(\cdot))) \in L^\infty(\mathbb{R}, \mathbb{R}^{n\times n}) \times L^\infty(\mathbb{R}, \mathbb{R}^{n\times m}).$$

Define

$$\Omega = \mathrm{cls}\,\{\omega \mid \mathrm{w}(\cdot) \text{ is a path of } \{\mathrm{W_t}\}\},$$

where the closure is taken with respect to the product of the weak-* topologies on $L^\infty(\mathbb{R}, \mathbb{R}^{n\times n})$ resp. $L^\infty(\mathbb{R}, \mathbb{R}^{n\times m})$. It follows from our assumption that $\{A(w), B(w) \mid w \in \mathcal{W}\}$ is bounded in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ that Ω is compact. Moreover Ω is invariant under shift flow defined by

$$(T_t\omega)(s) = \omega(t+s) \qquad (\omega \in \Omega;\ t, s \in \mathbb{R}).$$

The pair $(\Omega, \{T_t\})$ defines a topological flow, or continuous dynamical system, because the map $T : \Omega \times \mathbb{R} \to \Omega$: $(\omega, t) \to T_t\omega$ is continuous.

Next let $\mathcal{W}$ be the path space of the stationary ergodic process $\{W_t\}$, and let η be the corresponding probability measure on $\mathcal{W}$. Let $i : \mathcal{W} \to \Omega$: $w(\cdot) \to \omega$ be the natural map, and let μ be the image measure on Ω. Then μ is a Radon measure on Ω which is ergodic with respect to the flow $(\Omega, \{T_t\})$ [6].

CONVENTION 2.3. We redefine Ω to be the topological support of the measure μ.

This convention clearly entails no loss of generality if one is interested in properties of the control process (1.2) which are valid for almost all paths $w(\cdot)$.

We now consider the family of control processes

$$x' = A(T_t(\omega))x + B(T_t(\omega))u \tag*{$(2.1)_\omega$}$$

where ω ranges over Ω. Here we have abused notation and written

$$A(T_t(\omega)) \equiv A(w(t)), \quad B(T_t(\omega)) \equiv B(w(t)).$$

Of course one can write down bounded Borel functions $A : \Omega \to \mathbb{R}^{n\times n}$, $B : \Omega \to \mathbb{R}^{n\times m}$ such that, for each $\omega \in \Omega$, equation $(2.1)_\omega$ coincides with (1.1) with path $w(\cdot)$. However for present purposes we can simply identify $t \to A(T_t(\omega))$ resp. $t \to B(T_t(\omega))$ with the first resp. the second component of $\omega \in L^\infty(\mathbb{R}, \mathbb{R}^{n\times n}) \times L^\infty(\mathbb{R}, \mathbb{R}^{n\times m})$.

We emphasize that, in what follows, the only hypotheses on Ω which we will need are: (i) that it is weak*-compact and translation invariant in $L^\infty(\mathbb{R}, \mathbb{R}^{n\times n}) \times L^\infty(\mathbb{R}, \mathbb{R}^{n\times m})$; (ii) that it is the topological support of the ergodic measure μ.

DEFINITION 2.4. The family of control processes $\{(2.1)_\omega \mid \omega \in \Omega\}$ is said to be *uniformly* locally null controllable if there is a $T > 0$ and a neighborhood V of $0 \in \mathbb{R}^n$ such that each $x_0 \in V$ can be steered to zero in time T by the process $(2.1)_\omega$ $(\omega \in \Omega)$.

In § 3 we will study the concept of uniform local null controllability. We will use a theorem of [7] which we now recall.

DEFINITION 2.5. The flow $(\Omega, \{T_t\})$ is called *minimal* or *uniformly recurrent* if every orbit $\{T_t(\omega) \mid t \in \mathbb{R}\}$ is dense in Ω $(\omega \in \Omega)$.

An equivalent definition is that the only nonempty compact invariant subset of Ω is Ω itself. See [5] for a detailed discussion of the theory of minimal sets. It is easy to see that, since Ω is the topological support of the ergodic measure μ, the orbit $\{T_t(\omega) \mid t \in \mathbb{R}\}$ is dense in Ω for μ-a.e. $\omega \in \Omega$. However minimality is a much more restrictive condition.

The result of [7] which we will use is the following (Theorem 2.10 of [7]).

THEOREM 2.6. *Suppose that the flow $(\Omega, \{T_t\})$ is minimal. Suppose that the process $(2.1)_{\omega_0}$ is locally null controllable for a single point $\omega_0 \in \Omega$. Then the family $\{(2.1)_\omega \mid \omega \in \Omega\}$ is uniformly locally null controllable.*

3. LOCAL NULL CONTROLLABILITY

We study the family of control processes $(2.1)_\omega$ where Ω is the topological support of an ergodic measure μ. Our goal is to generalize Theorem 2.6 to this situation.

Our starting point is a result of Barmish-Schmitendorf, which also began developments in [7]. For a subset U of $\mathbb{R}^m$ the *support function* $H_U : \mathbb{R}^m \to \mathbb{R}$ is defined by

$$H_U(\alpha) = \sup\{<\alpha, u>\mid u \in U\},$$

where $<,>$ denotes the Euclidean inner product on $\mathbb{R}^m$. Let $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R} \to \mathbb{R}^{n\times m}$ be locally integrable matrix functions, and consider the control process

$$x' = A(t)x + B(t)u. \tag{3.1}$$

Here the control function $u : \mathbb{R} \to U$ is any measurable U-valued function. The set U is assumed to be compact and to contain $0 \in \mathbb{R}^m$.

THEOREM 3.1. [13] *The following are equivalent.*

(a) *The process (3.1) is locally null controllable.*
(b) *There exists $\epsilon > 0$ such that*

$$\int_0^\infty H_U(B^*(s)z^*(s))\,ds \geq \epsilon,$$

where B^* is the transpose of B and $z^*(t)$ is any solution of the adjoint system $z' = -A^*(t)z$ such that $\| z^*(0) \| = 1$.

REMARK 3.2. It follows from the Barmish-Schmitendorf proof of Theorem 3.1 that, if $\epsilon > 0$ is a number for which 3.1 (b) holds, then each $x_0 \in \mathbb{R}^n$ with $\| x_0 \| < \epsilon/2$ can be steered to zero by the process (3.1).

Now we can state and prove the main result of this section. As in the first two sections of the paper, $U \subset \mathbb{R}^m$ is a compact convex subset containing the origin. We consider the family of control processes

$$x' = A(T_t(\omega))x + B(T_t(\omega))u \tag*{$(3.2)_\omega$}$$

where ω ranges over Ω.

THEOREM 3.3. *Suppose that each minimal subset $M \subset \Omega$ contains a point ω_0 such that the process $(3.2)_{\omega_0}$ is locally null controllable. Then the family $\{(3.2)_\omega \mid \omega \in \Omega\}$ of control processes is uniformly locally null controllable.*

Proof. First note that, by Theorem 2.8, the conclusion of Theorem 3.3 holds over every minimal subset $M \subset \Omega$.

Suppose now for contradiction that there exists $\omega \in \Omega$ such that $(3.2)_\omega$ is not locally null controllable . By Theorem 3.1 there exists a sequence $\{z_n^*(0)\}$ of vectors of norm 1 in $\mathbb{R}^n$ such that the corresponding solutions $z_n(t)$ of the adjoint equation $z' = -A^*(T_t(\omega))z$ satisfy:

$$\int_0^\infty H_U(B^*(T_t(\omega))z_n^*(t))\,dt < \frac{1}{n}.$$

Passing to a subsequence and using Fatou's Lemma (note that $H_U \geq 0$ since $0 \in U$), we conclude that there exists $z^*(0) \in \mathbb{R}^n$ of norm 1 such that, if $z^*(t)$ is the corresponding solution of the adjoint equation:

$$\int_0^\infty H_U(B^*(T_t(\omega))z^*(t))\,dt = 0.$$

Next let $t_n \to \infty$ be a sequence such that $T_{t_n}(\omega)$ converges to a point $\widetilde{\omega}$ in a minimal subset of Ω. Elementary arguments of topological dynamics [5] show that such a sequence exists. We have

$$\begin{aligned} 0 &= \int_0^\infty H_U(B^*(T_t(\omega))z^*(t))\,dt \\ &= \int_0^{t_n} H_U(B^*(T_t(\omega))z^*(t))\,dt + \int_{t_n}^\infty H_U(B^*(T_t(\omega))z^*(t))\,dt \\ &\geq \int_0^\infty H_U(B^*(T_{t+t_n}(\omega))z^*(t+t_n))\,dt. \end{aligned}$$

Hence writing $\omega_n = T_{t_n}(\omega)$:

$$0 = \int_0^\infty H_U(B^*(T_t(\omega_n))\widetilde{z}_n(t))\,dt \tag{3.3}$$

where $\widetilde{z}_n(t)$ is the solution of the adjoint system $z' = -A^*(T_t(\omega_n))z$ which satisfies $\widetilde{z}_n(0) = \dfrac{z^*(t_n)}{\|z^*(t_n)\|}$.

Passing to a subsequence, we can assume that $\widetilde{z}_n(0)$ converges to a point $\widetilde{z}(0) \in \mathbb{R}^n$ of norm 1. Let $\widetilde{z}(t)$ be the solution of the adjoint equation $z' = -A^*(T_t(\widetilde{\omega}))z$ with initial condition $\widetilde{z}(0)$. We cannot apply Fubini's theorem directly to (3.3) to conclude that $\displaystyle\int_0^\infty H_U(B^*(T_t(\widetilde{\omega}))\widetilde{z}(t))\,dt = 0$

because it is not clear that $H_U(B^*(T_t(\omega_n))\widetilde{z}_n(t))$ converges pointwise to $H_U(B^*(T_t(\widetilde{\omega}))\widetilde{z}(t))$. However we can apply the theory of measurable selections [4]. Write $b_n(t) = B(T_t(\omega_n))$, $\widetilde{b}(t) = B(T_t(\widetilde{\omega}))$, and choose $T > 0$. We have first of all:

$$\begin{aligned} 0 &= \int_0^T H_U(b_n^*(t)\widetilde{z}_n(t))\,dt = \int_0^T \sup_{\alpha\in U} < \alpha, b_n^*(t)\widetilde{z}_n(t) > \,dt \\ &\geq \sup_{u\in\mathcal{U}} \int_0^T < u(t), b_n^*(t)\widetilde{z}_n(t) > \,dt \end{aligned}$$

where $\mathcal{U}$ is the set of all measurable mappings $u : [0,T] \to U$. Hence

$$\int_0^T < u(t), b_n^*(t)\widetilde{z}_n(t) > \,dt \leq 0 \qquad (u \in \mathcal{U}).$$

Now use the compactness of U, the uniform convergence of $\widetilde{z}_n(t)$ to $\widetilde{z}(t)$ on $[0,T]$, the uniform boundedness of $\{b_n^*\}$, and the weak-* convergence of b_n^* to b^* to obtain

$$0 \geq \int_0^T < u(t), b_n^*(t)\widetilde{z}_n(t) > \,dt \;\to\; \int_0^T < u(t), \widetilde{b}^*(t)\widetilde{z}(t) > \,dt$$

for each $u \in \mathcal{U}$. Hence using measurable selection theory [4]:

$$\begin{aligned} 0 &\geq \sup_{u\in\mathcal{U}} \int_0^T < u(t), \widetilde{b}^*(t)\widetilde{z}(t) > \,dt = \int_0^T \sup_{\alpha\in U} < \alpha, \widetilde{b}^*(t)\widetilde{z}(t) > \,dt \\ &= \int_0^T H_U(\widetilde{b}^*(t)\widetilde{z}(t))\,dt \geq 0, \end{aligned}$$

and since this holds for every $T > 0$ we get

$$\int_0^\infty H_U(B^*(T_t(\widetilde{\omega}))\widetilde{z}(t))\,dt = 0.$$

This contradicts Theorem 3.1 and the Theorem 2.10 of [7] referred to above. Hence $(3.2)_\omega$ is locally null controllable for each $\omega \in \Omega$.

We remark that the proof of Theorem 2.10 in [7] tacitly assumes that B is continuous as a function of ω. To generalize the proof to the case of the measurable B considered here requires only the use of the measurable selection theorem as just illustrated.

We finish the proof of Theorem 3.3 by proving the *uniform* local null controllability. For this we use Remarks 2.2 and 3.2: it suffices to find $\epsilon > 0$ such that, for each $\omega \in \Omega$ and each solution $z^*(t)$ of the adjoint system $z' = -A^*(T_t(\omega))z$ with $\|z^*(0)\| = 1$, there holds

$$\int_0^\infty H_U(B^*(T_t(\omega))z^*(t))\,dt \geq \epsilon.$$

Assume for contradiction that there are sequences $\{z_n^*(0)\} \subset \mathbb{R}^n$ of norm 1 and $\{\omega_n\} \subset \Omega$ such that $\int_0^\infty H_U(B^*(T_t(\omega_n))z^*(t))\,dt < 1/n$. Passing to convergent subsequences and using the measurable selection theorem as above, we can find $\widetilde{\omega} \in \Omega$ and $\widetilde{z}(0) \in \mathbb{R}^n$ of norm 1 such that, if $\widetilde{z}(t)$ is the corresponding solution of $z' = -A^*(T_t(\widetilde{\omega}))z$, then

$$\int_0^\infty H_U(B^*(T_t(\widetilde{\omega}))\widetilde{z}(t))\,dt = 0.$$

But then $(3.2)_{\widetilde{\omega}}$ is not locally null controllable, and this contradicts the first part of the proof. So Theorem 3.3 is verified. $\square$

4. Global Null Controllability

Consider for a moment the initial value problem

$$\begin{aligned} x' &= A(t)x \qquad x \in \mathbb{R}^n \\ x(0) &= x_0 \neq 0 \end{aligned}$$

where $A(t)$ is a locally-integrable matrix function defined on $\mathbb{R}$. Letting $x(t, x_0)$ be the corresponding solution, the *Lyapounov exponent* is given by

$$\lambda(x_0) = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \| x(t, x_0) \|.$$

Of course this definition can be applied to each equation in the ergodic family

$$x' = A(T_t(\omega))x \tag{4.1}$$

where ω ranges over Ω. The well-known theorem of Oseledec [10] states that, for μ-a.e. $\omega \in \Omega$, there are finitely many Lyapounov exponents $\lambda_1, \dots, \lambda_k$ where $k \leq n$ and the exponents do not depend on ω. Furthermore there is a measurable decomposition of the product bundle $\Omega \times \mathbb{R}^n$

$$\Omega \times \mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$$

into invariant measurable subbundles $V_1, \dots, V_k$ where $(\omega, x_0) \in V_r$ if and only if either $x_0 = 0$ or $\lim_{t\to\pm\infty} t^{-1} \ln \| x(t; x_0, \omega) \| = \lambda_r$. See [10, 9] for details.

Next we review a basic construction which will be useful in developing our theory. Write $\Phi(\omega, t)$ for the $n \times n$ matrix solution of (4.1) which is the $n \times n$ identity at $t = 0$. We note that, by a convenient smoothing trick due to Ellis [5], there is a change of variables $x = P_1(T_t(\omega))y$ with continuous invertible coefficient function $P_1 : \Omega \to \mathbb{R}^{n\times n}$ such that the transformed coefficient matrix

$$P_1^{-1} A P_1 - P_1^{-1} \frac{dP_1}{dt}$$

is a *continuous* function of ω. The introduction of such a change of variables clearly has no effect on the local or global controllability properties of the processes $(3.2)_\omega$. So we can and will assume WLOG that $A : \Omega \to \mathbb{R}^{n\times n}$ is a continuous function. (However B cannot be made continuous in this way).

Turning to the promised construction, let g be an element of the orthogonal group $O(n)$. We can use the Gram-Schmidt procedure to write

$$\Phi(\omega, t) = G(\omega, g, t) \cdot E(\omega, g, t)$$

where $G(\omega, g, t) \in O(n)$ and $E(\omega, g, t)$ is a triangular matrix with zeros above the main diagonal and positive diagonal entries. Write $z = (\omega, g) \in \Omega \times O(n)$. Using the "cocycle identity" $\Phi(\omega, t, s) = \Phi(T_t(\omega), s)\Phi(\omega, t)$, one checks that the maps $\widehat{T}_t : z = (\omega, g) \to (T_t(\omega), G(\omega, g, t))$ define a flow on $\Omega \times O(n)$. Furthermore the continuous matrix function

$$e(z) = \frac{d}{dt} E(z, t)\bigg|_{t=0}$$

is lower triangular, and $E(z,t)$ is the $n\times n$ matrix solution of

$$x' = e(\widehat{T}_t(z))x \tag{4.2}$$

which satisfies $E(z,0)=I$. See [9, 8].

Next let $\pi:\Omega\times O(n)\to\Omega$: $(\omega,g)\to\omega$ be the natural projection. Let $\omega_0\in\Omega$ be a point whose orbit $\{T_t(\omega_0)\mid t\in\mathbb{R}\}$ is dense in Ω, and let $g_0\in O(n)$. Then the orbit closure $Z_1=\mathrm{cls}\{\widehat{\mathrm{T}}_{\mathrm{t}}(\omega_0,\mathrm{g}_0)\mid \mathrm{t}\in\mathbb{R}\}$ has the property that $\pi(Z_1)=\Omega$. Let ν be a Radon probability measure on Z_1 which is ergodic with respect to $\{\widehat{T}_t\}$ and whose image $\pi(\nu)=\mu$; such a measure always exists. Define Z to be the topological support of ν.

We can lift the family $(3.2)_\omega$ to a family of processes on Z by simply defining $\widehat{A}(z)=A(\pi(z))$, $\widehat{B}(z)=B(\pi(z))$. Define $P:Z\to O(n)$: $P(z)=g$ if $z=(\omega,g)$. Introduce the orthogonal change of variables

$$x=P(\widehat{T}(z))y$$

in the family $(3.2)_\omega$ lifted to Z. The result is

$$y'=e(\widehat{T}_t(z))y+b(\widehat{T}(z))u \tag{4.3}$$

where $e(\cdot)$ is the function introduced above and $b(z)=P^{-1}(z)\widehat{B}(z)$. Clearly the local/global null controllability properties of the process (4.3) are the same as those of the process $(3.2)_\omega$ with $\omega=\pi(z)$.

We proceed to analyze the family of processes (4.3). The first step is to state the following ergodic theoretic result, proved by Schneiberg [14].

THEOREM 4.1. *Let $f\in L^1(Z,\nu)$ be a function such that $\int_Z f\,d\nu=0$ and let $\epsilon>0$. Then for ν-a.e. $z\in Z$, there is a sequence $t_k\to\infty$ (which depends on z) such that*

$$\left|\int_0^{t_k} f(\widehat{T}_s(z))\,ds\right|<\epsilon.$$

We will use the following variant of Theorem 4.1.

PROPOSITION 4.2. *Let $f\in L^1(Z,\nu)$ be a function such that $\int_Z f\,d\nu\le 0$. Let $\widetilde{Z}$ be the set of $z\in Z$ such that given $\epsilon>0$, $T>0$, and $k\ge 1$, there are numbers $Q_j>T$ such that, if $S_0=0$ and $S_j=\sum_{i=1}^{j}Q_i$, then*

$$\int_0^{Q_j} f(\widehat{T}_s(\widehat{T}_{S_j-1}(z)))\,ds<\epsilon\quad (1\le j\le k).$$

Then $\nu(\widetilde{Z})=1$.

Proof. The statement of the Proposition follows from the Birkhoff ergodic theorem if $\int_Z f\,d\nu<0$. If $\int_Z f\,d\nu=0$, we fix ϵ, T and k, and we use Theorem 4.1 to choose $Q_j>T$ such that $\left|\int_0^{S_j} f(T_s(z))\,ds\right|<\epsilon/2$ for all $1\le j\le k$; here $z\in\widetilde{Z}$ and $S_j=Q_1+\cdots+Q_j$. This implies the statement of Proposition 4.2 for each $z\in\widetilde{Z}$ and completes the proof. □

Now we return to the family of processes $(3.2)_\omega$ and to the Lyapounov exponents $\lambda_1,\dots,\lambda_k$ of the ergodic family (4.1); recall that these exponents are constant μ-a.e. We will show that, if these exponents are all non-positive, and if the family $(3.2)_\omega$ is uniformly locally null controllable, then $(3.2)_\omega$ is globally null controllable for μ-a.e. $\omega \in \Omega$. That is, we will find a set $\Omega_* \subset \Omega$ with $\mu(\Omega_*) = 1$ such that, if $\omega \in \Omega_*$, then each vector $x_0 \in \mathbb{R}^n$ can be steered to zero by $(3.2)_\omega$. (The time T which it takes to steer x_0 to zero may depend on ω and x_0.)

First let e_{ij} $(1 \leq i, j \leq n)$ be the entries of the matrix function $e : Z \to \mathbb{R}^{n\times n}$, thus $e_{ij} = 0$ if $i < j$. It is known (e.g., [9]) that the Lyapounov exponents $\lambda_1, \ldots, \lambda_k$ are given by the mean values of the diagonal elements e_{ii} of e with respect to ν; therefore

$$\int_Z e_{ii}(z)\, d\nu(z) \leq 0 \qquad (1 \leq i \leq n).$$

Let $\widehat{Z}$ be the set of $z \in Z$ for which the conclusion of Proposition 4.2 holds for each $1 \leq i \leq n$, and let $Z_* = \cap\{\widehat{T}_n(\widehat{Z}) \mid n = 1, 2, 3, \ldots\}$. Then $\nu(Z_*) = 1$, and therefore $\Omega_* = \pi(Z_*)$ is μ-measurable and $\mu(\Omega_*) = 1$. We will show that $(3.2)_\omega$ is globally null controllable for each $\omega \in \Omega_*$.

We prove a preliminary steering lemma. Let us say that a vector $x_0 \in \mathbb{R}^n$ can be ω-steered to another vector $x_1 \in \mathbb{R}^n$ in time T if there is a measurable control $u : [0, T] \to U$ so that the solution $x(t)$ of

$$\begin{aligned} x' &= A(T_t(\omega))x + B(T_t(\omega))u \\ x(0) &= x_1, \end{aligned}$$

satisfies $x(T) = x_2$.

Lemma 4.3. *Let $\omega \in \Omega$. If a vector $x_0 \in \mathbb{R}^n$ can be ω-steered to a vector x_1, in time T_1, and if x_1 can be $T_{t_1}(\omega)$-steered to x_2 in time T_2, then x_0 can be ω-steered to x_2 in time $T_1 + T_2$.*

Proof. Let u_1 and u_2 be admissible controls which steer x_0 to x_1 and x_1 to x_2 respectively. Then the control

$$u(t) = \begin{cases} u_1(t) & 0 \leq t < T_1, \\ u_2(t - T_1) & T_1 \leq t \leq T_1 + T_2, \end{cases}$$

will ω-steer x_0 to x_2 in time $T_1 + T_2$. □

We now prove

Lemma 4.4. *Suppose that the family of control processes $\{(3.2)_\omega \mid \omega \in \Omega\}$ is uniformly locally null controllable (see § 3). Let $z \in Z_*$. There exists $\epsilon > 0$ such that, for each $a, y_2, \ldots, y_n \in \mathbb{R}$ with $|a| > 2\epsilon$, the vector $(a, y_2, \ldots, y_n)^t$ can be z-steered to a vector $(b, v_2, \ldots, v_n)^t$ with $|b| < |a| - \epsilon$.*

Proof. It is clear that the family of control processes (4.3) is uniformly locally null controllable. Choose $\epsilon > 0$ such that each vector y of length $\|y\| < 3\epsilon$ can be z-steered to 0 for each $z \in Z$.

Fix $z \in Z_*$, and let $\Psi_z(t)$ be the matrix solution of the homogeneous system (4.2)

$$y' = e(\widehat{T}_t(z))y = \begin{pmatrix} e_{11}(\widehat{T}_t(z)) & & \\ * & \ddots & 0 \\ & & e_{nn}(\widehat{T}_t(z)) \end{pmatrix} y$$

satisfying $\Psi_z(0) = I = n \times n$ identity matrix. The solution of (4.2) with initial condition $(a, y_2, \ldots, y_n)^t$ has the form

$$\begin{pmatrix} a \exp \int_0^t e_{11}(\widehat{T}_s(z))\, ds \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}.$$

Using Proposition 4.2, we can find a time $T_2 > 0$ such that

$$(|a| - 2\epsilon) \exp \int_0^{T_2} e_{11}(\widehat{T}_s(z))\, ds < |a| - \epsilon. \qquad (*)$$

Consider the control problem (4.3) with initial condition $(\pm 2\epsilon, 0, \ldots, 0)^t$ where the sign is the negative of $\operatorname{sgn} a$. There is a time $T_1 > 0$ and a control $u_1 : [0, T] \to U$ such that the solution $y(t)$ of (4.3) with this initial condition satisfies $y(T_1) = 0$. We can assume that $T_1 < T_2$.

Next we write

$$\begin{pmatrix} a \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a \pm 2\epsilon \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} \pm 2\epsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where again the sign is the negative of the $\operatorname{sgn} a$. If $\widetilde{y}(t)$ is the solution of the problem (4.3) with initial condition $(a, y_2, \ldots, y_n)^t$ and control $u(t)$ defined by

$$u(t) = \begin{cases} u_1(t) & 0 \le t \le T_1 \\ 0 & T_1 < t \le T_2, \end{cases}$$

then $\widetilde{y}(T_2)$ has the form

$$\widetilde{y}(T_2) = \begin{pmatrix} (a \pm 2\epsilon) \exp \int_0^{T_2} e_{11}(\widehat{T}_s(z))\, ds \\ \widetilde{y}_2(T_2) \\ \vdots \\ \widetilde{y}_n(T_2) \end{pmatrix},$$

where we have used the variation of parameters formula. Using $(*)$, we see that the first component of $\widetilde{y}(T_2)$ has magnitude less than $|a| - \epsilon$. This completes the proof of Lemma 4.4. $\square$

We now prove

THEOREM 4.5. *Suppose that the Lyapounov exponents of the process (3.2)$_\omega$ are non-positive and that the process is uniformly locally null controllable. Then for every $\omega \in \Omega_*$, the process (3.2)$_\omega$ is globally null controllable.*

Proof. It suffices to prove that, if $z \in Z_*$ is a point such that $\pi(z) = \omega$, then the process (4.3) is globally null controllable.

As in the proof of Lemma 4.4, choose $\epsilon > 0$ such that each vector $y \in \mathbb{R}^n$ of length less than 3ϵ can be z-steered to zero. Fix an initial vector $\widetilde{y} = (a, *, \ldots, *)^t \in \mathbb{R}^n$, and assume that $a \neq 0$. Applying Proposition 4.2, Lemma 4.3, and Lemma 4.4 at most $[a/\epsilon]$ times, we can find a time $S_1 > 0$ and a control $u_1 : [0, S_1] \to U$ such that, if $y_1(t)$ is the solution of (4.3) with control u_1 and initial condition $y_1(0) = \widetilde{y}$, then $y_1(S_1)$ has first component equal to zero. Increasing S_1 if necessary, we can assume that S_1 is an integer.

Now $T_{s_1}(z) \in Z_*$ by construction of Z_*. Using the triangular form of the equations (4.2), and arguing as we did in the proof of Lemma 4.4 and in the initial step of the present proof, we can find an integer $S_2 > 0$ and a control $u_2 : [0, S_2] \to U$ which will $T_{S_1}(z)$-steer the vector $y_1(S_1)$ to a vector $y_2(S_2)$ whose second component is zero. Repeating this reasoning n times, and applying Lemma 4.3 at the end, we obtain a control u with values in U which z-steers $\widetilde{y}$ to zero. We have proved Theorem 4.5. □

Remark 4.6. (1) Theorem 4.5 generalizes Theorem 3.2 of [7] in that we do not assume that the flow $(\Omega, \{T_t\})$ is minimal. On the other hand, our proof by triangularization is a direct generalization of the proof of the aforementioned [7, Theorem 3.2].

(2) Baranova [1] has give a proof of Theorem 4.5 which also uses the triangularization technique. She does not use the Schneiberg recurrence result quoted in Theorem 4.1.

We wish now to consider the situation when one or more of the Lyapounov exponents $\lambda_1, \ldots, \lambda_k$ with respect to μ is positive. In this case, global null controllability definitely does not hold if $\{w_t\} = 0$, i.e. if the background noise is not present. However, in the random case it may well happen that a dense set of processes $(3.2)_\omega$ is globally null controllable (always assuming uniform local null controllability ...). This phenomenon was studied in [8] when Ω is the topological support of the ergodic measure μ. We will give a general sufficient condition for the existence of such a dense set (Theorem 4.8). It should be pointed out, however, that the verification of the sufficient condition has only been efficiently carried out when μ is the *unique* ergodic measure on Ω (see [8]). But the uniqueness of μ implies that $(\Omega, \{T_t\})$ is minimal since we set $\Omega = \operatorname{supp} \mu$.

We need the following

Lemma 4.7. *Let $f : \Omega \to \mathbb{R}$ be a continuous function such that $\int_\Omega f\, d\mu \leq 0$. Let $\Omega_* = \{\omega \in \Omega \mid$ to each $\epsilon > 0$ there corresponds a sequence $t_n \to \infty$ such that*

$$\int_0^{t_n} f(T_s(\omega))\, ds < \epsilon.$$

Then Ω_* is residual in Ω, i.e. contains a countable intersection of open dense sets.

Proof. For integers $n \geq 1$, $N \geq 1$ consider the set $\Omega_{n,N} = \{\omega \in \Omega \mid \int_0^t f(T_s(\omega))\, ds \geq \frac{1}{n}$ for $t \geq N\}$. The set $\Omega_{n,N}$ is closed and one has

$$\Omega - \Omega_* = \bigcup_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \Omega_{n,N}.$$

So it suffices to show that no $\Omega_{n,N}$ contains an open set.

Suppose for contradiction that some $\Omega_{n,N}$ does contain an open set V. Then $\mu(V) > 0$ since Ω is the topological support of μ. So if $\int_\Omega f\, d\mu = 0$, we obtain a contradiction with Schneiberg's result (Theorem 4.1). Hence Ω_* is residual in this case. On the other hand if $\int_\Omega f\,d\mu < 0$, then by the Birkhoff ergodic theorem:

$$\lim_{t\to\infty}\int_0^t f(T_s(\omega))\,ds = -\infty$$

for almost all $\omega \in \Omega$. Thus $\mu(V) = 0$ in this case as well, a contradiction. So Ω_* is indeed residual in Ω. □

Now we state and prove our topological result on global null controllability. As before, let $Z \subset \Omega \times O(n)$ be a compact invariant set such that Z is the topological support of an ergodic lift ν of μ. We will again make reference to the equations (4.2) and the processes (4.3).

THEOREM 4.8. *Suppose that, for each diagonal element $e_{ii}(\cdot)$ of the matrix function e, there is an ergodic lift ν_i of μ with topological support Z such that $\int_Z e_{ii}\,d\nu_i \leq 0$ $(1 \leq i \leq n)$. Let $\Omega_* = \{\omega \in \Omega \mid$ the process $(3.2)_\omega$ is globally null controllable$\}$. Then Ω_* is dense in Ω.*

Proof. Apply Lemma 4.6 to conclude that $Z_i = \{z \in Z \mid$ for each $\epsilon > 0$ there is a sequence $t_n \to \infty$ with the property that $\int_0^{t_n} e_{ii}(\widehat{T}_s(z))\,ds < \epsilon\}$ is a residual subset of Z. Hence so is $\widetilde{Z} = \cap_{i=1}^n Z_i$. Let $\widetilde{\Omega} = \pi(\widetilde{Z})$. Repeating the proof of Theorem 4.5, we conclude that the process $(3.2)_\omega$ is globally null controllable for each $\omega \in \widetilde{\Omega}$. This completes the proof since $\widetilde{\Omega}$ is dense in Ω and $\widetilde{\Omega} \subset \Omega_*$. □

REMARK 4.9. (a) It is proved in [8] that, if Ω is minimal, then Ω_* is residual in Ω. This is because Z may then be chosen to be minimal, and the image $\pi(\widetilde{Z})$ of the residual set $\widetilde{Z}$ is residual in Ω [5].

(b) In [8], certain situations are considered in which the hypothesis of Theorem 4.8 can be verified. In general, if at least one Lyapounov exponent is ≤ 0 and if the flow $(Z, \{\widehat{T}_t\})$ is a "proximal extension" [5] of $(\Omega, \{T_t\})$, then one might conjecture that the hypothesis holds.

(c) If the hypothesis of Theorem 4.8 holds, and if at least one Lyapounov exponent λ_r is strictly positive, then a well-known result of Pandolfi [11] implies that $\mu(\Omega_*) = 0$. Thus Ω_* is measure-theoretically invisible. On the other hand, Ω_* is topologically quite large; for example if $\Omega_1 \subset \Omega$ is any residual set then $\Omega_* \cap \Omega_1 \neq \emptyset$. Thus for example Ω_* must contain points $\omega \in \Omega$ with dense orbit.

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