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# UNIQUENESS RESULTS FOR STOKES EQUATIONS AND THEIR CONSEQUENCES IN LINEAR AND NONLINEAR CONTROL PROBLEMS

CAROLINE FABRE

Abstract. The goal of this article is the study of the approximate controllability for two approximations of Navier Stokes equations with distributed controls. The method of proof combines a suitable linearization of the system with a fixed point argument. We then are led to study the approximate controllability of linear Stokes systems with potentials. We study both the case where there is no constraint on the control and the case where we search a control having one null component. In both cases, the problems is reduced to prove unique continuation results. This is done by means of Carleman estimates.

## 1. Introduction

The goal of this article is the study of the approximate controllability for two approximations of Navier Stokes equations with distributed controls. To be more precise, let us state the problem: consider an open bounded connected and regular set $\Omega$ of $\mathbb{R}^n$, $(n \geq 2)$, a time $T > 0$ and an open subset $\omega$ of $\Omega$. We write $Q = \Omega \times (0,T)$, $q = \omega \times (0,T)$, and $\Sigma = \partial\Omega \times (0,T)$. Let $H$ and $V$ be the closure in $L^2(\Omega)^n$ and in $H_0^1(\Omega)^n$ of $E = \{u \in C_0^\infty(\Omega)^n,$ $\operatorname{div} u = 0$ in $\Omega\}$ respectively.

Let $M > 0$ be an arbitrary positive constant that will be fixed all along the paper, we introduce a mapping $T_M \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n, \mathbb{R}^n)$, such that $T_M(s_1, \cdots, s_n) = (s_1, \cdots, s_n)$ if for every $i \in [1, n]$, one has $|s_i| \leq M$. In others words, $T_M$ coincides with the identity in the hypercube $[-M, M]^n$.

For $v = (v_1, \cdots, v_n) \in L^2(q)^n$, and $y^0 \in H$, we denote by $y = y(x,t) = (y_1(x,t), \cdots, y_n(x,t))$ the vector-valued solution of

$$\begin{cases} y' - \Delta y + \dfrac{\partial}{\partial x_k}(T_M(y)_k y) = \nabla p + v\chi_q & \text{in } Q \\ \operatorname{div}\, y = 0 & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(0) = y^0 & \text{in } \Omega \end{cases} \tag{1.1}$$

Université Paris 12-Val de Marne, U.F.R. Sciences, Département de Mathématiques, Av. du Général de Gaulle, 94010 Créteil Cedex and Centre de Mathématiques Appliquées, École Polytechnique, 91128 Palaiseau Cedex, France.
Email: cfabre@cmapx.polytechnique.fr.

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where $'$ denotes the derivative with respect to time and $\chi_q$ is the characteristic function of $q$. System (1.1) is in fact made of $3n+1$ equations and $\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(T_M(y)_k y_j)$ is the $j^{th}$ component of the vector $\frac{\partial}{\partial x_k}(T_M(y)_k y)$ where $T_M(y)_k$ is the $k^{th}$ component of $T_M(z)$.

System (1.1) can be viewed as a variant of the classical Navier-Stokes equations in which the quadratic nonlinearity has been truncated.

As $T_M \in C(\mathbb{R}^n, \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n, \mathbb{R}^n)$, one can prove (using a fixed point method) that for every $(y^0, v) \in H \times L^2(q)^n$, system (1.1) has a unique solution $(y,p) \in \big(C([0,T];H) \cap L^2(0,T;V)\big) \times \mathcal{D}'(Q)$ in the sense that $y$ is unique and the pressure $p$ is defined up to a time dependent distribution. Of course, the solution $y = y_M$ of (1.1) depends on the parameter $M$ which has been fixed here. One can remark that we would get the Navier Stokes equation if we had $T_M(y_M) = y_M$ for some $M$.

Our purpose is to study the reachable set at time $T$ which is defined for a fixed $y^0 \in H$ by

$$R(T) = \{y(x,T),\ v \in L^2(q)^n,\ y \,\text{solution of (1.1)}\}.$$

Clearly $R(T)$ is a subset of $H$ for any $T > 0$ and $y^0 \in H$. We will prove the following result:

THEOREM 1.1. *For every $M > 0$, every $y^0 \in H$ and every $T > 0$, the reachable set $R(T)$ is dense in $H$.*

REMARK 1.2.
(i) In [4], it has been proved for example that in two space dimensions and for the Navier Stokes equation, the vector space spanned by the reachable set is dense in $H$.

(ii) J.M. Coron proved in [1] the approximate controllability in a particular sense for the Navier Stokes equations in two space dimensions with different boundary conditions (said to be Navier conditions).

(iii) In [5], several results are proved: first, in one space dimension, it is proved that Burgers equation is not approximatively controllable. Then, the authors prove a local result which is roughly the following: if there is a solution of the Burgers equation which starts from $y^0$ and goes to $y^1$ at time $t = T$ then for initial data close enough to $y^0$, there exists a distributed control allowing to go to $y^1$ at time $t = T$. In this book, this local result is extended for the Navier Stokes equations in two space dimensions.

We will also replace the nonlinear term by $(T_M(y).\nabla)y$ instead of $\frac{\partial}{\partial x_k}(T_M(y)_k y)$. In that case, the system becomes (where . denotes the scalar product in $\mathbb{R}^n$)

$$\begin{cases} y' - \Delta y + (T_M(y).\nabla)y = \nabla p + v\chi_q & \text{in} \quad Q \\ \text{div } y = 0 \quad \text{in} \quad Q \\ y = 0 \quad \text{on} \quad \Sigma \\ y(0) = y^0 \in H \end{cases} \tag{1.2}$$

and we will prove:

THEOREM 1.3. *We suppose that $\omega$ is a neighbourhood of the boundary $\Gamma$. For every $M > 0$, every $y^0 \in H$ and every $T > 0$, the reachable set*

$$R(T) = \{y(x,T),\ v \in L^2(q)^n,\ y \ \textit{solution of}\ (1.2)\}$$

*is dense in $H$.*

The density in $H$ of the reachable set is, in general (*i.e.* for any open non empty subset $\omega$ of $\Omega$), an open problem which is related to the following regularity result concerning the pressure in linear Stokes systems: which minimum supplementary regularity conditions on $f \in L^2(0,T,V')$ are needed in order to ensure the existence of $(u,\pi) \in L^2(0,T,V) \times L^2_{loc}(Q)$ with $u(0) = 0$ and $u' - \Delta u = \nabla\pi + f$? The difficulty comes from the regularity of $\pi$ with respect to time and the only known result is that $f \in L^2(Q)^n$ implies $\pi \in L^2(0,T,H^1(\Omega))$.

In order to prove Theorems 1.1 and 1.3, we use a fixed point method together with a precise study of the approximate controllability for linear Stokes equations. As usual in linear cases, the approximate controllability property can be reduced (using Hahn-Banach theorem) to a unique continuation property concerning the solutions of the adjoint homogeneous problem.

We will set this uniqueness property in a more general setting which includes systems of linearized Navier-Stokes equations.

More precisely, we consider for $1 \le j,k,l \le n$, functions $a^j_{kl}, b^j_{kl} \in L^\infty(\Omega \times (-T,T))$ and functions $u = (u_1,\cdots,u_n)$ and $\pi$ solutions of the system

$$\begin{cases} \forall j,\ 1 \le j \le n, \\ u'_j - \Delta u_j + a^j_{kl}\dfrac{\partial u_l}{\partial x_k} + \dfrac{\partial}{\partial x_k}(b^j_{kl}u_l) = \dfrac{\partial \pi}{\partial x_j} \quad \text{in} \quad \Omega \times (-T,T) \\ \operatorname{div} u = 0 \quad \text{in} \quad \Omega \times (-T,T) \\ (u,\pi) \in L^2_{loc}(-T,T;H^1_{loc}(\Omega))^n \times L^2_{loc}(\Omega \times (-T,T)) \end{cases} \tag{1.3}$$

where the $j^{th}$ equation, according to the convention of summation of repeated indexes, has to be read for $1 \le j \le n$ :

$$u'_j - \Delta u_j + \sum_{k,l} a^j_{kl}\frac{\partial u_l}{\partial x_k} + \sum_{k,l}\frac{\partial}{\partial x_k}(b^j_{kl}u_l) = \frac{\partial \pi}{\partial x_j}.$$

We prove

THEOREM 1.4. *Let $a^j_{kl}$ and $b^j_{kl}$ be elements of $L^\infty_{loc}(\Omega \times (-T,T))$ for $1 \le j,k,l \le n$. If $(u,\pi)$ is a solution of (1.3) and if $u$ vanishes in an open non-empty subset $O$ of $\Omega \times (-T,T)$ then it vanishes in the whole horizontal component of $O$ in $\Omega \times (-T,T)$ which is the set*

$$C(O) = \{(x,t) \in \Omega \times (-T,T),\quad \exists x_0 \in \Omega,\quad (x_0,t) \in O\}.$$

Theorem 1.4 has been first proved by Saut and Temam (see [12]) when coefficients $a^j_{kl}$ and $b_{kl}$ are $C^1$ and $C^2$ respectively. Then it has been extended when $a^j_{kl} = a_k\delta_{lk} \in W^{1,\infty}$ and $b^j_{kl} = 0$ (which means that $u$ is solution of $u' - \Delta u - (a.\nabla)u = \nabla\pi$ with $a = (a_1, \cdots, a_n)$) by Fernandez-Cara and Real in [4]. These authors applied a Carleman estimate on the heat equation combined with a Carleman estimate for the Laplace operator in order to control the pressure which then satisfies an equation like $\Delta\pi = \text{div}\,((a.\nabla)u) \in L^2$.

In [3], we also studied the case where $u$ is solution of $u' - \Delta u - (a.\nabla)u = \nabla\pi$ with coefficients $a \in L^\infty(Q)^n$. Note that when $a \in L^\infty(Q)^n$, one can no longer use the usual Carleman inequality for the Laplace operator because of the lack of regularity of $a$. To overcome this difficulty, we proved a Carleman estimate for solutions $(\pi, f) \in H^1_{loc} \times L^2_{loc}$ of $\Delta\pi + L_1 f \in L^2$ where $L_1$ is a first order operator (for more details, see [3]). The argument of the proof of the unique continuation for the Stokes system with $a \in L^\infty(Q)^n$ then combined this new inequality, the usual one on the heat operator (stated by Saut and Scheurer in [11]) and a change of scale. In our case, the pressure satisfies an equation like $\Delta\pi + L_1 f + L_2 k \in L^2$ with functions $f$ and $k$ in $L^2$ (and no more), $L_1$ and $L_2$ being first and second order operators. Thus, the pressure will no more be in $H^1_{loc}$ and there is again a lack of estimate on $\pi$. Futhermore, the usual Carleman inequality for the heat operator cannot be applied on $u$ since the functions $b^j_{kl}$ do not possess sufficient regularity (this would require $b^j_{kl} \in W^{1,\infty}$ in space). One can only say that each component of $u$ satisfies a heat equation but with a right-hand side in $H^{-1}(Q)$ (in space and time) and no more even locally. We therefore need a new inequality in order to treat $u$.

Using the unique continuation property of Theorem 1.4, we will deduce the approximate controllability for linear Stokes systems. Note that for $y^0 \in H$ and $v \in L^2(q)^n$, using a variational method, one can show that there exists a unique vector-valued fonction $y \in L^2(0,T;V) \cap C([0,T];H)$ and a pressure $p \in \mathcal{D}'(Q)$ (defined up to a time dependent distribution) such that $(y,p)$ is solution of

$$\begin{cases} \forall j,\ 1 \le j \le n, \\ y'_j - \Delta y_j + a^j_{kl}\dfrac{\partial y_l}{\partial x_k} + \dfrac{\partial}{\partial x_k}(b^j_{kl}y_l) = \dfrac{\partial p}{\partial x_j} + v\chi_\omega \quad \text{in} \quad Q \\ \text{div}\ y = 0 \quad \text{in} \quad Q \\ y = 0 \quad \text{on} \quad \Sigma \\ y(0) = y^0 \quad \text{in} \quad \Omega. \end{cases} \tag{1.4}$$

We prove the following approximate controllability results for system (1.4):

PROPOSITION 1.5. *Let $\omega$ be a neighbourhood of the boundary $\Gamma$. Let $(a^j_{kl}, b^j_{kl}) \in L^\infty(Q)^2$ for $1 \le j,k,l \le n$. For every $y^0 \in H$, the reachable set at any time $T > 0$ defined by*

$$R(T) = \{y(x,T) \quad \textit{where } y \textit{ is solution of (1.4)},\ v \in L^2(q)^n\}$$

*is dense in $H$.*

PROPOSITION 1.6. *Let $\omega$ an open and non-empty subset of $\Omega$. Let $(a^j_{kl}, b^j_{kl}) \in L^\infty(Q)^2$ with*

$$\begin{cases} \text{if } n = 2, \quad \exists p_0 \in ]2, +\infty[, \quad \forall j,k,l \in \{1,\cdots,n\} \quad a^j_{kl} \in L^2(0,T,W^{1,p_0}(\Omega)) \\ \text{if } n > 2, \quad \forall j,k,l \in \{1,\cdots,n\} \quad a^j_{kl} \in L^2(0,T,W^{1,n}(\Omega)). \end{cases} \tag{1.5}$$

*For every $y^0 \in H$, the reachable set at any time $T > 0$ defined by*

$$R(T) = \{y(x,T), \quad \text{where } y \text{ is solution of (1.4) } v \in L^2(q)^n\}$$

*is dense in $H$.*

REMARK 1.7.

(i) The hypotheses on $\omega$ in Proposition 1.5 and (1.5) are made in order to ensure the existence of a pressure in $L^2_{loc}(Q)$ for the adjoint system of (1.4).

(ii) Conditions (1.5) can be replaced by

$$\begin{cases} \text{if } n = 2, \quad \exists p_0 \in ]2, +\infty[, \quad \forall j,k,l \in \{1,\cdots,n\} \quad a^j_{kl} \in H^1(0,T,L^{p_0}(\Omega)) \\ \text{if } n > 2, \quad \forall j,k,l \in \{1,\cdots,n\} \quad a^j_{kl} \in H^1(0,T,L^n(\Omega)). \end{cases}$$

Once the approximate controllability of this Stokes system is understood, one can ask if we can take controls with one component equal to zero. The problem reduces again to the underlying unique continuation property which can be formulated as follows: suppose that a solution $u$ of (1.3) satisfies $u_1 = \cdots = u_{n-1} = 0$ in $O$ then do we have $u = 0$ in $C(O)$?

We give a partial result which requires more conditions on the coefficients $a^j_{kl}$ and $b^j_{kl}$ and more regularity on $u$ and $\pi$. We will then give a counter-exemple when one of these conditions is not fulfilled. We consider functions $a^j_{kl}$ and $B^j_l$ in $L^\infty(\Omega \times (-T,T))$ with

$$\begin{cases} \forall k \in \{1,\cdots,n\}, \quad \dfrac{\partial a^n_{k,n}}{\partial x_n} = \dfrac{\partial B^n_n}{\partial x_n} = 0 \\ \forall (j,k) \in \{1,\cdots,n-1\} \times \{1,\cdots,n\}, \quad a^j_{kn} = B^j_n = 0. \end{cases} \tag{1.6}$$

We denote by $(u,\pi)$ a solution of

$$\begin{cases} u'_j - \Delta u_j + a^j_{kl} \dfrac{\partial u_l}{\partial x_k} + B^j_l u_l = \dfrac{\partial \pi}{\partial x_j} \quad \text{in} \quad \Omega \times (-T,T) \\ \text{div } u = 0 \quad \text{in} \quad \Omega \times (-T,T) \\ (u,\pi) \in L^2_{loc}(-T,T;H^2_{loc}(\Omega))^n \times L^2_{loc}(-T,T;H^1_{loc}(\Omega)). \end{cases} \tag{1.7}$$

We prove

THEOREM 1.8. *If the functions $a^j_{kl}$ and $B^j_l$ are in $L^\infty(\Omega \times (-T,T))$ and satisfy (1.6) and if $(u,\pi)$ is solution of (1.7) with $u_1 = \cdots = u_{n-1} = 0$ in an open set $O$ of $\Omega \times (-T,T)$ then $u_1 = \cdots = u_{n-1} = \frac{\partial u_n}{\partial x_n} = 0$ in $C(O)$.*

We will then deduce

Corollary 1.9. *Suppose that $\Omega$ is bounded and that $u$ belongs to $L^2(-T,T;H_0^1(\Omega))^n$ and is solution of (1.7) with $u_1 = \cdots = u_{n-1} = 0$ in an open set $O$ of $Q$. Then $u$ vanishes in $C(O)$.*

Theorem 1.8 is false when condition (1.6) is not fulfilled even in the stationary case: indeed, let $\Omega$ be the square $\Omega = ]-1,1[^2$ and let $O = [x_1 < 0] \cap \Omega$.

We consider

$$\begin{cases} u_1(x_1,x_2) = x_1^3 1_{(x_1>0)} \\ u_2(x_1,x_2) = -3x_1^2 x_2 1_{(x_1>0)} + 5 + x_1 \\ \pi(x_1,x_2) = 3x_1^2 1_{(x_1>0)} + x_2 \\ b(x_1,x_2) = \dfrac{1+6x_2}{5+x_1-3x_1^2x_2} 1_{(x_1>0)} + \dfrac{1}{5+x_1} 1_{(x_1<0)}. \end{cases} \tag{1.8}$$

We have $(u_1,u_2,\pi,b) \in H^2_{loc}(\Omega)^2 \times H^1_{loc}(\Omega) \times L^\infty(\Omega)$, div $u = 0$ in $\Omega$ and $u_1 = 0$ in $O$.

One can easily compute that

$$\Delta u_1 = \frac{\partial \pi}{\partial x_1}$$

and

$$\Delta u_2 + b u_2 = \frac{\partial \pi}{\partial x_2}.$$

Condition (1.6) fails since $B_2^2 = b$ and $\frac{\partial b}{\partial x_2} \neq 0$.

Remark 1.10.

(i) If $\Omega$ is not bounded, even when $a = B = 0$, Theorem 1.8 does not necessary imply Corollary 1.9 as one can see for $n = 2$, $\Omega = [(x_1,x_2), x_1 > 0]$, and $u_1 = 0$, $u_2 = x_1^2/2$ and $p = x_2$.

(ii) The controllability problem with controls having two zero components has been studied by J.-L. Lions and E. Zuazua in [15] in three space dimensions for the Stokes system and without potential. They proved a generic result and gave a counter-exemple to the underlying uniqueness property when $\Omega$ is a cylinder.

We now describe the plan of this article: in a second section we prove Theorem 1.4 and 1.8. The main tool (that we briefly recall) is the $h$-pseudo-differential calculus (as in [8]). The proofs of these theorems are based on the same idea as the one developped in [4]: we first state the Carleman inequalities by separating the low and high frequencies. In order to be clear, at each step, we will recall the previously known inequalities and the new ones. In a third section, using these unique continuation properties, we will deduce the approximate controllability for linear Stokes systems and, in particular, the case of the linearized Navier-Stokes equations will be considered. We then go back to our nonlinear problems and we prove Theorems 1.1 and 1.3 in the last section. The method is now standard and uses a fixed point argument (see for example [2] or [14].)

## 2. Unique continuation for Stokes equations

### 2.1. Preliminaries on pseudo-differential calculus

The results that we recall in this section are classical and we refer to the books of D. Robert [10] and X. St Raymond [9] for details.

We note $D_j = \frac{h}{i}\partial_j$, $\lambda(\xi) = (1+|\xi|^2)^{\frac{1}{2}}$, for $\xi \in \mathbb{R}^n$. The set $S^m$ (where $m \in \mathbb{Z}$) denotes the space of symbols defined on $\mathbb{R}^{2n}$ and which satisfy:

$$\forall \alpha, \beta \in \mathbb{N}^n, \quad \exists C_{\alpha,\beta} \ \forall x, \xi, h \quad |\partial_x^\alpha \partial_\xi^\beta a(x,\xi,h)| \leq C_{\alpha,\beta}\lambda(\xi)^{m-|\beta|}$$

and

$$\forall j \in \mathbb{N}, \quad \exists a_j(x,\xi) \in S^{m-j}, \quad a(x,\xi,h) \in \sum_{j=0}^{N} h^j a_j(x,\xi) + h^{N+1}S^{m-N-1}.$$

The principal symbol of $a$ is then $a_0$. If $a \in S^m$ and if $u$ is $C_0^\infty(\mathbb{R}^n)$, we define the pseudo-differential operator of order $m$

$$a(x,D,h)u(x) = \frac{1}{(2\pi h)^n}\int e^{ix.\frac{\xi}{h}} a(x,\xi,h)\hat{u}(\frac{\xi}{h})d\xi$$

(where $\hat{u}(\xi) = \int e^{-ix.\xi}u(x)dx$ is the Fourier transform of $u$). If $a \in S^m$, $a(x,D,h)$ maps $H^s(\mathbb{R}^n)$ in $H^{s-m}(\mathbb{R}^n)$. We write $E^m$ the space of pseudo-differential operators of order $m$ and $E^{-\infty} = \cap_{m\in Z}E^m$.

We write $|v|_0^2 = \int_{R^n}|v(x)|^2dx$, $(u,v) = \int_{R^n} u\bar{v}$ and $|v|_1^2 = |\lambda(D)v|_0^2$. Thus

$$|v|_1^2 = \frac{1}{(2\pi)^n}\int_{R^n}\lambda^2(h\xi)|\hat{u}(\xi)|^2d\xi = \int |v|^2dx + h^2\int|\nabla v|^2dx.$$

We recall that if $a \in E^m$ and $b \in E^l$ then $a(x,D)\circ b(x,D) \in E^{m+l}$ with a principal part $a_0b_0$. On the other hand, $\frac{i}{h}[a(x,D),b(x,D)] = \frac{i}{h}(a(x,D)\circ b(x,D) - b(x,D)\circ a(x,D)) \in E^{m+l-1}$ and its principal part is $\{a_0,b_0\} = \partial_\xi a_0\partial_x b_0 - \partial_x a_0 \partial_\xi b_0$.

We recall the Garding inequality:

Proposition 2.1. *Let $U$ be an open set of $\mathbb{R}^n$. If $a \in E^2$ satisfies:*

$$\exists c_1 > 0, \quad \forall (x,\xi) \in U\times\mathbb{R}^n, \quad Re\, a_0(x,\xi) \geq c_1\lambda(\xi)^2$$

*then for every compact set $K$ of $U$, there exists $h_K > 0$ such that*

$$\forall u \in H_0^1(K), \quad \forall h \in ]0,h_K], \qquad Re(a(x,D,h)u,u) \geq \frac{c_1}{4}|u|_1^2.$$

The following proposition concerns the inversion on the high frequencies of operators of order 2 which are elliptic on high frequencies.

PROPOSITION 2.2. *If $p \in E^2$ satisfies*

$$\exists c > 0 : \quad |p_0(x,\xi)| \geq c\lambda(\xi)^2 \quad \forall \xi, \ |\xi| \geq R, \quad \forall x \in \mathbb{R}^n$$

*then there exists $e \in E^{-2}$ and $\alpha \in E^{-\infty}$ such that $e \circ p = 1 + \alpha + hR$, where $R \in E^{-1}$, and $\alpha(x,\xi) = 0$ if $|\xi| \geq R$.*

As we are dealing with evolution equations, the operators under consideration will depend on a parameter (the time) and we will need uniform estimates. Let $I = [-T_0, T_0]$ be a subinterval of $\mathbb{R}$. We will say that a set of operators with symbols of order $m$, $\{a(t,x,\xi,h)\}_{t\in I}$, is in $L^\infty(I, E^m)$ if the constants $C_{\alpha,\beta}$ appearing in the definition of $S^m$ do not depend on $t \in I$. We recall that (see [10]) if $(a(t))_{t\in I} \in L^\infty(I, E^m)$ and if $(b(t))_{t\in I} \in L^\infty(I, E^l)$ then $(a(t) \circ b(t))_t \in L^\infty(I, E^{m+l})$ and we have

$$\forall s \in \mathbb{R}, \quad \exists C_s > 0 : \quad \forall t \in I, \ \forall u \in H^s, \ |a(t)u|_{s-m} \leq C_s |u|_s.$$

In the same way, we have the following Garding inequality: if $U$ is an open set of $\mathbb{R}^n$ and if $(a(t))_{t\in I} \in L^\infty(I, E^2)$ satisfies

$$\exists c_1 > 0 : \quad \forall t \in I, \quad \forall (x,\xi) \in U \times \mathbb{R}^n, \quad Re\, a_0(t,x,\xi) \geq c_1 \lambda(\xi)^2$$

then for every compact set $K$ of $U$, there exists $h_K > 0$ such that

$$\forall u \in H_0^1(K), \quad \forall (t,h) \in I \times ]0, h_K], \qquad Re(a(t,x,D,h)u, u) \geq \frac{c_1}{4} |u|_1^2.$$

Finally, we have the analogous of Proposition 2.2: if $(p(t))_t \in L^\infty(I, E^2)$ satisfies

$$\exists c, R > 0 : \quad |p_0(x,\xi)| \geq c\lambda(\xi)^2 \quad \forall \xi, \ |\xi| \geq R, \quad \forall (t,x) \in I \times \mathbb{R}^n$$

then there exist $(e(t))_t \in L^\infty(I, E^{-2})$ and $(\alpha(t))_t \in L^\infty(I, E^{-\infty})$ such that $e(t) \circ p(t) = 1 + \alpha(t) + hR(t)$, with $R(t) \in L^\infty(I, E^{-1})$ and $\alpha(t,x,\xi) = 0$ if $|\xi| \geq R$.

Let now $\varphi = \varphi(t,x) \in C_0^\infty(\mathbb{R}^{n+1})$. We write $\varphi_t(x) = \varphi(t,x)$ and $p(t)$ the operator of $E^2$ :

$$p(t)(x,D,h) = -h^2 e^{\frac{\varphi}{h}} \circ \Delta \circ e^{-\frac{\varphi}{h}}.$$

$p(t)$ has as principal symbol

$$p_0(t)(x,\xi) = \sum_{j=1}^{n} (\xi_j + i \frac{\partial \varphi}{\partial x_j})^2.$$

The adjoint operator $p^*(t)$ of $p(t)$ is also of order 2 and its principal symbol is the conjugate of $p_0(t)(x,\xi)$, $\bar{p}_0(t)(x,\xi)$. We then decompose the operator $p(t) = a(t) + ib(t)$ with ($a^*$ denotes the adjoint operator of $a$): $a(t) =$

$\frac{p(t)+p^*(t)}{2} \in L^\infty(-T,T;E^2)$ with principal part $a_0(t) = Re\, p_0(t)$ and $b(t) = \frac{p(t)-p^*(t)}{2i} \in L^\infty(-T,T;E^1)$ with principal part $b_0(t) = Im\, p_0(t)$.

Let $U_0$ be a bounded and open set of $\mathbb{R}^n$ and suppose that there exists $C_0 > 0$ such that $\varphi_0$ satisfies

$$(x,\xi) \in \bar{U}_0 \times \mathbb{R}^n, \;\; a_0(0)(x,\xi) = 0 \;\; \Rightarrow \;\; \{a_0(0), b_0(0)\}(x,\xi) \geq C_0. \tag{2.1}$$

Then there exist $\rho_0 > 0$ and $c_3, d' > 0$ such that for $|t| \leq \rho_0$, $\varphi_t$ satisfies

$$\begin{cases} (x,\xi) \in \bar{U}_0 \times \mathbb{R}^n, \quad a_0(t)(x,\xi) = 0 \quad \Rightarrow \quad \{a_0(t), b_0(t)\}(x,\xi) \geq \dfrac{C_0}{2}, \\ d'\lambda^{-2}(\xi) a_0(t)^2(x,\xi) + d' b_0(t)^2(x,\xi) + \{a_0(t), b_0(t)\}(x,\xi) \geq c_3 \lambda^2(\xi). \end{cases} \tag{2.2}$$

We now define general operators of order 1 and 2. For this let $s \in \mathbb{N}$ and for $f = (f_1, \cdots, f_s)$ we write

$$L_1(f) = \sum_{(j,k)\in\{1,\cdots,n\}\times\{1,\cdots,s\}} c_{jk} \frac{\partial f_k}{\partial x_j},$$

with coefficients $c_{jk} \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We have $L_1 \in E^1$.

On the other hand, for $k = (k_{lm})_{1\leq l,m\leq s}$, define

$$L_2(k) = \sum_{(l,m)\in\{1,\cdots,s\}^2,(l',m')\in\{1,\cdots,n\}^2} \frac{\partial}{\partial x_{l'}}\left(d_{lml'm'} \frac{\partial}{\partial x_{m'}} k_{lm}\right),$$

where $d_{lml'm'} \in C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$. We have $L_2 \in E^2$.

### 2.2. Carleman inequalities

In the sequel $c$ and $d$ will denote positive constants that may change from line to line and are independent of the parameter $h$.

As we have already said in the introduction, our problem to prove Theorems 1.4 and 1.8 comes from a lack of regularity on $u$ and $\pi$. For clarity, let us recall what is known: the usual Carleman inequality on the Laplace operator has been stated by Hormander in [6], it concerns the stationnary case and it is the following proposition

Proposition 2.3. *Let $U$ be an open and bounded set of $\mathbb{R}^n$ and $K$ be a compact set included in $U$. Let $\varphi = \varphi(x)$ be in $C_0^\infty(\mathbb{R}^n)$. If $\nabla\varphi$ does not vanish on $U$ and if*

$$\exists c_1 > 0, p_0(x,\xi) = 0 \text{ and } (x,\xi) \in U \times \mathbb{R}^n \Rightarrow \{Re\, p_0, Im\, p_0\}(x,\xi) \geq c_1, \tag{2.3}$$

*where $p_0$ is the principal symbol of $p = -h^2 e^{\frac{\varphi}{h}} \circ \Delta \circ e^{\frac{\varphi}{h}}$, then there exists $c > 0$ and $h_1 > 0$ such that for all $0 < h < h_1$ and all functions $y \in H_0^2(K)$ we have*

$$\int_K |y|^2 e^{2\frac{\varphi}{h}} dx + h^2 \int_K |\nabla y|^2 e^{2\frac{\varphi}{h}} dx \leq ch^3 \int_K |\Delta y|^2 e^{2\frac{\varphi}{h}} dx. \tag{2.4}$$

In [3] we could not use this last inequality since the pressure $\pi$ was not $H^2$ in space (even locally). We then proved the following (stated here in the evolution case)

PROPOSITION 2.4. *Let $U_0$ be an open and bounded set of $\mathbb{R}^n$. We suppose that $\nabla\varphi$ does not vanish on $U_0$ and that $\varphi$ satisfies (2.1). Then: there exists $\rho_0 > 0$ such that for every compact set $K$ in $U_0$, there exists $c > 0$ and $h_1 > 0$ such that for almost every $t \in (-\rho_0, \rho_0)$, and every $h \in ]0, h_1[$, we have*

$$\int_K |y|^2 e^{2\frac{\varphi}{h}} dx + h^2 \int_K |\nabla y|^2 e^{2\frac{\varphi}{h}} dx \leq hc \int_K |f|^2 e^{2\frac{\varphi}{h}} dx + ch^3 \int_K |\Delta y - L_1 f|^2 e^{2\frac{\varphi}{h}} dx \tag{2.5}$$

*for every $(y, f) \in L^2(-\rho_0, \rho_0; H_0^1(U_0)) \times L^2((-\rho_0, \rho_0) \times U_0)^s$ with $\Delta y - L_1 f \in L^2((-\rho_0, \rho_0) \times U_0)$, all these functions having compact support in $K$.*

If we compare these two inequalities in the stationary case, we see that we loose in power of $h$ but we need weaker norm of $f$. Of course, for $f = 0$, we find again Hörmander's inequality (2.4).

In our problem, the pressure will be solution of an equation like $\Delta\pi - L_1 f - L_2 k \in L^2(U)$, with $L_2$ a second order operator, and thus $\pi$ will not be any more in $H^1$ in space (even locally) and the above inequalities do not make sense. We are going to state a Carleman inequality where the only norm of $\pi$ appearing is the $L^2$ norm. We now prove

LEMMA 2.5. *Let $U_0$ be a bounded and open set of $\mathbb{R}^n$ and $U = U_0 \times (-\rho_0, \rho_0)$. We suppose that $\nabla\varphi$ does not vanish in $U_0$ and that $\varphi$ satisfies (2.1). Then, there exists $\rho_0 > 0$ such that for every compact set $K$ in $U_0$, there exist $h_1 > 0$ and $c > 0$ such that for every $h \in ]0, h_1[$, for almost every $t \in (-\rho_0, \rho_0)$, we have*

$$\int_K |y|^2 e^{2\frac{\varphi}{h}} dx \leq ch \int_K |f|^2 e^{2\frac{\varphi}{h}} dx + \frac{c}{h} \int_K |k|^2 e^{2\frac{\varphi}{h}} dx + ch^3 \int_K |\Delta y - L_1 f - L_2 k|^2 e^{2\frac{\varphi}{h}} dx \tag{2.6}$$

*for every $(y, f, k) \in L^2(U) \times L^2(U)^s \times L^2(U)^{s^2}$ with $\Delta y - L_1 f - L_2 k \in L^2(U)$, all these functions having compact supports in $K$.*

REMARK 2.6. This time, for $f = k = 0$, we do not find Hörmander's inequality since we loose the gradient of $y$.

*Proof of Lemma 2.5.*

Let $\rho_0$ be given by (2.1) and its consequences. We introduce as usual the $L^2$-functions $F = \Delta y - L_1 f - L_2 k$, $z = y\exp(\frac{\varphi}{h})$, $g_j = f_j \exp(\frac{\varphi}{h})$, $G = F\exp(\frac{\varphi}{h})$ and $r = k\exp(\frac{\varphi}{h})$. Equation $F = \Delta y - L_1 f - L_2 k$ is then equivalent to

$$p(t)z = -h^2 e^{\frac{\varphi}{h}} L_1(e^{-\frac{\varphi}{h}} g) - h^2 e^{\frac{\varphi}{h}} L_2(e^{-\frac{\varphi}{h}} r) - h^2 G,$$

thus

$$p(t)z = -h^2 L_1(g) - h^2[e^{\frac{\varphi}{h}}, L_1](e^{-\frac{\varphi}{h}} g) - h^2 L_2 r - h^2[e^{\frac{\varphi}{h}}, L_2](e^{-\frac{\varphi}{h}} r) - h^2 G.$$

By a simple computation, one has

$$R_0 u = -h[e^{\frac{\varphi}{h}}, L_1]e^{-\frac{\varphi}{h}} u = -h[e^{\frac{\varphi}{h}} L_1(e^{-\frac{\varphi}{h}} u) - L_1(u)] = c_{jk}\frac{\partial \varphi}{\partial x_k} u_j$$

and thus $R_0 \in L^\infty(-\rho_0, \rho_0; E^0)$. In the same way,

$$\begin{aligned} R_1 r &= -h^2[e^{\frac{\varphi}{h}}, L_2]e^{-\frac{\varphi}{h}} r \\ &= -h^2 e^{\frac{\varphi}{h}} \frac{\partial}{\partial x_{l'}}(d_{lml'm'}\frac{\partial}{\partial x_{m'}}(e^{-\frac{\varphi}{h}} r_{lm})) + h^2 \frac{\partial}{\partial x_{l'}}(d_{lml'm'}\frac{\partial}{\partial x_{m'}} r_{lm}) \\ &= -h^2 e^{\frac{\varphi}{h}} \frac{\partial}{\partial x_{l'}}(d_{lml'm'}\frac{\partial r_{lm}}{\partial x_{m'}} e^{-\frac{\varphi}{h}}) + h e^{\frac{\varphi}{h}} \frac{\partial}{\partial x_{l'}}(d_{lml'm'} r_{lm}\frac{\partial \varphi}{\partial x_{m'}} e^{-\frac{\varphi}{h}}) \\ &\qquad + h^2 \frac{\partial}{\partial x_{l'}}(d_{lml'm'}\frac{\partial}{\partial x_{m'}} r_{lm}) \\ &= h d_{lml'm'}\frac{\partial r_{lm}}{\partial x_{m'}}\frac{\partial \varphi}{\partial x_{l'}} + h\frac{\partial}{\partial x_{l'}}(d_{lml'm'} r_{lm}\frac{\partial \varphi}{\partial x_{m'}}) \\ &\qquad - d_{lml'm'} r_{lm}\frac{\partial \varphi}{\partial x_{m'}}\frac{\partial \varphi}{\partial x_{l'}} \end{aligned}$$

and thus $R_1 \in L^\infty(-\rho_0, \rho_0; E^1)$. With $iD_j = h\partial_j$, one obtains

$$p(z) = h\mathcal{L}_1(D)g + \mathcal{L}_2(D)r - h^2 G + hR_0(g) + R_1(r)$$

where $\mathcal{L}_1(D) \in L^\infty(-T, T; E^1)$, $\mathcal{L}_2(D) \in L^\infty(-T, T; E^2)$.

We first study the high frequencies. Let

$$V = \{\xi \in \mathbb{R}^n,\ \exists x \in \mathbb{R}^n, \text{ such that } p_0(t)(x, \xi) = 0\}.$$

We have $V \subset \bar{B}_{R^n}(0, c_0)$ where $c_0 = |\nabla\varphi|_{L^\infty(R^{n+1})}$. We consider $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \chi \le 1$ and $\chi = 1$ on a neighbourhood of $\bar{B}_{R^n}(0, 2c_0^2 + 2)$.

The function $\delta = 1 - \chi$ is in $S^0$ and $\delta(D) \circ p = p \circ \delta(D) + [\delta(D), p]$ with $[\delta(D), p] \in hL^\infty(-T, T; E^{-\infty})$.

There then exists $R_2 = R_2(t) \in L^\infty(-T, T; E^0)$ such that

$$\begin{aligned} p \circ \delta(D)z = h\delta(D) \circ \mathcal{L}_1(D)(g) + \delta(D) \circ \mathcal{L}_2(D)r - h^2\delta(D)G \\ + h\delta(D) \circ R_0(g) + \delta(D) \circ R_1(r) + hR_2 z. \end{aligned} \tag{2.7}$$

We have

$$\forall x \in \mathbb{R}^n,\ \ \forall \xi,\ |\xi|^2 \ge 2c_0^2 + 1, \quad Re(2p_0(t)(x, \xi)) \ge \lambda^2(\xi). \tag{2.8}$$

Using Proposition 2.2, there exists $e(t) \in L^\infty(-T, T; E^{-2})$, such that $e(t) \circ p(t) = 1 + \alpha(t) + hR_{-1}(t)$ with $R_{-1}(t) \in L^\infty(-T, T; E^{-1})$. Furthermore, $\alpha = 0$ on the support of $\delta$ and thus $\alpha \circ \delta(D) \in hL^\infty(-T, T; E^{-\infty})$.

There then exists $R'(t) \in L^\infty(-T,T;E^{-1})$ such that

$$\begin{aligned}\delta(D)z &= h(e\circ\delta(D)\circ\mathcal{L}_1(D)(g) + e\circ\delta(D)\circ\mathcal{L}_2(D)r - h^2 e\circ\delta(D)G \\ &+ h(e\circ\delta(D)\circ R_0)g + e\circ\delta(D)\circ R_1(r) + he\circ R_2 z - hR'(t)z.\end{aligned} \tag{2.9}$$

Since operators $e\circ\delta(D)\circ\mathcal{L}_1(D)$, $e\circ\delta(D)\circ\mathcal{L}_2(D)$, $e\circ\delta(D)$, $e\circ\delta(D)\circ R_0$, $e\circ\delta(D)\circ R_1$, and $e\circ R_2$ are in $L^\infty(-T,T;E^{-1})$, $L^\infty(-T,T;E^0)$, $L^\infty(-T,T;E^{-2})$, $L^\infty(-T,T;E^{-2})$, $L^\infty(-T,T;E^{-1})$ and $L^\infty(-T,T;E^{-2})$ respectively, there exists $c>0$ such that

$$|\delta(D)z|_0^2 \le c\big(h^2|g|_0^2 + |r|_0^2 + h^4|G|_0^2 + h^2|z|_0^2\big). \tag{2.10}$$

Let us now study $\chi(D)z$. As $\chi$ is compactly supported in $\xi$, $\chi(D)\in E^{-\infty}$ and $\chi(D)\circ p(t)\in p(t)\circ\chi(D) + hL^\infty(-T,T;E^{-\infty})$.

There then exists $R_{-\infty}(t)\in L^\infty(-T,T;E^{-\infty})$, such that

$$\begin{aligned}p\circ\chi(D)z &= h\chi(D)\circ\mathcal{L}_1(D)g + \chi(D)\circ\mathcal{L}_2(D)r - h^2\chi(D)G \\ &+ h\chi(D)\circ R_0(g) + \chi(D)\circ R_1(r) + hR_{-\infty}(t)(z).\end{aligned} \tag{2.11}$$

Consider $\beta\in C_0^\infty(U)$ such that $\beta = 1$ on $K$. One has $\chi(D)z = \beta(x)\chi(D)z + [\chi(D),\beta]z$ with $[\chi(D),\beta]\in hL^\infty(-T,T;E^{-\infty})$ and $v = \beta\chi(D)z\in C_0^\infty(U)$. Using the decomposition of $p(t)$ as $a(t)+ib(t)$, one has

$$\begin{aligned}|p(t)(v)|_0^2 = |a(t)(x,D)(v)|_0^2 &+ |b(t)(x,D)(v)|_0^2 \\ &+ i\big((a\circ b - b\circ a)v, v\big).\end{aligned} \tag{2.12}$$

Futhermore, $a\circ b - b\circ a = [a,b]\in L^\infty(-T,T;E^2)$ with principal symbol $\frac{h}{i}\{a_0,b_0\}$.

Using (2.1), we have for $|t|\le\rho_0$,

$$d'\lambda^{-2}(\xi)a_0(t)^2(x,\xi) + d'b_0(t)^2(x,\xi) + \{a_0(t),b_0(t)\}(x,\xi) \ge c_3\lambda^2(\xi). \tag{2.13}$$

We apply Garding's inequality on $U$, when $K$ is the support of $\beta$ and when the operator is $d'a(t)\circ\lambda^{-2}\circ a(t) + d'b(t)\circ b(t) + \frac{i}{h}[a(t),b(t)](x,D)$. We deduce that

$$i([a(t),b(t)]v,v) \ge \frac{c_3}{4}h|v|_1^2 - d'h|\lambda^{-1}\circ a(t)v|_0^2 - d'h|b(t)v|_0^2. \tag{2.14}$$

Taking into account (2.12), (2.14) and $|\lambda^{-1}a(t)(v)|_0^2 \le |a(t)(v)|_0^2$, we obtain for $h$ small enough

$$|p(t)(v)|_0^2 \ge h\frac{c_3}{4}|v|_1^2. \tag{2.15}$$

Using (2.15), the fact that $v=\beta\chi(D)z$ and

$$\begin{cases} |\beta\chi(D)z|_1^2 \ge \frac{1}{2}|\chi(D)z|_1^2 - |[\chi(D),\beta]z|_1^2 \ge \frac{1}{2}|\chi(D)z|_1^2 - h^2c|z|_0^2 \\ |p(\chi(D)z)|_0^2 \ge \frac{1}{2}|p(\beta\chi(D)z)|_0^2 - h^2c|z|_0^2, \end{cases}$$

we get for $h$ small enough:

$$|p(\chi(D)z)|_0^2 \geq h\frac{c_3}{16}|\chi(D)z|_1^2 - h^2 c|z|_0^2. \tag{2.16}$$

On the other hand,

$$\begin{aligned}|p(\chi(D)z)|_0^2 &\leq 8h^2|\chi(D)\circ\mathcal{L}_1(D)g|_0^2 + 8|\chi(D)\circ\mathcal{L}_2(D)r|_0^2 + 8h^4|\chi(D)G|_0^2 \\ &+ 8h^2|\chi(D)\circ R_0(g)|_0^2 + 8|\chi(D)\circ R_1(r)|_0^2 + 8h^2|R_{-\infty}(t)(z)|_0^2.\end{aligned} \tag{2.17}$$

As $\chi(D) \in E^{-\infty}$, all the operators appearing in the right hand side are in $L^\infty(-T,T;E^0)$, hence there exists $c > 0$ such that for $h$ small enough (2.16) et (2.17) imply

$$|\chi(D)z|_1^2 \leq c[h|g|_0^2 + h^3|G|_0^2 + \frac{1}{h}|r|_0^2 + h|z|_0^2]. \tag{2.18}$$

Combining (2.10) and (2.18), we obtain

$$\begin{aligned}|z|_0^2 &\leq 2(|\chi(D)z|_0^2 + |\delta(D)z|_0^2) \\ &\leq 2c[h|g|_0^2 + h^3|G|_0^2 + \frac{1}{h}|r|_0^2 + h|z|_0^2] \\ &\qquad + 2c\big(h^2|g|_0^2 + |r|_0^2 + h^4|G|_0^2 + h^2|z|_0^2\big),\end{aligned} \tag{2.19}$$

which proves the lemma. □

We now turn to the heat equation: in [11], J.C. Saut and B. Scheurer stated a Carleman inequality for a general heat equation with an explicit phase $\varphi$ which required for the solution $u$ to be $H^2_{loc}$ in space and for $u'$ to be in $L^2_{loc}(Q)$. Because of the lack of regularity of the coefficients $b^j_{kl}$, there is no hope that our solution $u$ of (1.3) possesses this regularity. We are, in the evolution case, in front of the same problem as we had in [3] in the stationnary case. Again for clarity, let us recall the usual Carleman inequality for the heat equation (see [11] for an explicit phase or [3] for any phase satisfying (2.1)):

PROPOSITION 2.7. *Let $U_0$ be an open and bounded set of $\mathbb{R}^n$. Suppose that $\varphi$ satisfies (2.1). Then there exists $\rho_0 > 0$, such that for every compact set $K$ included in $U_0$, there exist $h_1 > 0$ and $c > 0$ such that for every $h \in (0,h_1)$, one has*

$$\begin{aligned}&\int_{K\times(-\rho_0,\rho_0)} |z|^2 e^{2\frac{\varphi}{h}}\,dxdt + h^2\int_{K\times(-\rho_0,\rho_0)} |\nabla z|^2 e^{2\frac{\varphi}{h}}\,dxdt \leq \\ &\quad \leq h^3 c\int_{K\times(-\rho_0,\rho_0)} |z' - \Delta z|^2 e^{2\frac{\varphi}{h}}\,dxdt\end{aligned} \tag{2.20}$$

*for every $z \in L^2(-\rho_0,\rho_0;H^2_0(K)) \cap H^1_0(-\rho_0,\rho_0;L^2(K))$ with $z' - \Delta z \in L^2((-\rho_0,\rho_0)\times U_0)$.*

The inequality that we prove does not require so much regularity and extends the previous one. Recall that $L_1$ is a first order operator:

Lemma 2.8. *Let $U_0$ be a bounded and open set of $\mathbb{R}^n$. We suppose that $\varphi$ satisfies (2.1). Then there exists $\rho_0 > 0$, such that for every compact set $K$ included in $U_0$, there exist $h_1 > 0$ and $c > 0$ such that for every $h \in (0, h_1)$, one has*

$$\int_{K\times(-\rho_0,\rho_0)} |z|^2 e^{2\frac{\varphi}{h}} dxdt + h^2 \int_{K\times(-\rho_0,\rho_0)} |\nabla z|^2 e^{2\frac{\varphi}{h}} dxdt \leq$$
$$\leq ch \int_{K\times(-\rho_0,\rho_0)} |f|^2 e^{2\frac{\varphi}{h}} dxdt \tag{2.21}$$
$$+ h^3 c \int_{K\times(-\rho_0,\rho_0)} |z' - \Delta z - L_1 f|^2 e^{2\frac{\varphi}{h}} dxdt$$

*for every $(z, f) \in L^2(-\rho_0, \rho_0; H_0^1(K)) \times L^2(U_0 \times (-\rho_0, \rho_0))$ with $z' - \Delta z - L_1 f \in L^2(U_0 \times (-\rho_0, \rho_0))$, all these functions being compactly supported in $K \times (-\rho_0, \rho_0)$.*

Remark 2.9. This is the analogue of Proposition 2.4 for the evolution case.

*Proof of Lemma 2.8.*

We write again $v = ze^{\frac{\varphi}{h}}$, $H = (z' - \Delta z - L_1 f)e^{\frac{\varphi}{h}}$ and $q = fe^{\frac{\varphi}{h}}$. We then have

$$h^2 v' + p(t)v = h\varphi' v - h\Delta\varphi v + h\mathcal{L}_1(D)q + hR_0 v + h^2 H$$

where $R_0 = -h[e^{\frac{\varphi}{h}}, L_1]e^{-\frac{\varphi}{h}} \in L^\infty(-T, T; E^0)$. Again we study separately high and low frequencies. Operators $a(t)$ (which were defined by $(p(t) + p^*(t))/2$) satisfy the hypotheses of Proposition 2.2 hence they are invertible on the high frequencies and therefore (in the same way than in the previous lemma) there exist operators $d(t)$ of order $-2$ such that

$$d(t) \circ a(t) = 1 + \alpha(t) + hR(t)$$

with $R(t) \in L^\infty(-T, T; E^{-1})$ and $\alpha\delta(D) \in hL^\infty(-T, T; E^{-1})$.

Since $[a(t), \lambda^{-1}] \in hL^\infty(-T, T; E^0)$, there exists $c > 0$ such that

$$|\delta(D)v|_1 \leq |d(t) \circ a(t)\delta(D)v|_1 + ch|v|_0 \leq c(|a(t)\delta(D)v|_{-1} + h|v|_0)$$
$$\leq c(|\lambda^{-1}a(t)\delta(D)v|_0 + h|v|_0) \leq c(|a(t)(\lambda^{-1}\delta(D)v)|_0 + h|v|_0). \tag{2.22}$$

On an other hand, we have

$$h^2\lambda^{-1}\delta(D)v' + i\lambda^{-1}\delta(D)b(t)v + a(t)\lambda^{-1}\delta(D)v =$$
$$\lambda^{-1}\delta(D)[h\varphi' v - h\Delta\varphi v + h\mathcal{L}_1(D)q + hR_0 v + h^2 H] - [\lambda^{-1}\delta(D), a(t)]v \tag{2.23}$$

with $[\lambda^{-1}\delta(D), a(t)] \in hL^\infty(-T, T; E^0)$.

Let $u = \lambda^{-1}\delta(D)v$. Since $v$ belongs to $L^2(-\rho_0, \rho_0; H_0^1(K))$ and since $\lambda$ is in $L^\infty(E^{-1})$, the function $u$ is in $L^2(-\rho_0, \rho_0; H^2(\mathbb{R}^n))$. Furthermore, using the equation satisfied by $u$, we can see that $u$ is in $H_0^1(-\rho_0, \rho_0, L^2(\mathbb{R}^n))$. Taking the $L^2(\mathbb{R}^{n+1})$-norm, we then get

$$|h^2\lambda^{-1}\delta(D)v' + i\lambda^{-1}\delta(D)b(t)v|_0^2 + |a(t)\lambda^{-1}\delta(D)v|_0^2 +$$
$$2re(h^2\lambda^{-1}\delta(D)v', a(t)\lambda^{-1}\delta(D)v) + 2re(i\lambda^{-1}\delta(D)b(t)v, a(t)\lambda^{-1}\delta(D)v)$$
$$= |\lambda^{-1}\delta(D)[h\varphi' v - h\Delta\varphi v + h\mathcal{L}_1(D)q + hR_0 v + h^2 H]$$
$$- [\lambda^{-1}\delta(D), a(t)]v|_0^2. \tag{2.24}$$

As $\lambda^{-1}\delta(D)b(t) \in L^\infty(-T,T;E^0)$ and $a(t)\lambda^{-1}\delta(D) \in L^\infty(-T,T;E^1)$ we get

$$[\lambda^{-1}\delta(D)b(t), a(t)\lambda^{-1}\delta(D)] \in hL^\infty(-T,T;E^0)$$

which easily proves that

$$\begin{aligned} &|2re(i\lambda^{-1}\delta(D)b(t)v, a(t)\lambda^{-1}\delta(D)v)| \\ &= i([a(t)\lambda^{-1}\delta(D), \lambda^{-1}\delta(D)b(t)]v, v)| \leq ch|v|_0^2. \end{aligned} \tag{2.25}$$

Now,

$$\begin{aligned} 2re(h^2u', a(t)u) = 2h^2 &\int_{R^n\times(-\rho_0,\rho_0)} u'(-h^2\Delta u - |\nabla\varphi|^2 u)dxdt \\ &= -h^2 \int_{R^n\times(-\rho_0,\rho_0)} |u|^2(|\nabla\varphi|^2)'dxdt \leq ch^2|u|_0^2 \\ &\leq ch^2|v|_0^2. \end{aligned} \tag{2.26}$$

Combining (2.22), (2.24), (2.25) and (2.26), we obtain

$$|\delta(D)v|_1^2 \leq c[h|v|_0^2 + h^2|q|_0^2 + h^4|H|_0^2]. \tag{2.27}$$

To prove such an inequality on the low frequencies, one just has to follow what has been done in the previous lemma, the only change is that integrals are taken over $\mathbb{R}^n \times (-\rho_0, \rho_0)$ instead of $\mathbb{R}^n$ and that the added term $2h^2Re(\chi(D)v', a(t)\chi(D)v)$ can be bounded by $ch^2|\chi(D)v|_0^2$. (Remark that $\chi(D)v \in L^2(-\rho_0, \rho_0; H^2(\mathbb{R}^n)) \cap H_0^1(-\rho_0, \rho_0, L^2(\mathbb{R}^n))$ thanks to the equation satisfied by $\chi(D)v$ and to $\chi(D) \in E^{-\infty}$).

We then obtain

$$|\chi(D)v|_1^2 \leq c[h|q|_0^2 + h^2|H|_0^2 + h|v|_0^2]. \tag{2.28}$$

Combining (2.27) and (2.28), one can deduce the lemma. □

## 2.3. Proof of Theorem 1.4

We follow the steps of the proof of the main result in [3]. We denote by $B(r)$ the open ball in $\mathbb{R}^{n+1}$ centered at $(0,0)$ with radius $r$. In order to prove Theorem 1.4, it is sufficient to prove that if $u = 0$ in a half-neighbourhood (in $\mathbb{R}^{n+1}$) of $(0,0)$ such as

$$\{(x,t);\quad \psi(x,t) < 0\} \cap B(\rho)$$

where $\psi$ is $C^\infty$, $\psi(0,0) = 0$, $\nabla_x\psi(0,0) \neq 0$, then $u$ vanishes in a neighbourhood of $(0,0)$.

Without loss of generality, one can suppose that $\nabla_x\psi(0,0) = (0,\cdots,0,1)$. As in [3], we first prove a unique continuation property for a 'radius' $r = 1$ and small potentials $a_{kl}^j$ and $b_{kl}^j$. We write

$$W = \{(x,t),\quad |t| < 1,\quad |x| < 1\}$$

and

$$a = \max_{j,k,l} |a^j_{kl}|_{L^\infty(W)}, \quad b = \max_{j,k,l} |b^j_{kl}|_{L^\infty(W)}.$$

We then have

LEMMA 2.10. *There exists $M > 0$ such that for all $(u, \pi, a^j_{kl}, b^j_{kl}) \in L^2(-1,1; H^1(|x| < 1)) \times L^2(W) \times L^\infty(W)^{2n^3}$ with*

$$\begin{aligned} &(u,\pi) \text{ is solution of (1.3)} \\ &\sup(a,b) \le M \\ &u = 0 \ \text{ in } \ W \cap [x_n + M(|x'| + |t|) < 0] \end{aligned} \tag{2.29}$$

*one has $u = 0$ in a neighbourhood of $(0,0)$ in $\mathbb{R}^{n+1}$.*

*Proof of Lemma 2.10.*

The choice of $\varphi$ is the same as in [3] and it is

$$\varphi(x,t) = (x_n + |x'|^2 + t^2 - \delta)^2 \chi$$

where $\delta > 0$ has to be chosen and $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ satisfies $\chi = 1$ on $W$. We proved in [3] (Lemma 4.5) that

$$\begin{aligned} &\exists \delta > 0, \quad \exists r_0 > 0, \text{ such that } \varphi \text{ satisfies (2.1) on} \\ &U_0 = \{x, \ \ |x| < r_0\} \ \text{ with } \quad C_0 = \delta^2. \end{aligned} \tag{2.30}$$

In what follows, $\delta$ and $r_0$ are chosen such that (2.30) holds. We now apply Lemmas 2.5 and 2.8 with $U_0 = \{x, \ \ |x| < r_0\}$ in order to get the existence of $\rho_0 > 0$ such that the conclusions of this lemma are satisfied. We then fix $r_1 > 0$ small enough beside $\delta^2$, $r_0$ et $\rho_0$ such that $B(4r_1) \subset \{(t,x), \ \ |t| < \rho_0, \ \ |x| < r_0\}$ and we choose $\zeta \in C_0^\infty(B(r_1))$ with $\zeta = 1$ on $B(3r_1/4)$. We write $K = \{|x| \le r_0/2\}$, $\rho_1 = \rho_0/2$ and

$$\Sigma = \text{supp}\, [\nabla_{x,t}\zeta \cap \{x_n + M(|x'| + |t|) \ge 0\}].$$

Then there exists $M_1 > 0$ such that for all $M \in ]0, M_1]$, one has

$$\sup_{(x,t)\in\Sigma} \varphi(x,t) < \varphi(0,0) = \delta^2. \tag{2.31}$$

As $\nabla\pi = 0$ on $W \cap [x_n + M(|x'| + |t|) < 0]$, we can suppose that $\pi = 0$ on that set. We denote by $z = \zeta u$ and $q = \zeta\pi$. Since div $u = 0$, one has

$$\Delta\pi = \frac{\partial^2}{\partial x_j \partial x_k}(b^j_{kl} u_l) + \frac{\partial}{\partial x_j}(a^j_{kl} \frac{\partial u_l}{\partial x_k}).$$

After a simple computation, one can then prove that

$$\begin{aligned} \Delta q = \text{div}\,(2\nabla\zeta p) - \Delta\zeta p + \frac{\partial^2}{\partial x_j \partial x_k}(b^j_{kl} z_l) - \frac{\partial}{\partial x_j}(b^j_{kl} u_l \frac{\partial \zeta}{\partial x_k}) \\ - \frac{\partial}{\partial x_k}(\frac{\partial \zeta}{\partial x_j} b^j_{kl} u_l) + b^j_{kl} u_l \frac{\partial^2 \zeta}{\partial x_j \partial x_k} \\ + \frac{\partial}{\partial x_j}(a^j_{kl} \frac{\partial z_l}{\partial x_k}) - \frac{\partial}{\partial x_j}(a^j_{kl} u_l \frac{\partial \zeta}{\partial x_k}) - \frac{\partial \zeta}{\partial x_j} a^j_{kl} \frac{\partial u_l}{\partial x_k}. \end{aligned} \tag{2.32}$$

So we obtain an equation of the form

$$\Delta q - L_1 f - L_{2,jk} K_{jk} = F \in L^2 \tag{2.33}$$

with $s = n$, $L_1 = \mathrm{div}$ and

$$\begin{aligned} f^j &= 2\frac{\partial \zeta}{\partial x_j} p + b^j_{kl} u_l \frac{\partial \zeta}{\partial x_k} - b^k_{jl} u_l \frac{\partial \zeta}{\partial x_k} + a^j_{kl} \frac{\partial z_l}{\partial x_k} - a^j_{kl} u_l \frac{\partial \zeta}{\partial x_k} \\ &= a^j_{kl} \frac{\partial z_l}{\partial x_k} + f^j_0 \in L^2 \end{aligned} \tag{2.34}$$

where $f^j_0$ is in $L^2$ and is supported in $\Sigma$, and $L_{2,jk} = \frac{\partial^2}{\partial x_j \partial x_k}$, $K_{jk} = (b^j_{kl} z_l) \in L^2$ and finally

$$F = -\Delta \zeta p - \frac{\partial \zeta}{\partial x_j} a^j_{kl} \frac{\partial u_l}{\partial x_k} + b^j_{kl} u_l \frac{\partial^2 \zeta}{\partial x_k \partial x_j} \in L^2$$

with support in $\Sigma$.

On the other hand, we have for each $1 \leq j \leq n$,

$$\begin{aligned} z'_j - \Delta z_j &= \zeta(u'_j - \Delta u_j) + u_j \zeta' - 2\nabla\zeta.\nabla u_j - \Delta\zeta u_j \\ &= \frac{\partial}{\partial x_j} q - \frac{\partial \zeta}{\partial x_j} p + u_j \zeta' - 2\nabla\zeta.\nabla u_j - \Delta \zeta u_j - a^j_{kl} \frac{\partial z_l}{\partial x_k} + a^j_{kl} u_l \frac{\partial \zeta}{\partial x_k} \\ &\quad - \frac{\partial}{\partial x_k}(b^j_{kl} z_l) + \frac{\partial \zeta}{\partial x_k} b^j_{kl} u_l \\ &= L^j_1(q, g_1, \cdots, g_n) - a^j_{kl} \frac{\partial z_l}{\partial x_k} + H^j_0 \end{aligned} \tag{2.35}$$

with $s = n+1$, $L^j_1(g_0, g_1, \cdots, g_n) = \frac{\partial}{\partial x_j} g_0 - \frac{\partial}{\partial x_k} g_k$ and $g_k = b^j_{kl} z_l$ for $1 \leq k \leq n$, and $H^j_0 = -p\frac{\partial \zeta}{\partial x_j} - a^j_{kl} u_l \frac{\partial \zeta}{\partial x_k} + \frac{\partial \zeta}{\partial x_k} b^j_{kl} u_l + u_j \zeta' - 2\nabla\zeta.\nabla u_j - u_j \Delta \zeta \in L^2$ with support in $\Sigma$.

By Lemma 2.5 and 2.8, there exist $c > 0$ and $h_1 > 0$ such that for every $h \in ]0, h_1]$, we have

$$\begin{aligned} &\int |z|^2 e^{2\frac{\varphi}{h}} dx dt + h^2 \int |\nabla z|^2 e^{2\frac{\varphi}{h}} dx dt \\ &\leq ch^3 \int |a^j_{kl} \frac{\partial z_l}{\partial x_k}|^2 e^{2\frac{\varphi}{h}} + ch^3 \int |H_0|^2 e^{2\frac{\varphi}{h}} dx dt \\ &\quad + ch \int |q|^2 e^{2\frac{\varphi}{h}} dx dt + ch \int |b^j_{kl} z_l|^2 e^{2\frac{\varphi}{h}} dx dt \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} &\int |q|^2 e^{2\frac{\varphi}{h}} dx dt \leq ch \int |a^j_{kl} \frac{\partial z_l}{\partial x_k} + f^j_0|^2 e^{2\frac{\varphi}{h}} dx dt \\ &+ \frac{c}{h} \int |b^j_{kl} z_l|^2 e^{2\frac{\varphi}{h}} dx dt + ch^3 \int |F|^2 e^{2\frac{\varphi}{h}} dx dt. \end{aligned} \tag{2.37}$$

Combining these two inequalities, we deduce that there exist $c > 0$ such that for $h$ small enough

$$\int |z|^2 e^{2\frac{\varphi}{h}} dxdt + h^2 \int |\nabla z|^2 e^{2\frac{\varphi}{h}} dxdt \leq ch^2 a^2 \int |\nabla z|^2 e^{2\frac{\varphi}{h}} dxdt$$
$$+ cb^2 \int |z|^2 e^{2\frac{\varphi}{h}} dxdt + ch^3 \int (|H_0|^2 + h|F|^2) e^{2\frac{\varphi}{h}} dxdt \tag{2.38}$$
$$+ ch^2 \int |f_0|^2 e^{2\frac{\varphi}{h}} dxdt.$$

We choose $M_0 \in (0, M_1]$ satisfying $2cM_0^2 < 1$, in order that for $a \leq M_0$ and $b \leq M_0$, we get for $h > 0$ small enough,

$$\int |z|^2 e^{2\frac{\varphi}{h}} dxdt \leq \int G^2 e^{2\frac{\varphi}{h}} dxdt \tag{2.39}$$

with $G^2 = (|F|^2 + |H_0|^2) + |f_0|^2 \in L^1$ and

$$\text{supp } (G) \subset \Sigma. \tag{2.40}$$

Using (2.31) and letting $h \to 0$, we deduce that $z = 0$ in a neighbourhood of $(0, 0)$ which proves Lemma 2.10. □

We now make a change of scale and for this we note $\bar{u}(x,t) = u(\frac{x}{\lambda}, \frac{t}{\lambda^2})$, $\bar{a}^j_{kl}(x,t) = a^j_{kl}(\frac{x}{\lambda}, \frac{t}{\lambda^2})$, $\bar{b}^j_{kl}(x,t) = b^j_{kl}(\frac{x}{\lambda}, \frac{t}{\lambda^2})$, $\bar{\pi}(x,t) = \frac{1}{\lambda}\pi(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ and $\psi_\lambda(x,t)$ $= \lambda\psi(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ with $\lambda > 0$.

We then have $\bar{u} = 0$ in $\psi_\lambda(x,t) < 0$. As $\psi_\lambda(x,t) = x_n + O(\frac{|x|+|t|}{\lambda})$ and

$$\bar{u}'_j - \Delta \bar{u}_j + \frac{a^j_{kl}}{\lambda} \frac{\partial \bar{u}_l}{\partial x_k} + \frac{\partial}{\partial x_k}(\frac{b^j_{kl}}{\lambda} \bar{u}_l) = \frac{\partial \bar{\pi}}{\partial x_j},$$

hypotheses of Lemma 2.10 are satisfied for $\lambda$ large enough. We then deduce that $\bar{u}$ (hence $u$) vanishes in a neighbourhood of $(0, 0)$. □

Remark 2.11.

(i) We have already seen in [3] that even when $a^j_{kl} = b^j_{kl} = 0$ and in the stationnary case, the unique continuation property is false if we suppose that $u \in H^1_{loc}(\Omega)$ and $u = \frac{\partial u}{\partial \nu} = 0$ on $\gamma$ where $\gamma$ is an open part of the boundary of $\Omega$. However, one can remark that the natural condition in order to apply Holmgrem Theorem is $u = \frac{\partial u}{\partial \nu} + \pi\nu = 0$ on $\gamma$.

Using similar methods, as in the proof of Theorem 1.4, one can prove the following extension of Saut and Scheurer's result (see [11]) concerning the unique continuation for the heat equation:

Corollary 2.12. *Let $(a, b_1, \cdots, b_n) \in L^\infty(\Omega \times (-T, T))^{n+1}$ and let $u \in L^2_{loc}(-T, T; H^1_{loc}(\Omega))$ be solution of the following heat equation*

$$u' - \Delta u + au + \sum_{k \in \{1,\cdots,n\}} \frac{\partial}{\partial x_k}(b_k u) = 0 \text{ in } \Omega \times (-T, T)$$

*which vanishes in an open subset $O$ of $\Omega \times (-T, T)$. Then $u$ vanishes in the horizontal component of $O$ in $\Omega \times (-T, T)$.*

## 2.4. Proof of Theorem 1.8

The steps will be the same as in the proof of the previous theorem but we won't apply the Carleman inequalities to the same functions. As div $u=0$ in $Q$, we have $v_n = \frac{\partial u_n}{\partial x_n} = 0$ in $O$. Futhermore $v_n$ is solution of

$$v_n' - \Delta v_n + \frac{\partial}{\partial x_n}(a^n_{kl}\frac{\partial u_l}{\partial x_k}) + \frac{\partial}{\partial x_n}(B^n_l u_l) = \frac{\partial^2 \pi}{\partial x_n^2}.$$

On the other hand, since $a^i_{kn} = B^i_n = 0$ for $i<n$, the pressure $\pi$ is solution of

$$\Delta\pi = \sum_{i<n,l<n,k} \frac{\partial}{\partial x_i}(a^i_{kl}\frac{\partial u_l}{\partial x_k}) + \sum_{i<n,l<n} \frac{\partial}{\partial x_i}(B^i_l u_l)$$
$$+ \sum_{k,l} \frac{\partial}{\partial x_n}(a^n_{kl}\frac{\partial u_l}{\partial x_k}) + \sum_{l} \frac{\partial}{\partial x_n}(B^n_l u_l).$$

Thus we have

$$v_n' - \Delta v_n = \sum_{i<n,l<n,k} \frac{\partial}{\partial x_i}(a^i_{kl}\frac{\partial u_l}{\partial x_k}) + \sum_{i<n,l<n} \frac{\partial}{\partial x_i}(B^i_l u_l) - \Delta'\pi$$

where $-\Delta'$ is the Laplace operator with respect to the $n-1$ variables $x_1, \cdots, x_{n-1}$. With the same notation as in the proof of Theorem 1.4, we write $p_j = \frac{\partial \pi}{\partial x_j}$, $q_j = \zeta p_j$, $w = \zeta v_n$ and $w_j = \zeta u_j$ for $1 \le j < n$. We have

$$\Delta p_j = \sum_{l<n,i,k} \frac{\partial^2}{\partial x_j \partial x_i}(a^i_{kl}\frac{\partial u_l}{\partial x_k}) + \frac{\partial^2}{\partial x_j \partial x_n}(a^n_{kn}\frac{\partial u_n}{\partial x_k})$$
$$+ \sum_{l<n,i,k} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}(B^i_l u_l) + \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_n}(B^n_n u_n).$$

Since $\frac{\partial^2}{\partial x_j \partial x_n}(a^n_{kn}\frac{\partial u_n}{\partial x_k}) = \frac{\partial}{\partial x_j}(a^n_{kn}\frac{\partial v_n}{\partial x_k})$ and $\frac{\partial^2}{\partial x_j \partial x_n}(B^n_n u_n) = \frac{\partial}{\partial x_j}(B^n_n v_n)$, we get

$$\begin{aligned}
\Delta q_j = &-\Delta\zeta p_j + 2\mathrm{div}\,(\nabla\zeta p_j) \\
&+ \sum_{l<n,k,i}[\frac{\partial^2}{\partial x_j \partial x_i}(a^i_{kl}\frac{\partial w_l}{\partial x_k}) - \frac{\partial^2}{\partial x_j \partial x_i}(a^i_{kl} u_l \frac{\partial \zeta}{\partial x_k}) \\
&- \frac{\partial}{\partial x_j}(a^i_{kl}\frac{\partial u_l}{\partial x_k}\frac{\partial \zeta}{\partial x_i}) - \frac{\partial}{\partial x_i}(a^i_{kl}\frac{\partial u_l}{\partial x_k}\frac{\partial \zeta}{\partial x_j}) + a^i_{kl}\frac{\partial u_l}{\partial x_k}\frac{\partial^2 \zeta}{\partial x_i \partial x_j}] \\
&+ \sum_{k}[\frac{\partial}{\partial x_j}(a^n_{kn}\frac{\partial w}{\partial x_k}) - \frac{\partial}{\partial x_j}(a^n_{kn} v_n \frac{\partial \zeta}{\partial x_k}) - \frac{\partial \zeta}{\partial x_j} a^n_{kn}\frac{\partial v_n}{\partial x_k}] \\
&+ \sum_{l<n,i}[\frac{\partial^2}{\partial x_j \partial x_i}(B^i_l w_l) - \frac{\partial}{\partial x_j}(B^i_l u_l \frac{\partial \zeta}{\partial x_i}) \\
&- \frac{\partial}{\partial x_i}(B^i_l u_l \frac{\partial \zeta}{\partial x_j}) + B^i_l u_l \frac{\partial^2 \zeta}{\partial x_j \partial x_i}] \\
&+ \frac{\partial}{\partial x_j}(B^n_n w) - B^n_n v_n \frac{\partial \zeta}{\partial x_j}.
\end{aligned} \tag{2.41}$$

Applying Lemma 2.5 and since $v_n \in L^2(H^1(K))$, there exists $c > 0$ such that (with $|a|_\infty = \max_{k,l,j} |a^j_{kl}|_\infty$ and $|B|_\infty = \max_{l,i} |B^j_l|_\infty$)

$$\int |q_j|^2 e^{2\frac{\varphi}{h}} dxdt \leq \frac{c}{h}[\sum_{l<n}(|a|^2_\infty \int |\nabla w_l|^2 e^{2\frac{\varphi}{h}} dxdt + |B|^2_\infty \int |w_l|^2 e^{2\frac{\varphi}{h}} dxdt)]$$
$$+ ch[|a|^2_\infty \int |\nabla w|^2 e^{2\frac{\varphi}{h}} dxdt + |B|^2_\infty \int |w|^2 e^{2\frac{\varphi}{h}} dxdt] + \frac{c}{h} \int |G|^2 e^{2\frac{\varphi}{h}} dxdt \tag{2.42}$$

where $G \in L^2$ does not depend on $h$. Futhermore, since we have for $1 \leq j < n$, $p_j = v_n = u_j = 0$ on $W \cap [x_n + M(|x'| + |t|) < 0]$, $G$ is supported in $\Sigma$.

We have

$$\begin{cases} w' - \Delta w = \zeta' v_n + \Delta\zeta v_n - \mathrm{div}\,(2\nabla\zeta v_n) - \sum_{k<n} [\frac{\partial}{\partial x_k}(q_k) - \frac{\partial\zeta}{\partial x_k} p_k] \\ + \sum_{i<n,l<n} [\frac{\partial}{\partial x_i}(B^i_l w_l) - \frac{\partial\zeta}{\partial x_i} B^i_l u_l] \\ + \sum_{i<n,l<n,k} [\frac{\partial}{\partial x_i}(a^i_{kl} \frac{\partial w_l}{\partial x_k}) - \frac{\partial}{\partial x_i}(a^i_{kl} u_l \frac{\partial\zeta}{\partial x_k}) - a^i_{kl} \frac{\partial u_l}{\partial x_k} \frac{\partial\zeta}{\partial x_i}]. \end{cases} \tag{2.43}$$

Using Lemma 2.8, we deduce that there exists $c > 0$, such that for $h$ small enough,

$$\int |w|^2 e^{2\frac{\varphi}{h}} dxdt + h^2 \int |\nabla w|^2 e^{2\frac{\varphi}{h}} dxdt \leq ch[\sum_{l<n}(\int |q_l|^2 e^{2\frac{\varphi}{h}} dxdt$$
$$+ |a|^2_\infty \int |\nabla w_l|^2 e^{2\frac{\varphi}{h}} dxdt + |B|^2_\infty \int |w_l|^2 e^{2\frac{\varphi}{h}} dxdt) + \int |F|^2 e^{2\frac{\varphi}{h}} dxdt], \tag{2.44}$$

where $F \in L^2$ does not depend on $h$ and is supported in $\Sigma$.

Finally, we have

$$w'_j - \Delta w_j = u_j \zeta' - 2\nabla u_j . \nabla\zeta - \Delta\zeta u_j + q_j - \sum_{l<n} [B^j_l w_l - a^j_{kl} \frac{\partial w_l}{\partial x_k} + a^j_{kl} \frac{\partial\zeta}{x_k} u_l],$$

and thus, applying the usual Carleman inequality on the heat equation, we get for every $j < n$,

$$\int |w_j|^2 e^{2\frac{\varphi}{h}} dxdt + h^2 \int |\nabla w_j|^2 e^{2\frac{\varphi}{h}} dxdt \leq ch^3[\int |q_j|^2 e^{2\frac{\varphi}{h}} dxdt$$
$$+ \sum_{l<n} [|a|^2_\infty \int |\nabla w_l|^2 e^{2\frac{\varphi}{h}} dxdt + |B|^2_\infty \int |w_l|^2 e^{2\frac{\varphi}{h}} dxdt] \tag{2.45}$$
$$+ \int |H|^2 e^{2\frac{\varphi}{h}} dxdt],$$

where $H \in L^2$ does not depend on $h$ and is supported in $\Sigma$. Hence, for $h$ small enough

$$\begin{aligned}&\sum_{j<n}[\int |w_j|^2 e^{2\frac{\varphi}{h}}\,dxdt + h^2\int |\nabla w_j|^2 e^{2\frac{\varphi}{h}}\,dxdt]\\ &\le ch^3[\sum_{j<n}\int |q_j|^2 e^{2\frac{\varphi}{h}}\,dxdt + \int |H|^2 e^{2\frac{\varphi}{h}}\,dxdt].\end{aligned} \tag{2.46}$$

Combining (2.42) and (2.46), we deduce that there exists $d > 0$

$$\begin{aligned}&\sum_{j<n}[\int |w_j|^2 e^{2\frac{\varphi}{h}}\,dxdt + h^2\int |\nabla w_j|^2 e^{2\frac{\varphi}{h}}\,dxdt]\\ &\le d[h^2\sum_{j<n}|a|^2_\infty\int |\nabla w_j|^2 e^{2\frac{\varphi}{h}}\,dxdt\\ &+ h^2|B|^2_\infty\sum_{j<n}\int |w_j|^2 e^{2\frac{\varphi}{h}}\,dxdt + h^4|a|^2_\infty\int |\nabla w|^2 e^{2\frac{\varphi}{h}}\,dxdt\\ &+ h^4|B|^2_\infty\int |w|^2 e^{2\frac{\varphi}{h}}\,dxdt + h^2\int |G|^2 e^{2\frac{\varphi}{h}}\,dxdt\\ &+ h^3\int |H|^2 e^{2\frac{\varphi}{h}}\,dxdt.\end{aligned} \tag{2.47}$$

Thus, for $|a|_\infty$ small enough, there exists $d > 0$ such that for $h$ small enough

$$\begin{aligned}&\sum_{j<n}[\int |w_j|^2 e^{2\frac{\varphi}{h}}\,dxdt + h^2\int |\nabla w_j|^2 e^{2\frac{\varphi}{h}}\,dxdt]\\ &\qquad\le dh^4|a|^2_\infty\int |\nabla w|^2 e^{2\frac{\varphi}{h}}\,dxdt\\ &\qquad+ dh^4|B|^2_\infty\int |w|^2 e^{2\frac{\varphi}{h}}\,dxdt + h^2\int |G_1|^2 e^{2\frac{\varphi}{h}}\,dxdt,\end{aligned} \tag{2.48}$$

where $G_1 \in L^2$ does not depend on $h$ and is supported in $\Sigma$.

Combining now (2.42) and (2.44), we have

$$\begin{aligned}&\int |w|^2 e^{2\frac{\varphi}{h}}\,dxdt + h^2\int |\nabla w|^2 e^{2\frac{\varphi}{h}}\,dxdt \le d[|a|^2_\infty\sum_{j<n}\int |\nabla w_j|^2 e^{2\frac{\varphi}{h}}\,dxdt\\ &\quad+ |B|^2_\infty\sum_{j<n}\int |w_j|^2 e^{2\frac{\varphi}{h}}\,dxdt + h\int |F|^2 e^{2\frac{\varphi}{h}}\,dxdt\\ &\quad+ h^2|a|^2_\infty\int |\nabla w|^2 e^{2\frac{\varphi}{h}}\,dxdt + h^2|B|^2_\infty\int |w|^2 e^{2\frac{\varphi}{h}}\,dxdt\\ &\quad+ \int |G|^2 e^{2\frac{\varphi}{h}}\,dxdt].\end{aligned} \tag{2.49}$$

Thus, for $|a|_\infty$ small enough, there exists $d > 0$ such that for $h$ small enough,

$$\begin{aligned}&\int |w|^2 e^{2\frac{\varphi}{h}} dxdt + h^2 \int |\nabla w|^2 e^{2\frac{\varphi}{h}} dxdt \\ &\leq d[|a|_\infty^2 \sum_{j<n} \int |\nabla w_j|^2 e^{2\frac{\varphi}{h}} dxdt \\ &+ |B|_\infty^2 \sum_{j<n} \int |w_j|^2 e^{2\frac{\varphi}{h}} dxdt + h \int |F|^2 e^{2\frac{\varphi}{h}} dxdt \\ &+ \int |G|^2 e^{2\frac{\varphi}{h}}]dxdt.\end{aligned} \tag{2.50}$$

We finally combine (2.48) and (2.50) and we obtain

$$\begin{aligned}&\sum_{j<n}[\int |w_j|^2 e^{2\frac{\varphi}{h}} dxdt + h^2 \int |\nabla w_j|^2 e^{2\frac{\varphi}{h}} dxdt] \leq \\ &\leq dh^2(|a|_\infty^2 + |B|_\infty^2)|a|_\infty^2 \sum_{j<n} \int |\nabla w_j|^2 e^{2\frac{\varphi}{h}} dxdt \\ &+ dh^2(|a|_\infty^2 + |B|_\infty^2)|B|_\infty^2 \sum_{j<n} \int |w_j|^2 e^{2\frac{\varphi}{h}} dxdt \\ &+ d \int |G_2|^2 e^{2\frac{\varphi}{h}} dxdt,\end{aligned} \tag{2.51}$$

for some $d > 0$ and where $G_2 \in L^2$ does not depend on $h$ and is supported in $\Sigma$. We deduce that for $2d(|a|_\infty^2 + |B|_\infty^2)^2 < 1$ and for $h$ small enough,

$$\sum_{j<n}[\int |w_j|^2 e^{2\frac{\varphi}{h}} dxdt + h^2 \int |\nabla w_j|^2 e^{2\frac{\varphi}{h}} dxdt]t \leq 2d \int |G_2|^2 e^{2\frac{\varphi}{h}} dxdt \tag{2.52}$$

which proves that $w_j = u_j = 0$ in a neighbourhood of $(0,0)$ for $1 \leq j < n$ and Theorem 1.8 is proved in the case of small potentials $a$ and $B$ and radius 1. We then end with a change of scale as in the proof of Theorem 1.4. □

Let us now prove Corollary 1.9. Let $(x^0, t^0) \in C(O)$ with $(x^0, t^0) = (x'^0, x_n^0, t_0)$ and $x'^0 \in \mathbb{R}^{n-1}$, $x_n^0 \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. There exists $r > 0$ such that

$$O' = \{(x', x_n, t), \quad |x' - x'^0| < r, \quad |x_n - x_n^0| < r, \quad |t - t_0| < r\} \subset C(O).$$

Thus $\frac{\partial u_n}{\partial x_n} = 0$ in $C(O') = \Omega \times (t_0 - r, t_0 + r)$. We write $P = \{(x', t), \quad |x' - x'^0| < r, \quad |t - t_0| < r\}$ and, for every $t$, $\hat{u}_n(t)$ the extension by zero of $u_n(t)$ to $\mathbb{R}^n$. We have: $\hat{u}_n \in L^2(-T, T; H^1(\mathbb{R}^n))$. Let

$$v(x_n) = \int_P \hat{u}_n^2(x', x_n, t) dx' dt.$$

Since $\hat{u}_n(t) \in L^2(-T, T; H^1(\mathbb{R}^n))$, we have $v_n \in W^{1,1}(\mathbb{R})$ and we can write for every $z \in \mathbb{R}$,

$$v(x_n) = v(z) + \int_z^{x_n} \frac{\partial v_n}{\partial x_n} = 0.$$

As $\Omega$ is bounded, for each $x_n$, we choose $z$ such that $(x', z) \in \mathbb{R}^n - \Omega$ for every $x'$ with $|x' - x_0'| < r$ and this ends the proof of Corollary 1.9. □

# 3. Approximate controllability for linear systems

## 3.1. Internal controllability with controls without constraint

We first prove Proposition 1.5. Using the Hahn-Banach theorem, it is sufficient to prove that every solution of

$$\begin{cases} \forall j,\ 1 \leq j \leq n, \\ -\varphi_j' - \Delta\varphi_j - b_{kj}^l \dfrac{\partial \varphi_l}{\partial x_k} - \dfrac{\partial}{\partial x_k}(a_{kj}^l \varphi_l) = \dfrac{\partial \pi}{\partial x_j} \quad \text{in} \quad Q \\ \operatorname{div} \varphi = 0 \quad \text{in} \quad Q \\ \varphi = 0 \quad \text{on} \quad \Sigma \\ \varphi(T) = \varphi^0 \in H, \end{cases} \tag{3.1}$$

with $\varphi = 0$ in $\omega \times (0,T)$ satisfies $\varphi^0 = 0$.

This will be an easy consequence of Theorem 1.4 if we prove that there exists a pressure $\pi \in L^2_{loc}(Q)$ solution of (3.1). We then consider $u = (T-t)\varphi$, and

$$f_j = \varphi_j + (T-t) b_{kj}^l \frac{\partial \varphi_l}{\partial x_k} + \frac{\partial}{\partial x_k}((T-t) a_{kj}^l \varphi_l) \in L^2(0,T;H^{-1}(\Omega))^n.$$

The function $u$ is in $L^2(0,T;V)$ and satisfies

$$-u' - \Delta u = \nabla((T-t)\pi) + f, \quad \text{and} \quad u(T) = 0.$$

Furthermore, there exists a compact set $K$ such that $\Omega - \omega \subset Int(K) \subset K \subset \Omega$ with $\text{supp}(f(t)) \in K$ for almost every $t$. The existence of a pressure $\pi \in L^2_{loc}(Q)$ solution of (3.1) is then a consequence of the following

Lemma 3.1. *Let $K$ be a compact set included in $\Omega$. For every $f$ in $L^2(0,T;H^{-1}(\Omega))^n$ such that for almost every $t$, $supp f(t) \subset K$, there exists a pair $(u,p)$ in $L^2(0,T,H_0^1(\Omega))^n \times L^2(Q)$ with*

$$\int_\Omega p(x,t)dx = 0 \ \ a.e.\ t,$$

*solution of*

$$\begin{cases} u' - \Delta u = \nabla p + f \quad \text{in} \quad Q, \\ \operatorname{div} u = 0 \quad in \quad Q, \\ u = 0 \quad on \quad \Sigma, \\ u(0) = 0 \quad in \quad \Omega, \end{cases} \tag{3.2}$$

*Furthermore, the mapping $f \in L^2(0,T;H^{-1}(\Omega))^n$*

$$supp\, f(t) \subset K, a.e.\ in\, t \in (0,T) \to (u,p) \in L^2(0,T,H_0^1(\Omega))^n \times L^2(Q)$$

*is linear continuous.*

*Proof of Lemma 3.1.*

We consider a function $\xi = \xi(x) \in \mathcal{D}(\Omega)$ such that $0 \leq \xi \leq 1$ in $\Omega$ and $\xi = 1$ on $K$. We fix an open set $U$ with supp $\xi \subset U \subset \bar{U} \subset \Omega$. Then we have $\xi^2 f(t) = f(t) = \xi f(t)$ a.e. in $t$ and system (3.2) can be written

$$\begin{cases} u' - \Delta u = \nabla p + \xi^2 f & \text{in} \quad Q, \\ \operatorname{div} u = 0 \quad \text{in} \quad Q, \\ u = 0 \quad \text{on} \quad \Sigma, \\ u(0) = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{3.3}$$

Let $f_k \in L^2(Q)$ such that $f_k \to f$ in $L^2(0,T;H^{-1}(\Omega))^n$. Then $\xi f_k \to \xi f = f$ in $L^2(0,T;H^{-1}(\Omega))^n$ and for almost every t, supp $(\xi f_k(t)) \subset U$. We consider solutions $(u_k, p_k)$ of (3.3) with right-hand side $\xi^2 f_k$.

Using a density argument, it is enough to prove that there exists $c > 0$, such that for every $f \in L^2(Q)$, one can choose a pressure $p$ solution of (3.3) such that

$$|p|_{L^2(Q)} \leq c|\xi f|_{L^2(0,T;H^{-1}(\Omega))}.$$

If $f \in L^2(Q)$, the pressure $p$ can be taken in $L^2(0,T;H^1(\Omega))$. Furthermore, $\hat{p}(t) = p(t) - \frac{1}{|\Omega|}\int_\Omega p(x,t)dx$ (where $|\Omega|$ is the measure of $\Omega$) is also in $L^2(0,T;H^1(\Omega))$ and $(u,\hat{p})$ is solution of (3.3). Thus, we can suppose that for almost every $t$, $\int_\Omega p(x,t)dx = 0$.

For every $w \in \mathcal{D}(0,T;C^\infty(\Omega))$, we define $\hat{w}$ by

$$\hat{w}(x,t) = w(x,t) - \frac{1}{|\Omega|}\int_\Omega w(x,t)dx.$$

We have $(p,w) = (p,\hat{w})$ where ( , ) denotes the scalar product in $L^2(Q)$. Furthermore, $\hat{w} \in \mathcal{D}(0,T;C^\infty(\Omega))$.

Suppose for a moment that we have proved that for every $\hat{w} \in \mathcal{D}(0,T;C^\infty(\Omega))$, satisfying $\int_\Omega \hat{w}(x,t)dx = 0$ for every $t$, there exists a solution $V \in L^2(0,T;H^1(U))^n \cap L^2(Q)^n$ of

$$\begin{cases} -V' - \Delta V = \nabla \pi & \text{in} \quad Q, \\ \operatorname{div} V = \hat{w} \quad \text{in} \quad Q, \\ V = 0 \quad \text{on} \quad \Sigma, \\ V(T) = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{3.4}$$

and that there exists $c > 0$ such that for every $\hat{w} \in \mathcal{D}(0,T;C^\infty(\Omega))$, we have

$$|V|_{L^2(0,T;H^1(U))^n} + |V|_{L^2(Q)^n} \leq c|\hat{w}|_{L^2(Q)}. \tag{3.5}$$

We will then have

$$\begin{cases} |(p,w)| = |(p,\hat{w})| = |(\xi^2 f, V)| = |(\xi f, \xi V)| \\ \qquad \leq c|\xi f|_{L^2(0,T;H^{-1}(\Omega))^n}|\hat{w}|_{L^2(Q)} \\ \qquad \leq c|\xi f|_{L^2(0,T;H^{-1}(\Omega))^n}|w|_{L^2(Q)} \end{cases} \tag{3.6}$$

and thus we will obtain the existence of a constant $d>0$ such that

$$|p|_{L^2(Q)} \leq d|\xi f|_{L^2(0,T;H^{-1}(\Omega))^n}.$$

This inequality proves Lemma 3.1. □

Now, let us prove the existence of $V$. For $\hat{w} \in \mathcal{D}(0,T;C^\infty(\Omega))$, with $\int_\Omega \hat{w}(x,t)dx = 0$ for all $t$, there exists (see [13]) $\psi \in \mathcal{D}(0,T;H_0^1(\Omega))^n$ such that $\operatorname{div}\psi = \hat{w}$ and such that $\psi$ depends continuously on $\hat{w}$. We set $V = \psi + \theta_1 + \frac{\partial\theta_2}{\partial t}$ with

$$\begin{cases} -\theta_1' - \Delta\theta_1 = \nabla\pi_1 + \Delta\psi \quad \text{in} \quad Q, \\ \operatorname{div}\theta_1 = 0 \quad \text{in} \quad Q, \\ \theta_1 = 0 \quad \text{on} \quad \Sigma, \\ \theta_1(T) = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{3.7}$$

and

$$\begin{cases} -\theta_2' - \Delta\theta_2 = \nabla\pi_2 + \psi \quad \text{in} \quad Q, \\ \operatorname{div}\theta_2 = 0 \quad \text{in} \quad Q, \\ \theta_2 = 0 \quad \text{on} \quad \Sigma, \\ \theta_2(T) = 0 \quad \text{in} \quad \Omega. \end{cases} \tag{3.8}$$

It is classical that $\theta_1 \in L^2(0,T;H_0^1(\Omega))^n$ and that the operator

$$\psi \in L^2(0,T;H_0^1(\Omega))^n \to \theta_1 \in L^2(0,T;H_0^1(\Omega))^n$$

is linear continuous. Let us prove that $\theta_2 \in H^1(0,T;H^1(U))^n$. As $\psi \in L^2(0,T;H_0^1(\Omega))^n$, we know that

$$(\theta_2,\pi_2) \in H^1(0,T;L^2(\Omega))^n \times L^2(0,T;H^1(\Omega))$$

with the continuity of the corresponding linear map. We then consider $\rho \in \mathcal{D}(\Omega)$ with $0 \leq \rho \leq 1$ on $\Omega$ and $\rho = 1$ on $U$. We multiply equation (3.8) by $\rho\Delta\theta_2'$ and we integrate over $Q_t = \Omega \times (t,T)$. We obtain

$$\begin{cases} \displaystyle\int_{Q_t} \rho|\nabla\theta_2'|^2 + \frac{1}{2}\int_\Omega \rho|\Delta\theta_2(t)|^2 \\ \displaystyle = \frac{1}{2}\int_{Q_t} \Delta\rho|\theta_2'|^2 - \sum_j \int_{Q_t} \nabla\psi_j.\nabla\theta_{2,j}'\rho + \int_{Q_t} \Delta\rho\theta_2'.\psi \\ \displaystyle + \int_{Q_t} \theta_2'.(\nabla\rho.\nabla)\psi + \int_{Q_t} \operatorname{div}\psi\theta_2'.\nabla\rho + 2\int_{Q_t} \theta_2'.(\nabla\pi_2.\nabla)\nabla\rho \\ \displaystyle + \int_{Q_t} \theta_2'.\nabla\Delta\rho\pi_2. \end{cases} \tag{3.9}$$

Therefore there exists $c>0$ such that

$$\begin{aligned} &|\theta_1|_{L^2(0,T;H_0^1(\Omega))^n} + |\theta_2|_{H^1(0,T;H^1(U))^n} \\ &\leq c|\Delta\psi|_{L^2(0,T;H^{-1}(\Omega))^n} + |\psi|_{L^2(0,T;H_0^1(\Omega))^n} \\ &\leq c'|\hat{w}|_{L^2(Q)}. \end{aligned} \tag{3.10}$$

Furthermore, since $\psi \in \mathcal{D}(0,T;H_0^1(\Omega))^n$, the function $\theta_2$ is identically equal to zero on an interval $[T-\delta,T]$ with a $\delta > 0$. Hence $\theta_2'(T) = 0$ and $V(T) = 0$. We have constructed $V \in L^2(0,T;H^1(U))^n \cap L^2(Q)^n$ solution of (3.4) and (3.5) and this finishes the proofs of Lemma 3.1 and Proposition 1.5. □

REMARK 3.2. If $\xi$ is a function in $C^\infty(\Omega)$ with compact support $K$ included in $\Omega$, we define $E = \{f \in L^2(0,T;H^{-1}(\Omega))^n; \quad (1-\xi)f \in L^2(Q)^n\}$ with $|f|_E = |f|_{L^2(0,T;H^{-1}(\Omega))^n} + |(1-\xi)f|_{L^2(Q)^n}$. It is easy to prove that $E$ is a Banach space. Applying Lemma 3.1, for every $f \in E$, there exists a pair $(y,p)$ in $L^2(0,T,H_0^1(\Omega))^n \times L^2(Q)$ solution of (3.2) and the application $f \in E \to (y,p) \in L^2(0,T,H_0^1(\Omega))^n \times L^2(0,T;L^2(\Omega))$ is linear continuous.

In the same manner (considering $u = (T-t)\varphi$), Proposition 1.6 is a consequence of

LEMMA 3.3. *Let $a_{kj}^l \in L^\infty(Q)$ satisfying (1.5). Then, for every $f \in L^2(Q)^n$, there exists $(u,q) \in L^\infty(0,T,V) \times L^2(0,T,H^1(\Omega))$ solution of*

$$\begin{cases} \forall 1 \le j \le n, \\ u_j' - \Delta u_j + \dfrac{\partial}{\partial x_k}(a_{kj}^l u_l) = \dfrac{\partial q}{\partial x_j} + f_j \quad in \quad \Omega \times (0,T), \\ \operatorname{div} u = 0 \quad in \quad Q, \\ u = 0 \quad on \quad \Sigma, \\ u(0) = 0. \end{cases} \tag{3.11}$$

REMARK 3.4. Considering the function $(T-t)u$ and following the proof of the previous lemma, one can see that conditions (1.5) can be replaced by

$$\begin{cases} \text{if } n = 2 \quad \exists p_0 \in ]2,+\infty[, \quad \forall j,k,l \in \{1,\cdots,n\} \quad a_{kl}^j \in H^1(0,T,L^{p_0}(\Omega)) \\ \text{if } n > 2 \quad \forall j,k,l \in \{1,\cdots,n\} \quad a_{kl}^j \in H^1(0,T,L^n(\Omega)). \end{cases}$$

*Proof of Lemma 3.3*

We write $m = p_0$ if $n = 2$ and $m = n$ if $n > 2$ and we will denote by $d$ any constant which depends only of $\Omega$ and $m$. We first prove that there exists $c > 0$ such that for every $f \in L^2(Q)^n$ and regular potentials $a_{kj}^l \in L^\infty(0,T,C^\infty(\Omega))$, we have

$$||u||_{L^\infty(0,T,H^1(\Omega))^n} \le c|f|_{L^2(Q)^n} exp(\frac{1}{2}\sum_{j,k,l} ||\frac{\partial}{\partial x_k}a_{kj}^l||^2_{L^2(0,T;L^m(\Omega))}). \tag{3.12}$$

If $a_{kj}^l \in L^\infty(0,T,C^\infty(\Omega))$, we can write that $u$ is solution of $u_j' - \Delta u_j + \frac{\partial}{\partial x_k}(a_{kj}^l)u_l = \frac{\partial q}{\partial x_j} + h_j$ where

$$|h|_{L^2(Q)^n} \le d \sup_{j,k,l} |a_{kj}^l|_{L^\infty(Q)} |f|_{L^2(Q)^n},$$

and it is known that $u \in L^\infty(0,T;H_0^1(\Omega))^n$.

Multiplying the equation satisfied by $u_j$ by $u'_j$ and adding them for $1 \leq j \leq n$, we obtain with $Q_t = \Omega \times (0,t)$,

$$\begin{cases} \frac{1}{4}\int_{Q_t} |u'_j|^2 dxdt + \frac{1}{2}\int_\Omega |\nabla u_j(t)|^2 dx \\ \leq \frac{1}{2}\int_{Q_t} (\frac{\partial a^l_{kj}}{\partial x_k})^2 u_l^2 dxdt + \int_Q h_j^2 dxdt \\ \leq \int_Q h_j^2 dxdt + \frac{1}{2}\int_0^t ||\frac{\partial a^l_{kj}}{\partial x_k}(t)||^2_{L^m(\Omega)} ||u_l(t)||^2_{L^{2m/(m-2)}(\Omega)}. \end{cases} \tag{3.13}$$

The embedding $H^1(\Omega) \hookrightarrow L^\alpha(\Omega)$ is linear continuous for every $1 \leq \alpha < +\infty$ if $n = 2$ and $1 \leq \alpha \leq \frac{2n}{n-2}$ if $n \geq 3$. Thus, using Poincaré 's inequality, we have in both cases

$$||u_l(t)||_{L^{2m/(m-2)}(\Omega)} \leq d|\nabla u_l(t)|_{L^2(\Omega)}.$$

We deduce that

$$\frac{1}{2}\int_{Q_t} \sum_j |u'_j|^2 dxdt + \int_\Omega \sum_j |\nabla u_j(t)|^2 dxdt$$
$$\leq 2\int_Q h_j^2 dxdt + \int_0^t \sum_{jkl} ||\frac{\partial a^l_{kj}}{\partial x_k}(t)||^2_{L^m(\Omega)} \sum_l |\nabla u_l(t)|^2_{L^2(\Omega)}.$$

Inequality (3.12) is then a consequence of Gronwall's Lemma.

If the functions $a^l_{kj}$ satisfy the hypotheses of the lemma, we construct sequences of functions $A^{lr}_{kj}$ which belong to $L^\infty(0,T,C^\infty(\Omega))$ and such that $A^{lr}_{kj} \to a^l_{kj}$ weakly-* in $L^\infty(Q)$ and in $L^2(0,T;W^{1,m}(\Omega))$ for each $j,k,l$. For this, we consider an extension mapping $P$ which maps $L^\infty(\Omega)$ in $L^\infty(\mathbb{R}^n)$ and $W^{1,m}(\Omega)$ in $W^{1,m}(\mathbb{R}^n)$ with (recall that we suppose $\Omega$ to be regular and at least $C^1$)

$$|Pu|_{L^\infty(\mathbb{R}^n)} \leq c|u|_{L^\infty(\Omega)}$$

and

$$|Pu|_{W^{1,m}(\mathbb{R}^n)} \leq c|u|_{W^{1,m}(\Omega)}.$$

We set

$$A^{lr}_{kj}(x,t) = \rho_r * Pa^l_{kj}|_\Omega$$

where $\rho_r = \rho_r(x)$ is a regulazing sequence and the convolution is taken only in the space variables. We then have $A^{lr}_{kj} \to a^l_{kj}$ in $L^2(0,T;W^{1,m}(\Omega))$ and since one can easily see that $(A^{lr}_{kj})_r$ is bounded in $L^\infty(Q)$, they converge ( after extraction of a subsequence) in $L^\infty(Q)$ weak -* to $a^l_{kj}$. The corresponding solutions $u_r$ of (3.11) are then bounded in $L^2(0,T;V)$ and in $H^1(0,T;V')$. Hence they strongly converge in $L^2(Q)^n$. This allows us to pass to the limit in (3.11) and the limit of $u_r$ is the solution $u$ of (3.11) with the potentials $a^l_{kj}$. Since $u_r$ are bounded in $L^\infty(0,T,V)$, we have $u \in L^\infty(0,T;V)$. If $u \in L^\infty(0,T;V)$ and $a^l_{kj} \in L^\infty(Q) \cap L^2(0,T;W^{1,m}(\Omega))$, we have $a^l_{kj}u_l \in L^2(0,T,H^1(\Omega))$ and this ensures that the pressure is in $L^2(0,T,H^1(\Omega))$. □

EXAMPLE: THE LINEARIZED NAVIER STOKES EQUATIONS

The linearized Navier Stokes equation at the point $z$ is

$$\begin{cases} y' - \Delta y + (z.\nabla)y + (y.\nabla)z = \nabla p + v\chi_\omega \quad \text{in} \quad Q, \\ \operatorname{div} y = 0 \quad \text{in} \quad Q, \\ y = 0 \quad \text{on} \quad \Sigma, \\ y(0) = y^0 \quad \text{in} \quad \Omega. \end{cases} \tag{3.14}$$

This system is of the form (1.4) with $a^j_{kl} = \delta_{lj}z_k$ and $b^j_{kl} = \delta_{kl}z_j$. We then have

COROLLARY 3.5. *Suppose that* $z \in L^\infty(Q)^n \cap L^2(0,T;W^{1,m}(\Omega))^n$ *with* $\operatorname{div} z = 0$ *in* $Q$ *and where* $m > 2$ *if* $n = 2$ *and* $m = n$ *if* $n > 2$. *Then, every solution* $(y,p)$ *of*

$$\begin{cases} y' - \Delta y + (z.\nabla)y + (y.\nabla)z = \nabla p \quad \textit{in} \quad Q, \\ \operatorname{div} y = 0 \quad \textit{in} \quad Q, \\ y = 0 \quad \textit{on} \quad \Sigma, \\ y(0) = y^0 \end{cases} \tag{3.15}$$

*which vanishes in an open set* $O$ *of* $Q$ *vanishes in the horizontal component of* $O$.

COROLLARY 3.6. *Suppose that* $z \in L^\infty(Q)^n \cap L^2(0,T;W^{1,m}(\Omega))^n$ *with* $\operatorname{div} z = 0$ *in* $Q$ *and where* $m > 2$ *if* $n = 2$ *and* $m = n$ *if* $n > 2$. *Then, the reachable set*

$$R(T) = \{y(x,T), y \textit{ is solution of (3.14) with } v \in L^2(q)^n\}$$

*is dense in* $H$.

## 3.2. INTERNAL CONTROLLABILITY WITH CONTROLS HAVING A NULL COMPONENT

We study here the approximate controllability of the solutions of the system

$$\begin{cases} y'_j - \Delta y_j + \dfrac{\partial}{\partial x_k}(a^l_{kj}y_l) + B^l_j y_l = \dfrac{\partial p}{\partial x_j} + v\chi_q \quad \text{in } Q, \\ \operatorname{div} y = 0 \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0) = y^0 \quad \text{in } \Omega, \end{cases} \tag{3.16}$$

with controls having one null component. Using Theorem 1.8, we get (since, in this case, the pair $(u,\pi)$ solution of the adjoint problem, is in $L^2_{loc}(0,T,H^2(\Omega))^n \times L^2_{loc}(0,T;H^1(\Omega))$ by classical results).

PROPOSITION 3.7. *Let $a^j_{kl}$ and $B^j_l$ be in $L^\infty(Q)$ satisfying (1.6). For every $y^0 \in H$, and $T > 0$, the reachable set*

$$R(T) = \{y(x,T), \textit{where } y \textit{ is solution of } (3.16) \textit{ and } v = (v_1, \cdots, v_{n-1}, 0) \in L^2(q)^{n-1} \times \{0\}\}$$

*is dense in $H$.*

REMARK 3.8. Conditions (1.6) mean (in particular) that only $y_n$ appears in the equation satisfied by $y_n$.

### 3.3 BOUNDARY CONTROLLABILITY

We now end with some results on boundary control. We denote by $\nu$ the unit exterior normal vector of the boundary $\Gamma$. We consider the solutions $y$ in $L^2(0,T;H) \cap C([0,T],V')$ defined by a duality process of

$$\begin{cases} \forall 1 \leq j \leq n, \\ y'_j - \Delta y_j + a^l_{kj}\dfrac{\partial y_l}{\partial x_k} + \dfrac{\partial}{\partial x_k}(b^l_{kj} y_l) = \dfrac{\partial p}{\partial x_j} \quad \text{in } Q, \\ \operatorname{div} y = 0 \quad \text{in } Q, \\ y = v \in L^2(0,T;H^{-1/2}(\Gamma))^n, \\ y(0) = y^0 \in V' \end{cases} \tag{3.17}$$

when $v$ satisfies for almost every $t$,

$$\langle v(t), \nu \rangle = 0$$

where $\langle\ ,\ \rangle$ is the duality bracket between $H^{-1/2}(\Gamma)^n$ and $H^{1/2}(\Gamma)^n$.

REMARK 3.9. Even if one supposes that the initial data $y^0$ is in $H$, the reachable set is not a subset of $H$. Furthermore, solutions of the adjoint problems with initial data in $H$ do not necessarily possess a normal derivative in $L^2(\Gamma \times (0,T))$. This is why we consider solutions defined by a duality process with initial data in $V'$.

We prove

PROPOSITION 3.10. *For $1 \leq j,k,l \leq n$, let $b^j_{kl} \in L^\infty(Q)$ and $a^j_{kl} \in L^\infty(0,T;W^{1,\infty}(\Omega))$. For every $y^0 \in V'$ the reachable set of the solutions of (3.17) with controls in $L^2(0,T;H^{-1/2}(\Gamma))^n$ satisfying ( for almost every $t$ ) $\langle v(t), \nu\rangle = 0$, is dense in $V'$.*

*Proof of Proposition 3.10.*

We just have to apply Hahn-Banach theorem: if $u^0 \in V$ is orthogonal (in the duality sense) to the reachable set, we consider the solution $u \in L^2(0,T,V) \cap C([0,T];H)$ of

$$\begin{cases} \forall 1 \leq j \leq n, \\ -u'_j - \Delta u_j - b^j_{kl}\dfrac{\partial u_l}{\partial x_k} - \dfrac{\partial}{\partial x_k}(a^j_{kl} u_l) = \dfrac{\partial \pi}{\partial x_j} \quad \text{in } Q, \\ \operatorname{div} u = 0 \quad \text{in } Q, \\ u = 0 \quad \text{on } \Sigma, \\ u(T) = u^0 \quad \text{in } \Omega, \end{cases} \tag{3.18}$$

Using the regularity of the coefficients $a^j_{kl}$ and $b^j_{kl}$, we have $u \in L^2(0,T;H^2(\Omega))^n$ and $\pi \in L^2(0,T;H^1(\Omega))$. Furthermore, we have

$$u^0 \in V'^{\perp} \Leftrightarrow \frac{\partial u}{\partial \nu} + \pi\nu = 0 \quad \text{on} \quad \Sigma.$$

With this condition, the extension by 0 (in space) of $u$ and $\pi$ in a ball $B$ which intersects $\Gamma$ are still solutions of the homogeneous system but in $(\Omega \cup (B \cap \Omega^c)) \times (0,T)$. Since they vanish in $(B \cap \Omega^c) \cap (0,T)$, Theorem 1.4 implies that they vanish in $Q$ and thus $u^0 = 0$. □

## 4. Approximate controllability for nonlinear equations

### 4.1 Proof of Theorem 1.1

Once one knows that the linear Stokes equations perturbed by lower order terms are approximately controllable, the fixed point method that we have developped in [2] can be applied in order to study the non linear equation (1.1). Furthermore, Theorem 1.1 will be proved if we show that the closure of $R(T)$ contains $V$. For $z \in L^2(0,T;H)$, we denote by $b_k$ the $k^{th}$ component of the vector $T_M(z)$. Since $b_k \in L^\infty(Q)$, for $y^0 \in H$ and $v \in L^2(q)^n$, there exists $y = y(z,v) \in L^2(0,T,V) \cap C([0,T];H)$ solution of

$$\begin{cases} y' - \Delta y + \dfrac{\partial}{\partial x_k}(b_k y) = \nabla p + v\chi_q \quad \text{in } Q, \\ \operatorname{div} y = 0 \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0) = y^0 \quad \text{in } \Omega. \end{cases} \tag{4.1}$$

As $T_M(z) \in L^\infty(Q)^n$, we know (the unique continuation property needed here was proved in [3]) that the reachable set at time $T$ of the solutions of (3.19) is dense in $H$. Hence for every $y^1 \in V$, and $\alpha > 0$ there exists a control $v \in L^2(q)^n$ such that the solution $y$ satisfies

$$|y(T) - y^1|_H \leq \alpha.$$

Let us consider the control of $L^2(q)-$ minimum norm. For this, we decompose $y$ as $y = Y + Y_0$ where $Y = Y(z,v)$ and $Y_0 = Y_0(z)$ are solutions of

$$\begin{cases} Y' - \Delta Y + \dfrac{\partial}{\partial x_k}(b_k Y) = \nabla p + v\chi_q \quad \text{in} \quad Q, \\ \operatorname{div} Y = 0 \quad \text{in } Q, \\ Y = 0 \quad \text{on } \Sigma, \\ Y(0) = 0 \quad \text{in } \Omega, \end{cases} \tag{4.2}$$

and

$$\begin{cases} Y_0' - \Delta Y_0 + \dfrac{\partial}{\partial x_k}(b_k Y_0) = \nabla p_0 \quad \text{in } Q, \\ \operatorname{div} Y_0 = 0 \quad \text{in } Q, \\ Y_0 = 0 \quad \text{on } \Sigma, \\ Y_0(0) = y^0 \quad \text{in } \Omega, \end{cases} \tag{4.3}$$

It is classical (see [8]) that the control of minimum norm is given by $v = v(z) = \varphi = \varphi(z)$ where

$$\begin{cases} -\varphi' - \Delta\varphi - (T_M(z).\nabla)\varphi = \nabla\pi \quad \text{in } Q, \\ \operatorname{div} \varphi = 0 \quad \text{in } Q, \\ \varphi = 0 \quad \text{on } \Sigma, \\ \varphi(T) = \varphi^0 \quad \text{in } \Omega, \end{cases} \tag{4.4}$$

and $\varphi^0 \in H$ minimizes the functional $J_z$ over $H$ defined by

$$J_z(\psi^0) = \frac{1}{2}\int_q |\psi|^2 dxdt + \alpha|\psi^0|_H - \int_\Omega (y^1(x) - Y_0(x,T))\psi^0(x)dx \tag{4.5}$$

over the solutions $\psi$ of (4.4) with $\psi(T) = \psi^0$.

With this choice of controls $v$, we introduce the mapping

$$\begin{aligned} \lambda : L^2(0,T;H) &\to L^2(0,T;H) \\ z &\to y \end{aligned} \tag{4.6}$$

where $y$ is the solution of (4.1) with $v = \varphi = \varphi(z)$.

If we prove that $\lambda$ possesses a fixed point, $y$, then it will be solution of (1.1) and it will satisfy $|y(T) - y^1|_H \leq \alpha$ and thus Theorem 1.1 will be proved. For this, let us prove the

LEMMA 4.1. *The minimizers $\varphi^0(z)$ of $J_z$ over $H$ remain uniformly bounded in $H$ when $z$ describes $L^2(0,T;H)$.*

*Proof of Lemma 4.1.*

The proof of such results is classical and we give it for the sake of completeness. We argue by contradiction: suppose that there exists a sequence $(z_n)_n$ of functions in $L^2(0,T;H)$ such that the corresponding minimizers $\varphi_n^0$ of $J_{z_n}$ satisfy $|\varphi_n^0|_H \to +\infty$ when $n \to +\infty$.

For $\tilde\varphi_n^0 = \varphi_n^0/|\varphi_n^0|_H$, we write $\tilde\varphi_n$ and $\tilde\pi_n$ the solutions of (4.4) with $\tilde\varphi_n(T) = \tilde\varphi_n^0$. Since $|\tilde\varphi_n^0|_H = 1$, we can extract a subsequence (still denoted by $(\tilde\varphi_n^0)_n$) which weakly converges in $H$ to an element $\tilde\varphi^0 \in H$. Since $J_{z_n}(\varphi_n^0) \leq J_{z_n}(0) = 0$, we have

$$\frac{1}{2}|\varphi_n^0|_H \int_q |\tilde\varphi_n|^2 + \alpha \leq |y^1|_H + |Y_{0n}(T)|_H. \tag{4.7}$$

As the functions $b^n = T_M(z_n)$ are bounded in $L^\infty(Q)^n$, they weakly -* converge (after extraction of a subsequence) in this space to an element $b \in$

$L^\infty(Q)^n$. It is then easy to prove that the functions $Y_{0n}(z)$ (associated to the potential $b^n$) are bounded in $L^2(0,T;V)\cap C([0,T];H)$ and thus $(Y_{0n}(T))_n$ is bounded in $H$. We now let $n\to+\infty$ in (4.7) and we deduce that $\tilde\varphi = 0$ in $q$.

In order to prove that one can pass to the limit in the equation satisfied by $\tilde\varphi_n$, we use the continuity of the linear map

$$f\in L^2(Q)^n \to (w,p)\in (L^2(0,T;H^2(\Omega)^n\cap V)\cap H^1(0,T,L^2(\Omega))^n)\times L^2(0,T;H^1(\Omega)),$$

where $w$ is solution $w'-\Delta w=\nabla p+f$ in $Q$ and $w(0)=0$. This proves that $(T-t)\tilde\varphi_n$ is bounded in $L^2(0,T;H^2(\Omega))^n\cap H^1(0,T,L^2(\Omega))^n$ and thus $\tilde\varphi_n$ is bounded in $L^2(0,T-\delta;H^2(\Omega))^n\cap H^1(0,T-\delta,L^2(\Omega))^n$. After extraction of a subsequence, they strongly converge in $L^2(0,T-\delta;H^1(\Omega))$ to $\tilde\varphi$. The equation satisfied by $\tilde\varphi_n$ then passes to the limit on $\Omega\times(0,T-\delta)$ for every $\delta>0$ and we have for every $\delta>0$,

$$\begin{cases} -\tilde\varphi'-\Delta\tilde\varphi-(b.\nabla)\tilde\varphi=\nabla\pi \quad \text{in} \quad \Omega\times(0,T-\delta)\\ \operatorname{div}\tilde\varphi=0 \quad \text{in} \quad \Omega\times(0,T-\delta)\\ (\tilde\varphi,\pi)\in L^2((0,T-\delta);H^1(\Omega))\times L^2_{loc}(\Omega\times(0,T-\delta))\\ \tilde\varphi=0 \quad \text{in} \quad q \end{cases} \tag{4.8}$$

with $b_k\in L^\infty(Q)$. Using Theorem 1.1 of [3], we deduce that $\tilde\varphi=0$ in $Q$. Since $\tilde\varphi_n$ is bounded in $L^\infty(0,T;H)\cap H^1(0,T,V')$, we deduce that $\tilde\varphi_n^0$ converges to 0 in $V'$. Furthermore, $\tilde\varphi_n$ is bounded in $L^\infty(0,T-\delta;H^1)\cap H^1(\Omega\times(0,T-\delta))$ for every $\delta>0$. Thus we can suppose that $\tilde\varphi_n(0)\to 0$ in $H$.

We have
$$\frac{J_{z_n}(\varphi_n^0)}{|\varphi_n^0|_H}\geq \alpha-\int_\Omega (y^1-Y_{0n}(T))\tilde\varphi_n^0 dx.$$

Since $y^1\in V$ and $\int_\Omega Y_{0n}(T)\tilde\varphi_n^0dx=\int_\Omega y^0\tilde\varphi_n^0(0)$, we have that $\int_\Omega(y^1-Y_{0n}(T))\tilde\varphi_n^0dx$ goes to zero when $n\to+\infty$. The proof of Lemma 4.1 is complete since on another hand, $J_{z_n}(\varphi_n^0)\leq 0$.

We now end the proof of Theorem 1.1 applying Schauder's fixed point theorem. Since $T_M$ is continuous and bounded, $\lambda$ is continuous on $L^2(0,T;H)$. Using Lemma 4.1, when $z$ describes $L^2(0,T;H)$, the controls $\varphi(z)$ are bounded in $L^2(q)$ and, as the range of $T_M$ is bounded in $L^\infty(\mathbb{R}^n)^n$, one can easily prove that the range of $\lambda$ is bounded in $L^2(0,T;V)$. Using the variational definition of $Y$, one can easily prove that the functions $Y$ are uniformly bounded with respect to $z$ in $H^1(0,T,V')\cap L^2(0,T,V)$ and thus they describe a compact set $K_1$ of $L^2(0,T;H)$. This proves that the range of $\lambda$ is relatively compact in $L^2(0,T;H)$. We can now apply Schauder's fixed point theorem and this ends the proof of Theorem 1.1. □

## 4.2. Proof of Theorem 1.3

The method is the same as for the proof of Theorem 1.1. For $z \in L^2(0,T;H)$, we denote by $b_k$ the $k^{th}$ component of $T_M(z)$. Since $b_k \in L^\infty(Q)$, for $y^0 \in H$ and $v \in L^2(q)^n$, there exists $y = y(z,v) \in L^2(0,T,V) \cap C([0,T];H)$ solution of

$$\begin{cases} y' - \Delta y + (T_M(z).\nabla)y = \nabla p + v\chi_q \quad \text{in } Q, \\ \operatorname{div} y = 0 \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0) = y^0 \quad \text{in } \Omega. \end{cases} \tag{4.9}$$

As $T_M(z) \in L^\infty(Q)^n$, we know that the reachable set at time $T$ of the solutions of (4.9) is dense in $H$. Hence for every $y^1 \in H$ and $\alpha > 0$ there exists a control $v \in L^2(q)^n$ such that the solution $y$ satisfies

$$|y(T) - y^1|_H \leq \alpha.$$

We now consider the control of $L^2(q)-$ minimum norm. For this, we decompose $y$ as $y = Y + Y_0$ where $Y = Y(z,v)$ and $Y_0 = Y_0(z)$ are solutions of

$$\begin{cases} Y' - \Delta Y + (T_M(z).\nabla)Y = \nabla p + v\chi_q \quad \text{in } Q, \\ \operatorname{div} Y = 0 \quad \text{in } Q, \\ Y = 0 \quad \text{on } \Sigma, \\ Y(0) = 0 \quad \text{in } \Omega, \end{cases} \tag{4.10}$$

and

$$\begin{cases} Y_0' - \Delta Y_0 + (T_M(z).\nabla)Y_0 = \nabla p_0 \quad \text{in } Q, \\ \operatorname{div} Y_0 = 0 \quad \text{in } Q, \\ Y_0 = 0 \quad \text{on } \Sigma, \\ Y_0(0) = y^0 \quad \text{in } \Omega. \end{cases} \tag{4.11}$$

The control of minimum norm is given by $v = v(z) = \varphi = \varphi(z)$ where

$$\begin{cases} -\varphi' - \Delta\varphi - \dfrac{\partial}{\partial x_k}(T_M(z)_k\varphi) = \nabla\pi \quad \text{in } Q, \\ \operatorname{div} \varphi = 0 \quad \text{in } Q, \\ \varphi = 0 \quad \text{on } \Sigma, \\ \varphi(T) = \varphi^0 \quad \text{in } \Omega, \end{cases} \tag{4.12}$$

where $\varphi^0 \in H$ minimizes the functional $J_z$ over $H$ defined by

$$J_z(\psi^0) = \frac{1}{2}\int_q |\psi|^2 dxdt + \alpha|\psi^0|_H - \int_\Omega (y^1(x) - Y_0(x,T))\psi^0(x)dx \tag{4.13}$$

over the solutions $\psi$ of (4.12) with $\psi(T) = \psi^0$.

With this choice of controls $v$, we introduce the mapping

$$\begin{aligned} \lambda : L^2(0,T;H) &\to L^2(0,T;H) \\ z &\to y \end{aligned} \tag{4.14}$$

where $y$ is the solution of (4.9) with $v = \varphi = \varphi(z)$ and we prove that $\lambda$ possesses a fixed point $y$.

LEMMA 4.2. *When $z$ describes $L^2(0,T;H)$, the minimizers $\varphi^0(z)$ of $J_z$ over $H$ remain bounded in $H$.*

*Proof of Lemma 4.2.*

Suppose that there exists a sequence $(z_n)_n$ of functions in $L^2(0,T;H)$ such that the corresponding minimizers $\varphi_n^0$ of $J_{z_n}$ satisfy $|\varphi_n^0|_H \to +\infty$ when $n \to +\infty$.

For $\tilde{\varphi}_n^0 = \varphi_n^0/|\varphi_n^0|_H$, we write $\tilde{\varphi}_n$ the solution of (4.12) with $\tilde{\varphi}_n(T) = \tilde{\varphi}_n^0$. Since $|\tilde{\varphi}_n^0|_H = 1$, we can extract a subsequence (still denoted by $(\tilde{\varphi}_n^0)_n$) which weakly converges in $H$ to an element $\tilde{\varphi}^0 \in H$. Since $J_{z_n}(\varphi_n^0) \le J_{z_n}(0) = 0$, we have

$$\frac{1}{2}|\varphi_n^0|_H \int_q |\tilde{\varphi}_n|^2 + \alpha \le |y^1|_H + |Y_{0n}(T)|_H. \tag{4.15}$$

As the functions $b^n = T_M(z_n)$ are bounded in $L^\infty(Q)^n$, they weakly -* converge (after extraction of a subsequence) in this space to an element $b \in L^\infty(Q)^n$. It is then easy to prove that the functions $Y_{0n}(z)$ are bounded in $L^2(0,T;V) \cap C[0,T];H)$ and thus $(Y_{0n}(T))_n$ is bounded in $H$. We now let $n \to +\infty$ in (4.15) and we deduce that $\tilde{\varphi} = 0$ in $q$.

In order to prove that one can pass to the limit in the equation satisfied by $\tilde{\varphi}_n$, it is sufficient to come back to the variational definition of $\tilde{\varphi}_n$ using that as $(\tilde{\varphi}_n')$ is bounded in $L^2(0,T;V')$ and $(\tilde{\varphi}_n)$ is bounded in $L^2(0,T;V)$, we have the strong convergence of $(\tilde{\varphi}_n)$ in $L^2(0,T;H)$. We can then write that $\tilde{\varphi}$ is solution of

$$\begin{cases} -\tilde{\varphi} - \Delta\tilde{\varphi} - \dfrac{\partial}{\partial x_k}(b_k\tilde{\varphi}) = \nabla\pi \quad \text{in } Q, \\ \operatorname{div}\tilde{\varphi} = 0 \quad \text{in } Q, \\ \tilde{\varphi} \in L^2(0,T;H^1(\Omega)), \\ \tilde{\varphi} = 0 \quad \text{in } q, \end{cases} \tag{4.16}$$

with $b_k \in L^\infty(Q)$. Using Lemma 3.1, we deduce that $\pi \in L^2_{loc}(Q)$. Then Theorem 1.4 implies that $\tilde{\varphi} = 0$ in $Q$ and thus $\tilde{\varphi}_n^0$ weakly converges in $H$ to 0.

We have

$$\frac{J_{z_n}(\varphi_n^0)}{|\varphi_n^0|_H} \ge \alpha - \int_\Omega (y^1 - Y_{0n}(T))\tilde{\varphi}_n^0 dx.$$

We prove that the term $\int_\Omega (y^1 - Y_{0n}(T))\tilde{\varphi}_n^0 dx$ tends to zero when $n \to +\infty$ : Indeed, we can see that $(Y_{0n}(T))_n$ strongly converges (after extraction of a subsequence) in $H$. For this, we decompose again $Y_{0n}$ in $Y_{0n} = w^1 + w_n^2$

with

$$\begin{cases} w^{1'} - \Delta w^1 = \nabla \pi^1 \quad \text{in } Q, \\ \operatorname{div} w^1 = 0 \quad \text{in } Q, \\ w^1 = 0 \quad \text{on } \Sigma, \\ w^1(0) = Y_{0n}(0) = y^0 \quad \text{in } \Omega, \end{cases} \tag{4.17}$$

and

$$\begin{cases} w_n^{2'} - \Delta w_n^2 = \nabla \pi_n^2 - (b_n.\nabla) Y_{0n} \quad \text{in } Q, \\ \operatorname{div} w_n^2 = 0 \ \text{in } \ Q, \\ w_n^2 = 0 \quad \text{on } \Sigma, \\ w_n^2(0) = 0 \quad \text{in } \Omega, \end{cases} \tag{4.18}$$

The result then comes from the fact that the mapping $f \in L^2(Q)^n \to w \in H^1(0,T;H) \cap C(0,T;V)$, where $w$ is the solution of $w' - \Delta w = \nabla \pi + f$ in $Q$ and $w(0) = 0$ in $\Omega$ is linear continuous. This finishes the proof of Lemma 4.2. □

In order to complete the proof of Theorem 1.3, we apply Schauder's fixed point theorem. Since $T_M$ is continuous and bounded, $\lambda$ is continuous on $L^2(0,T;H)$. Using Lemma 3.1, when $z$ describes $L^2(0,T;H)$, the controls $\varphi(z)$ are bounded in $L^2(q)$ and as the range of $T_M$ is bounded $L^\infty(\mathbb{R}^n, \mathbb{R}^n)$, one can easily prove that the range of $\lambda$ is bounded in $L^2(0,T;V)$. We still have to prove that this range is relatively compact in $L^2(0,T;H)$. We have already written $y = Y_0 + Y$ which we decompose again in $y = u + w + Y$ where

$$\begin{cases} u' - \Delta u = \nabla \pi^1 \quad \text{in } Q, \\ \operatorname{div} u = 0 \quad \text{in } Q, \\ u = 0 \quad \text{on } \Sigma, \\ u(0) = y^0 \quad \text{in } \Omega. \end{cases} \tag{4.19}$$

The function $w$, which is in $L^2(0,T;V) \cap C([0,T],H)$, satisfies

$$w' - \Delta w = \nabla \pi^2 - (T_M(z).\nabla) Y_0 \quad \text{in} \quad Q$$

and $w(0) = 0$ in $\Omega$. Since the function $(T_M(z).\nabla)Y_0$ are uniformly bounded in $L^2(Q)^n$ with respect to $z$, solutions $w$ stay in a bounded set of $H^1(0,T,H) \cap L^2(0,T,V)$ and thus in a compact set $K_1$ of $L^2(0,T;H)$. The same argument proves that functions $Y$ also leave in a compact set $K_2$ of $L^2(0,T;H)$. Functions $y$ then leave in $u + K_1 + K_2$ which proves that the range of $\lambda$ is relativly compact in $L^2(0,T;H)$. We can now apply Schauder's fix point theorem and this ends the proof of Theorem 1.3.

Remark 4.3. If $\omega$ is any open subset of $\Omega$, one can see that the only missing argument in order to prove Theorem 1.3 is that there exists a pressure $\pi$ in $L^2_{loc}(Q)$ solution of (4.16).

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