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ON $\mathcal{L}_2$ PERFORMANCE INDUCED BY FEEDBACKS WITH MULTIPLE SATURATIONS

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Abstract. Multi-level saturation feedbacks induce nonlinear disturbance-to-state $\mathcal{L}_2$ stability for nonlinear systems in feedforward form. This class of systems includes linear systems with actuator constraints.

Notation • A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{K}_+$ ($\gamma \in \mathcal{K}_+$) if it is continuous and nondecreasing. It is of class-$\mathcal{K}$ if, in addition, it is zero at zero. It is of class-$\mathcal{K}_\infty$ if, moreover, it is strictly increasing and unbounded. For $\gamma \in \mathcal{K}_\infty$, its inverse is another function of class-$\mathcal{K}_\infty$ and is denoted γ^{-1}.

• By abuse of notation we will often write the vector $(x^T, u^T)^T$ as (x, u).

• A measurable signal $v : [0, \infty) \to \mathbb{R}^m$ is said to belong to $\mathcal{L}_2$ or $v \in \mathcal{L}_2$ (respectively, belong to $\mathcal{L}_\infty$ or $v \in \mathcal{L}_\infty$) if the quantity

$$||v||_2 := \sqrt{\int_0^\infty |v(t)|^2 dt} \qquad (\text{resp. } ||v||_\infty := \text{ess. sup}_{t \geq 0}|v(t)|\ )$$

is finite.

• The function sat : $\mathbb{R}^n \to \mathbb{R}^n$ is defined as $\text{sat}(x) = \dfrac{x}{\max\{1, |x|\}}$.

• More generally, a function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is said to be a saturation function if it is differentiable at the origin and there exist $K > 0$, $b > 0$ such that, for all $u, v \in \mathbb{R}^m$,

1. $|\sigma(u + v) - \sigma(u)| \leq \min\{K|v|, b\}$,
2. $|\sigma(u) - u| \leq K u^T \sigma(u)$.

• For a nonlinear control system with state space $x \in \mathbb{R}^n$, an m-level saturation feedback is any feedback of the form

$$u = \sigma(Fx + v) \tag{0.1}$$

where F is a matrix of appropriate dimension, σ is a saturation function and v is an $(m - 1)$-level saturation feedback. A zero-level saturation feedback is the identically zero function.

• For a strictly positive real number a and a vector $v \in \mathbb{R}^m$, where m is an arbitrary strictly positive integer, $\pi(v, a) = v - a\,\text{sat}\left(\dfrac{v}{a}\right)$. Also, $\pi(v, 0) = v$.

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1. Introduction

Recently, it has been shown that multi-level saturation feedback (see the notation section) can be effectively used to stabilize the origin of linear systems subjected to actuator constraints (see [5], [4], [10]). In fact, these feedbacks have been shown to induce the property that, in the presence of additive disturbances converging to a sufficiently small ball, the state converges to a proportionally small ball.

In this paper, we will establish nonlinear disturbance-to-state $\mathcal{L}_2$ stability using multi-level saturation feedback. This duplicates the result of [3] where nonlinear $\mathcal{L}_2$ stability is established using a nonlinear controller that statically schedules a family of linear $\mathcal{H}_\infty$ controllers (c.f. [9]). Our result is a corollary of a nonlinear $\mathcal{L}_2$ performance result for a general class of so-called nonlinear feedforward systems. This class of systems includes, for example, "the ball and beam" (see [6]), the "PVTOL" and "inverted pendulum on a cart" (see [8]), and linear systems with limits on the magnitude and n derivatives of the input (see [7]).

The proof of our main result is obtained with two tools. Initially, we use a Lyapunov argument to establish nonlinear disturbance-to-state $\mathcal{L}_2$ stability for critically stable, stabilizable linear control systems with actuator saturation when a certain passive linear feedback is used. This result draws on the proof techniques used in [2]. A small gain argument is then used to show that the stability is robust to input-driven additive, dynamic perturbations that satisfy certain $\mathcal{L}_2$ stability properties. Ultimately, this robust stabilization result is used iteratively in controller design for nonlinear feedforward systems.

The paper is organized as follows. In section 2 we summarize our main results on $\mathcal{L}_2$ performance while the subsequent sections are dedicated to the proofs. In section 3 we present a robust stabilization result (lemma 3.2) for critically stable, stabilizable linear systems with additive, dynamic perturbations driven by the input. This result is the basis for an inductive proof of our main results (theorems 2.2 and 2.3). Using the proof, we are able to show how the main results extend to cover, for example, results for linear systems with exponentially unstable modes. This is done at the end of section 3. The proof of the robust stabilization result (lemma 3.2) is given in section 4. Its proof uses a small gain argument together with a result (lemma 4.1) concerning nonlinear disturbance-to-state $\mathcal{L}_2$ stabilization by passive linear feedback for critically stable, stabilizable linear systems with actuator saturation. The proof of lemma 4.1 is given in section 5. Finally, we will provide some concluding remarks.

2. Main Results

Nonlinear feedforward systems: The main result of this paper applies to nonlinear feedforward control systems, i.e. systems of the form

$$\begin{aligned}\dot{x}_1 &= A_1x_1 + f_1(x_2,\dots,x_p,u,d)\\ \dot{x}_2 &= A_2x_2 + f_2(x_3,\dots,x_p,u,d)\\ &\vdots\\ \dot{x}_{p-1} &= A_{p-1}x_{p-1} + f_{p-1}(x_p,u,d)\\ \dot{x}_p &= A_px_p + f_p(u,d)\end{aligned} \tag{2.1}$$

where $x_i \in \mathbb{R}^{n_i}$. Define $n = n_1 + \ldots + n_p$ and $X_i = (x_i^T,\dots,x_p^T)^T$. For system (2.1), our standing assumption will be the following.

Assumption 2.1. The f_i are locally Lipschitz and zero at zero, the Jacobian linearization at the origin and with $d \equiv 0$ exists and is stabilizable, the A_i are critically stable, and for each i there exists a class-$\mathcal{K}_+$ function φ_i such that [1]

$$|f_i(X_{i+1},u,d) - f_i(X_{i+1},u,0)| \leq \varphi_i(|(X_{i+1},u)|)|d|\,. \tag{2.2}$$

With this assumption, the class of systems we are considering is slightly less general than that considered in [10]. In particular, the condition (2.2) was not assumed and the x_p subsystem was allowed to be more general in [10]. (See the end of section 3 for a discussion of the case where the x_p subsystem is more general.) Under assumption 2.1, we can show that there exists a p-level saturation feedback that induces nonlinear $\mathcal{L}_2$ stability from d to the state x. For simplicity, we will only consider multiple saturation feedbacks that are nested, as defined in the notation section of this paper. But, more general combinations (see [4] and [10]) could also be used to give the same result. Our result is summarized in the theorem below. The proof will be given in section 3. A control synthesis algorithm is given after the statement of the theorem.

Theorem 2.2. *For the system in (2.1) satisfying assumption 2.1, there exist a p-level saturation feedback $\alpha(\cdot)$ and class-$\mathcal{K}$ functions γ_2°, γ_2^d, γ_∞° and γ_∞^d such that, for each $d \in \mathcal{L}_2$ and each $x_\circ \in \mathbb{R}^n$, the trajectory of the system (2.1) with $u = \alpha(x)$ and $x(0) = x_\circ$ exists for all $t \geq 0$ and satisfies:*

$$\begin{aligned}||x||_2 &\leq \max\{\,\gamma_2^\circ(|x_\circ|)\,,\ \gamma_2^d(||d||_2)\,\}\\ ||x||_\infty &\leq \max\{\,\gamma_\infty^\circ(|x_\circ|)\,,\ \gamma_\infty^d(||d||_2)\,\}\,.\end{aligned} \tag{2.3}$$

This result implies that the origin is globally asymptotically stable when $d \equiv 0$. Indeed, global stability follows from the second inequality. Then, with $x, \dot{x} \in \mathcal{L}_\infty$ and $x \in \mathcal{L}_2$ (from the first inequality) it follows from Barbalat's lemma (see [1, Lemma 4.4]) that x converges to zero.

We now summarize the control synthesis algorithm. Throughout, $i = 1,\dots,p$. The matrices $(\mathcal{A}_i,\mathcal{B}_i)$ will represent the linear approximation of

[1] To make sense out of the case $i = p$, define X_{p+1} to be a vector of dimension zero.

the X_i subsystem with $d \equiv 0$ (where X_i is defined above assumption 2.1), i.e., with $d \equiv 0$ and ignoring higher order terms in X_i and u we have

$$\dot{X}_i = \mathcal{A}_i X_i + \mathcal{B}_i u \ . \tag{2.4}$$

Let $v_i(x) = -\mathcal{B}_i^T P_i X_i$ where P_i is a positive definite, symmetric matrix satisfying

$$\bar{\mathcal{A}}_i^T P_i + P_i \bar{\mathcal{A}}_i \leq 0 \tag{2.5}$$

with

$$\bar{\mathcal{A}}_i = \frac{\partial}{\partial X_i} \left[\mathcal{A}_i X_i + \mathcal{B}_i \sum_{j=i+1}^{p} v_j(x) \right] \ . \tag{2.6}$$

(One first determines P_p, which depends only on $(\mathcal{A}_p, \mathcal{B}_p)$, then v_p, then P_{p-1}, then v_{p-1}, etc.) Next let $w_0(x) \equiv 0$ and

$$w_i(x) = \lambda_i \sigma_i \left(\frac{v_i(x) + w_{i-1}(x)}{\lambda_i} \right) \tag{2.7}$$

where the σ_i are saturation functions. With the parameters $\lambda_i > 0$ adjusted appropriately (guided by the proof of theorem 2.2), the control law $\alpha(x)$ is the p-level saturation feedback $w_p(x)$.

Linear systems with actuator saturation: We now state the corollary of the above theorem for linear control systems of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + w_1 \\ y &= Cx + w_2 \end{aligned} \tag{2.8}$$

where $x \in \mathbb{R}^n$, $w = (w_1^T, w_2^T)^T \in \mathcal{L}_2$. The result is for stabilizable, detectable systems that may be open loop unstable but are not open loop exponentially unstable. Open loop exponentially unstable systems will be discussed at the end of section 3.

THEOREM 2.3. *Consider the system (2.8). If (A, B) is stabilizable, (C, A) is detectable and the eigenvalues of A have nonpositive real part then, with L chosen so that $A + LC$ is Hurwitz, there exists a p-level saturation feedback $\alpha(\cdot)$ ($p \leq n$) and class-$\mathcal{K}$ functions γ_2°, γ_2^e, γ_2^w, γ_∞°, γ_∞^e and γ_∞^w such that, using the dynamic feedback*

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ u &= \alpha(\hat{x}) \ , \end{aligned} \tag{2.9}$$

for each $w \in \mathcal{L}_2$, each $x_\circ \in \mathbb{R}^n$ and each $\hat{x}_\circ \in \mathbb{R}^n$, the trajectory of the system (2.8)-(2.9) with $(x(0), \hat{x}(0)) = (x_\circ, \hat{x}_\circ)$ exists for all $t \geq 0$ and satisfies

$$\begin{aligned} \|x\|_2 &\leq \max\{ \gamma_2^\circ(|x_\circ|) \ , \ \gamma_2^e(|x_\circ - \hat{x}_\circ|) \ , \ \gamma_2^w(\|w\|_2) \} \\ \|x\|_\infty &\leq \max\{ \gamma_\infty^\circ(|x_\circ|) \ , \ \gamma_\infty^e(|x_\circ - \hat{x}_\circ|) \ , \ \gamma_\infty^w(\|w\|_2) \} \ . \end{aligned} \tag{2.10}$$

PROOF. Defining $e = x - \hat{x}$, we get

$$\dot{e} = (A + LC)e + w_1 - Lw_2 \ . \tag{2.11}$$

Since $(A+LC)$ is Hurwitz, there exist strictly positive real numbers β and $\bar{\gamma}$ such that

$$||e||_2 \leq \max\{\beta|e(0)|, \bar{\gamma}||w||_2\} \ . \tag{2.12}$$

Next, since (A, B) is stabilizable and the eigenvalues of A have nonpositive real part, there exists a coordinate transformation $z = Tx$ so that TAT^{-1} is upper triangular with p critically stable matrices on the diagonal for some $p \leq n$. In the z coordinates, the system is in the form of system (2.1) and satisfies assumption 2.1. Picking $u = \alpha(\hat{x})$ where α is an appropriate p-level saturation feedback and using the fact that α is globally Lipschitz, the result follows from theorem 2.2. □

REMARK 2.4. As pointed out in [4], if the result holds for $u = \alpha(\hat{x})$ then the same results holds (qualitatively) for $u = \lambda\alpha(\hat{x}/\lambda)$. This follows by working in the coordinates $\bar{x} = x/\lambda$. The consequence of this observation is that nonlinear $\mathcal{L}_2$ stability can be achieved with a p-level saturation that is arbitrarily small in magnitude. In fact, the general feedforward result could be used to prove a similar result with arbitrarily small bounds on the input magnitude and any number of its derivatives (c.f. [10]).

REMARK 2.5. The stability gain from w to the state must be, in general, super-linear at infinity when using a bounded control. Otherwise, a small gain argument would give $\mathcal{L}_2$ stability for small perturbations including those that moved the open loop poles from the imaginary axis into the open right half plane. This is not possible since even global asymptotic stabilization with bounded controls is not possible in this case.

REMARK 2.6. The result is robust to small (dynamic) uncertainty in how the input affects the dynamics. (Such perturbations cannot move the open loop pole locations.) We have chosen not to state this result here but the results should be transparent after digesting the proof. The analogous (restricted) $\mathcal{L}_\infty$ stability results have been presented in [10].

3. PROVING THEOREM 2.2

The proof of theorem 2.2 is by induction on a robust stabilization result for critically stable linear systems with additive, dynamic disturbances driven by the input. To state this result compactly, we make a preliminary definition. In the definition, the $\mathcal{L}_2$-norm of a disturbance's distance to a ball of a certain radius is related to the $\mathcal{L}_2$-norm of the output's distance to a ball of a related radius.

DEFINITION 3.1. The output y of a dynamical system

$$\begin{array}{rcl} \dot{x} &=& f(x, d_1, d_2) \\ y &=& h(x, d_1, d_2) \end{array} \tag{3.1}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $d_1 \in \mathbb{R}^{p_1}$, $d_2 \in \mathbb{R}^{p_2}$ is said to satisfy the induction hypothesis if there exist strictly positive real numbers δ, M, L and $\bar{\gamma}$ and class-$\mathcal{K}$ functions γ_2°, $\gamma_2^{d_2}$, γ_∞° and $\gamma_\infty^{d_2}$ such that, for all $a \geq 0$ and d_1 satisfying $||d_1||_\infty \leq M$ and $||\pi(d_1, a)||_2 < \infty$, all $d_2 \in \mathcal{L}_2$ and all $x_\circ \in \mathbb{R}^n$, the trajectories of the system (3.1) with $x(0) = x_\circ$ satisfy [2]:

$$\begin{array}{rcl} \|\pi(y,\delta a)\|_2 & \leq & \max\left\{ \gamma_2^\circ(|x_\circ|) \ , \ \bar{\gamma}\|\pi(d_1,a)\|_2 \ , \ \gamma_2^{d_2}(\|d_2\|_2) \right\} \\ \|y\|_\infty & \leq & \max\left\{ \gamma_\infty^\circ(|x_\circ|) \ , L\|d_1\|_\infty \ , \ \gamma_\infty^{d_2}(\|d_2\|_2) \right\} . \end{array} \tag{3.2}$$

The following lemma on robust stabilization will be used to prove theorem 2.2. The result parallels [10, Theorem 3] where an analogous result is stated in terms of $\mathcal{L}_\infty$ properties. The proof of lemma 3.2 is in section 4.

LEMMA 3.2. *Consider the locally Lipschitz control system*

$$\begin{array}{rcl} \dot{x}_1 & = & Ax_1 + Bu + g(x_2,u,d) \\ \dot{x}_2 & = & f(x_2,u,d) \ . \end{array} \tag{3.3}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. Suppose

1. *(A,B) is stabilizable and there exists $P = P^T > 0$ such that $A^TP + PA \leq 0$, i.e., A is critically stable,*
2. *the state x_2 satisfies the induction hypothesis with $d_1 := u$ and $d_2 := d$,*
3. *there exists a function φ of class-$\mathcal{K}_+$ such that*
$$|g(x_2,u,d) - g(x_2,u,0)| \leq \varphi(|(x_2,u)|)|d| \ ,$$
4. $$\lim_{|(x_2,u)|\to 0} \frac{|g(x_2,u,0)|}{|(x_2,u)|} = 0 \ .$$

Let σ be a saturation function. Then there exists a strictly positive real number λ^* such that, with the control

$$u = \lambda\sigma\left(\frac{-B^TPx_1 + v}{\lambda}\right) \tag{3.4}$$

where $\lambda \in (0,\lambda^*]$, the output x_1 for the (x_1,x_2) system satisfies the induction hypothesis with $d_1 := v$ and $d_2 := d$.

PROOF OF THEOREM 2.2.

The proof is by induction. Apply lemma 3.2 to the x_p subsystem (x_p is to be identified with x_1 in the lemma and there is no x_2 subsystem in this case) to see that, with a control of the form (3.4) where $B = B_p = \frac{\partial f_p}{\partial u}|_{d,u=0}$ and $\lambda = \lambda_p$ sufficiently small, the state x_p satisfies the induction hypothesis with $d_1 := v$ and $d_2 := d$. It can be easily shown that $A_p - B_pB_p^TP_p$ is Hurwitz and, with the properties of a saturation function and the fact that A_{p-1} is critically stable, it follows that the linearization of the X_{p-1} subsystem with v as control is stabilizable and open loop critically stable.

We analyze the X_i subsystem, for $i = 1,\ldots,p-1$, by first making a copy of the X_{i+1} subsystem, the state of which we denote by $\tilde{X}_{i+1}$, i.e.,

$$\dot{\tilde{X}}_{i+1} = F_{i+1}(\tilde{X}_{i+1},v,d) \ , \qquad \tilde{X}_{i+1}(0) = X_{i+1}(0) \ . \tag{3.5}$$

Let (A_i,B_i) represent the Jacobian linearization of the X_i subsystem and let the function $g_i(X_{i+1},v,d)$ be given by

$$g_i(X_{i+1},v,d) := F_i(X_{i+1},v,d) - (A_iX_i + B_iv) \ . \tag{3.6}$$

[2]See the notation section for the definition of $\pi(v,a)$.

The fact that g_i is independent of x_i follows from the structure of feedforward systems. Notice that $\lim_{|(X_{i+1},v)|\to 0} \frac{|g_i(X_{i+1},v,0)|}{|(X_{i+1},v)|} = 0$. We can now write the X_i system as

$$\begin{aligned} \dot{X}_i &= A_i X_i + B_i v + g_i(\tilde{X}_{i+1}, v, d) \\ \dot{\tilde{X}}_{i+1} &= F_{i+1}(\tilde{X}_{i+1}, v, d) \ . \end{aligned} \tag{3.7}$$

Since (A_i, B_i) is stabilizable and A_i is a critically stable matrix, we can again apply lemma 3.2 (X_i is associated with x_1, $\tilde{X}_{i+1}$ is associated with x_2 and v is associated with u) to get that, under a control v of the form (3.4) with $\lambda = \lambda_i$ sufficiently small (and with v on the right hand side of (3.4) replaced by w), the state X_i satisfies the induction hypothesis with $d_1 := w$ and $d_2 := d$. When $i \geq 2$, it follows from the structure of the system (2.1) and the properties of σ that the linearization of the X_{i-1} subsystem with w as control is stabilizable and open loop critically stable. So, theorem 2.2 follows by induction. □

Extensions of theorem 2.2.

From the proof of theorem 2.2, we see all that is required for the x_p subsystem is the existence of a control $u = \alpha(x_p, v)$ that is differentiable at the origin, locally exponentially stabilizes the origin of the x_p subsystem when $d \equiv 0$, the linearization is controllable through v, and the state x_p satisfies the induction hypothesis. So, if we rewrite the x_p subsystem in the more general form

$$\dot{x}_p = \tilde{f}_p(x_p, u, d_p) \tag{3.8}$$

and assume the existence of such an $\alpha(x_p, v)$ then we arrive at the conclusion of theorem 2.2. We will see from the proof of lemma 3.2 that if this feedback is such that x_p only satisfies the induction hypothesis for $|x_p(0)|$ sufficiently small and d_p with sufficiently small $\mathcal{L}_2$ norm, those restrictions can be carried through during the iteration to get the conclusion of theorem 2.2 but with restrictions on $|x_p(0)|$ and the $\mathcal{L}_2$-norm of d_p.

A special case where this discussion is relevant is when the x_p subsystem contains the exponentially unstable open loop modes of a linear system and when the actuators of the linear system saturate, i.e.,

$$\dot{x}_p = A_p x_p + B_p \sigma(u) + d_p \tag{3.9}$$

where the eigenvalues of A_p have positive real part and σ is a saturation function. In this case, because σ is bounded, no feedback exists so that x_p satisfies the induction hypothesis for all $x_p(0)$ and all $d_p \in \mathcal{L}_2$. But, x_p does satisfy the hypothesis, at least for $|x_p(0)|$ sufficiently small and d_p with sufficiently small $\mathcal{L}_2$ norm, when $u = Fx_p + v$ where $u = Fx_p$ is locally exponentially stabilizing when $d_p \equiv 0$. So, a local version of theorem 2.3 holds when A has eigenvalues with positive real part.

4. Proving lemma 3.2

The key piece in proving lemma 3.2 is the following result for critically stable, stabilizable linear systems with input saturation and additive disturbances. It shows that a particular passive feedback induces the type of stability described in the induction hypothesis.

LEMMA 4.1. *Let (A, B) be stabilizable, let A be such that there exists $P = P^T > 0$ satisfying $A^T P + PA \leq 0$ and let σ be a saturation function. Then the state of the system*

$$\dot{x} = Ax + B\sigma\left(-B^T Px\right) + d_1 + d_2 \tag{4.1}$$

satisfies the induction hypothesis.

REMARK 4.2. This result for the case where $d_2 \equiv 0$ and $a = 0$ (in the definition of the induction hypothesis) was already reported in [2]. The proof of lemma 4.1, given in section 5, draws on the proof technique used in [2].

We will combine lemma 4.1 with small gain arguments to prove lemma 3.2. Before we do, we need some preliminary facts.

FACT 1. Let $c \geq 0$. If $|w| \leq \max\{|w_1|, |w_2|, |w_3|\}$ then

$$||\pi(w,c)||_2 \leq \sqrt{3} \cdot \max\left\{ ||\pi(w_1,c)||_2 \ , \ ||\pi(w_2,c)||_2 \ , \ ||\pi(w_3,c)||_2 \right\} \ .$$

PROOF. We have

$$\begin{aligned} |\pi(w,c)| &= |w|\left(1 - \frac{c}{\max\{c,|w|\}}\right) \\ &\leq \max\left\{|w_1|,|w_2|,|w_3|\right\}\left(1 - \frac{c}{\max\{c,|w_1|,|w_2|,|w_3|\}}\right) \\ &= \max\left\{|\pi(w_1,c)| \ , \ |\pi(w_2,c)| \ , \ |\pi(w_3,c)|\right\} \ . \end{aligned} \tag{4.2}$$

Thus

$$\begin{aligned} |\pi(w,c)|^2 &\leq \max\left\{|\pi(w_1,c)|^2 \ , \ |\pi(w_2,c)|^2 \ , \ |\pi(w_3,c)|^2\right\} \\ &\leq |\pi(w_1,c)|^2 + |\pi(w_2,c)|^2 + |\pi(w_3,c)|^2 \ . \end{aligned} \tag{4.3}$$

We integrate both sides, use the fact that $a + b + c \leq 3 \cdot \max\{a, b, c\}$ for positive numbers a, b, c and then take the square root on both sides to obtain the result.

□

FACT 2. If $0 < a_1 < a_2$ then $||\pi(w, a_2)||_2 \leq ||\pi(w, a_1)||_2$.

PROOF. We have

$$\begin{aligned} |\pi(w,a_2)| &= |w|\left(1 - \frac{1}{\max\{1,|w|/a_2\}}\right) \\ &\leq |w|\left(1 - \frac{1}{\max\{1,|w|/a_1\}}\right) \\ &= |\pi(w,a_1)| \ . \end{aligned} \tag{4.4}$$

□

FACT 3. Let $c \geq 0$ and $k > 0$. Then $\pi(kw, c) = k\pi(w, c/k)$.

PROOF. We have

$$\pi(kw, c) = kw - c\,\mathrm{sat}\left(\frac{kw}{c}\right) = k\left[w - c/k\,\mathrm{sat}\left(\frac{w}{c/k}\right)\right] = k\pi(w, c/k)\ . \tag{4.5}$$

□

PROOF OF LEMMA 3.2.

Let the given saturation function σ be parameterized by the strictly positive real numbers K_σ and b (see notation section). For the given A, B and σ, let $(\delta_1, M_1, L_1, \bar{\gamma}_1)$ be the set of constants of the induction hypothesis that follows from lemma 4.1. For the given x_2 subsystem, let $(\delta_2, M_2, L_2, \bar{\gamma}_2)$ be the set of constants of the induction hypothesis given by assumption. We will use γ as a generic class-$\mathcal{K}$ function. All norms below should be thought of as norms on truncated signals. The norms on $[0, \infty)$, once this is shown to be the maximal interval of definition, can be obtained from the limiting process as the truncation time goes to infinity.

We write the x_1 subsystem as

$$\dot{x}_1 = Ax_1 + B\lambda\sigma\left(\frac{-B^T P x_1}{\lambda}\right) + d_1 + d_2 \tag{4.6}$$

where

$$\begin{aligned} d_1 &= B\lambda\left[\sigma\left(\frac{-B^TPx_1 + v}{\lambda}\right) - \sigma\left(\frac{-B^TPx_1}{\lambda}\right)\right] \\ &\qquad + g\left(\delta_2\lambda b\,\mathrm{sat}\left(\frac{x_2}{\delta_2\lambda b}\right), u, 0\right) \\ d_2 &= g(x_2, u, d) - g\left(\delta_2\lambda b\,\mathrm{sat}\left(\frac{x_2}{\delta_2\lambda b}\right), u, 0\right)\ . \end{aligned} \tag{4.7}$$

Our goal is to find a suitable $\lambda^* > 0$ so that the lemma holds. This will be achieved by choosing

$$\lambda^* := \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \tag{4.8}$$

where the λ_i are strictly positive real numbers to be specified.

Let $\lambda_1 = M_2/b$. Then $\lambda \in (0, \lambda^*]$ guarantees that, on the maximal interval of definition, $||u||_\infty \leq M_2$. Then, since x_2 satisfies the induction hypothesis and since $d \in \mathcal{L}_2$, $x_2 \in \mathcal{L}_\infty$ on the maximal interval of definition. This, together with the form of the differential equation, implies that the maximal interval of definition is $[0, \infty)$.

Let λ_2 be a strictly positive real number such that, for all $0 < \lambda \leq \lambda_2$,

$$|x_2| \leq \delta_2\lambda b\,, \quad |u| \leq \lambda b \Longrightarrow |g(x_2, u, 0)| < \lambda M_1\ . \tag{4.9}$$

Such a λ_2 exists from the fourth assumption of the lemma. Then $\lambda \in (0, \lambda^*]$ guarantees that there exists $\tilde{M} > 0$ such that $||v||_\infty \leq \tilde{M}$ implies $||d_1||_\infty \leq \lambda M_1$.

Now, with u given in (3.4), there exist a strictly positive real number K, class-$\mathcal{K}_+$ functions φ and φ_2 and a class-$\mathcal{K}$ function ρ such that

$$\begin{aligned} |d_1| &\leq \max\left\{K|v|, \rho(\lambda b)|x_2|, \rho(\lambda b)|x_1|\right\} \\ |d_2| &\leq \varphi\left(|(x_2,u)|\right)|d| + \varphi_2\left(|(x_2,u)|\right)|\pi(x_2,\delta_2\lambda b)| \ . \end{aligned} \tag{4.10}$$

So, using facts 1 and 3,

$$||\pi(d_1,a)||_2 \leq \sqrt{3}\cdot\max\left\{K\left\|\pi\left(v,\frac{a}{K}\right)\right\|_2, \rho(\lambda b)\left\|\pi\left(x_2,\frac{a}{\rho(\lambda b)}\right)\right\|_2, \rho(\lambda b)\left\|\pi\left(x_1,\frac{a}{\rho(\lambda b)}\right)\right\|_2\right\} . \tag{4.11}$$

Since x_2 satisfies the induction hypothesis,

$$\left\|\pi\left(x_2,\frac{a}{\rho(\lambda b)}\right)\right\|_2 \leq \max\left\{\gamma(|x_{2_\circ}|), \bar{\gamma}_2\left\|\pi\left(u,\frac{a}{\rho(\lambda b)\delta_2}\right)\right\|_2, \gamma\left(||d||_2\right)\right\} . \tag{4.12}$$

Then, combining (4.11) and (4.12) and using (3.4), the fact that σ is globally Lipschitz and fact 2, there exist strictly positive real numbers K_1 and K_2 such that

$$||\pi(d_1,a)||_2 \leq \max\left\{\gamma(|x_{2_\circ}|), K_1\left\|\pi\left(v,\frac{a}{K_1}\right)\right\|_2, K_2\rho(\lambda b)\left\|\pi\left(x_1,\frac{a}{K_2\rho(\lambda b)}\right)\right\|_2, \gamma(||d||_2)\right\} . \tag{4.13}$$

Now, using the scalings $x \to x/\lambda$, $d_1 \to d_1/\lambda$, $d_2 \to d_2/\lambda$, and using lemma 4.1 and fact 3 we have, for $||v||_\infty \leq \tilde{M}$,

$$\begin{aligned} ||\pi(x_1,\delta_1 a)||_2 &\leq \max\left\{\lambda\gamma\left(\frac{|x_{1_\circ}|}{\lambda}\right), \bar{\gamma}_1||\pi(d_1,a)||_2, \lambda\gamma\left(\frac{||d_2||_2}{\lambda}\right)\right\} \\ ||x_1||_\infty &\leq \max\left\{\lambda\gamma\left(\frac{|x_{1_\circ}|}{\lambda}\right), L_1||d_1||_\infty, \lambda\gamma\left(\frac{||d_2||_2}{\lambda}\right)\right\} . \end{aligned} \tag{4.14}$$

Let λ_3 satisfy

$$\rho(\lambda_3 b)K_2 < \min\left\{\frac{1}{\bar{\gamma}_1}, \frac{1}{\delta_1}\right\} . \tag{4.15}$$

Then, combining (4.13) and the first inequality of (4.14) and using fact 2, we have for $\lambda \in (0,\lambda^*]$ that

$$||\pi(x_1,\delta_1 a)||_2 \leq \max\left\{\lambda\gamma\left(\frac{|x_{1_\circ}|}{\lambda}\right), \bar{\gamma}_1\gamma(|x_{2_\circ}|), \bar{\gamma}_1 K_1\left\|\pi\left(v,\frac{a}{K_1}\right)\right\|_2, \bar{\gamma}_1\gamma(||d||_2), \lambda\gamma\left(\frac{||d_2||_2}{\lambda}\right)\right\} . \tag{4.16}$$

Using (4.10) and the bound on $||\pi(x_2,\delta_2\lambda b)||_2$ from the induction hypothesis satisfied by x_2 we have

$$||d_2||_2 \leq \varphi\left(||(x_2,u)||_\infty\right)||d||_2 + \varphi_2\left(||(x_2,u)||_\infty\right)\max\left\{\gamma(|x_{2\circ}|), \gamma(||d||_2)\right\} . \tag{4.17}$$

First inserting into (4.17) the bound on $||x_2||_\infty$ from the induction hypothesis and, in turn, the bound $||u||_\infty \leq \lambda b$ and then inserting the resulting bound on $||d_2||_2$ into (4.16), we get the form of bound on $||\pi(x_1, \delta_1 a)||_2$ claimed in the lemma.

For the bound on $||x_1||_\infty$, we repeat the above argument using

$$||d_1||_\infty \leq \max\left\{K||v||_\infty\,,\, \rho(\lambda b)||x_2||_\infty\,,\, \rho(\lambda b)||x_1||_\infty\right\} \tag{4.18}$$

instead of (4.11) and using, from the induction hypothesis on x_2,

$$||x_2||_\infty \leq \max\left\{\gamma(|x_{2_\circ}|)\,,\, L_2||u||_\infty\,,\, \gamma\left(||d||_2\right)\right\} \tag{4.19}$$

instead of (4.12). We then can assert, as we did before (4.13), the existence of strictly positive real numbers K_1 and K_2 such that (compare with (4.13))

$$||d_1||_\infty \leq \max\left\{\gamma(|x_{2_\circ}|)\,,\, K_1||v||_\infty\,,\, K_2\rho(\lambda b)||x_1||_\infty\;\,, \gamma(||d||_2)\right\}\,. \tag{4.20}$$

Using the second of the inequalities in (4.14) and letting λ_4 satisfy

$$\rho(\lambda_4 b)K_2 < \frac{1}{L_1} \tag{4.21}$$

we have that $\lambda \in (0, \lambda^*]$ implies (compare with (4.16))

$$\begin{aligned} ||x_1||_\infty \leq \max\Bigg\{\; &\lambda\gamma\left(\frac{|x_{1_\circ}|}{\lambda}\right)\;,\; L_1\gamma(|x_{2_\circ}|)\,, \\ & L_1K_1||v||_\infty\,,\, L_1\gamma(||d||_\infty)\,,\, \lambda\gamma\left(\frac{||d_2||_2}{\lambda}\right)\Bigg\}\,. \end{aligned} \tag{4.22}$$

Then using the bound on $||d_2||_2$ established above we get the form of the bound on $||x_1||_\infty$ claimed in the lemma.

We conclude that the induction hypothesis is satisfied by $\tilde{M}$, $\tilde{L} = L_1K_1$, $\tilde{\bar{\gamma}} = \bar{\gamma}_1 K_1$ and $\tilde{\delta} = \delta_1 K_1$. □

5. Proof of lemma 4.1

FACT 4. Under the conditions of lemma 4.1, there exists $P_2 = P_2^T > 0$ such that

$$(A - BB^TP)^TP_2 + P_2(A - BB^TP) = -I\;.$$

A preliminary energy function: Consider a function of the form

$$V(x) = \int_0^{x^TPx} \alpha_1(s)ds + \int_0^{x^TP_2x} \alpha_2(s)ds \tag{5.1}$$

where α_1 and α_2 are continuous, nonnegative functions to be specified. When we consider the time derivative of $V(x)$ along the trajectories of (4.1) we find, using $||d_1||_\infty \leq M$, completely squares and using the second inequality in the definition of a saturation function, that there exists strictly positive real numbers c_1, c_2 and $\tilde{K}$ such that

$$\begin{aligned}\dot{V} \leq\ & \alpha_1(x^T Px)\left[2x^T PB\sigma(-B^T Px) + 2x^T P(d_1+d_2)\right] + \\ & \alpha_2(x^T P_2 x)\left[-x^T x + 2x^T P_2 B\left(\sigma(-B^T Px) + B^T Px\right)\right. \\ & \qquad\qquad \left. + \ 2x^T P_2(d_1+d_2)\right] \\ \leq\ & -\alpha_1(x^T Px)\left(-2x^T PB\sigma(-B^T Px)\right) \\ & +2M\lambda_{\max}(P)|x|\alpha_1(x^T Px) + |x|^2\alpha_1^2(x^T Px) + c_1|d_2|^2 \\ & -x^T x\alpha_2(x^T P_2 x) + \tilde{K}|x|\left(-2x^T PB\sigma\left(-B^T Px\right)\right)\alpha_2\left(x^T P_2 x\right) \\ & +\alpha_2\left(x^T P_2 x\right)2\lambda_{max}(P_2)|x||d_1| + \alpha_2^2(x^T P_2 x)|x|^2 + c_2|d_2|^2 \ . \end{aligned} \tag{5.2}$$

If M, α_1 and α_2 can be chosen so that the following three inequalities are satisfied:

$$\tilde{K}|x|\alpha_2(x^T P_2 x) \leq \alpha_1(x^T Px) \tag{5.3}$$

$$2M\lambda_{max}(P)\alpha_1(x^T Px) \leq 0.5|x|\alpha_2(x^T P_2 x) \tag{5.4}$$

$$\alpha_1^2(x^T Px) + \alpha_2^2(x^T P_2 x) \leq 0.25\alpha_2(x^T P_2 x) \tag{5.5}$$

then

$$\dot{V} \leq \alpha_2(x^T P_2 x)\left[-0.25|x|^2 + 2\lambda_{max}(P_2)|x||d_1|\right] + \tilde{c}|d_2|^2 \tag{5.6}$$

where $\tilde{c} = c_1 + c_2$. To satisfy inequalities (5.3)-(5.5), we choose

$$\alpha_2(s) = \frac{1}{4\left(\dfrac{\tilde{K}^2 s}{\mu^3\lambda_{max}(P_2)} + 1\right)} \tag{5.7}$$

where

$$\mu = \frac{\lambda_{min}(P)\lambda_{min}(P_2)}{\lambda_{max}(P)\lambda_{max}(P_2)} \tag{5.8}$$

and we choose

$$\alpha_1(s) = \tilde{K}\sqrt{\frac{s}{\lambda_{min}(P)}}\alpha_2\left(\frac{\lambda_{min}(P_2)s}{\lambda_{max}(P)}\right) \ . \tag{5.9}$$

Notice that if $0 \leq b \leq a$ then $\alpha_2(a) \leq \alpha_2(b)$ but also $\alpha_2(\mu s) \leq \dfrac{1}{\mu}\alpha_2(s)$ (since $\mu \leq 1$). We then have

$$\tilde{K}|x|\alpha_2(x^T P_2 x) \leq \tilde{K}\sqrt{\frac{x^T Px}{\lambda_{min}(P)}}\alpha_2\left(\frac{\lambda_{min}(P_2)}{\lambda_{max}(P)}x^T Px\right) = \alpha_1(x^T Px) \tag{5.10}$$

so that (5.3) is satisfied. Also,

$$\begin{aligned}\alpha_1(x^T Px) \ &\leq\ \tilde{K}\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}|x|\alpha_2(\mu x^T P_2 x) \\ &\leq\ \tilde{K}\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}|x|\frac{1}{\mu}\alpha_2(x^T P_2 x) \ . \end{aligned} \tag{5.11}$$

So (5.4) is satisfied if $M > 0$ is such that

$$2M\lambda_{max}(P)\tilde{K}\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\frac{1}{\mu} \leq 0.5 \ . \tag{5.12}$$

Finally,

$$\begin{aligned} \alpha_1^2(x^TPx) + \alpha_2^2(x^TP_2x) &\leq \left(\tilde{K}^2\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\frac{|x|^2}{\mu^2} + 1\right)\alpha_2^2(x^TP_2x) \\ &\leq \left(\frac{\tilde{K}^2}{\mu^3}\frac{\lambda_{min}(P_2)}{\lambda_{max}(P_2)}|x|^2 + 1\right)\alpha_2^2(x^TP_2x) \\ &\leq \left(\frac{\tilde{K}^2}{\mu^3}\frac{x^TP_2x}{\lambda_{max}(P_2)} + 1\right)\alpha_2^2(x^TP_2x) \\ &\leq 0.25\alpha_2(x^TP_2x) \end{aligned} \tag{5.13}$$

so that (5.5) is satisfied.

<u>Modifying V to get an $\mathcal{L}_\infty$ bound in terms of $\mathcal{L}_2/\mathcal{L}_\infty$ bounds:</u>
Returning to (5.6), note that for any $a > 0$ we can write

$$\dot{V} \leq \alpha_2(x^TP_2x)\left[-0.25|x|^2 + 2\lambda_{max}(P_2)|x|\left(|\pi(d_1,a)| + \left|a\,\mathrm{sat}\left(\frac{d_1}{a}\right)\right|\right)\right] + \tilde{c}|d_2|^2 \ . \tag{5.14}$$

From this and the fact that $||d_1||_\infty \leq M$ it follows, by completing squares, that

$$|x| \geq 16\lambda_{max}(P_2)\min\{a, M\} =: b_0(a) \tag{5.15}$$

implies

$$\dot{V} \leq \alpha_2(x^TP_2x)\left[-\frac{1}{16}|x|^2 + 16\lambda_{max}(P_2)^2\,|\pi(d_1,a)|^2\right] + \tilde{c}|d_2|^2 \ . \tag{5.16}$$

Now we define

$$\alpha(r) = \int_0^r [\alpha_1(s) + \alpha_2(s)]\,ds \tag{5.17}$$

and note that $\alpha \in \mathcal{K}_\infty$ while there exist strictly positive real numbers κ_1, κ_2 such that

$$\alpha\left(\kappa_1|x|^2\right) \leq V(x) \leq \alpha\left(\kappa_2|x|^2\right) \ . \tag{5.18}$$

Define

$$b_1(a) := \alpha\left(\kappa_2 b_0^2(a)\right) \ , \tag{5.19}$$

let ϵ be a strictly positive real number and define

$$b_2(a) := \alpha\left(\kappa_2(1+\epsilon)b_0^2(a)\right) \ . \tag{5.20}$$

Also define

$$\delta := \sqrt{\frac{\kappa_2}{\kappa_1}}16\lambda_{max}(P_2)\sqrt{1+\epsilon} \ . \tag{5.21}$$

Notice that $b_2(a) > b_1(a)$ for $a > 0$ and $V(x) \leq b_2(a)$ implies $|x| \leq \delta a$. Define

$$W(x,a) := \alpha_{3_a}(V(x)) \tag{5.22}$$

where $\alpha_{3_a}(s)$ is a smooth, nondecreasing function such that $\alpha_{3_a}(s) = 0$ when $s \in [0, b_1(a)]$ and $\alpha_{3_a}(s) = s$ when $s \in [b_2(a), \infty)$. Notice that $V(x) - b_2(a) \leq W(x,a) \leq V(x)$ and, since $b_2(a)$ is bounded, the derivative of α_{3_a} can be bounded independent of a. It then follows from (5.16) and considering the two case $|x| \geq \delta a$ and $|x| < \delta a$ that there exist strictly positive real numbers κ_3 and κ_4 such that, for each $a \geq 0$,

$$\dot{W} \leq \alpha_2(x^T P_2 x)\left[-\frac{1}{16}\left|\pi(x,\delta a)\right|^2 + \kappa_3\left|\pi(d_1,a)\right|^2\right] + \kappa_4|d_2|^2 \ . \tag{5.23}$$

We let $a = ||d_1||_\infty$ and we obtain

$$\dot{W} \leq \kappa_4|d_2|^2 \ . \tag{5.24}$$

Integrating both sides we have

$$W(x(T),a) \leq \kappa_4||d_2||_2^2 + W(x(0),a) \ . \tag{5.25}$$

In other words,

$$V(x(T)) \leq \kappa_4||d_2||_2^2 + V(x(0)) + b_2(a) \tag{5.26}$$

or

$$\alpha\left(\kappa_1|x(T)|^2\right) \leq \kappa_4||d_2||_2^2 + \alpha\left(\kappa_2|x(0)|^2\right) + b_2(a) \ . \tag{5.27}$$

Since, on intervals of the form $[0, s_\circ]$ where $s_\circ > 0$, α_1 and α_2 are upper bounded and α_2 is positive and bounded away from zero, there exist strictly positive real numbers ℓ_1 and ℓ_2 such that

$$\alpha^{-1}(s) \leq \ell_1 s \qquad \forall s \leq 3b_2(M) \tag{5.28}$$

and

$$\alpha(s) \leq \ell_2 s \qquad \forall s \leq \kappa_1\delta^2 M^2 \ . \tag{5.29}$$

It follows from manipulating (5.27) that, with $L = \delta\sqrt{3\ell_1\ell_2}$, there exist class-$K$ functions γ_∞° and $\gamma_\infty^{d_2}$ such that

$$\begin{aligned} |x(T)| &\leq \max\left\{\gamma_\infty^\circ(|x(0)|), \gamma_\infty^{d_2}(||d_2||_2), \sqrt{\frac{1}{\kappa_1}\alpha^{-1}\left(3b_2(a)\right)}\right\} \\ &\leq \max\left\{\gamma_\infty^\circ(|x(0)|), \gamma_\infty^{d_2}(||d_2||_2), \sqrt{\frac{\ell_1 3}{\kappa_1}\alpha\left(\kappa_1\delta^2 a^2\right)}\right\} \\ &\leq \max\left\{\gamma_\infty^\circ(|x(0)|), \gamma_\infty^{d_2}(||d_2||_2), \sqrt{\ell_1 3\ell_2\delta^2 a^2}\right\} \\ &\leq \max\left\{\gamma_\infty^\circ(|x(0)|), \gamma_\infty^{d_2}(||d_2||_2), La\right\} \ . \end{aligned} \tag{5.30}$$

The $\mathcal{L}_\infty$ inequality of the induction hypothesis follows from taking the supremum over T on both sides.

Further modifying V to get the nonlinear $\mathcal{L}_2$ bound:

Let $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ be strictly positive real numbers satisfying

$$\alpha\left(\tilde{\kappa}_1 x^T P_2 x\right) \leq V(x) \leq \alpha\left(\tilde{\kappa}_2 x^T P_2 x\right) \ , \tag{5.31}$$

where α was defined in (5.17). Define $\gamma(s) := \dfrac{1}{\alpha_2(s)}$ and note that $\gamma(s)$ is nondecreasing. Also define

$$\rho(s) := \gamma\left(\frac{1}{\tilde{\kappa}_2}\alpha^{-1}(s)\right) . \tag{5.32}$$

Then, since $W(x,a) \leq V(x)$ it follows that

$$\begin{aligned} \rho(W(x,a))\alpha_2(x^T P_2 x) &= \gamma\left(\frac{1}{\tilde{\kappa}_2}\alpha^{-1}(W(x,a))\right)\alpha_2(x^T P_2 x) \\ &\leq \gamma(x^T P_2 x)\alpha_2(x^T P_2 x) = 1 \end{aligned} \tag{5.33}$$

and since $W(x,a) = V(x)$ when $|x| \geq \delta a$ it follows that, when $|x| \geq \delta a$,

$$\begin{aligned} \rho(W(x,a))\alpha_2(x^T P_2 x) &= \gamma\left(\frac{1}{\tilde{\kappa}_2}\alpha^{-1}(V(x))\right)\alpha_2(x^T P_2 x) \\ &\geq \gamma\left(\frac{\tilde{\kappa}_1}{\tilde{\kappa}_2}x^T P_2 x\right)\alpha_2(x^T P_2 x) \\ &\geq \frac{\tilde{\kappa}_1}{\tilde{\kappa}_2}\gamma(x^T P_2 x)\alpha_2(x^T P_2 x) \\ &= \frac{\tilde{\kappa}_1}{\tilde{\kappa}_2} . \end{aligned} \tag{5.34}$$

Defining

$$U(x,a) = \int_0^{W(x,a)} \rho(s)ds \tag{5.35}$$

and using (5.23) we get

$$\dot{U} \leq -\frac{\tilde{\kappa}_1}{\tilde{\kappa}_2 16}|\pi\,(x,\delta a)\,|^2 + \kappa_3|\pi\,(d_1,a)\,|^2 + \rho(V(x))\kappa_4|d_2|^2 . \tag{5.36}$$

Assuming that $\pi(d_1,a) \in \mathcal{L}_2$ and $d_2 \in \mathcal{L}_2$, which from above gives that $V(x) \in \mathcal{L}_\infty$, and integrating both sides we get

$$\frac{\tilde{\kappa}_1}{\tilde{\kappa}_2 16}||\pi\,(x,\delta a)\,||_2^2 \leq \kappa_3||\pi\,(d_1,a)\,||_2^2 + \rho(||V(x)||_\infty)\kappa_4||d_2||_2^2 + U(x(0),a) . \tag{5.37}$$

After using (5.18) and (5.30) to bound $||V(x)||_\infty$ in terms of $|x(0)|$, $||d_2||_2$ and M (the upper bound on $||d_1||_\infty$), and noting that, from the properties of $W(x,a)$, there exists $\alpha_4 \in \mathcal{K}_\infty$ such that $U(x,a) \leq \alpha_4(|x|)$, we get that the $\mathcal{L}_2$ inequality of the induction hypothesis holds with

$$\bar{\gamma} = \sqrt{\frac{48\tilde{\kappa}_2\kappa_3}{\tilde{\kappa}_1}} . \tag{5.38}$$

□

6. Conclusion

In this paper we have constructed a nonlinear disturbance-to-state $\mathcal{L}_2$ gain for systems in nonlinear feedforward form that are controlled using multi-level saturation feedback. When specialized to linear systems with input saturation, the results are qualitatively the same as those reported in [3] where linear systems with saturation are controlled by a static scheduling of a family of linear $\mathcal{H}_\infty$ controllers. It is not yet clear how the $\mathcal{L}_2$ performance of the two schemes compares quantitatively. It would also be interesting to see if any general connections can be made between nonlinear $\mathcal{L}_2$ stability and the type of $\mathcal{L}_\infty$ stability studied in [10].

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