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# Dual pairs, spherical harmonics and a Capelli identity in quantum group theory

*Dedicated to Professor Hisaaki Yoshizawa on his 70th birthday*

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**Abstract.** A quantum analogue of the dual pair $(\mathfrak{sl}_2, \mathfrak{o}_n)$ is constructed and its detailed investigation is worked out. The main results include a quantum version of the theory of spherical harmonics, the Capelli identity, and the first fundamental theorem of invariant theory. Also a description of Casimir elements of the twisted $q$-deformed algebra $\mathsf{U}_q(\mathfrak{o}_n)$ is given in relation to the Reflection Equations.

**Key words:** Quantum group, Capelli identity, dual pair, quantum spherical harmonics, oscillator representation, Casimir element, reflection equation, double commutant theorem, invariant theory.

## Introduction

At the earliest stage of quantum group theory, as an analogue of the Schur-Weyl reciprocity, a dual pair already appeared in the study [J]. Further developement of Howe duality nevertheless seems to have been only implicit in the context of quantum groups since then. In this paper, we present a new step, an example of $q$-deformed dual pair $(\mathfrak{sl}_2, \mathfrak{o}_n)$ and its Capelli Identity.

Attempting once to define a $q$-analogue of the dual pairs, however, we must face first the widely-known fact that the standard process of $q$-deformations of the universal enveloping algebras of Lie algebras is not compatible with inclusion maps. Against this difficulty, we proceed with the following principle: start from the oscillator representation $\omega$; make its $n$-fold tensor power representation $\omega^{\otimes n}$; then see what appears as the commutant of the representation. Applying this strategy to $\mathsf{U}_q(\mathfrak{sl}_2)$, we get as a result in the commutant of $\omega^{\otimes n}$, a representation of a $q$-deformation of $\mathsf{U}(\mathfrak{o}_n)$, which is *different* from the standard one that Drinfeld and Jimbo defined. The $q$-deformed algebra $\mathsf{U}_q(\mathfrak{o}_n)$ appeared here turns out to be what Gavrilik and Klimyk defined in [GK]. This algebra is not a Hopf algebra but plays an important role also in the study of quantized homogeneous spaces (cf. [N4]).

Incidentally for the Howe duality $(\mathfrak{sp}_{2m}, \mathfrak{o}_n)$ in the case of Yangian, similar twisted objects introduced in [O] naturally come into the story [Na2].

There are quite a few various points of view to look at this new dual pair, as this is a quantum counterpart of the spherical harmonics. Our main focus here is on the Capelli identity, which is an explicit identity between two central elements of these two algebras in duality. Through this study, we are naturally led to basic investigation about the irreducible decomposition of the $q$-commutative algebra under $\mathrm{U}_q(\mathfrak{o}_n)$ as well as the description of the central elements of $\mathrm{U}_q(\mathfrak{o}_n)$. Our argument is first along the classical theory, and give yet another invariant-theoretic reasoning as in the spirit of [H1]. We note that we encounter non-classical computations there. Whereas the Capelli identity we treat here is classically the easiest, for example, its quantum counterpart shows unexpected interest and difficulty both in the formulation and proof.

Here is an overview of our paper. The first two sections are to give our formulation and the statement of main results. In Section 1 we introduce the oscillator representation of the quantized $\mathfrak{sl}_2$ and realize its tensor power on the space of $q$-commutative ring. In Section 2 we observe that the $q$-deformed algebra $\mathrm{U}_q(\mathfrak{o}_n)$ of Gavrilik and Klimyk appears in the commutant of $n$-fold tensor power of the oscillator representation. Further we define the Casimir operator for this algebra and state our Capelli identity in this case. The sections that follow are mainly devoted to its proof. First in Section 3, we describe the irreducible decomposition of the $q$-commutative ring under the action of $\mathrm{U}_q(\mathfrak{o}_n)$. This is along a quite analogous way to classical discussion using the zonal spherical polynomials. One key here is a Frobenius reciprocity, which is formulated with the notion of almost homogeneity introduced in [U]. In Section 4, we show that the Casimir element is central. With these preliminaries, we prove the Capelli identity in Section 5. The proof is representation theoretic and is done by the comparison of eigenvalues of the two operators in question. This shows a clear contrast with the proof of another type of the Capelli identity treated in [NUW1]. In Section 6, we look at the situation from the dual pair point of view as in [H1] and discuss the double commutant property. Our guiding principle here is the first fundamental theorem of invariants, and we establish it in a special form under the formulation of algebras with Hopf algebra symmetry. A further discussion on the central elements of $\mathrm{U}_q(\mathfrak{o}_n)$ is given in Section 7. There reflection equations control the commutation relations among the elements of $\mathrm{U}_q(\mathfrak{o}_n)$. In the Appendix, we give a similar-looking identity to the Capelli identity we gave. This itself does not represent as clear meaning as the Capelli identity does. But it is related to another realization of $q$-deformed orthogonal Lie algebra in [N4] so that it should play some role in the analysis of quantum homogeneous spaces. In some sense, this *mock* one, which has a simpler form, may be regarded as a first approximation to the real one.

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## 1. The oscillator representation and its tensor powers

We first recall the definition of $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$. This is an associative algebra over the ground field $\mathbb{K} = \mathbb{Q}(q)$ generated by the elements $e, f, k^{\pm 1}$ with the relations

$$ke = q^4 ek, \qquad kf = q^{-4} fk;$$

$$ef - fe = \frac{k - k^{-1}}{q^2 - q^{-2}} = \frac{1}{[2]} \frac{k - k^{-1}}{q - q^{-1}}.$$

Here [2] stands for the basic number $q + q^{-1}$.

Their comultiplication rule is given by

$$\begin{aligned}
\Delta(e) &= e \otimes 1 + k^{-1} \otimes e, \\
\Delta(f) &= f \otimes k + 1 \otimes f, \\
\Delta(k) &= k \otimes k.
\end{aligned}$$

The oscillator representation of $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$ is realized on the polynomial ring $\mathbb{K}[x]$ of one variable. For this definition we introduce some notations. Let $\gamma$ be the algebra automorphism of $\mathbb{K}[x]$ given by $\gamma: x \mapsto qx$. The $q$-difference operator $\partial = \partial_q$ is then defined as

$$\partial = \partial_q = x^{-1} \frac{\gamma - \gamma^{-1}}{q - q^{-1}}.$$

Using these, we define the following three operators:

$$\omega(e) = \frac{x^2}{[2]}, \qquad \omega(f) = -\frac{\partial^2}{[2]}, \qquad \omega(k) = q\gamma^2.$$

An easy calculation shows that the commutator $[\partial^2, x^2]$ of $\partial^2$ and $x^2$ yields

$$[\partial^2, x^2] = [2] \, \frac{q\gamma^2 - q^{-1}\gamma^{-2}}{q - q^{-1}},$$

so that the $\omega$ above gives us a left $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$-module structure on $\mathbb{K}[x]$. We call this the *oscillator representation*, or more precisely a $q$-oscillator representation. Note that this is not irreducible but breaks into two irreducible components which consist of the polynomials respectively of even and odd degrees.

Since the algebra $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$ has a Hopf algebra structure, we can thereby define the $n$-fold tensor power $\omega^{\otimes n}$ of the representation $\omega$. It is convenient, and even natural from the geometrical point of view, to identify the representation space of $\omega^{\otimes n}$ with the $q$-commutative ring of $n$ variables.

We denote simply by $\mathsf{A} = \mathbb{K}[x_1, \ldots, x_n]$ the $q$-commutative ring with the relations $x_i x_j = q x_j x_i$ for $i < j$. Then using the multiplication in $\mathsf{A}$, we identify naturally

$$\mathbb{K}[x_1] \otimes \cdots \otimes \mathbb{K}[x_n] \xrightarrow{\sim} \mathbb{K}[x_1, \ldots, x_n].$$

Through this identification and the action $\omega$ on $\mathbb{K}[x] \simeq \mathbb{K}[x_i]$, we have an action $\omega^{\otimes n}$ of $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$ on $\mathsf{A}$ by the following comultiplication rule:

$$\begin{cases} \Delta(e) = e \otimes 1 + k^{-1} \otimes e, \\ \Delta(f) = f \otimes k + 1 \otimes f, \\ \Delta(k) = k \otimes k. \end{cases}$$

In order to describe this action, we introduce the notations on $q$-difference operators as follows. Let $\gamma_i$ be the algebra automorphism of $\mathsf{A}$ given by $\gamma_i \colon x_j \mapsto q^{\delta_{ij}} x_j$. We put $\gamma = \gamma_1 \cdots \gamma_n$. The $q$-difference operator $\partial_i = \partial_{i,q}$ is defined as

$$\partial_i = x_i^{\circ -1} \frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}}.$$

Here for an element $a \in \mathsf{A}$, we denote its left or right multiplication operator by $a$ or $a^\circ$ respectively. We put for brevity

$$\begin{cases} Q \ = \ x_1^2 + q^{-1} x_2^2 + \cdots + q^{-n+1} x_n^2, \\ \Delta \ = \ q^{n-1} \partial_1^2 + q^{n-2} \partial_2^2 + \cdots + \partial_n^2. \end{cases}$$

With this notation, we have

$$\begin{aligned} \omega^{\otimes n}(e) \ &= \ \frac{1}{[2]} Q, \qquad \omega^{\otimes n}(f) = -\frac{1}{[2]} \Delta, \\ \omega^{\otimes n}(k) \ &= \ q^n \gamma_1^2 \cdots \gamma_n^2 = q^n \gamma^2. \end{aligned}$$

Let us introduce here a notation: $\{a\} = (a - a^{-1})/(q - q^{-1})$ and $[\alpha] = \{q^\alpha\} = (q^\alpha - q^{-\alpha})/(q - q^{-1})$. From the defining relations above, for a positive integer $s$, we can easily derive the following commutation formula:

$$[e^s, f] = \frac{[2s]}{[2]^2} e^{s-1} \{q^{2(s-1)} k\}.$$

In fact, by the derivation rule, we have

$$\begin{aligned}
[e^s, f] &= [e,f]e^{s-1} + e[e,f]e^{s-2} + \cdots + e^{s-1}[e,f] \\
&= \frac{1}{[2]}(\{k\}e^{s-1} + e\{k\}e^{s-2} + \cdots + e^{s-1}\{k\}) \\
&= \frac{1}{[2]}e^{s-1}(\{k\} + \{q^4k\} + \cdots + \{q^{4(s-1)}k\}) \\
&= \frac{1}{[2]}e^{s-1}\left(\frac{q^{4s}-1}{q^4-1}k - \frac{q^{-4s}-1}{q^{-4}-1}k^{-1}\right) \\
&= \frac{1}{[2]}e^{s-1}\frac{q^{2s}-q^{-2s}}{q^2-q^{-2}}(q^{2(s-1)}k - q^{-2(s-1)}k^{-1}).
\end{aligned}$$

Applying this to the representation $\omega^{\otimes n}$, we obtain

LEMMA 1.1 ($b$-function)

$$\Delta Q^s - Q^s\Delta = Q^{s-1}[2s]\{q^{2s-2+n}\gamma^2\}, \tag{1}$$

$$\Delta(Q^s) = Q^{s-1}[2s][2s-2+n]. \tag{2}$$

*Remark.* We have actually several choices for the realization of the tensor power $\omega^{\otimes n}$, depending both on the comultiplication rule of $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$ and on the way of identification of the algebra $\mathsf{A}$ with the representation space of $\omega^{\otimes n}$. These alternatives give us essentially the same objects but a slight modification in the form of operators.

## 2. A $q$-deformed enveloping algebra and the Capelli identity

In this section we state our main results. For their proofs, we indiate the exact places to look in the sections that follow.

On the space $\mathsf{A}$ of the $q$-commutative ring, we have another action $\rho$ of the quantized enveloping algebra $\mathsf{U}_q(\mathfrak{gl}_n)$ from the left. Let $L^\pm = (L^\pm_{ij})_{1\leqslant i,j\leqslant n}$ ($L^\pm_{ij} \in \mathsf{U}_q(\mathfrak{gl}_n)$) denote the $L$-operators corresponding to the constant $R$-matrices $R^\pm$ acting on the tensor product of two vector representations of $\mathsf{U}_q(\mathfrak{gl}_n)$. Explicitly they are given by

$$R^\pm = \sum_{i,j=1}^n q^{\pm\delta_{ij}} e_{ii}\otimes e_{jj} \pm (q-q^{-1})\sum_{i\lessgtr j} e_{ij}\otimes e_{ji}.$$

Here $e_{ij}$ is the matrix units with restpect to the standard basis of $n$-dimensional vector space. For more detailed definition of $U_q(\mathfrak{gl}_n)$ and $L$-operators, see [NUW1]. In terms of $L$-operators, the action $\rho$ is written as

$$\begin{aligned}
\rho(L_{ii}^{+}) &= \rho((L_{ii}^{-})^{-1}) = \gamma_i, \quad (i = 1, \dots, n), \\
\rho(L_{ji}^{+}) &= (q - q^{-1}) x_i^{\circ} \partial_j, \quad (j < i), \\
\rho(L_{ji}^{-}) &= -(q - q^{-1}) x_i^{\circ} \partial_j, \quad (i < j).
\end{aligned}$$

Note that under this action, the multiplication $\mathsf{A} \otimes \mathsf{A} \to \mathsf{A}$ of $\mathsf{A}$ is a $U_q(\mathfrak{gl}_n)$-homomorphism.

We now set $\Theta_j$ in $U_q(\mathfrak{gl}_n)$ for $j = 1, \dots, n-1$ as

$$\Theta_j = (q - q^{-1})^{-1} q^{-\varepsilon_j} (L_{j\,j+1}^{+} - qS(L_{j+1\,j}^{-})),$$

where $S$ is the antipode and $q^{\varepsilon_j} = L_{jj}^{+}$. From the definition of $S$, we see

$$S(L_{j+1j}^{-}) = -q^{-1+\varepsilon_j+\varepsilon_{j+1}} L_{j+1\,j}^{-}, \tag{2.1}$$

$$S(L_{jj+1}^{+}) = -q^{1-\varepsilon_j-\varepsilon_{j+1}} L_{j\,j+1}^{+}, \tag{2.2}$$

so that we have another expression of $\Theta_j$ as

$$\Theta_j = (q - q^{-1})^{-1} (q^{-\varepsilon_j} L_{j\,j+1}^{+} + q^{\varepsilon_{j+1}} L_{j+1\,j}^{-}). \tag{2.3}$$

These elements represent the generators of the $q$-deformed algebra $U_q(\mathfrak{o}_n)$ which is discussed in [GK] and [N4]. To be precise, we define here $U_q(\mathfrak{o}_n)$ as the associative algebra with $n-1$ generators $\Pi_j$ ($j = 1, \dots, n-1$) subject to the relations

$$\begin{cases} [\Pi_i, \Pi_j] = 0 \quad \text{if} \quad |i - j| > 1, \\ \Pi_i^2 \Pi_j - (q + q^{-1}) \Pi_i \Pi_j \Pi_i + \Pi_j \Pi_i^2 = -\Pi_j \quad \text{if} \quad |i - j| = 1. \end{cases} \tag{2.4}$$

For the proof that $\Theta_j$'s satisfy these relations, see Theorem 7.4.

If we let them act on $\mathsf{A}$ through $\rho$ and put $\theta_j = \rho(\Theta_j)$, then they take the form

$$\begin{aligned}
\theta_j &= x_{j+1}^{\circ} \gamma_j^{-1} \partial_j - x_j^{\circ} \gamma_{j+1} \partial_{j+1} \\
&= (x_{j+1} \partial_j - x_j \partial_{j+1}) \gamma_1 \cdots \gamma_{j-1} \gamma_{j+2}^{-1} \cdots \gamma_n^{-1}.
\end{aligned}$$

This is seen from the definition and the formula (2.3) above. We will give a direct proof in Proposition 4.3.1 that these $\theta_j$'s satisfy the relations (2.4).

The point is that they commute with the action $\omega^{\otimes n}$ of $U_{q^2}(\mathfrak{sl}_2)$.

PROPOSITION. *The operators $\theta_j$'s commute with the action of* $U_{q^2}(\mathfrak{sl}_2)$.

*Proof.* See Section 3.1, Proposition 3.1.6.

Furthermore in a suitable big subalgebra of $q$-difference operators, $\mathsf{U}_q(\mathfrak{o}_n)$ forms essentially the commutant of the action $\omega^{\otimes n}$ of $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$, and vice versa (see Section 6, Theorems 6.1 and 6.3 for details). With this fact we may well regard the pair $(\mathsf{U}_{q^2}(\mathfrak{sl}_2), \mathsf{U}_q(\mathfrak{o}_n))$ as a quantum analogue of the dual pair $(\mathfrak{sl}_2, \mathfrak{o}_n)$.

To define the Casimir element of $\mathsf{U}_q(\mathfrak{o}_n)$, we make appropriate elements corresponding not only to the generators but also to the whole Lie algebra $\mathfrak{o}_n$. Let us define the elements $\Pi_{ji}^{\pm}$ for $1 \leqslant i < j \leqslant n$ inductively as

$$\begin{cases} \Pi_{i+1\,i}^{\pm} = \Pi_i, \\ \Pi_{ji}^{\pm} = \Pi_{jk}^{\pm}\Pi_{ki}^{\pm} - q^{\pm 1}\Pi_{ki}^{\pm}\Pi_{jk}^{\pm}, \quad (i < k < j). \end{cases}$$

The choice of $k$ does not affect on the definition as long as $i < k < j$, hence well-defined (see Proposition 4.2). When we consider the acion of $\mathsf{U}_q(\mathfrak{o}_n)$ on the space $\mathsf{A}$, we denote by $\theta_{ji}^{\pm}$ the corresponding operator for $\Pi_{ji}^{\pm}$. Though their explicit form is a bit complicated, we can write them down (cf. Proposition 4.3.1 (2)):

$$\begin{aligned} \theta_{ji}^{+} &= (x_j\partial_i\gamma_{i+1}^{-1}\cdots\gamma_{j-1}^{-1} - q^{j-i-1}x_i\partial_j\gamma_{i+1}\cdots\gamma_{j-1}) \\ &\quad \times\gamma_1\cdots\gamma_{i-1}\gamma_{j+1}^{-1}\cdots\gamma_n^{-1} + (q-q^{-1})\sum_{i<k<j} q^{j-k}x_k^2\partial_i\partial_j \\ &\quad \times(\gamma_1\cdots\gamma_{i-1})^2\gamma_i\gamma_j^{-1}(\gamma_{j+1}^{-1}\cdots\gamma_n^{-1})^2, \\ \theta_{ji}^{-} &= (x_j\partial_i\gamma_{i+1}\cdots\gamma_{j-1} - q^{i-j+1}x_i\partial_j\gamma_{i+1}^{-1}\cdots\gamma_{j-1}^{-1}) \\ &\quad \times\gamma_1\cdots\gamma_{i-1}\gamma_{j+1}^{-1}\cdots\gamma_n^{-1} - (q-q^{-1})\sum_{i<k<j} q^{i-k}x_ix_j\partial_k^2 \\ &\quad \times(\gamma_1\cdots\gamma_{i-1})^2\gamma_i\gamma_j^{-1}(\gamma_{j+1}^{-1}\cdots\gamma_n^{-1})^2. \end{aligned}$$

From this we can see that $\theta_{ji}^{\pm}$ is no more 'first order' but is a 'second order' difference operator if $|i-j| > 1$. With these elements, the (second order) Casimir element $\mathcal{C}$ of $\mathsf{U}_q(\mathfrak{o}_n)$ and its representation $\mathcal{C}_{\mathsf{A}}$ are defined as

$$\begin{aligned} \mathcal{C} &= \sum_{i<j} q^{n-i-j+1}\Pi_{ji}^{-}\Pi_{ji}^{+}, \\ \mathcal{C}_{\mathsf{A}} &= \textstyle\sum_{i<j} q^{n-i-j+1}\theta_{ji}^{-}\theta_{ji}^{+} = \sum_{i<j} q^{-n+i+j-1}\theta_{ji}^{+}\theta_{ji}^{-}. \end{aligned}$$

THEOREM. *The element $\mathcal{C}$ is in the center of* $\mathsf{U}_q(\mathfrak{o}_n)$.

The proof is given in Theorem 4.2 (see also Section 7). Further details about the Casimir element together with $\Pi_{ji}^{\pm}$'s and their representation $\theta_{ji}^{\pm}$'s will be discussed in Section 4 and Section 7.

Our Capelli identity is now in order (see Section 5 for its proof).

THEOREM (Capelli Identity). *The following equality holds:*

$$Q\,\Delta - \{\gamma\}\{q^{n-2}\gamma\} = \mathcal{C}_{\mathsf{A}}.$$

*Here we used the notation* $\{a\} = (a - a^{-1})/(q - q^{-1})$.

Note that the left-hand side of the identity is essentially the Casimir element of $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$ modulo Euler operators. In other words, this identity expresses explicitly the coincidence of the central elements of the two algebras $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$ and $\mathsf{U}_q(\mathfrak{o}_n)$ which are in duality. This looks quite analogous in its form to the classical case (see e.g. [HU], [Wy, p. 292]). Note, however, in checking it by a straightforward calculation based on the explicit formulas above, which can be carried out in principle, we come across quite a few tricky cancellations. The proof we give in this paper is instead based on representation theory, so that we need to develop it for clarity.

The proofs of the statements above will be given in the following four sections.

## 3. Spherical harmonics under the $q$-deformed algebra $\mathsf{U}_q(\mathfrak{o}_n)$

In this section, we describe the $\mathsf{U}_q(\mathfrak{o}_n)$-module structure of the $q$-commutative algebra $\mathsf{A} = \mathbb{K}[x_1, \ldots, x_n]$. This is exactly an analogy of the theory of spherical harmonics. Later we will give some double commutant discussion, which gives us a slightly different reasoning from what is going to be done this section. The way we proceed here is along a quite similar discussion to the classical case.

Let us introduce the space of harmonics:

$$\mathsf{H} = \{\varphi \in \mathsf{A};\ \Delta(\varphi) = 0\}.$$

This breaks up into graded pieces according to the grading of $\mathsf{A}$:

$$\mathsf{H} = \bigoplus_{m \geqslant 0} \mathsf{H}_m; \quad \mathsf{H}_m = \mathsf{H} \cap \mathsf{A}_m,$$

$$\mathsf{A}_m = \{\varphi \in \mathsf{A};\ q^{\varepsilon}\,.\varphi = q^m \varphi\}.$$

The following three facts are our first goal, which is an analogy with the elementary part of theory of spherical harmonics:

THEOREM (Analogue of spherical harmonics).

(1) *The fixed point algebra* $\mathsf{A}^{\mathsf{U}_q(\mathfrak{o}_n)}$ *is generated by the element* $Q$ *if* $n \geqslant 2$.
(2) *The algebra* $\mathsf{A}$ *is decomposed into the tensor product of the harmonics* $\mathsf{H}$ *and the fixed point algebra* $\mathsf{A}^{\mathsf{U}_q(\mathfrak{o}_n)}$*:*

$$\mathsf{A} \underset{\text{multiplication}}{\xleftarrow{\sim}} \mathsf{A}^{\mathsf{U}_q(\mathfrak{o}_n)} \otimes \mathsf{H},$$

*or more precisely*

$$\mathsf{A}_m = \bigoplus_{0 \leqslant 2j \leqslant m} Q^j \mathsf{H}_{m-2j}.$$

(3) *The space* $\mathsf{H}_m$ *of harmonics of homogeneous degree* $m$ *is irreducible under the action of* $\mathsf{U}_q(\mathfrak{o}_n)$ *for* $n \geqslant 3$. *Furthermore* $\mathsf{H}_m$*'s are mutually distinct.*

*Remark.* For the case $n = 1$, since the algebra $\mathsf{U}_q(\mathfrak{o}_n)$ is trivial, the assertion (1) of the Theorem is not true. In this case, the harmonics are just of the form $ax + b$, and other two assertions are not literally true.

For $n = 2$, the harmonics $\mathsf{H}_m$ are two dimensional except for $m = 0$, and break into two irreducibles when we extend our based field $\mathbb{K}$ by adjoining $\sqrt{-1}$. Explicitly we have for $m \geqslant 1$,

$$\mathsf{H}_m = \mathbb{K}\, z^{[m]} \oplus \mathbb{K}\, \bar{z}^{[m]},$$

where $z^{[m]} = z_0 z_1 \cdots z_{m-1}$, $\bar{z}^{[m]} = \bar{z}_0 \bar{z}_1 \cdots \bar{z}_{m-1}$ with $z_r = x_1 + \sqrt{-1} q^r x_2$ and $\bar{z}_r = x_1 - \sqrt{-1} q^r x_2$ (cf. Proposition 5.2.1).

These exceptions are parallel to the classical case, where the difference from the cases $n \geqslant 3$ disappears when we consider the orthogonal group $O_n$ instead of its Lie algebra.

### 3.1. Analogue of Fischer inner product

To pursue a further analogy with the classical theory, we introduce here an inner product on $\mathsf{A} = \mathbb{K}[x_1, \ldots, x_n]$. For this purpose, we define another set of difference operators $\partial_i^\circ$ defined by

$$\partial_i^\circ = x_i^{-1} \frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}}.$$

Its difference from the operator $\partial_i$ lies in the direction from which division by $x_i$ is made: $\partial_i^\circ$ is from the left whereas $\partial_i$ is from the right. Explicitly it makes $\partial_i^\circ = \partial_i \omega_i^{-1}$, because $x_i = x_i^\circ \omega_i$ with $\omega_i = \gamma_1^{-1} \cdots \gamma_{i-1}^{-1} \gamma_{i+1} \cdots \gamma_n$. It should be noted also that $\partial_i^\circ$'s are $q$-commutative while $\partial_i$'s are $q^{-1}$-commutative: $\partial_i^\circ \partial_j^\circ = q \partial_j^\circ \partial_i^\circ$ for $i < j$. We therefore have a natural ring homomorphism over $\mathbb{K}$ as $x_i \mapsto \partial_i^\circ$. Let us denote this homomorphism by $\varphi \mapsto \varphi(\partial^\circ)$ for $\varphi \in \mathsf{A}$. Now we define an analogue of Fischer inner product by

$$(\varphi|\psi) = \varphi(\partial^\circ)\psi|_{x=0},$$

where $|_{x=0}$ stands for the linear form that takes out the constant term. Note that it is a well-defined ring homomorphism from $\mathsf{A}$ to $\mathbb{K}$.

It is not hard to see that the monomials form an orthogonal basis. To be explicit, we can compute the inner product as

$$(x^\alpha|x^\alpha) = [\alpha]!\, q^{\Sigma_{i<j}\alpha_i\alpha_j}.$$

Here we used an abbreviated notation $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $[\alpha]! = [\alpha_1]!\cdots[\alpha_n]!$; $[a]! = [a][a-1]\cdots[1]$. From this calculation, the bilinear form is seen to be symmetric. Furthermore, since the function field $\mathbb{K} = \mathbb{Q}(q)$ is formally real, or by an argument of specialization of $q$, we can prove that the inner product is anisotropic. When we extend our base field as $\mathbb{K} = \mathbb{C}(q)$, we make the bilinear form into hermitian form with respect to the complex conjugation.

With respect to this inner product, we consider the adjoint $a^\dagger$ for an operator $a \in \mathrm{End}_{\mathbb{K}}(\mathsf{A})$ defined by $(a^\dagger\varphi|\psi) = (\varphi|a\psi)$. It is clear that $(ab)^\dagger = b^\dagger a^\dagger$. Since the inner product is symmetric (or hermitian), the adjoint is involutory: $(a^\dagger)^\dagger = a$.

LEMMA 3.1.1. *We have the following formulas* for the adjoint.

$$\gamma_i^\dagger = \gamma_i; \qquad x_i^{\circ\dagger} = \partial_i^\circ, \quad \partial_i^{\circ\dagger} = x_i^\circ; \qquad x_i^\dagger = \partial_i, \quad \partial_i^\dagger = x_i.$$

*In particular,*

$$Q\dagger = q^{-n+1}\Delta, \qquad \Delta^\dagger = q^{n-1}Q,$$

$$\theta_j^\dagger = -\theta_j, \qquad \theta_{ji}^{\pm\dagger} = -q^{\pm(j-i-1)}\theta_{ji}^{\mp}.$$

*Proof.* The first three formulas are obvious from the definition. The next two follow from them and the relations $\partial_i^\circ = \partial_i\omega_i^{-1}$ and $x_i = x_i^\circ\omega_i$ with $\omega_i = \gamma_1^{-1}\cdots\gamma_{i-1}^{-1}\gamma_{i+1}\cdots\gamma_n$. The assertions for $Q$ and $\Delta$ are clear from these. The formula for $\theta_j$ is also easy to see, because we have an expression $\theta_j = (x_{j+1}\partial_j - x_j\partial_{j+1})\gamma_1\cdots\gamma_{i-1}\gamma_{i+2}^{-1}\cdots\gamma_n^{-1}$. The last assertion is shown by induction from the recursive definition of the $\theta_{ji}^{\pm}$'s. The first step is just $\theta_j^\dagger = -\theta_j$. Taking the adjoint of the recursive formula, we have for $i < k < j$

$$\theta_{ji}^{\pm\dagger} = \theta_{ki}^{\pm\dagger}\theta_{jk}^{\pm\dagger} - q^{\pm1}\theta_{jk}^{\pm\dagger}\theta_{ki}^{\pm\dagger} = -q^{\pm1}(\theta_{jk}^{\pm\dagger}\theta_{ki}^{\pm\dagger} - q^{\mp1}\theta_{ki}^{\pm\dagger}\theta_{jk}^{\pm\dagger}).$$

Plugging the induction assumption $\theta_{jk}^{\pm\dagger} = -q^{\pm(j-k-1)}\theta_{jk}^{\pm}$ and $\theta_{ki}^{\pm\dagger} = -q^{\pm(k-i-1)}\theta_{ki}^{\pm}$ into this formula, we see

$$\theta_{ji}^{\pm\dagger} = -q^{\pm(j-i-1)}(\theta_{jk}^{\pm}\theta_{ki}^{\pm} - q^{\mp1}\theta_{ki}^{\pm}\theta_{jk}^{\pm}) = -q^{\pm(j-i-1)}\theta_{ji}^{\mp},$$

as desired.

From this lemma, we obtain the following two important conclusions:

PROPOSITION 3.1.2. *The representation of* $\mathsf{U}_q(\mathfrak{o}_n)$ *on* $\mathsf{A} = \mathbb{K}[x_1, \cdots, x_n]$ *is completely reducible.*

*Proof.* Since the generators $\theta_j$'s of $\mathsf{U}_q(\mathfrak{o}_n)$ are skew-symmetric with respect to the inner product, the orthogonal complement of a $\mathsf{U}_q(\mathfrak{o}_n)$-submodule is also a $\mathsf{U}_q(\mathfrak{o}_n)$-submodule.

PROPOSITION 3.1.3. *The space* $\mathsf{A}_m$ *of homogenesous polynomials of degree* $m$ *breaks into the direct sum* $\mathsf{A}_m = \mathsf{H}_m \bigoplus Q\mathsf{A}_{m-2}$. *Furthermore the direct sum is orthogonal direct sum with respect to the inner product* $(\cdot|\cdot)$.

*Proof.* Recall $\mathsf{H}_m = \mathrm{Ker}(\Delta\colon \mathsf{A}_m \to \mathsf{A}_{m-2})$. Then the assertion follows from $\Delta^\dagger = q^{n-1}Q$. In fact, from this we see easily $\mathsf{H}_m = \mathrm{Im}(Q\colon \mathsf{A}_{m-2} \to \mathsf{A}_m)^\perp$. Since our inner product is anisotropic, this shows $\mathsf{A}_m = \mathsf{H}_m \bigoplus Q\mathsf{A}_{m-2}$.

COROLLARY 3.1.4. *The dimension of the space* $\mathsf{H}_m$ *of harmonics is the same as in the classical case:*

$$\dim \mathsf{H}_m = \dim \mathsf{A}_m - \dim \mathsf{A}_{m-2} = \frac{(n+2m-2)\,(n+m-3)!}{(n-2)!\,m!}.$$

*This number is strictly increasing in* $m$, *if* $n \geqslant 3$.

*Remarks 3.1.5.* (1) We have a more general inner product for the finite dimensional representations of $\mathsf{U}_q(\mathfrak{gl}_n)$ (cf. [N4]). Using this fact, the complete reducibility can be seen to hold in the following form: if $V$ is a representation of $\mathsf{U}_q(\mathfrak{o}_n)$ liftable up to $\mathsf{U}_q(\mathfrak{gl}_n)$, then $V$ is completely reducible.

(2) By a successive use of Proposition 3.1.3, we get the second assertion of Theorem.

(3) The Proposition 3.1.3 can be proved actually without using the inner product. We have only to prove $\mathrm{Im}\, Q \cap \mathrm{Ker}\, \Delta = \{0\}$. In fact, this implies $\Delta\colon \mathsf{A}_m \to \mathsf{A}_{m-2}$ is injective on $Q\mathsf{A}_{m-2} \simeq \mathsf{A}_{m-2}$, hence surjective. In other words, $Q\mathsf{A}_{m-2}$ gives a section for $\Delta\colon \mathsf{A}_m \to \mathsf{A}_{m-2}$, so that $\mathsf{A}_m$ is a direct sum of $Q\mathsf{A}_{m-2}$ and the kernel $\mathsf{H}_m$ of the Laplacian.

Suppose now $\varphi \in Q\mathsf{A}_{m-2} \cap \mathsf{H}_m$ and take the maximal integer $s$ such that $\varphi = Q^s\psi$ with some $\psi \neq 0$. Using the identity (1) in Lemma 1.1, we see

$$0 = \Delta(\varphi) = \Delta(Q^s\psi) = Q^s \cdot \Delta(\psi) + Q^{s-1}[2s][2s-2+n+2(m-2s)]\psi.$$

This shows, as $[2s][2s-2+n+2(m-2s)]$ is non-zero, $\psi$ can be divided by $Q$ once again, which contradicts the maximality of the integer $s$.

We complete here the proof of the following

PROPOSITION 3.1.6. *The operators* $\theta_j$*'s commute with the action of* $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$*:*

$$[\theta_j, Q] = 0, \qquad [\theta_j, \Delta] = 0, \quad \text{and} \quad [\theta_j, \gamma] = 0.$$

*In particular,* $\theta_j \,.\, Q^s = 0$ *for any non-negative integer* $s$.

*Proof.* The last assertion is obvious. To see the first one, we note for $\varphi \in \mathsf{A}$, from the comultiplication rule for $\theta_j$,

$$\theta_j \,.\, (Q\varphi) = (\theta_j \,.\, Q)(\gamma_j^{-1}\gamma_{j+1}\varphi) + Q \cdot (\theta_j \,.\, \varphi)$$

or as operators $[\theta_j, Q] = (\theta_j \,.\, Q)\gamma_j^{-1}\gamma_{j+1}$. We have thus only to show $\theta_j \,.\, Q = 0$ for the first assertion. This is easily checked by a simple calculation. In fact, it reduces to the case for $n = 2$ and essentially to $(x_2\partial_1 - x_1\partial_2) \,.\, (x_1^2 + q^{-1}x_2^2) = [2](x_2x_1 - q^{-1}x_1x_2) = 0$. The second assertion comes from the first, because $\Delta$ and $Q$ are mutual adjoint up to constant multiple.

*Remark.* We have another reasoning of the fact that the second assertion follows from the first one and vice versa. The proof is clearly reduced to the case for $n = 2$, where the Capelli identity is easy to prove, as we will see in Section 5. The Capelli identity then shows the product of $Q$ and $\Delta$ commutes with $\theta$.

### 3.2. Invariant Polynomials and Almost Homogeneity

Let us consider a little bit bigger algebra $\mathsf{U}_q(\mathfrak{go}_n)$, which is, by definition, gotten by adding (central) elements $q^{\pm\varepsilon}$ to $\mathsf{U}_q(\mathfrak{o}_n)$. The point is the action of $\mathsf{U}_q(\mathfrak{go}_n)$ is *almost homogeneous* in the sense of [U]. In fact, denoting by $\chi_n$ the algebra homomorphism from $\mathsf{A}$ to $\mathbb{K}$ given by the 'evaluation' at the point $(0, \ldots, 0, 1)$, we can show that $\chi_n(a\varphi) = 0$ for all $a \in \mathsf{U}_q(\mathfrak{go}_n)$ implies $\varphi = 0$ (see Proposition 11 in [U]). This fact will be essentially used later for the proof of assertion (3) of the Theorem, i.e., the irreducibility of the harmonics $\mathsf{H}_m$. Also we see this to be of use for the first assertion of the Theorem.

PROPOSITION 3.2. *The fixed point subalgebras are determined as follows.*

(1) *For* $n \geqslant 2$, $\mathsf{A}^{\mathsf{U}_q(\mathfrak{o}_n)} = \mathbb{K}[Q]$.
(2) *For* $n \geqslant 3$, $\mathsf{A}^{\mathsf{U}_q(\mathfrak{o}_{n-1})} = \mathbb{K}[Q, x_n]$. *Here* $\mathsf{U}_q(\mathfrak{o}_{n-1})$ *is the subalgebra of* $\mathsf{U}_q(\mathfrak{o}_n)$ *generated by* $\theta_1, \cdots, \theta_{n-2}$.

*Proof.* For (1), take a homogeneous $\varphi \in \mathsf{A}_m^{\mathsf{U}_q(\mathfrak{o}_n)}$. Then $\varphi$ is a relative invariant for $\mathsf{U}_q(\mathfrak{go}_n)$. By the almost homogeneity of the action of $\mathsf{U}_q(\mathfrak{go}_n)$, such a $\varphi$ is uniquely determined from $m$ up to constant multiple, i.e., $\mathsf{A}_m^{\mathsf{U}_q(\mathfrak{o}_n)}$ is at most one dimensional. When $m$ is even, by the Proposition 3.1.6 above, the power $Q^{m/2}$ certainly gives a non-zoro element in $\mathsf{A}_m^{\mathsf{U}_q(\mathfrak{o}_n)}$.

What remains to show is now $\mathsf{A}_m^{\mathsf{U}_q(\mathfrak{o}_n)} = \{0\}$ for $m$ odd. Consider algebra automorphisms given by $r_i\colon x_j \mapsto (-1)^{\delta_{ij}}x_j$. These $r_i$'s normalize $\mathsf{U}_q(\mathfrak{o}_n)$. In fact, we have $\theta_j r_i = -r_i\theta_j$ for $j = i, i+1$ and $\theta_j r_i = r_i\theta_j$ for $j \neq i, i+1$. It follows from this, $r_i\varphi$ is also fixed by $\mathsf{U}_q(\mathfrak{o}_n)$ for any $i$, so that by the uniqueness of $\mathsf{A}_m^{\mathsf{U}_q(\mathfrak{o}_n)}$ our $\varphi$ is a joint eigenvector for $r_i$'s. Since $r_i$ is involutory, the eigenvalue of $r_i$ is

$\pm 1$. Note then the product $r_1 \cdots r_n$ changes $\varphi$ into $-\varphi$, as $m$ is odd, whence there exists an $r_i$ such that $r_i\varphi = -\varphi$. This means that $\varphi$ contains only odd degree terms with respect to $x_i$. We may thus assume the form of $\varphi$ as

$$\varphi = \sum_{0\leqslant j\leqslant \ell} \varphi_j x_i^{2j+1},$$

where $m = 2\ell + 1$ and $\varphi_j$ does not contain $x_i$. Compute $\theta_i\varphi$ (if $i = n$, use $\theta_{i-1}$ instead, and make suitable changes):

$$\begin{aligned}\theta_i\varphi &= \sum_{0\leqslant j\leqslant \ell} ([2j+1]\varphi_j x_{i+1} x_i^{2j} - q^{-2j-1}(\gamma_{i+1}\partial_{i+1}\varphi_j)x_i^{2j+2}) \\ &= -q^{-2\ell-1}(\gamma_{i+1}\partial_{i+1}\varphi_\ell)x_i^{2\ell+2} \\ &\quad + \sum_{1\leqslant j\leqslant \ell} ([2j+1]\varphi_j x_{i+1} - q^{-2j+1}\gamma_{i+1}\partial_{i+1}\varphi_j)x_i^{2j} + \varphi_0 x_{i+1}.\end{aligned}$$

We see now from this $\varphi_0 = 0$ and $[2j+1]\varphi_j x_{i+1} - q^{-2j+1}\gamma_{i+1}\partial_{i+1}\varphi_{j-1} = 0$ for $j = 1, \dots, \ell$, because $\theta_i\varphi = 0$. Thus all the coefficients $\varphi_j$ must be zero. Hence the first assertion.

For (2), note $\mathbb{K}[x_1, \dots, x_n] \simeq \mathbb{K}[x_1, \dots, x_{n-1}] \otimes \mathbb{K}[x_n]$ as $\mathsf{U}_q(\mathfrak{o}_n)$-modules and the action on $\mathbb{K}[x_n]$ is trivial. From the result (1), for $n \geqslant 3$ we have $\mathsf{A}^{\mathsf{U}_q(\mathfrak{o}_{n-1})} \simeq \mathbb{K}[Q_{n-1}] \otimes \mathbb{K}[x_n]$, where $Q_{n-1} = x_1^2 + q^{-1}x_2^2 + \cdots + q^{-n+2}x_{n-1}^2$. Recalling $Q = Q_{n-1} + q^{-n+1}x_n^2$, we come to the conclusion.

*Remark.* More general discussions are given in Theorem 3.1 in [N4] for the irreducible representations of $\mathsf{U}_q(\mathfrak{gl}_n)$ with fixed vectors by certain $q$-deformations of $\mathfrak{o}_n$ or $\mathfrak{sp}_n$. Later in Section 6.1, we will use a special case of this theorem.

### 3.3. Zonal Polynomials and Irreduciblity of Harmonics

For the assertion (3) of our Theorem, we prove the following two propositions.

PROPOSITION 3.3.1. *For every non-zero finite dimensional* $\mathsf{U}_q(\mathfrak{o}_n)$*-submodule* $\mathsf{Z}$ *of* $\mathsf{A}$, *its fixed point space* $\mathsf{Z}^{\mathsf{U}_q(\mathfrak{o}_{n-1})}$ *under* $\mathsf{U}_q(\mathfrak{o}_{n-1})$ *is non-zero.*

PROPOSITION 3.3.2. *Assume* $n \geqslant 3$. *Then the space* $\mathsf{H}_m^{\mathsf{U}_q(\mathfrak{o}_{n-1})}$ *of zonal polynomials of homogeneous degree* $m$ *is one dimensional.*

The assertion clearly follows from those two together with the complete reducibility of the $\mathsf{U}_q(\mathfrak{o}_n)$-module $\mathsf{A}$. The fact that $\mathsf{H}_m$'s are mutually distinct comes from the comparison of the dimensions (see Corollary 3.1.4). For Proposion 3.3.1, we make use of the almost homogeneity. Though the discussion below is essentially the same as Proposition 6 in [U], we will give it for completeness.

Using the inner product above, we can embed A into its dual A′, on which $U_q(\mathfrak{gl}_n)$ acts by contragredient action. Pulling this back, we have another action $\nu$ of $U_q(\mathfrak{gl}_n)$ on A as: $\langle \nu(a)\varphi, \psi\rangle = (\varphi|S(a).\psi)$, where $\varphi, \psi \in A$ and $a \in U_q(\mathfrak{gl}_n)$, and $S$ is the antipode. Roughly speaking, what we get is $\nu(a) = S(a)^\dagger$. To distinguish these two acions, we denote the new module by $A^\nu$. Recall the comultiplication rule $\Delta(\theta_j) = \theta_j \otimes q^{\varepsilon_j - \varepsilon_{j+1}} + 1 \otimes \theta_j$, whence the antipode is $S(\theta_j) = -\theta_j q^{-\varepsilon_j + \varepsilon_{j+1}}$. From these, the explicit action of $\theta_j$ on the tensor product $A \otimes A^\nu$ is given by

$$\theta_j(\varphi \otimes \psi) = \theta_j\varphi \otimes \gamma_j^{-1}\gamma_{j+1}\psi + \varphi \otimes \gamma_j^{-1}\gamma_{j+1}\theta_j\psi,$$

for $\varphi, \psi \in A$.

LEMMA 3.3.3. *Let* Z *be a finite dimensional* $U_q(\mathfrak{o}_n)$*-submodule of* A. *Take any basis* $e_\alpha$ *and its dual basis* $e^\alpha$, *i.e.,* $(e_\alpha|e^\beta) = \delta_\alpha^\beta$. *Then* $\mathcal{I}_Z = \sum_\alpha e_\alpha \otimes e^\alpha \in A \otimes A^\nu$ *is invariant under* $U_q(\mathfrak{go}_n)$.

*Proof.* It might seem obvious from the definition of contragredient. However, since $U_q(\mathfrak{go}_n)$ is not a Hopf algebra, we give a proof to make sure. It is clear for the action of $q^\varepsilon$, because $q^\varepsilon \mathcal{I}_Z = \sum_\alpha \gamma e_\alpha \otimes \gamma^{-1} e^\alpha$. For $\theta_j$, we have by definition

$$\theta_j \mathcal{I}_Z = \sum_\alpha \theta_j e_\alpha \otimes \gamma_j^{-1}\gamma_{j+1} e^\alpha + \sum_\beta e_\beta \otimes \gamma_j^{-1}\gamma_{j+1}\theta_j e^\beta.$$

Note that we have no reason to expect $\gamma_j^{-1}\gamma_{j+1}e^\alpha$ to sit inside Z. However, for fixed $j$, $\gamma_j^{-1}\gamma_{j+1}\theta_j e^\beta$ is in $\gamma_j^{-1}\gamma_{j+1}Z$, so that it can be written by the basis $\gamma_j^{-1}\gamma_{j+1}e^\alpha$ as

$$\begin{aligned}
\gamma_j^{-1}\gamma_{j+1}\theta_j e^\beta &= \sum_\alpha (\gamma_j\gamma_{j+1}^{-1}e_\alpha|\gamma_j^{-1}\gamma_{j+1}\theta_j e^\beta)\gamma_j^{-1}\gamma_{j+1}e^\alpha \\
&= \sum_\alpha (e_\alpha|\theta_j e^\beta)\gamma_j^{-1}\gamma_{j+1}e^\alpha \\
&= -\sum_\alpha (\theta_j e_\alpha|e^\beta)\gamma_j^{-1}\gamma_{j+1}e^\alpha.
\end{aligned}$$

On the other hand, we have $\theta_j e_\alpha = \sum_\beta (\theta_j e_\alpha|e^\beta)e_\beta$. Plugging these into the summations, we get respectively $\sum_{\alpha,\beta} (\theta_j e_\alpha|e^\beta)e_\beta \otimes \gamma_j^{-1}\gamma_{j+1}e^\alpha$ for the first summation and its negative for the second, hence it vanishes in total.

Now consider a linear map $r: A \otimes A^\nu \to A$ defined by $r(\varphi \otimes \psi) = \chi_n(\psi)\varphi$, where $\chi_n$ is the evaluation map $\chi_n: A \to \mathbb{K}$ at the point $(0, \dots, 0, 1)$. For the Proposition 3.3.1, it suffiecs to show the following two:

(A) The image $r(\mathcal{I}_Z)$ is in $Z^{U_q(\mathfrak{o}_{n-1})}$.
(B) The map r is injective on the space $(A \otimes A^\nu)^{U_q(\mathfrak{go}_n)}$ of fixed points.

The first one is easy to see. Note first from the definition $\chi_n(\gamma_j^{-1}\gamma_{j+1}\varphi) = \chi_n(\varphi)$ and $\chi_n(\theta_j\varphi) = 0$ for $j+1 < n$. In fact, $\theta_j\varphi$ contains positive powers of $x_j$ or $x_{j+1}$ in this case. Then as we saw above $\theta_j\mathcal{I}_{\mathsf{Z}} = 0$, so that for $j \leqslant n-2$

$$\begin{aligned} 0 = \mathsf{r}(\theta_j\mathcal{I}_{\mathsf{Z}}) &= \sum_\alpha \chi_n(\gamma_j^{-1}\gamma_{j+1}e^\alpha)\theta_j e_\alpha + \sum_\alpha \chi_n(\gamma_j^{-1}\gamma_{j+1}\theta_j e^\alpha)e_\alpha \\ &= \sum_\alpha \chi_n(e^\alpha)\theta_j e_\alpha = \theta_j \mathsf{r}(\mathcal{I}_{\mathsf{Z}}). \end{aligned}$$

Thus $\mathsf{r}(\mathcal{I}_{\mathsf{Z}})$ is fixed by $\mathsf{U}_q(\mathfrak{o}_{n-1})$. By definition, $\mathsf{r}(\mathcal{I}_{\mathsf{Z}}) = \sum_\alpha \chi_n(e^\alpha)e_\alpha$ is clearly in $\mathsf{Z}$.

For the second assertion, we employ Theorem 5 in [U]. Note the assumption there that $\mathsf{U}$ is a left coideal of a Hopf algebra is just to make sense for the tensor product. In our case both $\mathsf{A}$ and $\mathsf{A}^\nu$ are actually $\mathsf{U}_q(\mathfrak{gl}_n)$-modules, so that the proof of Theorem 5 is applicable here without any change. One thing to be cleared is that $\mathsf{A}^\nu$ is also almost homogeneous under $\mathsf{U}_q(\mathfrak{go}_n)$. Its direct proof is quite parallel to Proposition 11 in [U].

We now determine the space $\mathsf{H}_m^{\mathsf{U}_q(\mathfrak{o}_{n-1})}$ of zonal polynomials and show that it is one dimensional for $n \geqslant 3$. Let us take $\varphi \in \mathsf{H}_m^{\mathsf{U}_q(\mathfrak{o}_{n-1})}$. By Proposition 3.2, it is of the form

$$\varphi = \sum_{0 \leqslant 2j \leqslant m} c_j\, Q_{n-1}^j x_n^{m-2j},$$

where $c_j \in \mathbb{K}$ and $Q_{n-1} = x_1^2 + q^{-1}x_2^2 + \cdots + q^{-n+2}x_{n-1}^2$. The conditions posed on $c_j$'s are from the equation $\Delta(\varphi) = 0$. For this computation, put $\Delta_{n-1} = q^{n-2}\partial_1^2 + q^{n-3}\partial_2^2 + \cdots + \partial_{n-1}^2$, i.e., $\Delta = q\Delta_{n-1} + \partial_n^2$. Let us compute $\Delta(Q_{n-1}^s x_n^t)$ for non-negative integers $s, t$:

$$\begin{aligned} \Delta(Q_{n-1}^s x_n^t) &= (q\Delta_{n-1} + \partial_n^2)(Q_{n-1}^s x_n^t) \\ &= q^{2t+1}\Delta_{n-1}(Q_{n-1}^s)\cdot x_n^t + Q_{n-1}^s \cdot \partial_n^2(x_n^t) \\ &= q^{2t+1}[2s][2s-3+n]Q_{n-1}^{s-1}x_n^t + [t][t-1]Q_{n-1}^s x_n^{t-2}. \end{aligned}$$

Here we used the formula for $b$-function (see Lemma 1.1(2)). From this computation, comparing the coefficient of $Q_{n-1}^{j-1}x_n^{m-2j}$ in the equation $\Delta(\varphi) = 0$, we get the relations between the adjacent coefficients:

$$[m-2j+2][m-2j+1]c_{j-1} + q^{2(m-2j)+1}[2j][2j-3+n]c_j = 0.$$

Since $q^{2(m-2j)+1}[2j][2j-3+n]$ does not vanish for $j \geqslant 1$, as we have assumed $n \geqslant 3$, all the coefficients $c_j$'s are completely determined from $c_0$ by these relations. Thus $\mathsf{H}_m^{\mathsf{U}_q(\mathfrak{o}_{n-1})}$ is one dimensional.

*Remark 3.3.4.* To be more explicit, the coefficients are determined as

$$c_j = (-)^j q^{j(2j-2m+1)} \frac{\beta_1(\frac{m}{2} - j + 1) \cdots \beta_1(\frac{m}{2})}{\beta_{n-1}(j) \cdots \beta_{n-1}(1)} c_0,$$

where $\beta_\ell(s) = [2s][2s + \ell - 2]$ with $\ell = 1, n-1$. Thus with the notation $(a;q)_j = (1-a)(1-qa)\cdots(1-q^{j-1}a)$, we have the zonal polynomial $\varphi_m$ normalized as $c_0 = 1$

$$\varphi_m = \sum_{0 \leqslant 2j \leqslant m} (-)^j \frac{(q^{-2m};q^4)_j\,(q^{-2m+2};q^4)_j}{(q^4;q^4)_j\,(q^{2n-2};q^4)_j} q^{j(2j-2m+1)} q^{j(2m+n)} Q_{n-1}^j x_n^{m-2j}.$$

Using the relation $Q_{n-1}x_n = q^2 x_n Q_{n-1}$, and putting $u = q^{n-1}Qx_n^{-2}$, we can rewrite this as

$$\begin{aligned}
\varphi_m &= \sum_{0 \leqslant 2j \leqslant m} (-)^j \frac{(q^{-2m};q^4)_j\,(q^{-2m+2};q^4)_j}{(q^4;q^4)_j\,(q^{2n-2};q^4)_j} q^{j(n+3)} (Q_{n-1}x_n^{-2})^j \cdot x_n^m \\
&= \sum_{0 \leqslant 2j \leqslant m} \frac{(q^{-2m};q^4)_j\,(q^{-2m+2};q^4)_j}{(q^4;q^4)_j\,(q^{2n-2};q^4)_j} q^{4j}(1-u)^j \cdot x_n^m.
\end{aligned}$$

If we write this zonal spherical polynomial $\varphi_m$ in terms of $Q$ and $x_n$ as

$$\varphi = \sum_{0 \leqslant 2j \leqslant m} C_j\, Q^j x_n^{m-2j},$$

then the relations between coefficents are seen from Lemma 1.1(2) as

$$[m-2j+2][m-2j+1]C_{j-1} + [2j][n+2m-2j-2]C_j = 0.$$

From this we have another expression of $\varphi_m$ as

$$\varphi_m = \sum_{0 \leqslant 2j \leqslant m} \frac{(q^{-2m};q^4)_j\,(q^{-2m+2};q^4)_j}{(q^4;q^4)_j\,(q^{-2n-4m+8};q^4)_j} q^{-j(n-5)} Q^j x_n^{m-2j}.$$

## 4. The Casimir element of $\mathrm{U}_q(\mathfrak{o}_n)$

### 4.1. Preliminary Calculations

In this section, we complete the proofs of the facts on the Casimir element of $\mathrm{U}_q(\mathfrak{o}_n)$ stated in Section 2. We start with some formulas on commutators. For $a \in \mathbb{K}$, define the '$a$-commutator' by $[x,y]_a = xy - ayx$. As usual, we omit the subscript in the notation of the commutator bracket for $a = 1$.

LEMMA 4.1.1. *For $a, b, c \in \mathbb{K}$, we have*

(1) $[x, yz]_{ab} = [x, y]_a z + az[x, z]_b$,

(2) $[x, [y, z]_c]_{ab} = [[x, y]_a, z]_{bc} + a[y, [x, z]_b]_{a^{-1}c}$,

(3) $[[x, y]_a, z]_{bc} = [x, [y, z]_c]_{ab} + c[[x, z]_b, y]_{c^{-1}a}$.

*Proof.* The first formula is just an easy calculation. For the second, replace $y$ and $z$ together with $a$ and $b$ in the first one. Then subtract it multiplied by $c$ from the first. The third one is the same as the second, because $a[y, [x, z]_b]_{a^{-1}c} = c[[x, z]_b, y]_{c^{-1}a}$.

In the first stage, we proceed in a little bit abstract way.

PROPOSITION 4.1.2. *Assume we are given a set of elements $A_j$ $(j = 1, \dots, n-1)$ satisfying $[A_i, A_j] = 0$ for $|i-j| > 1$. Define $A_{ji}$ for $n \geqslant j > i \geqslant 1$ inductively by*

$$A_{i+1\,i} = A_i, \qquad A_{ji} = A_{j\,j-1}A_{j-1\,i} - qA_{j-1\,i}A_{j\,j-1}, \quad (|j-i| \geqslant 2).$$

*Then*

(1) *If $j > k > \ell > i$, then the elements $A_{jk}$ and $A_{\ell i}$ commute with each other.*
(2) *For $j > k > i$, we have $A_{ji} = [A_{jk}, A_{ki}]_q$. In other words, in the inductive definition of $A_{ji}$, we are allowed to take any $k$ in between $j$ and $i$.*

*Proof.* We prove the assertion (1) by induction on $h = (j-k) + (\ell - i)$. The first step $h = 2$ is nothing but our assumption. For $h > 2$, by changing the letter if necessary, we may assume $\ell - i \geqslant 2$ and $k > \ell$ or $i > j$ without loss of generality. From the formula (2) in the Lemma 4.1.1 above, we see

$$\begin{aligned} [A_{jk}, A_{\ell i}] &= [A_{jk}, [A_{\ell\,\ell-1}, A_{\ell-1\,i}]_q] \\ &= [[A_{jk}, A_{\ell\,\ell-1}], A_{\ell-1\,i}]_q + [A_{\ell\,\ell-1}, [A_{jk}, A_{\ell-1\,i}]]_q. \end{aligned}$$

Then by the induction assumption, the inner commutator brackets vanish in both terms.

For (2), suppose $j > k \geqslant \ell > i$. Then it suffices to show $[A_{jk}, A_{ki}]_q = [A_{j\ell}, A_{\ell i}]_q$ by induction on $j - i$. The first step $j - i = 2$ is trivial, because it implies $k = \ell$. For $j - i > 2$, we may assume $A_{ki} = [A_{k\ell}, A_{\ell i}]_q$ and $A_{j\ell} = [A_{jk}, A_{k\ell}]_q$. Plugging the first one in the left-hand side then using the second, we see

$$\begin{aligned} [A_{jk}, A_{ki}]_q &= [A_{jk}, [A_{k\ell}, A_{\ell i}]_q]_q \\ &= [[A_{jk}, A_{k\ell}]_q, A_{\ell i}]_q + q[A_{k\ell}, [A_{jk}, A_{\ell i}]] \\ &= [A_{j\ell}, A_{\ell i}]_q + q[A_{k\ell}, [A_{jk}, A_{\ell i}]]. \end{aligned}$$

Here we used the formula with $a = c = q, b = 1$. The second term vanishes by the assertion (1) because $k > \ell$. This completes the proof.

Up to here, what we used is only the first defining relations of $\mathrm{U}_q(\mathfrak{o}_n)$. The second relations (Serre relations) yield the Proposition 4.1.3 below. To prove that the Casimir element is central, however, we need only (1) and (2) in the Proposition.

PROPOSITION 4.1.3. *We keep to the assumptions and notation on $A_j$ and $A_{ji}$ in the previous Proposition. We assume in addition for $|i-j|=1$,*

$$A_i^2 A_j - (q+q^{-1})A_i A_j A_i + A_j A_i^2 = -A_j.$$

*Then*

(1) *For $j>k+1, k>i$, we have the following.*

$$[A_{k+1\,k}, A_{k+1\,i}]_{q^{-1}} = -A_{ki}, \qquad [A_{jk}, A_{k+1\,k}]_{q^{-1}} = -A_{j\,k+1}.$$

(2) *Assume $q^2 \neq -1$ hereafter. If $j>k+1, k>i$, then the elements $A_{ji}$ and $A_{k+1\,k}$ commute with each other.*

(3) *For $j>k>i$, we have in general*

$$[A_{jk}, A_{ji}]_{q^{-1}} = -q^{j-k-1}A_{ki}, \qquad [A_{ji}, A_{ki}]_{q^{-1}} = -q^{k-i-1}A_{jk}.$$

(4) *Furthermore for $j>k>\ell>i$, the elements $A_{ji}$ and $A_{k\ell}$ commute with each other.*

*Proof.* Note that our assumption above can be written as

$$[A_i, [A_i, A_j]_q]_{q^{-1}} = [[A_j, A_i]_q, A_i]_{q^{-1}} = -A_j$$

for $j=i\pm 1$. These formulas give us the first step of the induction for (1). Taking an $\ell$ with $k>\ell>i$, we have

$$\begin{aligned}[A_{k+1\,k}, A_{k+1\,i}]_{q^{-1}} &= [A_{k+1\,k}, [A_{k+1\,\ell}, A_{\ell i}]_q]_{q^{-1}} \\ &= [[A_{k+1\,k}, A_{k+1\,\ell}]_{q^{-1}}, A_{\ell i}]_q \\ &\quad + q^{-1}[A_{k+1\,\ell}, [A_{k+1\,k}, A_{\ell i}]]_{q^2}.\end{aligned}$$

Here we have put $a=q^{-1}, b=1, c=q$ in Lemma 4.1.1(2). Use then the induction assumption $[A_{k+1\,k}, A_{k+1\,\ell}]_{q^{-1}} = -A_{k\ell}$ in the first term and observe the second term vanish by the previous Proposition 4.1.2(1). Then we come to the conclusion for the first formula of (1). The proof of the other formula is similar.

For (2), assume $j>k+1, k>i$. Putting $a=b=q^{-1}, c=q$ in Lemma 4.1.1 (2), we have

$$\begin{aligned}
[A_{k+1\,k}, A_{ji}]_{q^{-2}} &= [A_{k+1\,k}, [A_{j\,k+1}, A_{k+1\,i}]_q]_{q^{-2}} \\
&= [[A_{k+1\,k}, A_{j\,k+1}]_{q^{-1}}, A_{k+1\,i}] \\
&\quad +q^{-1}[A_{j\,k+1}, [A_{k+1\,k}, A_{k+1\,i}]_{q^{-1}}]_{q^2} \\
&= -q^{-1}[[A_{j\,k+1}, A_{k+1\,k}]_q, A_{k+1\,i}] - q^{-1}[A_{j\,k+1}, A_{ki}]_{q^2} \\
&= [A_{j\,k}, A_{ki}] - q^{-1}[A_{j\,k+1}, A_{ki}]_{q^2}.
\end{aligned}$$

Here we used the first formula of (1) for the second term. For the first term, we continue by applying the Lemma 4.1.1(3) with $a = q^{-1}, b = c = q$ as

$$\begin{aligned}
[A_{j\,k}, A_{ki}] &= [[A_{jk}, A_{k+1\,k}]_{q^{-1}}, A_{ki}]_{q^2} + q^{-1}[A_{k+1\,k}, [A_{jk}, A_{ki}]_q]_{q^2} \\
&= -[A_{j\,k+1}, A_{ki}]_{q^2} - q^{-1}[A_{k+1\,k}, A_{ji}]_{q^2}.
\end{aligned}$$

Here we used the second formula of (1) for the first term. With these two calculations, we obtain

$$\begin{aligned}
[A_{k+1\,k}, A_{ji}]_{q^{-2}} &= q^{-1}([A_{jk+1}, A_{ki}]_{q^2} \\
&\quad +q^{-1}[A_{k+1\,k}, A_{ji}]_{q^2}) - q^{-1}[A_{j\,k+1}, A_{ki}]_{q^2} \\
&= -q^{-2}[A_{k+1\,k}, A_{ji}]_{q^2} = [A_{ji}, A_{k+1\,k}]_{q^{-2}}.
\end{aligned}$$

This means $(1 + q^{-2})(A_{k+1\,k}A_{ji} - A_{ji}A_{k+1\,k}) = 0$. Hence the assertion (2).

We prove the first formula of (3) by induction on $j - k$. The case $j - k = 1$ is just (1). Assume $j - k > 1$ and $[A_{j\,k+1}, A_{ji}]_{q^{-1}} = -q^{j-k-2}A_{ki}$. Using the formula (3) in the Lemma with $a = q, b = q^{-1}, c = 1$, we see from (1) and (2),

$$\begin{aligned}
[A_{jk}, A_{ji}]_{q^{-1}} &= [[A_{j\,k+1}, A_{k+1\,k}]_q, A_{ji}]_{q^{-1}} \\
&= [A_{j\,k+1}, [A_{k+1\,k}, A_{ji}]_q] + [[A_{j\,k+1}, A_{ji}]_{q^{-1}}, A_{k+1\,k}]_q \\
&= -q^{j-k-2}[A_{k+1\,i}, A_{k+1\,k}]_q \\
&= -q^{j-k-1}[A_{k+1\,k}, A_{k+1\,i}]_{q^{-1}} = -q^{j-k-1}A_{ki}
\end{aligned}$$

as desired. The other formula can be similarly shown.

Now that the formulas in (3) are proved, the assertion (4) can be derived in a quite parallel way as (2). To avoid the repetion, we give a slightly different-looking proof instead. For $j > k > \ell > i$, we compute the commutator $[A_{j\ell}, A_{ki}]$ in two ways. Putting $a = q^{-1}, b = c = q$, or $a = b = q, c = q^{-1}$ in the formula (2) in

Lemma 4.1.1, we get respectively

$$\begin{aligned}
[A_{j\ell}, A_{ki}] &= [A_{j\ell}, [A_{k\ell}, A_{\ell i}]_q] \\
&= [[A_{j\ell}, A_{k\ell}]_{q^{-1}}, A_{\ell i}]_{q^2} + q^{-1}[A_{k\ell}, [A_{j\ell}, A_{\ell i}]_q]_{q^2} \\
&= -q^{k-l-1}[A_{jk}, A_{\ell i}]_{q^2} + q^{-1}[A_{k\ell}, A_{ji}]_{q^2}
\end{aligned}$$

and

$$\begin{aligned}
[A_{j\ell}, A_{ki}] &= [[A_{jk}, A_{k\ell}]_q, A_{\ell i}] \\
&= [A_{jk}, [A_{k\ell}, A_{\ell i}]_{q^{-1}}]_{q^2} + q^{-1}[[A_{jk}, A_{ki}]_q, A_{k\ell}]_{q^2} \\
&= -q^{k-l-1}[A_{jk}, A_{\ell i}]_{q^2} + q^{-1}[A_{ji}, A_{k\ell}]_{q^2}.
\end{aligned}$$

We have thus $[A_{k\ell}, A_{ji}]_{q^2} = [A_{ji}, A_{k\ell}]_{q^2}$, or $(1+q^2)[A_{k\ell}, A_{ji}] = 0$, the conclusion.

### 4.2. Centrality of the Casimir element

With these preliminaries, we prove the Casimir element is central in $\mathrm{U}_q(\mathfrak{o}_n)$. Note that in the above, replacing $q$ with $q^{-1}$ if necessary, we already have the commutation relations for $\Pi^{\pm}_{ji}$ in our hands.

THEOREM 4.2. *In the algebra $\mathrm{U}_q(\mathfrak{o}_n)$, the following two elements are central.*

$$\mathcal{C} = \sum_{i<j} q^{n-i-j+1}\Pi^{-}_{ji}\Pi^{+}_{ji}, \qquad \mathcal{C}' = \sum_{i<j} q^{-n+i+j-1}\Pi^{+}_{ji}\Pi^{-}_{ji}.$$

Here $\Pi_j$'s are the generators and $\Pi^{\pm}_{ji}$ for $1 \leqslant i < j \leqslant n$ are inductively defined by

$$\begin{cases} \Pi^{\pm}_{i+1\,i} = \Pi_i, \\ \Pi^{\pm}_{ji} = \Pi^{\pm}_{jk}\Pi^{\pm}_{ki} - q^{\pm 1}\Pi^{\pm}_{ki}\Pi^{\pm}_{jk}, \quad (i < k < j). \end{cases}$$

*Proof.* We show $\Pi_k = \Pi^{+}_{k+1\,k} = \Pi^{-}_{k+1\,k}$ commutes with $\mathcal{C}$ for $1 \leqslant k < n$. From the Propositions, the element $\Pi^{\pm}_{ji}$ commute with $\Pi_k$ if its indices satisfy either one of the conditions $i > k+1$ or $k > j$ or $j > k+1, k > i$. The commutator $[\Pi_k, \mathcal{C}]$ thereby reduces to

$$\begin{aligned}
[\Pi_k, \mathcal{C}] = &\sum_{k>i} q^{n-i-k+1}[\Pi_k, \Pi^{-}_{ki}\Pi^{+}_{ki}] \\
&+ \sum_{k>i} q^{n-i-k}[\Pi_k, \Pi^{-}_{k+1\,i}\Pi^{+}_{k+1\,i}]
\end{aligned}$$

$$+ \sum_{j>k+1} q^{n-k-j+1}[\Pi_k, \Pi^-_{jk}\Pi^+_{jk}]$$

$$+ \sum_{j>k+1} q^{n-k-j}[\Pi_k, \Pi^-_{j\,k+1}\Pi^+_{j\,k+1}].$$

To compute these, use the formulas $[x, yz] = [x, y]_{q^{-1}}z + q^{-1}y[x, z]_q = [x, y]_q z + qy[x, z]_{q^{-1}}$. Then from the Proposition 4.1.3 (1), we see

$$\begin{aligned}
[\Pi_k, \Pi^-_{ki}\Pi^+_{ki}] &= \Pi^-_{k+1\,i}\Pi^+_{ki} + q^{-1}\Pi^-_{ki}\Pi^+_{k+1\,i},\\
[\Pi_k, \Pi^-_{k+1\,i}\Pi^+_{k+1\,i}] &= -\Pi^-_{ki}\Pi^+_{k+1\,i} - q\Pi^-_{k+1\,i}\Pi^+_{ki},\\
[\Pi_k, \Pi^-_{jk}\Pi^+_{jk}] &= -q^{-1}\Pi^-_{j\,k+1}\Pi^+_{jk} - \Pi^-_{jk}\Pi^+_{j\,k+1},\\
[\Pi_k, \Pi^-_{j\,k+1}\Pi^+_{j\,k+1}] &= q\Pi^-_{j\,k}\Pi^+_{j\,k+1} + \Pi^-_{j\,k+1}\Pi^+_{jk}.
\end{aligned}$$

From these, the first two and the latter two summations are respectively seen to cancell out. Hence $C$ is central. The proof is quite similar for $C'$. Thus proved the Theorem.

*Remarks.* (1) We see $C$ and $C'$ are transformed to each other by an automorphism. Actually two elements $C$ and $C'$ should be identical.

(2) In the proof above, we used Proposition 4.1.3, where $q^2 \neq -1$ is assumed, so that the same condition is implicitly posed in Theorem 4.2. This restriction, however, can be actually removed. See Corollary 7.5(2) and Remark 7.9(2).

### 4.3. EXPRESSION OF $\theta^\pm_{ji}$ AS $q$-DIFFERENCE OPERATORS

In the rest of this section, we prove that the elements $\theta_j = x^\circ_{j+1}\gamma_j^{-1}\partial_j - x^\circ_j\gamma_{j+1}\partial_{j+1} = x_{j+1}\gamma_{j+1}\partial^\circ_j - x_j\gamma_j^{-1}\partial^\circ_{j+1}$ certainly give the representation of $U_q(\mathfrak{o}_n)$ and compute the explicit form of $\theta^\pm_{ji}$. See Section 6.3, for the meaning of those $\theta^\pm_{ji}$'s from the invariant theoretic point of view.

PROPOSITION 4.3.1. (1) *The elements* $\theta_j = x^\circ_{j+1}\gamma_j^{-1}\partial_j - x^\circ_j\gamma_{j+1}\partial_{j+1}$ *satisfies the following*

$$\begin{cases} [\theta_i, \theta_j] = 0 \quad \text{if} \quad |i-j| > 1,\\ \theta_i^2\theta_j - (q+q^{-1})\theta_i\theta_j\theta_i + \theta_j\theta_i^2 = -\theta_j \quad \text{if} \quad i = j \pm 1. \end{cases}$$

(2) *For the elements* $\theta^\pm_{ji}$ *inductively defined as*

$$\begin{cases} \theta^\pm_{i+1\,i} = \theta_i,\\ \theta^\pm_{ji} = \theta^\pm_{jk}\theta^\pm_{ki} - q^{\pm 1}\theta^\pm_{ki}\theta^\pm_{jk}, \quad (i<k<j), \end{cases}$$

*we have the following expressions:*

$$\theta^+_{ji} = x_j\gamma_j\partial^\circ_i - q^{j-i-1}x_i\gamma_i^{-1}\partial^\circ_j + q^{-1}(q-q^{-1})\sum_{i<k<j} q^{j-k}x_k^2\partial^\circ_i\partial^\circ_j,$$

$$\theta^-_{ji} = x^\circ_j\gamma_i^{-1}\partial_i - q^{i-j+1}x^\circ_i\gamma_j\partial_j - q(q-q^{-1})\sum_{i<k<j} q^{i-k}x^\circ_i x^\circ_j\partial_k^2.$$

For the proof, we prepare some calculations:

LEMMA 4.3.2. *For $i \neq j, j+1$, the element $\theta_j$ commutes with all the $\gamma_i$, $x_i$, $x^\circ_i$, $\partial_i$ and $\partial^\circ_i$. For $i = j, j+1$, we have the following relations:*

$$\begin{aligned}
&[\theta_j, x_j] = x_{j+1}\gamma_j^{-1}\gamma_{j+1}, && [\theta_j, x_{j+1}] = -x_j\gamma_j^{-1}\gamma_{j+1};\\
&[\theta_j, \partial_j] = \gamma_j^{-1}\gamma_{j+1}\partial_{j+1}, && [\theta_j, \partial_{j+1}] = -\gamma_j^{-1}\gamma_{j+1}\partial_j;\\
&[\theta_j, x^\circ_j]_{q^{-1}} = x^\circ_{j+1}, && [\theta_j, x^\circ_{j+1}]_q = -x^\circ_j;\\
&[\theta_j, \partial^\circ_j]_q = q\partial^\circ_{j+1}, && [\theta_j, \partial^\circ_{j+1}]_{q^{-1}} = -q^{-1}\partial^\circ_j;
\end{aligned}$$

*and also*

$$\begin{aligned}
&[\theta_j, \gamma_j]_q = q(q-q^{-1})x_j\partial^\circ_{j+1}, && [\theta_j, \gamma_{j+1}]_q = -q(q-q^{-1})x^\circ_{j+1}\gamma_j^{-1}\gamma_{j+1}\partial_j;\\
&[\theta_j, \gamma_j]_{q^{-1}} = (q-q^{-1})x^\circ_{j+1}\partial_j, && [\theta_j, \gamma_{j+1}]_{q^{-1}} = -(q-q^{-1})x_j\gamma_j^{-1}\gamma_{j+1}\partial^\circ_{j+1};\\
&[\theta_j, \gamma_{j+1}^{-1}]_q = (q-q^{-1})x^\circ_j\partial_{j+1}, && [\theta_j, \gamma_j^{-1}]_q = -(q-q^{-1})x_{j+1}\gamma_j^{-1}\gamma_{j+1}\partial^\circ_j;\\
&[\theta_j, \gamma_{j+1}^{-1}]_{q^{-1}} = q^{-1}(q-q^{-1})x_{j+1}\partial^\circ_j, && [\theta_j, \gamma_j^{-1}]_{q^{-1}} = -q^{-1}(q-q^{-1})x^\circ_j\gamma_j^{-1}\gamma_{j+1}\partial_{j+1}.
\end{aligned}$$

*Furthermore we have*

$$\begin{aligned}
&[\theta_j, x_j\gamma_j]_q = x_{j+1}\gamma_{j+1} + q(q-q^{-1})x_j^2\partial^\circ_{j+1},\\
&[\theta_j, [\theta_j, x_j\gamma_j]_q]_{q^{-1}} = -x_j\gamma_j,\\
&[\theta_j, [\theta_j, \gamma_{j+1}\partial_{j+1}]_q]_{q^{-1}} = -\gamma_{j+1}\partial_{j+1}.
\end{aligned}$$

*Proof.* From the comultiplication rule of $\theta_j$, we have for $\varphi, \psi \in \mathsf{A}$,

$$\theta_j(\varphi\psi) = (\theta_j\varphi)(\gamma_j^{-1}\gamma_{j+1}\psi) + \varphi\cdot(\theta_j\psi),$$

which gives the commutation relation between $\theta_j$ and left or right multiplication operators. In particular,noting $\theta_j\,.\,x_j = x_{j+1}$, $\theta_j\,.\,x_{j+1} = -x_j$, we get the foumulas for multiplication by $x_j$ and $x_{j+1}$. The formulas for $q$-differance operators are seen from these by making the adjoint $\dagger$ with respect to the Fischer inner product (see Lemma 3.1.1).

The commutation relations of $\theta_j$ and $\gamma_j^{\pm 1}, \gamma_{j+1}^{\pm 1}$ are just simple calculations. Actually they form pairs which can be transformed to each other by the adjoint $\dagger$. Moreover the relation $[\theta_j, \gamma_j^{-1}\gamma_{j+1}] = 0$ reduces the number of formulas to be checked into 2. With this reason, it suffices to show the first two, and their proofs are easy.

The last three formulas are applications of these. First one is immediate:

$$\begin{aligned}[\theta_j, x_j\gamma_j]_q &= [\theta_j, x_j]\gamma_j + x_j[\theta_j, \gamma_j]_q \\ &= x_{j+1}\gamma_{j+1} + q(q-q^{-1})x_j^2\partial_{j+1}^{\circ}.\end{aligned}$$

Noting then $[\theta_j, x_j^2] = (\theta_j \,.\, x_j^2)\gamma_j^{-1}\gamma_{j+1} = q^{-1}[2]x_jx_{j+1}\gamma_j^{-1}\gamma_{j+1}$, we prove the second one as

$$\begin{aligned}[\theta_j, [\theta, x_{j+1}\gamma_{j+1}]_q]_{q^{-1}} &= [\theta_j, x_{j+1}\gamma_{j+1} + q(q-q^{-1})x_j^2\partial_{j+1}^{\circ}]_{q^{-1}} \\ &= [\theta_j, x_{j+1}\gamma_{j+1}]_{q^{-1}} + [\theta_j, x_j^2]\partial_{j+1}^{\circ} + x_j^2[\theta_j, \partial_{j+1}^{\circ}]_{q^{-1}} \\ &= -x_j\gamma_j^{-1}\gamma_{j+1}^2 - q^{-1}(q-q^{-1})x_jx_{j+1}\gamma_j^{-1}\gamma_{j+1}\partial_{j+1}^{\circ} \\ &\quad +(q^2-q^{-2})x_jx_{j+1}\gamma_j^{-1}\gamma_{j+1}\partial_{j+1}^{\circ} - q^{-1}x_j^2\partial_j^{\circ} \\ &= -x_j\gamma_j^{-1}\gamma_{j+1}^2 + x_j\gamma_j^{-1}(\gamma_{j+1}^2 - 1) - x_j(\gamma_j - \gamma_j^{-1}) \\ &= -x_j\gamma_j.\end{aligned}$$

The last formula is proved similarly. In fact, starting from

$$\begin{aligned}[\theta_j, \gamma_{j+1}\partial_{j+1}]_q &= [\theta_j, \gamma_{j+1}]_q\partial_{j+1} + q\gamma_{j+1}[\theta_j, \partial_{j+1}] \\ &= -q\gamma_j^{-1}\partial_j + (q-q^{-1})x_j^{\circ}\partial_{j+1}^2,\end{aligned}$$

we have

$$\begin{aligned}[\theta_j, [\theta_j, \gamma_{j+1}\partial_{j+1}]_q]_{q^{-1}} &= [\theta_j, -q\gamma_j^{-1}\partial_j + (q-q^{-1})x_j^{\circ}\partial_{j+1}^2]_{q^{-1}} \\ &= -q[\theta_j, -q\gamma_j^{-1}]_{q^{-1}}\partial_j - \gamma_j^{-1}[\theta_j, \partial_j]_{q^{-1}} \\ &\quad +(q-q^{-1})[\theta_j, x_j^{\circ}]_{q^{-1}}\partial_{j+1}^2 + q^{-1}(q-q^{-1}) \\ &\quad \times x_j^{\circ}[\theta_j, \partial_{j+1}^2]_{q^{-1}} \\ &= (q-q^{-1})x_j^{\circ}\gamma_j^{-1}\gamma_{j+1}\partial_{j+1}\partial_j - \gamma_j^{-2}\gamma_{j+1}\partial_{j+1} \\ &\quad +(q-q^{-1})x_j^{\circ}\partial_{j+1}^2 - (q-q^{-1})[2]x_j^{\circ}\gamma_j^{-1}\gamma_{j+1}\partial_j\partial_{j+1} \\ &= q^2(1-\gamma_j^{-2})\gamma_{j+1}\partial_{j+1} - \gamma_j^{-2}\gamma_{j+1}\partial_{j+1}\end{aligned}$$

$$
\begin{aligned}
&+(\gamma_{j+1}-\gamma_{j+1}^{-1})\partial_{j+1}-(q^2+1)(1-\gamma_j^{-2})\gamma_{j+1}\partial_{j+1}\\
=\ &-\gamma_{j+1}\partial_{j+1}.
\end{aligned}
$$

Here we used $[\theta_j,\partial_{j+1}^2] = [\theta_j,\partial_{j+1}]\partial_{j+1}+\partial_{j+1}[\theta_j,\partial_{j+1}] = -q[2]\gamma_j^{-1}\gamma_{j+1} \times \partial_j\partial_{j+1}$.

*Remark.* See also Remark 6.3.1(3) for a meaning of the first eight foumulas from the representation-theoretic point of view.

*Proof of Proposition 4.3.1.* For the assertion (1), since the first formula in (1) is clear, we have only to show the second. It amounts to the same as the following two equalities:

$$
[\theta_j,[\theta_j,\theta_{j-1}]_q]_{q^{-1}}=-\theta_{j-1},\qquad [\theta_j,[\theta_j,\theta_{j+1}]_q]_{q^{-1}}=-\theta_{j+1}.
$$

The first one is proved by using the formulas in Lemma 4.3.2 as

$$
\begin{aligned}
[\theta_j,[\theta_j,\theta_{j-1}]_q]_{q^{-1}} &= [\theta_j,[\theta_j,x_j\gamma_j\partial_{j-1}^\circ-x_{j-1}\gamma_{j-1}^{-1}\partial_j^\circ]_{q^{-1}}\\
&= [\theta_j,[\theta_j,x_j\gamma_j]_{q^{-1}}\partial_{j-1}^\circ-x_{j-1}\gamma_{j-1}^{-1}[\theta_j,[\theta_j,\partial_j^\circ]_{q^{-1}}\\
&= -x_j\gamma_j\partial_{j-1}^\circ+x_{j-1}\gamma_{j-1}^{-1}\partial_j^\circ\\
&= -\theta_{j-1}.
\end{aligned}
$$

The latter can be shown similarly again by Lemma 4.3.2,

$$
\begin{aligned}
[\theta_j,[\theta_j,\theta_{j+1}]_q]_{q^{-1}} &= [\theta_j,[\theta_j,x_{j+2}^\circ\gamma_{j+1}^{-1}\partial_{j+1}-x_{j+1}^\circ\gamma_{j+2}\partial_{j+2}]_q]_{q^{-1}}\\
&= x_{j+2}^\circ[\theta_j,[\theta_j,\gamma_{j+1}^{-1}\partial_{j+1}]_q]_{q^{-1}}\\
&\quad -[\theta_j,[\theta_j,x_{j+1}^\circ]_q]_{q^{-1}}\gamma_{j+2}\partial_{j+2}\\
&= -x_{j+2}^\circ\gamma_{j+1}^{-1}\partial_{j+1}+x_{j+1}^\circ\gamma_{j+2}\partial_{j+2}\\
&= -\theta_{j+1}.
\end{aligned}
$$

We now prove the assertion (2). Let us show the first formula by induction. Since $\partial_i^\circ$ and $x_i\gamma_i^{-1}$ commute with $\theta_j$, we have from the Lemma above

$$
\begin{aligned}
[\theta_j,x_j\gamma_j\partial_i^\circ]_q &= [\theta_j,x_j\gamma_j]_q\partial_i^\circ = x_{j+1}\gamma_{j+1}\partial_i^\circ+(q-q^{-1})x_j^2\partial_i^\circ\partial_{j+1}^\circ,\\
[\theta_j,x_i\gamma_i^{-1}\partial_j^\circ]_q &= x_i\gamma_i^{-1}[\theta_j,\partial_j^\circ]_q = qx_i\gamma_i^{-1}\partial_{j+1}^\circ.
\end{aligned}
$$

Similarly, since $x_k^2\partial_i^\circ$ commutes with $\theta_j$, we see

$$
\left[\theta_j,\sum_{i<k<j}q^{j-k}x_k^2\partial_i^\circ\partial_j^\circ\right]_q = \sum_{i<k<j}q^{j-k+1}x_k^2\partial_i^\circ\partial_{j+1}^\circ.
$$

Summing up those results with suitable factors, we get the desired expression for $\theta^+_{j+1\,i} = [\theta_j, \theta^+_{ji}]_q$. The latter formula can be obtained from the first by making the adjoint $\dagger$ with respect to the Fischer inner product (see Section 3.1).

## 5. Proof of the Capelli identity

### 5.1. CAPELLI IDENTITY FOR $n = 2$

We treat first the case for $n = 2$ separately. This case is quite simple and the direct computation will not meet any difficulty. Recall the operators in question: $Q = x_1^2 + q^{-1}x_2^2$ and $\Delta = q\partial_1^2 + \partial_2^2$. Furthermore we denote the Euler degree operator $\{\gamma\}$ by $E$. Note that this can be written as $E = x_1\partial_1 + x_2\partial_2$. The generator of $\mathsf{U}_q(\mathfrak{o}_2)$ is the element $\theta_1 = \theta = x_2\partial_1 - x_1\partial_2$ (see the formula in Section 2). The commutation relations between (left) multiplication operators and $q$-difference oprerators are

$$\partial_1 x_1 - q x_1\partial_1 = \gamma_1^{-1}\gamma_2, \qquad \partial_2 x_2 - q^{-1}x_2\partial_2 = \gamma_1^{-1}\gamma_2.$$

LEMMA 5.1. (Capelli identity for $n = 2$)

$$Q\Delta - E^2 = \theta^2.$$

*Proof.* We prove the equality $Q\Delta = E^2 + \theta^2$. Since $E$ and $\theta$ commute, the right-hand side has a factorization as $E^2 + \theta^2 = (E - \sqrt{-1}\theta)(E + \sqrt{-1}\theta)$. The each factor is then further factorized into

$$\begin{aligned} E + \sqrt{-1}\theta &= x_1\partial_1 + x_2\partial_2 + \sqrt{-1}(x_2\partial_1 - x_1\partial_2) \\ &= (x_1 + \sqrt{-1}\,x_2)(\partial_1 - \sqrt{-1}\partial_2), \\ E - \sqrt{-1}\theta &= (x_1 - \sqrt{-1}\,x_2)(\partial_1 + \sqrt{-1}\partial_2). \end{aligned}$$

Using the commutation relations $x_1x_2 = qx_2x_1$ and $q\partial_1\partial_2 = \partial_2\partial_1$, we have the following factorization:

$$Q = (x_1 + \sqrt{-1}\,x_2)(x_1 - \sqrt{-1}q^{-1}\,x_2),$$
$$\Delta = (q\partial_1 - \sqrt{-1}\partial_2)(\partial_1 + \sqrt{-1}\partial_2).$$

From the commutation relations between multiplication and $q$-difference operators above, we see

$$\begin{aligned} &(x_1 - \sqrt{-1}q^{-1}\,x_2)(q\partial_1 - \sqrt{-1}\partial_2) \\ &= qx_1\partial_1 - \sqrt{-1}x_1\partial_2 - \sqrt{-1}x_2\partial_1 - q^{-1}\,x_2\partial_2 \\ &= \partial_1x_1 - \gamma_1^{-1}\gamma_2 - \sqrt{-1}x_1\partial_2 - \sqrt{-1}x_2\partial_1 - \partial_2x_2 + \gamma_1^{-1}\gamma_2 \\ &= (\partial_1 - \sqrt{-1}\partial_2)(x_1 - \sqrt{-1}\,x_2). \end{aligned}$$

Then our assertion follows from those factorization formulas.

*Remarks.* (1) By the commutation relations above, we can of course check the formula by direct high-school calculations:

$$\begin{aligned}
Q\Delta &= (x_1^2+q^{-1}x_2^2)(q\partial_1^2+\partial_2^2) = qx_1^2\partial_1^2 + x_2^2\partial_1^2 + x_1^2\partial_2^2 + q^{-1}x_2^2\partial_2^2,\\
E^2 &= (x_1\partial_1 + x_2\partial_2)^2 = x_1\partial_1x_1\partial_1 + x_1\partial_1x_2\partial_2 + x_2\partial_2x_1\partial_1 + x_2\partial_2x_2\partial_2\\
&= (qx_1^2\partial_1^2 + x_1\gamma_1^{-1}\gamma_2\partial_1) + (x_1x_2\partial_1\partial_2 + x_2x_1\partial_2\partial_1)\\
&\quad +(q^{-1}x_2^2\partial_2^2 + x_2\gamma_1^{-1}\gamma_2\partial_2),\\
\theta^2 &= (x_2\partial_1 - x_1\partial_2)^2 = x_2^2\partial_1^2 - x_2\partial_1x_1\partial_2 - x_1\partial_2x_2\partial_1 + x_1^2\partial_2^2\\
&= x_2^2\partial_1^2 - (qx_2x_1\partial_2\partial_1 + x_2\gamma_1^{-1}\gamma_2\partial_2)\\
&\quad -(q^{-1}x_1x_2\partial_2\partial_1 + x_1\gamma_1^{-1}\gamma_2\partial_1) + x_1^2\partial_2^2.
\end{aligned}$$

(2) There is another a little bit more sophisticated proof of this formula based on the exterior calculus (see the Appendix). Note also

$$\begin{bmatrix} Q & \{\gamma\} \\ \{q^2\gamma\} & \Delta \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ q\partial_1 & \partial_2 \end{bmatrix}\begin{bmatrix} x_1 & \partial_1 \\ q^{-1}x_2 & \partial_2 \end{bmatrix}.$$

### 5.2. HIGHEST WEIGHT VECTORS IN HARMONICS

In Section 3, we saw the irreducible decomposition of the space $\mathsf{A} = \mathbb{K}[x_1, \ldots, x_n]$ of $q$-commutative ring under the action of $\mathsf{U}_q(\mathfrak{o}_n)$. For $n \geqslant 3$, the space of harmonics $\mathsf{H}_m$ is shown to be irreducible. We give here a special element in $\mathsf{H}_m$, a highest weight vector in a suitable sense. Later we will use this vector for the comparison of two central elements, hence for the Capelli identity.

We put $z_r = x_1 + \sqrt{-1}q^r x_2$ and $z^{[m]} = z_0 z_1 \cdots z_{m-1}$.

PROPOSITION 5.2.1. *The shifted power* $z^{[m]} = (x_1 + \sqrt{-1}\,x_2)\cdots(x_1 + \sqrt{-1}\,q^{m-1}x_2)$ *is*

(1) *harmonic:* $\Delta(z^{[m]}) = 0$, *and*
(2) *a joint eigenvector for* $\theta_1$ *and* $\theta_j$ *with* $j \geqslant 3$:

$$\theta_1 \,.\, z^{[m]} = -\sqrt{-1}[m]\, z^{[m]}, \qquad \theta_j \,.\, z^{[m]} = 0, \quad (j \geqslant 3).$$

For this proof, we prepare a lemma.

LEMMA 5.2.2. *The actions of* $\partial_j$*'s on the shifted power are given by*

$$\partial_1 z^{[m]} = [m]\, z_1 \cdots z_{m-1}; \quad \partial_2 z^{[m]} = \sqrt{-1}\,[m]\, z_1 \cdots z_{m-1}.$$

*Proof.* It suffices to prove these formulas for $n = 2$. Recall the following (twisted) Leibniz rule:

$$\begin{aligned}\partial_1(\varphi\psi) &= (\gamma_1^{-1}\varphi)(\partial_1\psi) + (\partial_1\varphi)(\gamma\psi),\\ \partial_2(\varphi\psi) &= (\gamma_2\varphi)(\partial_2\psi) + (\partial_2\varphi)(\gamma^{-1}\psi).\end{aligned}$$

Then we can immediately see our assertions from these by noting $\gamma_1^{-1}z_r = q^{-1}z_{r+1}$; $\gamma_2 z_r = z_{r+1}$. In fact, with the convention that the empty product represents 1,

$$\begin{aligned}\partial_1 z^{[m]} &= \sum_{r=0}^{m-1}\gamma_1^{-1}(z_0\cdots z_{r-1})\cdot\partial_1 z_r\cdot\gamma(z_{r+1}\cdots z_{m-1})\\ &= \sum_{r=0}^{m-1} q^{-r}(z_1\cdots z_r)\cdot q^{m-r-1}(z_{r+1}\cdots z_{m-1})\\ &= z_1\cdots z_{m-1}\sum_{r=0}^{m-1} q^{m-2r-1}\end{aligned}$$

and similarly

$$\begin{aligned}\partial_2 z^{[m]} &= \sum_{r=0}^{m-1}\gamma(z_0\cdots z_{r-1})\cdot\partial_2 z_r\cdot\gamma^{-1}(z_{r+1}\cdots z_{m-1})\\ &= \sum_{r=0}^{m-1}(z_1\cdots z_r)\cdot\sqrt{-1}q^r\cdot q^{-(m-r-1)}(z_{r+1}\cdots z_{m-1})\\ &= \sqrt{-1}\,z_1\cdots z_{m-1}\sum_{r=0}^{m-1} q^{-m+2r+1}.\end{aligned}$$

*Proof of Proposition 5.2.1.* Since the last equalities $\theta_j.z^{[m]} = 0$ for $j \geqslant 3$ are obvious, we may assume $n = 2$ for the computations. For (1), as seen above, we have a factorization $\Delta = (q\partial_1 - \sqrt{-1}\,\partial_2)(\partial_1 + \sqrt{-1}\,\partial_2)$. We have thereby only to prove $(\partial_1 + \sqrt{-1}\,\partial_2)(z^{[m]}) = 0$, which equality is clearly seen from the Lemma.

For (2), recalling $\theta_1 = x_2\partial_1 - x_1\partial_2$, we see from the Lemma that

$$\begin{aligned}\theta_1\,.\,z^{[m]} &= (x_2\partial_1 - x_1\partial_2)z^{[m]} = (x_2 - \sqrt{-1}\,x_1)[m]z_1\cdots z_{m-1}\\ &= -\sqrt{-1}\,[m]z_0z_1\cdots z_{m-1} = -\sqrt{-1}\,[m]z^{[m]},\end{aligned}$$

as desired. Thus completes the proof.

*Remark 5.2.3.* There are several different ways to prove the formulas (1) and (2) in Proposition 5.2.1. For example, we have the commutation relations $z_r z_s = z_{s+1} z_{r-1}$ and $\theta . z_r = -\sqrt{-1}\, q^r z_{-r}$. Together with these, using the (non-standard) Leibniz rule $\theta(\varphi\psi) = (\gamma_1^{-2}\varphi)(\theta\psi) + (\theta\varphi)(\gamma\psi)$, we get the formula (2).

Another way to prove is through the binomial expansion.

$$(x_1 + \lambda x_2)(x_1 + \lambda q x_2) \cdots (x_1 + \lambda q^{n-1} x_2) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-1)/2} x_1^{n-r} x_2^r \lambda^r,$$

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}; \qquad [a]! = [a] \cdots [1].$$

## 5.3. Proof of the Capelli Identity – General Case

With these preparations, we now give a proof of the Capelli identity. As we have proved it for the case $n = 2$ above, we assume $n \geqslant 3$ hereafter. Recall that the space of harmonics $\mathsf{H}_m$ is irreducible under $\mathsf{U}_q(\mathfrak{o}_n)$ for $n \geqslant 3$, so that the space $\mathsf{M}_m = \bigoplus_{j \geqslant 0} Q^j \mathsf{H}_m$ is irreducible under the joint action of $\mathsf{U}_q(\mathfrak{o}_n)$ and $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$. Since both sides of the Capelli identity commute with this joint action, by Schur's lemma, they must be constant on the space $\mathsf{M}_m$. (Note the irreducibility did not depend on the base field. To apply the Schur's lemma correctly, we at first need to make a field extension, which process eventually turns to be unnecessary. Other way to see this is: note first there is only one $\mathsf{U}_q(\mathfrak{o}_{n-1})$-fixed vector, the zonal spherical polynomial in $\mathsf{H}_m$, so that it is an eigenvector of the two central elements. Since $\mathsf{M}_m$ is generated from that zonal spherical polynomial under the joint action of $\mathsf{U}_q(\mathfrak{o}_n)$ and $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$, the central elements must be scalar on this space.)

For our Capelli identity, it thus suffices to show the eigenvalues of the two operators $Q\Delta - \{\gamma\}\{q^{n-2}\gamma\}$ and $\mathcal{C}_{\mathsf{A}}$ coincide on the shifted power $z^{[m]} = (x_1 + \sqrt{-1}\, x_2) \cdots (x_1 + \sqrt{-1}\, q^{m-1} x_2) \in \mathsf{M}_m$. Here we recall that $\mathcal{C}_{\mathsf{A}}$ denotes the Casimir operator $\mathcal{C}$ with an emphasis of its representation space $\mathsf{A}$. The computation $(Q\Delta - \{\gamma\}\{q^{n-2}\gamma\}) z^{[m]}$ is easy, because we have already seen that $z^{[m]}$ is harmonic:

$$(Q\Delta - \{\gamma\}\{q^{n-2}\gamma\}) z^{[m]} = -[m][m+n-2] z^{[m]}.$$

On the other hand, since $z^{[m]}$ is an eigenvector for $\mathcal{C}_{\mathsf{A}}$ with a form $x_1^m + \cdots$ (see the binomial expansion in the Remark above), it is sufficient to look at only the coefficient of this top term $x_1^m$ in the computation of $\mathcal{C}_{\mathsf{A}} z^{[m]}$. Now recall the definition

$$\mathcal{C}_{\mathsf{A}} = \sum_{i<j} q^{n-i-j+1} \theta_{ji}^{-} \theta_{ji}^{+},$$

and the expression Proposition 4.3.1 (2)

$$\theta^+_{ji} = x_j\gamma_j\partial^\circ_i - q^{j-i-1}x_i\gamma_i^{-1}\partial^\circ_j + q^{-1}(q-q^{-1})\sum_{i<k<j} q^{j-k}x_k^2\partial^\circ_i\partial^\circ_j,$$

$$\theta^-_{ji} = x^\circ_j\gamma_i^{-1}\partial_i - q^{i-j+1}x^\circ_i\gamma_j\partial_j - q(q-q^{-1})\sum_{i<k<j} q^{i-k}x^\circ_i x^\circ_j\partial^2_k.$$

Then in those many terms, what we need for $C_{\mathsf{A}}z^{[m]}$ are only the terms $\theta^\pm_{ji}$ whose indices includes 1 or 2. In fact, otherwise they contain the difference operators with respect to $x_i$ with $i > 2$, so that their actions end up with zero. We have actually a further reduction to the terms $\theta^\pm_{j1}$'s, because we have concentrated on looking at the coefficient of $x_1^m$, for in the expression of $\theta^-_{ji}$ with $i > 2$ contains only $x_i$ with $i > 1$ as multiplications. With these reasonings, our computation now reduces to

$$\sum_{2\leqslant j\leqslant n} q^{n-j}\theta^-_{j1}\theta^+_{j1}z^{[m]}$$

and further to

$$q^{n-2}\theta^-_{21}\theta^+_{21}x_1^m - \sum_{3\leqslant j\leqslant n} q^{n-j}q^{-j+2}x^\circ_1\gamma_j\partial_j x_j\gamma_j\partial_1 x_1^m.$$

The first term can be computed by the Capelli identity for $n = 2$: $q^{n-2}\theta^-_{21}\theta^+_{21}z^{[m]} = -q^{n-2}[m]^2z^{[m]}$. From the second summation, the contribution to the term $x_1^m$ amounts to

$$-\sum_{3\leqslant j\leqslant n} q^{-m+n-2j+1}[m]x_1^m = -q^{-m}[m][n-2]x_1^m$$

because $\partial_j x_j - qx_j\partial_j = \gamma_1^{-1}\cdots\gamma_j^{-1}\gamma_{j+1}\cdots\gamma_n$ and $x^\circ_1\gamma_1^{-1}\partial_1 = qx^\circ_1\partial_1\gamma_1^{-1}$. Then we come to the conclusion from a simpe identity $q^{n-2}[m] + q^{-m}[n-2] = [m+n-2]$. Hence the proof of our Capelli identity.

*Remark.* The identity $Q\,\Delta - \{\gamma\}\{q^{n-2}\gamma\} = C'_{\mathsf{A}} = \sum_{i<j} q^{-n+i+j-1}\theta^+_{ji}\theta^-_{ji}$ also holds quite similarly.

## 6. A quantum analogue of the dual pair, the commutant of $\omega^{\otimes n}$

Let $\mathfrak{D} = \mathbb{K}[x_1,\dots,x_n,\partial_1,\dots,\partial_n,\gamma_1^{\pm1},\dots,\gamma_n^{\pm1}]$ be the ring of $q$-difference operators. It is no doubt natural to ask whether the double commutant property holds between the two algebras $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$ and $\mathsf{U}_q(\mathfrak{o}_n)$ within this $\mathfrak{D}$. However, not alike the classical cases, the adjoint action contains infinite dimensional parts, so that the problems become more complicated. This kind of difficulty occurs quite commonly when one considers quantum group actions.

In this section, we will not attack the double commutant problem in the above form itself, but choose suitably big subalgebras of $\mathfrak{D}$ consisting of only finite dimensional parts under $\mathrm{U}_{q^2}(\mathfrak{sl}_2)$ or $\mathrm{U}_q(\mathfrak{o}_n)$ and determine their fixed point subalgebras. Instead of loosing a most generality, we will gain a clarification of the meaning of the elements $\theta_{ji}^{\pm}$ through this approach. We note also that our result below will suffice for the analogy of spherical harmonics from an operator-theoretic approach.

In the discussions below, we follow an analogous way as classical ones. Since we have not yet got well-developed general theories there, our discussion still have to go slightly longer ways.

### 6.1. A DETERMINATION OF $\mathrm{U}_q(o_n)$-INVARIANT DIFFERENCE OPERATORS

Let us take a subalgebra $\mathfrak{D}_0 = \mathbb{K}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ of $\mathfrak{D}$. This is an algebra with $\mathrm{U}_q(\mathfrak{gl}_n)$-symmetry under the adjoint action, and is generated by the vector representation and its contragredient. We denote by $V(\lambda)$ the irreducible representation of $\mathrm{U}_q(\mathfrak{gl}_n)$ with highest weight $\lambda$. Then $\varepsilon_1, \ldots, \varepsilon_n$ being the canonical basis for the weight lattice, vector representation or its contragredient is $V(\varepsilon_1)$ or $V(-\varepsilon_n)$, respectively.

We will show the following.

THEOREM 6.1. *The invariant difference operators in* $\mathfrak{D}_0 = \mathbb{K}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ *under* $\mathrm{U}_q(\mathfrak{o}_n)$ *are determined as follows:*

(1) *When* $n \geqslant 3$, *we have*

$$\mathfrak{D}_0^{\mathrm{U}_q(\mathfrak{o}_n)} = \mathbb{K}[Q, \Delta, \gamma^{\pm 1}] = \omega^{\otimes n}(\mathrm{U}_{q^2}(\mathfrak{sl}_2))[\gamma^{\pm 1}].$$

(2) *When* $n = 2$, *we have*

$$\mathfrak{D}_0^{\mathrm{U}_q(\mathfrak{o}_n)} = \mathbb{K}[Q, \Delta, \gamma^{\pm 1}, \theta] = \omega^{\otimes n}(\mathrm{U}_{q^2}(\mathfrak{sl}_2))[\gamma^{\pm 1}, \theta].$$

*Remark.* Since $\mathrm{U}_q(\mathfrak{o}_n)$ is a right coideal of the Hopf algebra $\mathrm{U}_q(\mathfrak{gl}_n)$, the fixed point algebra $\mathfrak{D}_0^{\mathrm{U}_q(\mathfrak{o}_n)}$ under the adjoint action coincides with the set of operators in $\mathfrak{D}_0$ commuting with the action of $\mathrm{U}_q(\mathfrak{o}_n)$ on $\mathrm{A} = \mathbb{K}[x_1, \ldots, x_n]$.

*Proof.* First we consider a little bit smaller algebra $\mathfrak{D}_{00} = \mathbb{K}[x_1, \ldots, x_n, \gamma\partial_1, \ldots, \gamma\partial_n]$. Note $\mathbb{K}[x_1, \ldots, x_n, \gamma\partial_1, \ldots, \gamma\partial_n, \gamma^{\pm 1}] = \mathbb{K}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ and $\gamma$ is invariant under $\mathrm{U}_q(\mathfrak{o}_n)$, so that algebra $\mathfrak{D}_{00}$ will suffice for our purpose. The reason why we make use $\gamma\partial_j$ instead of $\partial_j$ is in the commutation relations:

$$(\gamma\partial_j)x_j = q^2 x_j(\gamma\partial_j) + (q^2 - 1)\sum_{\ell > j} x_\ell(\gamma\partial_\ell) + 1.$$

Giving the degree 1 both to $x_j$ and $\gamma\partial_j$, we make $\mathbb{K}[x_1,\dots,x_n,\gamma\partial_1,\dots,\gamma\partial_n]$ into a filtered algebra. Recall the decomposition

$$\mathbb{K}[x_1,\dots,x_n]=\bigoplus_{\ell=0}^{\infty}V(\ell\varepsilon_1),\quad \mathbb{K}[\gamma\partial_1,\dots,\gamma\partial_n]=\bigoplus_{m=0}^{\infty}V(-m\varepsilon_n)$$

and that the multiplication gives the isomorphism $\mathbb{K}[x_1,\dots,x_n]\otimes\mathbb{K}[\gamma\partial_1,\dots,\gamma\partial_n]\simeq\mathfrak{D}_{00}$. Then transferring to the graded algebra, we have the isomorphism for every degree $d$

$$\bigoplus_{d=\ell+m}V(\ell\varepsilon_1)\otimes V(-m\varepsilon_n)\xrightarrow{\sim}gr_d(\mathfrak{D}_{00})$$

as $\mathsf{U}_q(\mathfrak{gl}_n)$-modules. Using the Clebsch–Gordan rule

$$V(\ell\varepsilon_1)\otimes V(-m\varepsilon_n)=\bigoplus_{k=0}^{\min(\ell,m)}V((\ell-k)\varepsilon_1-(m-k)\varepsilon_n),$$

we have essentially obtained the irreducible decomposition of $\mathfrak{D}_{00}$ under the action of $\mathsf{U}_q(\mathfrak{gl}_n)$. More explicitly, the description of irreducible component is as follows. It is easy to find the highest weight vector $P_{\ell,m,k}$ in $\mathfrak{D}_{00}$ corresponding to this $V((\ell-k)\varepsilon_1-(m-k)\varepsilon_n)$ in degree $d=\ell+m$ part as:

$$P_{\ell,m,k}=x_1^{\ell-k}\left(\sum_{1\leqslant i\leqslant n}x_i\gamma\partial_i\right)^k(\gamma\partial_n)^{m-k}.$$

For $0\leqslant k\leqslant\min(\ell,k)$, we write $V_{\ell,m,k}$ as the $\mathsf{U}_q(\mathfrak{gl}_n)$-module generated by $P_{\ell,m,k}$. These are the irreducible components of $\mathfrak{D}_{00}$.

Next step is to find the fixed vectors under $\mathsf{U}_q(\mathfrak{o}_n)$ in the irreducible representation $V(\lambda)$ of $\mathsf{U}_q(\mathfrak{gl}_n)$. It is determined in Theorem 3.1 in [N4] as

(A) In every irreducible representation $V(\lambda)$ of $\mathsf{U}_q(\mathfrak{gl}_n)$, the fixed vectors $V(\lambda)^{\mathsf{U}_q(\mathfrak{o}_n)}$ under $\mathsf{U}_q(\mathfrak{o}_n)$ is at most one dimensional.
(B) The fixed vectors $V(\lambda)^{\mathsf{U}_q(\mathfrak{o}_n)}$ are non-zero if and only if $\lambda_j-\lambda_{j+1}$ is even for all $j$.

Applying this criterion to $V(\ell\varepsilon_1-m\varepsilon_n)$, we see its $\mathsf{U}_q(\mathfrak{o}_n)$-fixed vectors is one dimensional if and only if

(i) for $n\geqslant 3$, both $\ell$ and $m$ are even;
(ii) for $n=2$, the parity of $\ell$ and $m$ are the same.

Assume now first $n\geqslant 3$. Our criterion tells that $V_{\ell,m,k}\simeq V((\ell-k)\varepsilon_1-(m-k)\varepsilon_n)$ contains non-zero $\mathsf{U}_q(\mathfrak{o}_n)$-fixed vectors only when both $\ell-k$,

$m-k$ are even. In this case, we can give the fixed vector in $V_{\ell,m,k}$ explicitly as

$$Q^{(\ell-k)/2}\left(\sum_{1\leqslant i\leqslant n} x_i\gamma\partial_i\right)^k(\gamma^2\Delta)^{(m-k)/2}.$$

Thus the assertion (1) of the Theorem is proved.

For $n=2$, since $\mathrm{U}_q(\mathfrak{o}_2)=\mathbb{K}[\theta]$ is commutative, $V(\varepsilon_1-\varepsilon_2)$ has another typical invariant $\gamma\theta = q(x_2\gamma\partial_1 - x_1\gamma\partial_2)$. Then in the irreducible component $V_{\ell,m,k}$ with the same parity of $\ell-k$ and $m-k$, we can find the fixed vector as

$$Q^{(\ell-m)/2}(\gamma\theta)^{m-k}\left(\sum_{1\leqslant i\leqslant n} x_i\gamma\partial_i\right)^k, \quad (\ell\geqslant m),$$

$$(\gamma\theta)^{m-k}\left(\sum_{1\leqslant i\leqslant n} x_i\gamma\partial_i\right)^k(\gamma^2\Delta)^{(m-\ell)/2}, \quad (\ell\leqslant m).$$

This completes the proof.

### 6.2. THE FIRST FUNDAMENTAL THEOREM FOR $\mathrm{U}_q(sl_2)$

Here we establish an analogy of the first fundamental theorem of invariants for $\mathfrak{sl}_2$. For the later use, we first recall the comultiplication rule of $\mathrm{U}_q(\mathfrak{sl}_2)$:

$$\begin{cases}\Delta(e)=e\otimes 1+k^{-1}\otimes e,\\ \Delta(f)=f\otimes k+1\otimes f,\\ \Delta(k)=k\otimes k.\end{cases}$$

Correspondingly the antipode $S$ is given by

$$S(e)=-ke, \qquad S(f)=-fk^{-1}, \qquad S(k)=k^{-1}.$$

Let $\mathfrak{A}$ be a left $\mathrm{U}_q(\mathfrak{sl}_2)$-module. Then for $\Phi\in\mathrm{End}_{\mathbb{K}}(\mathfrak{A})$ and $a\in\mathrm{U}_q(\mathfrak{sl}_2)$, the adjoint action $\mathrm{ad}(a)$ on $\Phi$ is in general defined by $\mathrm{ad}(a)\Phi=\sum_\kappa a_\kappa^{(1)}\Phi S(a_\kappa^{(2)})$, where the comultiplication is given as $\Delta(a)=\sum_\kappa a_\kappa^{(1)}\otimes a_\kappa^{(2)}$. We note by definition

$$\mathrm{ad}(e)\Phi=e\Phi-k^{-1}\Phi ke, \quad \mathrm{ad}(f)\Phi=(f\Phi-\Phi f)k^{-1}.$$

When a left $\mathrm{U}_q(\mathfrak{sl}_2)$-module $\mathfrak{A}$ is an algebra, we call it with $\mathrm{U}_q(\mathfrak{sl}_2)$-symmetry if the unit $\mathbb{K}\to\mathfrak{A}$ and the multiplication $\mathfrak{A}\otimes\mathfrak{A}\to\mathfrak{A}$ are $\mathrm{U}_q(\mathfrak{sl}_2)$-homomorphisms. Our setting for the first fundamental theorem is the following. Assume we are given an algebra $\mathfrak{A} = \mathbb{K}[\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n]$ with $\mathsf{U}_q(\mathfrak{sl}_2)$-symmetry such that $V_j = \mathbb{K}\xi_j \oplus \mathbb{K}\eta_j$ bears the standard 2 dimensional (vector) representation for all $1 \leqslant j \leqslant n$:

$$e\xi_j = 0, \qquad e\eta_j = \xi_j; \qquad f\xi_j = \eta_j, \qquad f\eta_j = 0;$$

$$k\xi_j = q\xi_j, \qquad k\eta_j = q^{-1}\eta_j.$$

We have then typical invariants, the $q^{-1}$-determinant of $2 \times 2$ minors, $\zeta_{ij} = \xi_i\eta_j - q^{-1}\eta_i\xi_j$ for $1 \leqslant i, j \leqslant n$. Put $\mathfrak{B} = \mathbb{K}[\zeta_{ij}; 1 \leqslant i, j \leqslant n]$, the subalgebra of $\mathfrak{A}$ generated by these invariants. Our theorem is now stated as

THEOREM 6.2. *In addition to the setting above, we assume the property*

$$\sum_{1 \leqslant i \leqslant n} \xi_j \mathfrak{B} = \sum_{1 \leqslant i \leqslant n} \mathfrak{B}\xi_j. \tag{C}$$

*Then the set of invariants under* $\mathsf{U}_q(\mathfrak{sl}_2)$ *in* $\mathfrak{A}$ *is genetated by the typical invariants* $\zeta_{ij}$*'s, i.e.,*

$$\mathbb{K}[\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n]^{\mathsf{U}_q(\mathfrak{sl}_2)} = \mathfrak{B} = \mathbb{K}[\zeta_{ij}; 1 \leqslant i, j \leqslant n].$$

*Proof.* We will describe the set of highest weight vectors in $\mathfrak{A}$ in two steps.

*First Step.* Denote by $V(\ell)$ the $(\ell+1)$-dimensional irreducible representation of $\mathsf{U}_q(\mathfrak{sl}_2)$ of highest weight $\ell$. Translating the Clebsch-Gordan rule $V(1) \otimes V(\ell) \simeq V(\ell+1) \oplus V(\ell-1)$ to our algebra $\mathfrak{A}$, we introduce the following two operators:

$$E_j = \xi_j, \qquad F_j = \xi_j f - q^{-1}\eta_j\{k\}.$$

Here $\{a\} = (a - a^{-1})/(q - q^{-1})$ and $\xi_j$ and $\eta_j$ are considered as left multiplication operators. It is easily checked that for a highest vector $u$ with weight $\ell$, the resulting vector $E_j u$ or $F_j u$ is either zero or a highest weight vector respectively with weight $\ell + 1$ or $\ell - 1$. Recall $\mathfrak{A}$ is generated by the vector subspace $V_1 + \cdots + V_n$ and consider a filtration on $\mathfrak{A}$ induced from these generators. With this filtration, it is inductively shown from the meaning of $E_j$ and $F_j$ that all the possible highest weight vectors in $\mathfrak{A}$ are obtained by successive applications of operators $E_j$'s and $F_j$'s to 1. Thus we have proved

*Claim* (1). The subalgebra of $\mathfrak{A}$ killed by $e$ is spanned by the elements

$$E_{j_1}^{\epsilon_1} \cdots E_{j_d}^{\epsilon_d} \cdot 1$$

with $d \geqslant 0$, $1 \leqslant j_1, \dots, j_d \leqslant n$ and $\epsilon_\kappa = \pm 1$. Here we used $E_j^+ = E_j$ and $E_j^- = F_j$. The weight of the element of the above form is $\sum_{\kappa=1}^{d} \epsilon_\kappa$.

*Second Step*. So far the assumption (C) in the theorem has not been used. From (C), for fixed $\alpha, i, j$, we can find $\varphi_\beta \in \mathfrak{B}$ such that

$$\xi_\alpha \zeta_{ij} = \sum_\beta \varphi_\beta \xi_\beta.$$

Applying here the adjoint action $\mathrm{ad}(f)$ of $f$ to this equality, we also get

$$\eta_\alpha \zeta_{ij} = \sum_\beta \varphi_\beta \eta_\beta.$$

In fact, note $\mathrm{ad}(f)\Phi = (f\Phi - \Phi f)k^{-1}$, so that $\mathrm{ad}(f)\xi_j = \eta_j$ and $\mathrm{ad}(f)\varphi = \varphi$ for any $\varphi \in \mathfrak{B}$. From the two formulas above, we see first $E_\alpha \zeta_{ij} = \sum_\beta \varphi_\beta E_\beta$ and

$$\begin{aligned} F_\alpha \zeta_{ij} &= \xi_\alpha f \zeta_{ij} - q^{-1}\eta_\alpha\{k\}\zeta_{ij} = \xi_\alpha\zeta_{ij} f - q^{-1}\eta_\alpha\zeta_{ij}\{k\} \\ &= \sum_\beta \varphi_\beta(\xi_\beta f - q^{-1}\eta_\beta\{k\}) = \sum_\beta \varphi_\beta F_\beta. \end{aligned}$$

This implies

$$\sum_{1\leqslant j\leqslant n} E_j \mathfrak{B} = \sum_{1\leqslant j\leqslant n} \mathfrak{B} E_j, \qquad \sum_{1\leqslant j\leqslant n} F_j \mathfrak{B} = \sum_{1\leqslant j\leqslant n} \mathfrak{B} F_j. \tag{C$'$}$$

An easy calculation shows the following commutation relations among $E_i$'s, $F_j$'s, and the typical invariants $\zeta_{ij}$'s:

$$\begin{aligned} E_iF_j - F_iE_j &= \xi_i\xi_j f - q^{-1}\xi_i\eta_j\{k\} - \xi_i f \xi_j + q^{-1}\eta_i\{k\}\xi_j \\ &= -\xi_i\eta_j(q^{-1}\{k\} + k) + q^{-1}\eta_i\xi_j\{qk\} \\ &= -(\xi_i\eta_j - q\eta_i\xi_j)\{qk\} = -\zeta_{ij}\{qk\}. \end{aligned}$$

Using this, in a polynomial in $E_i$'s and $F_j$'s with coefficients in $\mathfrak{B}[k, k^{-1}]$, we can rearrange the order of product of $E_i$'s and $F_j$'s. Together with (C$'$), we then have

$$\mathbb{K}[E_1, \ldots, E_n, F_1, \ldots, F_n] \subseteq \mathbb{K}[E_1, \ldots, E_n] \cdot \mathfrak{B}[k, k^{-1}][F_1, \ldots, F_n].$$

As we have shown in the first step, a highest weight vector is a linear span of elements of the form $E_{j_1}^{\epsilon_1} \cdots E_{j_d}^{\epsilon_d} \cdot 1$ with $\epsilon_\kappa = \pm 1$, $E_j^+ = E_j$ and $E_j^- = F_j$. By the above reduction, we can rewrite $E_{j_1}^{\epsilon_1} \cdots E_{j_d}^{\epsilon_d}$ into a sum of elements in $\mathbb{K}[E_1, \ldots, E_n] \cdot \mathfrak{B}[k, k^{-1}][F_1, \ldots, F_n]$. Note $F_j$ kills 1 and that the operator $E_j$ is the left multiplication of $\xi_j$. We see thus $E_{j_1}^{\epsilon_1} \cdots E_{j_d}^{\epsilon_d} \cdot 1$ can be written as an element in $\mathbb{K}[\xi_1, \ldots, \xi_n]\mathfrak{B}$. Taking account of the weight, we have obtained

*Claim* (2).Under the assumption (C), the set of highest weight vectors of weight $\ell$ in $\mathfrak{A}$ coinscides with

$$\sum_{j_1,\cdots,j_\ell} \xi_{j_1}\cdots\xi_{j_\ell}\mathfrak{B}.$$

In particular, the fixed point subalgebra $\mathfrak{A}^{\mathsf{U}_q(\mathfrak{sl}_2)}$ coincides with the algebra $\mathfrak{B}$ of typical invariants.

Hence the assertion of the first fundamental theorem.

### 6.3. A DETERMINATION OF $\mathsf{U}_{q^2}(sl_2)$-INVARIANT DIFFERENCE OPERATORS

We will here apply the first fundamental theorem above to determine the commutant of $\omega^{\otimes n}(\mathsf{U}_{q^2}(\mathfrak{sl}_2))$ in a suitable subalgebras of $\mathfrak{D}$. Note under the adjoint action, $\mathfrak{D}$ is an algebra with $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$-symmetry, whereas $\mathsf{A} = \mathbb{K}[x_1,\ldots,x_n]$ is not so under the action $\omega^{\otimes n}$. The adjoint action to $\omega^{\otimes n}$ is given for $\Phi \in \mathrm{End}_{\mathbb{K}}(\mathsf{A})$ by

$$\begin{aligned}
\mathrm{ad}(e)\Phi &= \frac{1}{[2]}(Q\Phi - \gamma^{-2}\Phi\gamma^2 Q),\\
\mathrm{ad}(f)\Phi &= \frac{1}{[2]}([\Phi,\Delta]\cdot q^{-n}\gamma^{-2}),\\
\mathrm{ad}(k)\Phi &= \gamma^2\Phi\gamma^{-2}.
\end{aligned}$$

Let us specify subalgebras of $\mathfrak{D}$, which are generated by $n$ copies of standard representation of $\mathsf{U}_{q^2}(\mathfrak{sl}_2)$. One choice is to start from the $q$-difference operators $\partial_j^\circ = x_j^{-1}\cdot(\gamma_j - \gamma_j^{-1})/(q-q^{-1})$, which clearly commute with $\partial_i$'s, so does with $\Delta$. We have therefore,

$$\mathrm{ad}(f)\partial_j^\circ = 0, \qquad \mathrm{ad}(k)\partial_j^\circ = q^{-2}\partial_j^\circ,$$

a lowest weight vector, which possibly generates the standard representation. Note here, if $\mathrm{ad}(e)^2\partial_j^\circ = 0$ holds, the elements $\eta_j^+ = \partial_j^\circ$ and $\xi_j^+ = \mathrm{ad}(e)\eta_j^+$ certainly generate the 2 dimensional representation. Since $[2]^2\mathrm{ad}(e)^2\partial_j^\circ = Q^2\partial_j^\circ - (q^2 + q^{-2})Q\partial_j^\circ Q + \partial_j^\circ Q^2$, we show the right-hand side vanishes. Using the notation

$$Q_{<j} = \sum_{\ell<j} q^{-\ell+1}x_\ell^2 \quad\text{and}\quad Q_{>j} = \sum_{\ell>j} q^{-\ell+1}x_\ell^2,$$

we have first $Q_{<j}\partial_j^\circ = q^{-2}\partial_j^\circ Q_{<j}$ and $Q_{>j}\partial_j^\circ = q^2\partial_j^\circ Q_{>j}$, so that

$$\begin{aligned}
&Q^2\partial_j^\circ - (q^2+q^{-2})Q\partial_j^\circ Q + \partial_j^\circ Q^2\\
&\quad = (Q_{<j}+Q_{>j})\partial_j^\circ(q^2Q_{<j}+q^{-2}Q_{>j})\\
&\qquad - (q^2+q^{-2})(Q_{<j}+Q_{>j})\partial_j^\circ(Q_{<j}+Q_{>j})\\
&\qquad + (q^{-2}Q_{<j}+q^2Q_{>j})\partial_j^\circ(Q_{<j}+Q_{>j})\\
&\qquad + q^{-j+1}(x_j^2(Q_{<j}+Q_{>j})\partial_j^\circ + (Q_{<j}+Q_{>j})x_j^2\partial_j^\circ)\\
&\qquad - q^{-j+1}(q^2+q^{-2})(x_j^2\partial_j^\circ(Q_{<j}+Q_{>j}) + (Q_{<j}+Q_{>j})\partial_j^\circ x_j^2)\\
&\qquad + q^{-j+1}(\partial_j^\circ x_j^2(Q_{<j}+Q_{>j}) + \partial_j^\circ(Q_{<j}+Q_{>j})x_j^2).
\end{aligned}$$

Here in the first three terms, expand them and look at the coefficients. Then $Q_{<j}\partial_j^\circ Q_{<j}$ and $Q_{>j}\partial_j^\circ Q_{>j}$ vanish and the coefficients of $Q_{<j}\partial_j^\circ Q_{>j}$ and $Q_{>j}\partial_j^\circ Q_{<j}$ are respectively $q^{-2}-q^2$ and its negative. Noting $Q_{<j}\partial_j^\circ Q_{>j} = Q_{>j}\partial_j^\circ Q_{<j}$, we see all those terms vanish. The remaining terms devided by $q^{-j+1}$ sum up to

$$\begin{aligned}
&x_j\{\gamma_j\}(q^{-2}Q_{<j}+q^2Q_{>j}) + x_j\{\gamma_j\}(q^2Q_{<j}+q^{-2}Q_{>j})\\
&\quad -(q^2+q^{-2})x_j\{\gamma_j\}(Q_{<j}+Q_{>j}) - (q^2+q^{-2})(Q_{<j}+Q_{>j})x_j\{q^2\gamma_j\}\\
&\quad +(q^{-2}Q_{<j}+q^2Q_{>j})x_j\{q^2\gamma_j\} + (q^2Q_{<j}+q^{-2}Q_{>j})x_j\{q^2\gamma_j\},
\end{aligned}$$

in which the sum of the first or latter three terms respectively ends up to zero. Hence the assertion.

An easier calculation shows

$$\begin{aligned}
\xi_j^+ &= \mathrm{ad}(e)\eta_j^+ = \frac{1}{[2]}(Q\partial_j^\circ - q^2\partial_j^\circ Q)\\
&= -q^2\left(q^{-j+1}x_j\gamma_j + (q-q^{-1})\left(\sum_{\ell<j} q^{-\ell+1}x_\ell^2\right)\partial_j^\circ\right).
\end{aligned}$$

We denote by $\mathfrak{D}_+$ the subalgebra of $\mathfrak{D}$ generated by these $\eta_j^+ = \partial_j^\circ$ and $\xi_j^+ = \mathrm{ad}(e)\eta_j^+$. The typical invariant $\zeta_{ij}^+$ here in $\mathfrak{D}_+$ is calculated from

$$\begin{aligned}
\zeta_{ij}^+ &= \xi_i^+\eta_j^+ - q^{-2}\eta_i^+\xi_j^+\\
&= \frac{1}{[2]}(Q\partial_i^\circ\partial_j^\circ - (q^2+q^{-2})\partial_i^\circ Q\partial_j^\circ + \partial_i^\circ\partial_j^\circ Q)
\end{aligned}$$

by a bit complicated but similar calcualtion to the above. The result is for $i<j$,

$$\zeta_{ij}^+ = q^{-j}\theta_{ji}^+, \qquad \zeta_{ji}^+ = -q^3\zeta_{ij}^+ = -q^{-j+3}\theta_{ji}^+, \qquad \zeta_{ii}^+ = q^{-i+1}.$$

Thus we naturally meet with the elements $\theta^{+}_{ji}$ again. We see from the definition and the computations in Section 4 that the ring $\mathfrak{B}_+$ of typical invariants in $\mathfrak{D}_+$ is generated by $\theta_j$'s.

Another choice is to start from the right multiplication $x^\circ_j$, which clearly commutes with $Q$. Putting $\xi^-_j = x^\circ_j\gamma$, we see

$$\begin{aligned}\mathrm{ad}(e)\xi^-_j &= \frac{1}{[2]}(Qx^\circ_j\gamma - \gamma^{-2}x^\circ_j\gamma^3 Q) \\ &= \frac{1}{[2]}(Qx^\circ_j - x^\circ_j Q)\gamma = 0.\end{aligned}$$

From the definition, we see $q^{2n}[2]^2\,\mathrm{ad}(f)^2(x_j\gamma) = (\Delta^2\partial_j - (q^2+q^{-2})\Delta\partial_j\Delta + \partial_j\Delta^2)\gamma^{-1}$. With respect to the Fischer inner product in Section 3, this is essentially the adjoint $(\mathrm{ad}(e)^2\partial^\circ_j)^\dagger$, which we have shown to vanish. Thus again the two elements $\xi^-_j$ and $\eta^-_j = \mathrm{ad}(f)\xi^-_j$ give the standard representation. We denote by $\mathfrak{D}_-$ the algebra generated by these $2n$ elements. The expression of $\eta^-_j = \mathrm{ad}(f)\xi^-_j = \mathrm{ad}(f)(x^\circ_j\gamma)$ is observed from

$$\begin{aligned}\mathrm{ad}(f)(x^\circ_j\gamma) &= \frac{1}{[2]}q^{-n-2}(x^\circ_j\Delta - q^2\Delta x^\circ_j)\gamma^{-1} \\ &= \frac{1}{[2]}q^{-3}(Q\partial^\circ_j - q^2\partial^\circ_j Q)^\dagger\gamma^{-1} \\ &= q^{-3}(\mathrm{ad}(e)\eta^+_j)^\dagger\gamma^{-1} = q^{-3}(\xi^+_j)^\dagger\gamma^{-1} \\ &= -q^{-2}\gamma^{-1}\left(q^{-j+1}\gamma_j\partial_j + q^{-n+1}(q-q^{-1})x^\circ_j\textstyle\sum_{\ell<j} q^{n-\ell}\partial^2_\ell\right).\end{aligned}$$

Thus $\xi^-_j = \eta^{+\dagger}_j\gamma$ and $\eta^-_j = q^{-4}\gamma^{-1}\xi^{+\dagger}_j$ hold and the relation of typical invariants between the $\mathfrak{D}_+$ and $\mathfrak{D}_-$ are given by

$$\begin{aligned}\zeta^-_{ij} &= \xi^-_i\eta^-_j - q^{-2}\eta^-_i\xi^-_j = q^{-4}(\eta^{+\dagger}_i\xi^{+\dagger}_j - q^{-2}\xi^{+\dagger}_i\eta^{+\dagger}_j) \\ &= q^{-4}(\xi^+_j\eta^+_i - q^{-2}\eta^+_j\xi^+_i)^\dagger = q^{-4}\zeta^{+\dagger}_{ji}.\end{aligned}$$

From the computations of the typical invariants for $\mathfrak{D}_+$ and Lemma 3.1.1, we obtain for $i<j$

$$\zeta^-_{ij} = q^{-i-2}\theta^-_{ji}, \qquad \zeta^-_{ji} = -q^{-3}\zeta^-_{ij} = -q^{-i-5}\theta^-_{ji}, \qquad \zeta^-_{ii} = q^{-i-3}.$$

By the same reason as for $\mathfrak{D}_+$, the ring $\mathfrak{B}_-$ of typical invariants in $\mathfrak{D}_-$ is generated by $\theta_j$'s, whence it coincides with $\mathfrak{B}_+$.

To apply the first fundamental theorem established above, we need to check the condition (C) for the algebras $\mathfrak{D}_+, \mathfrak{D}_-$. This is seen from Lemma 4.3.2, which tells

$$\begin{cases} \theta_j \xi_j^- - q^{-1}\xi_j^- \theta_j = \xi_{j+1}^-, \\ \theta_j \xi_{j+1}^- - q\xi_{j+1}^- \theta_j = -\xi_j^-, \\ \theta_j \xi_i^- - \xi_i^- \theta_j = 0 \quad (i \neq j, j+1), \end{cases}$$

the condition (C) for the algebra $\mathfrak{D}_-$, and also

$$\begin{cases} \theta_j \eta_j^+ - q\eta_j^+ \theta_j = q\eta_{j+1}^+, \\ \theta_j \eta_{j+1}^+ - q^{-1}\eta_{j+1}^+ \theta_j = -q^{-1}\eta_j^+, \\ \theta_j \eta_i^+ - \eta_i^+ \theta_j = 0 \quad (i \neq j, j+1), \end{cases}$$

which assures the condition (C) for $\mathfrak{D}_+$, because $\eta_j$'s and $\xi_j$'s are transformed with each other by the adjoint action of $U_{q^2}(\mathfrak{sl}_2)$.

Thus by the first fundamental theorem, we come to the following

THEOREM 6.3. *The set of invariant difference operators in $\mathfrak{D}_+$ or $\mathfrak{D}_-$ is generated by the typical invariants:*

$$\mathfrak{D}_+^{U_{q^2}(\mathfrak{sl}_2)} = \mathfrak{D}_-^{U_{q^2}(\mathfrak{sl}_2)} = \mathbb{K}[\theta_1, \cdots, \theta_{n-1}].$$

*Remarks 6.3.1.* (1) In the above, when we transfer from $\mathfrak{D}_+$ to $\mathfrak{D}_-$, we utilize the adjoint $\dagger$ with respect to the Fischer inner product. Note, however, they are not transferred by $\dagger$ with the action of $U_{q^2}(\mathfrak{sl}_2)$.

(2) There are several choices for the comultiplication rule of $U_{q^2}(\mathfrak{sl}_2)$. If we use other comultiplication rule, then the adjoint action accordingly changes, so that the corresponding subalgebras $\mathfrak{D}_+$ and $\mathfrak{D}_-$ also get changed.

For example, if we adopt the comultiplication $\tilde{\Delta}$ as

$$\begin{cases} \tilde{\Delta}(e) &= e \otimes k^{-1} + 1 \otimes e, \\ \tilde{\Delta}(f) &= f \otimes 1 + k \otimes f, \\ \tilde{\Delta}(k) &= k \otimes k, \end{cases}$$

then the realization of the tensor power of the oscillator representation gets accordingly changed as

$$\tilde{\omega}^{\otimes n}(e) = \frac{1}{[2]} Q^\bullet, \qquad \tilde{\omega}^{\otimes n}(f) = -\frac{1}{[2]} \Delta^\bullet,$$

$$\tilde{\omega}^{\otimes n}(k) = q^n \gamma_1^2 \cdots \gamma_n^2 = q^n \gamma^2,$$

with

$$\begin{cases} Q^\bullet &= q^{-n+1}x_1^{\circ 2} + q^{-n+2}x_2^{\circ 2} + \cdots + x_n^{\circ 2}, \\ \Delta^\bullet &= \partial_1^{\circ 2} + q\partial_2^{\circ 2} + \cdots + q^{n-1}\partial_n^{\circ 2}. \end{cases}$$

Also the adjoint action $\widetilde{\mathrm{ad}}$ changes as

$$\begin{aligned} \widetilde{\mathrm{ad}}(e)\Phi &= \frac{1}{[2]}([Q^\bullet, \Phi] \cdot q^n\gamma^2), \\ \widetilde{\mathrm{ad}}(f)\Phi &= \frac{1}{[2]}(\gamma^2\Phi\gamma^{-2}\Delta^\bullet - \Delta^\bullet\Phi), \\ \widetilde{\mathrm{ad}}(k)\Phi &= \gamma^2\Phi\gamma^{-2}. \end{aligned}$$

With this change, we have another sublagebras $\tilde{\mathfrak{D}}_+$ and $\tilde{\mathfrak{D}}_-$ generated respectively by

$$\tilde{\eta}_j^+ = \gamma^{-1}\partial_j, \qquad \tilde{\xi}_j^+ = -q^{j-1}x_j^\circ\gamma_j^{-1}\gamma + (q-q^{-1})\left(\sum_{\ell<j} q^{\ell-1}x_\ell^{\circ 2}\right)\partial_j\gamma;$$

$$\tilde{\xi}_j^- = x_j, \qquad \tilde{\eta}_j^- = -q^{j-1}\gamma_j^{-1}\partial_j^\circ + (q-q^{-1})x_j\sum_{\ell<j} q^{\ell-1}\partial_\ell^{\circ 2}.$$

The typical invariants in this case are defined as $\tilde{\xi}_i\tilde{\eta}_j - q^2\tilde{\eta}_i\tilde{\xi}_j$.

(3) In checking the condition (C) for $\mathfrak{D}_+$ and $\mathfrak{D}_-$, we made use of the commutation relations in Lemma 4.3.2. Here we look at those formulas in relation to the adjoint action of $\mathsf{U}_q(\mathfrak{gl}_n)$.

Recall first that $\mathsf{A} = \mathbb{K}[x_1, \ldots, x_n]$ is an algebra with $\mathsf{U}_q(\mathfrak{gl}_n)$-symmetry. Then the usual adjoint action on $\mathrm{End}_{\mathbb{K}}(\mathsf{A})$ is compatible with the original action on $\mathsf{A}$ under the identification to the left multiplication operators: $\mathrm{ad}(a)\varphi = a\,.\,\varphi$ for $\varphi \in \mathsf{A}$. Also for the 'left' $q$-difference operators $\partial_j$, this adjoint action gives the contragredient representation.

To get a harmony with the right multiplications, however, we need to make a flip on the comultiplication of $\mathsf{U}_q(\mathfrak{gl}_n)$. Denote by $\mathrm{ad}'$ the adjoint action on $\mathrm{End}_{\mathbb{K}}(\mathsf{A})$ under the flipped comultiplication: $\mathrm{ad}'(a)\Phi = \sum_\kappa a_\kappa^{(2)}\Phi S^{-1}(a_\kappa^{(1)})$, where $a \in \mathsf{U}_q(\mathfrak{gl}_n)$, $\Phi \in \mathrm{End}_{\mathbb{K}}(\mathsf{A})$, and $\Delta(a) = \sum_\kappa a_\kappa^{(1)} \otimes a_\kappa^{(2)}$. Then we have $\mathrm{ad}'(a)(\Phi\Psi) = \sum_\kappa(\mathrm{ad}'(a_\kappa^{(2)})\Phi)(\mathrm{ad}'(a_\kappa^{(1)})\Psi)$ and $\mathrm{ad}'(a)(\varphi^\circ) = (a\,.\,\varphi)^\circ$. Here for $\varphi \in \mathsf{A}$, we denote by $\varphi^\circ$ the right multiplication operator. Recall the action of $\mathsf{U}_q(\mathfrak{o}_n)$ on $\mathsf{A}$ is defined through $\mathsf{U}_q(\mathfrak{gl}_n)$ and the generators $\theta_j$ are given by $\theta_j = (q-q^{-1})^{-1}q^{-\varepsilon_j}(L_{j\,j+1}^+ -$

$qS(L^-_{j+1\,j})$), so that $\Delta(\theta_j) = \theta_j \otimes q^{\varepsilon_j-\varepsilon_{j+1}} + 1 \otimes \theta_j$ and $S^{-1}(\theta_j) = -q^{-\varepsilon_j+\varepsilon_{j+1}}\theta_j$. Applying the formula above, we see for example

$$\mathrm{ad}'(\theta_j)(x^\circ_i) = \begin{cases} x^\circ_{i+1} & (i=j), \\ -x^\circ_{i-1} & (i=j+1), \\ 0 & (i \neq j, j+1). \end{cases}$$

This gives some formulas in Lemma 4.3.2.

With a quite similar reasoning, since the set of 'right' $q$-difference operators $\partial^\circ_j$ bears the contragredient of the standard representation of $\mathrm{U}_q(\mathfrak{gl}_n)$ under ad$'$ action, we get other formulas in Lemma 4.3.2.

## 7. Central elements of $\mathrm{U}_q(o_n)$ and reflection equations

Behind the fact that the Casimir element $\mathcal{C}$ is central, we have a multiplicative structure and reflection equations. In this section, we explain that mechanism and apply it to get further higher degree central elements of $\mathrm{U}_q(\mathfrak{o}_n)$.

In this section we work in the algebra $\mathrm{U}_q(\mathfrak{gl}_n)$. Let us introduce the following four matrices:

$$K^+ := S(L^+)J^tS(L^-), \qquad K^- := {}^tL^+JL^-;$$

$$\tilde{K}^+ := S(L^-)J^tS(L^+), \qquad \tilde{K}^- := {}^tL^-JL^+.$$

Here $L^\pm$ is the $L$-operators of $\mathrm{U}_q(\mathfrak{gl}_n)$ in matrix form and $J$ is $\mathrm{diag}(q^{n-1}, q^{n-2}, \ldots, 1)$, which corresponds to the quadratic form defining the quantum $\mathfrak{o}_n$. Since $L^+$ (resp. $L^-$) is upper (resp. lower) triangular, the matrices $K^+, \tilde{K}^-$ (resp. $K^-, \tilde{K}^+$) are upper (resp. lower) triangular. It is also readily seen that their diagonal components coincide with $J$. The first claim is that the subdiagonal elements of these matrices $K^\pm$'s coincide with the elements $\Theta_j = (q-q^{-1})^{-1}q^{-\varepsilon_j}(L^+_{j\,j+1} - qS(L^-_{j+1\,j}))$ introduced in Section 2:

LEMMA 7.1. *Let $K^\pm_{ij}$ and $\tilde{K}^\pm_{ij}$ denote the $(i,j)$-component of the matrices $K^\pm$ and $\tilde{K}^\pm$ respectively. Then*

$$\begin{aligned} K^+_{j\,j+1} &= -K^-_{j+1\,j} = -(q-q^{-1})q^{n-j-1}\Theta_j, \\ \tilde{K}^+_{j+1\,j} &= -\tilde{K}^-_{j\,j+1} = -(q-q^{-1})q^{n-j}\Theta_j. \end{aligned}$$

*Proof.* As the proofs are all parallel, we only prove one case. By definition we have

$$K^-_{j+1\,j} = ({}^tL^+JL^-)_{j+1\,j} = L^+_{j+1\,j+1}q^{n-j-1}L^-_{j+1\,j} + L^+_{j\,j+1}q^{n-j}L^-_{jj}.$$

Then the expression (2.3) of $\Theta_j$ proves the assertion.

As we will see later in Theorem 7.4, these $K^{\pm}_{ji}$'s give a realization of $U_q(\mathfrak{o}_n)$ in $U_q(\mathfrak{gl}_n)$. The basis of this fact is in the following reflection equations, which also play the central role over the rest of this section.

PROPOSITION 7.2. (Reflection Equations). *We have the following:*

$$K_2^{-}\,{}^{t_1}R_{12}^{+}K_1^{-}R_{12}^{\pm} = R_{12}^{\pm}K_1^{-}\,{}^{t_1}R_{12}^{+}K_2^{-}, \tag{7.2.1}$$

$$K_2^{+}\,{}^{t_1}R_{21}^{+}K_1^{+}R_{21}^{\pm} = R_{21}^{\pm}K_1^{+}\,{}^{t_1}R_{21}^{+}K_2^{+}, \tag{7.2.2}$$

$$K_1^{+}\,{}^{t_2}R_{12}^{-}\tilde{K}_2^{+}R_{12}^{+} = R_{12}^{+}\tilde{K}_2^{+}\,{}^{t_2}R_{12}^{-}K_1^{+} \tag{7.2.3}$$

*and*

$${}^{t_1}R_{12}^{+}K_1^{-}R_{12}^{+}\tilde{K}_2^{+}J_2^{-2} = \tilde{K}_2^{+}J_2^{-2}R_{12}^{+}K_1^{-}\,{}^{t_1}R_{12}^{+}. \tag{7.2.4}$$

*Here ${}^{t_1}R_{12}^{\pm}$ or ${}^{t_2}R_{12}^{\pm}$ denotes respectively the transposition of $R_{12}^{\pm}$ with respect to the first or the second component in the tensor product space. Also we followed the convention on subscripts to indicate the tensor components.*

The following Lemma gives the reflection equations satisfised by $J$, which we use in the proof of Proposition 7.2. Its proof is just a calculation using the explicit form of the $R$-matirces, so we omit it.

LEMMA 7.3. *The constant matrix $J$ satisfies the following two equations:*

$$R_{12}^{+}J_1\,{}^{t_1}R_{12}^{+}J_2 = J_2\,{}^{t_1}R_{12}^{+}J_1R_{12}^{+}, \tag{7.3.1}$$

$$R_{12}^{+}J_2\,{}^{t_2}R_{12}^{-}J_1 = J_1\,{}^{t_2}R_{12}^{-}J_2R_{12}^{+}. \tag{7.3.2}$$

*Proof of Proposition 7.2.*

For (7.2.1)–(7.2.3): Since their proofs are all similar, we will only prove the equation (7.2.3). We recall the commutation relations (Yang–Baxter equations) for the $L$-operators:

$$R^{+}L_1^{\varepsilon}L_2^{\varepsilon} = L_2^{\varepsilon}L_1^{\varepsilon}R^{+} \quad (\varepsilon = \pm) \quad \text{and} \quad R^{+}L_1^{+}L_2^{-} = L_2^{-}L_1^{+}R^{+}.$$

Based on these relations, our calculation goes as follows:

$$\begin{aligned}
&K_1^{+}\,{}^{t_2}R_{12}^{-}\tilde{K}_2^{+}R_{12}^{+}\\
&= {}^{t}\tilde{L}_1^{-}J_1\tilde{L}_1^{+}\,{}^{t_2}R_{12}^{-}\,{}^{t}\tilde{L}_2^{+}J_2\tilde{L}_2^{-}R_{12}^{+}\\
&= {}^{t}\tilde{L}_1^{-}\,{}^{t}\tilde{L}_2^{+}J_1\,{}^{t_2}R_{12}^{-}J_2\tilde{L}_1^{+}\tilde{L}_2^{-}R_{12}^{+} \quad (\because\ \tilde{L}_1^{+}\,{}^{t_2}R_{12}^{-}\,{}^{t}\tilde{L}_2^{+} = {}^{t}\tilde{L}_2^{+}\,{}^{t_2}R_{12}^{-}\tilde{L}_1^{+})\\
&= {}^{t}\tilde{L}_1^{-}\,{}^{t}\tilde{L}_2^{+}J_1\,{}^{t_2}R_{12}^{-}J_2R_{12}^{+}\tilde{L}_2^{-}\tilde{L}_1^{+} \quad (\because\ \tilde{L}_1^{+}\tilde{L}_2^{-}R_{12}^{+} = R_{12}^{+}\tilde{L}_2^{-}\tilde{L}_1^{+})\\
&= {}^{t}\tilde{L}_1^{-}\,{}^{t}\tilde{L}_2^{+}R_{12}^{+}J_2\,{}^{t_2}R_{12}^{-}\tilde{L}_2^{-}J_1\tilde{L}_1^{+} \quad (\because\ (7.3.2))\\
&= R_{12}^{+}\,{}^{t}\tilde{L}_2^{+}J_2\,{}^{t}\tilde{L}_1^{-}\,{}^{t_2}R_{12}^{-}\tilde{L}_2^{-}J_1\tilde{L}_1^{+} \quad (\because\ {}^{t}\tilde{L}_1^{-}\,{}^{t}\tilde{L}_2^{+}R_{12}^{+} = R_{12}^{+}\,{}^{t}\tilde{L}_2^{+}\,{}^{t}\tilde{L}_1^{-})\\
&= R_{12}^{+}\tilde{K}_2^{+}\,{}^{t_2}R_{12}^{-}K_1^{+} \quad (\because\ {}^{t}\tilde{L}_1^{-}\,{}^{t_2}R_{12}^{-}\tilde{L}_2^{-} = \tilde{L}_2^{-}\,{}^{t_2}R_{12}^{-}\,{}^{t}\tilde{L}_1^{-})
\end{aligned}$$

In the course of the above calculation, we freely used the fact that $J_1$ commutes with $M_2$ for any operator $M$, because $J$ is a constant matrix. For the proof of (7.2.1) and (7.2.2), use the formula (7.3.1) instead of (7.3.2).

For (7.2.4): This follows from (7.2.1) and a simple relation $K^-\tilde{K}^+ = J^2$. In fact, (7.2.4) is derived from (7.2.1) by conjugation under $K_2^-$. Recall a basic fact on the square of the antipode $S$ (cf. [RTF, Th 4]):

$$S^2(L^\pm) = J^2 L^\pm J^{-2}. \tag{7.4}$$

These together with the relations $L^-S(L^-) = I$ and ${}^tS^2(L^+){}^tS(L^+) = {}^tS(L^+S(L^+)) = I$, we see that

$$K^-\tilde{K}^+ = {}^tL^+ J^2\, {}^tS(L^+) = J^2\, {}^tS^2(L^+)\, {}^tS(L^+) = J^2.$$

This proves our assertion.

*Remark.* There are other types of reflection equations among $\tilde{K}^\pm$ themselves, $K^-$ and $\tilde{K}^-$ and so on. Since they are not necessary for our later discussion, we will not write them down here.

The above reflection equations have many important consequences. The first one is the following:

THEOREM 7.4. *The map*

$$\Theta\colon \mathrm{U}_q(\mathfrak{o}_n) \ni \Pi_i \longmapsto \Theta_i \in \mathrm{U}_q(\mathfrak{gl}_n)$$

*can be extended to an algebra homomorphism of* $\mathrm{U}_q(\mathfrak{o}_n)$ *to* $\mathrm{U}_q(\mathfrak{gl}_n)$. *More precisely,* $\Theta_i$*'s satisfy the following:*

$$\begin{cases} [\Theta_i, \Theta_j] = 0 \quad \textit{if} \quad |i-j| > 1, \\ \Theta_i^2\Theta_j - (q+q^{-1})\Theta_i\Theta_j\Theta_i + \Theta_j\Theta_i^2 = -\Theta_j \quad \textit{if} \quad |i-j| = 1. \end{cases} \tag{7.4.1}$$

*Further if we define the elements* $\Theta_{ji}^\pm$ *for* $j > i$ *by the formula* $\Theta_{ji}^\pm = \Theta(\Pi_{ji}^\pm)$, *then they are expressed by* $K_{ji}^\pm$*'s and/or* $\tilde{K}_{ji}^\pm$*'s as follows:*

$$\begin{cases} \Theta_{ji}^+ = -(q-q^{-1})^{-1}q^{-n+j}K_{ij}^+, \\ \Theta_{ji}^- = (q-q^{-1})^{-1}q^{-n+i+1}K_{ji}^-, \end{cases} \tag{7.4.2}$$

$$\begin{cases} \Theta_{ji}^+ = -(q-q^{-1})^{-1}q^{-n+j-1}\tilde{K}_{ji}^+, \\ \Theta_{ji}^- = (q-q^{-1})^{-1}q^{-n+i}\tilde{K}_{ij}^-. \end{cases} \tag{7.4.3}$$

*Proof.* The components of the $R$-matrices are by definition given as

$$(R^+)^{\alpha\beta}_{\alpha\beta} = q^{\delta_{\alpha\beta}}, \qquad (R^+)^{\beta\alpha}_{\alpha\beta} = (q-q^{-1})\,\delta(\alpha>\beta),$$

and the others are all zero. Here we used the notation:

$$\delta(i>j) = \begin{cases} 1 & \text{for} \quad i>j, \\ 0 & \text{otherwise.} \end{cases}$$

Let us compute the both sides of the reflecton equation (7.2.1): the $\binom{\ell,m}{\alpha,\beta}$-component of the left-hand side reads as

$$\begin{aligned}(K_2^{-\ t_1}R_{12}^+K_1^-R_{12}^+)^{\ell m}_{\alpha\beta} &= q^{\delta_{\alpha\beta}+\delta_{\ell\beta}}K^-_{m\beta}K^-_{\ell\alpha} + (q-q^{-1})\delta(\alpha>\beta)q^{\delta_{\ell\alpha}}K^-_{m\alpha}K^-_{\ell\beta} \\ &\quad +(q-q^{-1})\delta(\ell>\beta)q^{\delta_{\alpha\beta}}K^-_{m\ell}K^-_{\beta\alpha} \\ &\quad +(q-q^{-1})^2\delta(\alpha>\beta)\delta(\ell>\alpha)K^-_{m\ell}K^-_{\alpha\beta};\end{aligned}$$

and the right-hand side is

$$\begin{aligned}(R_{12}^+K_1^{-\ t_1}R_{12}^+K_2^-)^{\ell m}_{\alpha\beta} &= q^{\delta_{\alpha m}+\delta_{\ell m}}K^-_{\ell\alpha}K^-_{m\beta} + (q-q^{-1})\delta(m>\alpha)q^{\delta_{\ell m}}K^-_{\ell m}K^-_{\alpha\beta} \\ &\quad +(q-q^{-1})\delta(m>\ell)q^{\delta_{\alpha\ell}}K^-_{m\alpha}K^-_{\ell\beta} \\ &\quad +(q-q^{-1})^2\delta(m>\ell)\delta(\ell>\alpha)K^-_{m\ell}K^-_{\alpha\beta}.\end{aligned}$$

Equating these two with $m=\beta+1, \ell=\alpha+1, \beta>\alpha+1$, we see that $\Theta_\beta$ and $\Theta_\alpha$ commutes wtih each other for $\beta-\alpha>1$, because $K^-$ is a lower triangular matrix and $K^-_{j+1\,j} = (q-q^{-1})q^{n-j-1}\Theta_j$ as shown in Lemma 7.1. Similarly, putting $m=\beta+1, \ell=\beta+2, \alpha=\beta$, we obtain the second relation in (7.4.1). Thus proved the first assertion.

From the first assertion, we see that the image $\Theta^\pm_{ji}$ of $\Pi^\pm_{ji}$ under the homomorphism $\Theta$ are recursively given by

$$\begin{cases} \Theta^\pm_{i+1\,i} = \Theta_i, \\ \Theta^\pm_{ji} = \Theta^\pm_{jk}\Theta^\pm_{ki} - q^{\pm 1}\Theta^\pm_{ki}\Theta^\pm_{jk} \quad (i<k<j). \end{cases}$$

Then the second assertion will be proved if $K^\pm_{ji}$'s and $\tilde{K}^\pm_{ji}$'s with suitable correction factors are shown to satisfy the same recursion formula above. This is also a consequence of the reflection equation: put $m>\beta=\ell>\alpha$, then the relation

$$qK^-_{m\beta}K^-_{\beta\alpha} = K^-_{m\beta}K^-_{\beta\alpha} + (q-q^{-1})K^-_{m\alpha}K^-_{\beta\beta}$$

comes out. Since $K^-_{\beta\beta} = q^{n-\beta}$, this coincides with the recursion above. This proves (7.4.2). The proof for (7.4.3) is similar.

COROLLARY 7.5. (1) *The subalgebra generated by the elements* $K^{\pm}_{ji}$*'s coincides with the image* $\Theta(\mathrm{U}_q(\mathfrak{o}_n))$, *whence generated by* $\Theta_j$*'s.*

(2) *The image of the Casimir element* $\mathcal{C}$ *in* $\mathrm{U}_q(\mathfrak{gl}_n)$ *is expressed as*

$$(q-q^{-1})^2 q^{n-2}\Theta(\mathcal{C}) = \mathrm{Tr}(K^- - J)(J - K^+).$$

*Remark.* Those $K^{\pm}$ and $\tilde{K}^{\mp}$ are actually transformed to each other under an involution that transforms $L^{\pm}$ to $\tilde{L}^{\pm} = {}^tS(L^{\mp})$. More precisely, let $*$ be an anti-automorphism defined by $(L^{\pm}_{ij})^* = \tilde{L}^{\pm}_{ij}$ and $q^* = q$. Then we have

$$(K^-_{ij})^* = K^+_{ji} \quad \text{and} \quad (\tilde{K}^-_{ij})^* = \tilde{K}^+_{ji}.$$

Note that this $*$ gives a Hopf $*$-structure on $\mathrm{U}_q(\mathfrak{gl}_n)$ with $q$ 'real'.

Another important consequence of the reflection equations is a description of central elements of higher degrees.

THEOREM 7.6. *For a positive integer* $m$, *put* $X^{[m]} = X(J^{-2}X)^{m-1}$ *with* $X = K^-K^+$. *Then the following equality holds:*

$$(\tilde{K}^+_2 J_2^{-2})^{-1}\, {}^{t_1}R^+_{12} X^{[m]}_1\, {}^{t_2}R^-_{12} (\tilde{K}^+_2 J_2^{-2}) = R^+_{12} X^{[m]}_1 J_2^2 (R^+_{12})^{-1} J_2^{-2}.$$

*As before* ${}^{t_1}R^{\pm}_{12}$ *or* ${}^{t_2}R^{\pm}_{12}$ *denotes respectively the transposition of* $R^{\pm}_{12}$ *with respect to the first or the second component in the tensor space. Taking the trace* $\mathrm{Tr}^{(1)}$ *of the both sides with respect to the first component, we have*

$$(\tilde{K}^+ J^{-2})^{-1}\mathrm{Tr}(X^{[m]}_1)(\tilde{K}^+ J^{-2}) = \mathrm{Tr}(X^{[m]}_1).$$

*In particular,* $\mathrm{Tr}\, X^{[m]}$ *commutes with the subalgebra* $\Theta(\mathrm{U}_q(\mathfrak{o}_n))$ *of* $\mathrm{U}_q(\mathfrak{gl}_n)$.

The following two lemmas are small calculations for the theorem.

LEMMA 7.7. *The following commutation relations between the R-matrices and* $J$ *hold:*

$$(J_1^{-2} R^+_{12} J_1^2)^{-1} = J_2^2 (R^+_{12})^{-1} J_2^{-2}, \tag{7.7.1}$$

$$({}^{t_1}R^+_{12} J_1^2)^{-1} = {}^{t_2}R^-_{12} J_1^{-2}. \tag{7.7.2}$$

LEMMA 7.8. *For any* $n \times n$ *matrix* $X$ *we have*

$$\mathrm{Tr}^{(1)}({}^{t_1}R^+_{12} X_1\, {}^{t_2}R^-_{12}) = (\mathrm{Tr}\, X) I, \tag{7.8.1}$$

$$\mathrm{Tr}^{(1)}(R^+_{12} X_1 J_2^2 (R^+_{12})^{-1} J_2^{-2}) = (\mathrm{Tr}\, X) I. \tag{7.8.2}$$

*Proof of Lemmas 7.7 and 7.8.* The equality (7.7.1) follows from the commutativity of $J_1 J_2$ and $R^+$, which is easy to see.

For any $n \times n$ matrix $X$, by a direct computation we obtain

$$\begin{aligned}({}^{t_1}R^+_{12} X_1\, {}^{t_2}R^-_{12})^{xy}_{zw} &= \sum_{\alpha,\beta,\gamma,\delta} ({}^{t_1}R^+_{12})^{xy}_{\alpha\beta} X_{\alpha\gamma}\delta_{\beta\delta}({}^{t_2}R^-_{12})^{\gamma\delta}_{zw} \\ &= q^{\delta_{xy}}\{X_{xz}q^{-\delta_{zw}}\delta_{yw} - (q-q^{-1})X_{xy}\delta(y>z)\delta_{zw}\} \\ &\quad +(q-q^{-1})\delta_{xy}\left\{X_{wz}q^{-\delta_{zw}}\delta(x>w)\right. \\ &\quad \left. -(q-q^{-1})\sum_{x>\alpha} X_{\alpha\alpha}\delta_{zw}\delta(\alpha>z)\right\}.\end{aligned}$$

For (7.7.2), put here $X = J$. Then the sum of geometric progression leads to the desired result.

For (7.8.1), putting $z = x$, we get

$$({}^{t_1}R^+_{12} X_1\, {}^{t_2}R^-_{12})^{xy}_{xw} = X_{xx}\delta_{yw} - (q-q^{-1})X_{wy}\delta(y>w)(\delta_{xw} - \delta_{xy}).$$

From this we have

$$\operatorname{Tr}^{(1)}({}^{t_1}R^+_{12} X_1\, {}^{t_2}R^-_{12})^{y}_{w} = (\operatorname{Tr} X)\delta_{yw}.$$

This completes the proof of (7.8.1). The proof of (7.8.2) is similar.

*Remark.* Similar type of trace identities also hold for the quantum trace $\operatorname{Tr}_q$ defined by $\operatorname{Tr}_q X = \operatorname{Tr} JX$.

*Proof of Theorem 7.6.* We observe first that

$$\begin{aligned}{}^{t_1}R^+_{12} X_1\, {}^{t_2}R^-_{12}\tilde{K}^+_2 R^+_{12} &= {}^{t_1}R^+_{12} K^-_1 K^+_1\, {}^{t_2}R^-_{12}\tilde{K}^+_2 R^+_{12} \\ &= {}^{t_1}R^+_{12} K^-_1 R^+_{12}\tilde{K}^+_2 R^+_{12}\, {}^{t_2}R^-_{12} K^+_1 \quad (\because (7.2.3)) \\ &= \tilde{K}^+_2 J_2^{-2} R^+_{12} K^-_1\, {}^{t_1}R^+_{12} J_2^2\, {}^{t_2}R^-_{12} K^+_1 \quad (\because (7.2.4)).\end{aligned}$$

Hence we have

$$\begin{aligned}&(\tilde{K}^+_2 J_2^{-2})^{-1}({}^{t_1}R^+_{12} X_1\, {}^{t_2}R^-_{12})(\tilde{K}^+_2 J_2^{-2}) \\ &= R^+_{12} K^-_1\, {}^{t_1}R^+_{12} J_2^2\, {}^{t_2}R^-_{12} K^+_1 (R^+_{12})^{-1} J_2^{-2}.\end{aligned}$$

Using a formula ${}^{t_1}R^+_{12} J_2^2\, {}^{t_2}R^-_{12} = J_2^2$, which is equivalent to (7.7.2), we see the right-hand side of this formula turns to

$$(\tilde{K}^+_2 J_2^{-2})^{-1}({}^{t_1}R^+_{12} X_1\, {}^{t_2}R^-_{12})(\tilde{K}^+_2 J_2^{-2}) = R^+_{12} X_1 J_2^2 (R^+_{12})^{-1} J_2^{-2}. \tag{7.9}$$

Thus we get to the first formula

$$\begin{aligned}
&(\tilde{K}_2^+ J_2^{-2})^{-1}({}^{t_1}R_{12}^+ X_1^{[m]}\,{}^{t_2}R_{12}^-)(\tilde{K}_2^+ J_2^{-2}) \\
&= (\tilde{K}_2^+ J_2^{-2})^{-1}({}^{t_1}R_{12}^+ X_1\,{}^{t_2}R_{12}^-)J_1^{-2} \\
&\quad\times ({}^{t_1}R_{12}^+ X_1\,{}^{t_2}R_{12}^-)J_1^{-2}\cdots({}^{t_1}R_{12}^+ X_1\,{}^{t_2}R_{12}^-)(\tilde{K}_2^+ J_2^{-2}) \quad (\because (7.7.2)) \\
&= (R_{12}^+ X_1 J_2^2 (R_{12}^+)^{-1} J_2^{-2})^{[m]} \quad (\because (7.9)) \\
&= R_{12}^+ X_1^{[m]} J_2^2 (R_{12}^+)^{-1} J_2^{-2} \quad (\because (7.7.1))
\end{aligned}$$

as desired.

The second formula in Theorem 7.6 follows from the first one and (7.8.2). It then proves that $\mathrm{Tr}\,X^{[m]}$ commutes with all the elements $K_{ji}^{\pm}$, because they are written by $\Theta_j$'s which are the subdiagonal components of the matrix $\tilde{K}^+$ (see Lemma 7.1 and Corollary 7.5 (1)).

*Remark* 7.9. (1) The homomorphism $\Theta\colon \mathrm{U}_q(\mathfrak{o}_n) \to \mathrm{U}_q(\mathfrak{gl}_n)$ is actually injective. This can be proved by a careful discussion based on the Diamond Lemma (cf. [B]). The elements we gave in Theorem 7.6 are, therefore, said to be central in $\mathrm{U}_q(\mathfrak{o}_n)$.

(2) Noting the formula in Corollary 7.5 (2),

$$(q - q^{-1})^2 q^{n-2}\Theta(C) = \frac{q^{2n} - 1}{q - 1} - \mathrm{Tr}(K^- K^+)$$

and the Remark (1) above, we see an alternating proof via Theorem 7.6 that the Casimir is in the center.

## Appendix: Mock Capelli identity

The Capelli identity we got has the definite meaning that it equates two central elements of the algebra in duality. As we saw, however, its right-hand side seems pretty complicated when it is written down in terms of $q$-difference operators. It contains 'fourth order' operators in general. Here we give some similar-looking identity, which contains only 'second order' difference operators in both sides. Though it has some relations with additive realization of the analogue of $\mathfrak{o}_n$ in [N4], its meaning is still obscure. Instead it has certainly a merit of simplicity. It is interesting to compare these two formulas.

Let us put $\vartheta_{ji}^{\pm} = x_j\partial_i - q^{\pm(j-i-1)}x_i\partial_j$ for $i < j$. These are sort of 'principal part' for the $\theta_{ji}^{\pm}$. We have then a *mock* Capelli identity as follows.

PROPOSITION. (Mock Capelli identity)

$$Q\Delta - \{\gamma\}\{q^{n-2}\gamma\} = \sum_{i<j} q^{n-i-j+1}\vartheta_{ji}^{-}\vartheta_{ji}^{+}.$$

*Proof.* We give here a proof partly based on an exterior calculus. First we show a *fake* Capelli identity, which can be derived naturally from an exterior calculus, though it still needs a further computations to reach the mock Capelli identity. Let us consider the exterior algebra $\Lambda$ generated by the two elements $\mathsf{e}, \mathsf{f}$: $\mathsf{e}^2 = 0, \mathsf{f}^2 = 0, \mathsf{ef} + \mathsf{fe} = 0$. We extend this to $\mathrm{End}_{\mathbb{K}}(\mathsf{A}) \otimes \Lambda$ endowed with the natural algebra structure: the elements in $\mathrm{End}_{\mathbb{K}}(\mathsf{A})$ commute with $\mathsf{e}$ and $\mathsf{f}$. Put $\omega_i = \gamma_1^{-1} \cdots \gamma_{i-1}^{-1}\gamma_i \cdots \gamma_n$ for $i = 1, \ldots, n+1$. The following are easily seen from the definition

$$x_i\partial_i = (q - q^{-1})^{-1}(\omega_i - \omega_{i+1}), \qquad \partial_i x_i = (q - q^{-1})^{-1}(q\omega_i - q^{-1}\omega_{i+1}).$$

Let us introduce elements $\vartheta_{ji}^{\pm(a)} = x_j\partial_i - q^{\pm(j-i-1+a)}x_i\partial_j$ with shifts in the exponents. Then the fake Capelli identity is

LEMMA (Fake Capelli identity).

$$\begin{aligned} Q\Delta - \{\gamma\}\{q^n\gamma\} &= \sum_{i<j} q^{n+1-i-j}\vartheta_{ji}^{-(2)}\vartheta_{ji}^{+(2)} \\ &\quad - \sum_i q^{n-2i+1}x_i\partial_i(\omega_i + \omega_{i+1}). \end{aligned}$$

*Proof.* Put $\mathsf{w}_i = q^{-i+1}x_i\mathsf{e} + \partial_i\mathsf{f} \quad (i = 1, \ldots, n)$ and

$$\mathsf{u} = \sum_{1\leqslant i\leqslant n} x_i\mathsf{w}_i, \qquad \mathsf{v} = \sum_{1\leqslant i\leqslant n} q^{n-i}\partial_i\mathsf{w}_i.$$

Then from the definition, it is not hard to see

$$\mathsf{u} = Q\mathsf{e} + \{\gamma\}\mathsf{f}, \qquad \mathsf{v} = \{q^n\gamma\}\mathsf{e} + \Delta\mathsf{f}.$$

Here we used the equality $\sum_{i=1}^n q^{n-2i+1}\partial_i x_i = \{q^n\gamma\}$, which is easily checked (see below). Multiplying these $\mathsf{u}$ and $\mathsf{v}$, we get on one hand

$$\mathsf{uv} = (Q\Delta - \{\gamma\}\{q^n\gamma\})\mathsf{ef}.$$

On the other hand this should be expressed through $\mathsf{w}_i$'s. To carry it out, we need some calculations:

LEMMA. *Putting in general* $\mathsf{w}_i^{(a)} = q^{1-i+a}x_i\mathsf{e} + \partial_i\mathsf{f}$, *we have the following.*

(1) *For* $i \neq j$,

$$\mathsf{w}_i^{(a)}\mathsf{w}_j^{(b)} = (q^{-i+1+a}x_i\partial_j - q^{-j+1+b}x_j\partial_i)\mathsf{ef}.$$

*In particular, we have*

$$\mathsf{w}_i^{(a)}\mathsf{w}_j^{(b)} = -\mathsf{w}_j^{(b)}\mathsf{w}_i^{(a)}, \qquad \mathsf{w}_i^{(a)}\mathsf{w}_j^{(b)} = -q^{-j+1+b}\vartheta_{ji}^{+(a-b+1)}\mathsf{ef}.$$

(2) *For* $i < j$,

$$\mathsf{w}_i^{(a)}\partial_j = q^{-1}\partial_j\mathsf{w}_i^{(a+1)}, \qquad \mathsf{w}_j^{(a)}\partial_i = q\partial_i\mathsf{w}_j^{(a-1)}.$$

(3) *For* $i = j$:

$$\mathsf{w}_i\partial_i\mathsf{w}_i = -q^{-i+1}\partial_i(\omega_i + \omega_{i+1})\mathsf{ef}.$$

*Proof.* Except for (3), those formulas can be checked directly from the definitions. For (3), recall first $\partial_i x_i - q^{-1}x_i\partial_i = \omega_i$ and $\partial_i x_i - q x_i\partial_i = \omega_{i+1}$. We see from this

$$\begin{aligned}\mathsf{w}_i\partial_i &= q^{1-i}x_i\partial_i\mathsf{e} + (\partial_i)^2\mathsf{f}\\ &= q^{2-i}(\partial_i x_i - \omega_i)\mathsf{e} + (\partial_i)^2\mathsf{f}\\ &= \partial_i(q^{2-i}x_i\mathsf{e} + \partial_i\mathsf{f}) - q^{2-i}\omega_i\mathsf{e}.\end{aligned}$$

Multiply $\mathsf{w}_i$ from the right and note $\mathsf{ew}_i = \partial_i\mathsf{ef}$. Then we have

$$\begin{aligned}\mathsf{w}_i\partial_i\mathsf{w}_i &= q^{1-i}\partial_i(qx_i\partial_i - \partial_i x_i)\mathsf{ef} - q^{2-i}\omega_i\partial_i\mathsf{ef}\\ &= -q^{1-i}\partial_i\omega_{i+1}\mathsf{ef} - q^{1-i}\partial_i\omega_i\mathsf{ef}.\end{aligned}$$

Hence the Lemma.

Given these formulas, we can proceed now to the computation of uv as

$$\begin{aligned}\mathsf{uv} &= \sum_{1\leqslant i,j\leqslant n} q^{n-j}x_i\mathsf{w}_i\partial_j\mathsf{w}_j\\ &= \sum_{i<j} q^{n-j}x_i\mathsf{w}_i\partial_j\mathsf{w}_j + \sum_{i<j} q^{n-i}x_j\mathsf{w}_j\partial_i\mathsf{w}_i + \sum_i q^{n-i}x_i\mathsf{w}_i\partial_i\mathsf{w}_i\\ &= \sum_{i<j} q^{n-j-1}x_i\partial_j\mathsf{w}_i^{(1)}\mathsf{w}_j + \sum_{i<j} q^{n-i+1}x_j\partial_i\mathsf{w}_j^{(-1)}\mathsf{w}_i\\ &\quad + \sum_i q^{n-i}x_i\mathsf{w}_i\partial_i\mathsf{w}_i\\ &= \sum_{i<j}(-q^{n-2j}x_i\partial_j\vartheta_{ji}^{+(2)} + q^{n-i-j+1}x_j\partial_i\vartheta_{ji}^{+(2)})\mathsf{ef}\\ &\quad - \sum_i q^{n-2i+1}x_i\partial_i(\omega_i + \omega_{i+1})\mathsf{ef}.\end{aligned}$$

This proves the fake Capelli identity.

*Transition from the fake to mock Capelli identity*: Note first $\vartheta_{ji}^{\pm(2)} = \vartheta_{ji}^{\pm} \mp q^{\pm(j-i)}(q-q^{-1})x_i\partial_j$. Plugging this into the part of $\sum_{i<j}$ of the right-hand side of fake Capelli identity, we get

$$\begin{aligned}\sum_{i<j} &= \sum_{i<j} q^{n-i-j+1}(\vartheta_{ji}^{-} + (q-q^{-1})q^{-j+i}x_i\partial_j)(\vartheta_{ji}^{+} - (q-q^{-1})q^{j-i}x_i\partial_j) \\ &= \sum_{i<j} q^{n-i-j+1}\vartheta_{ji}^{-}\vartheta_{ji}^{+} + (q-q^{-1})\sum_{i<j} q^{n-i-j+1}(q^{-j+i}x_i\partial_j\vartheta_{ji}^{+} \\ &\quad -q^{j-i}\vartheta_{ji}^{-}x_i\partial_j - (q-q^{-1})(x_i\partial_j)^2).\end{aligned}$$

Here sum up the following two in the second summation:

$$\begin{aligned} q^{-j+i}x_i\partial_j\vartheta_{ji}^{+} &= x_i\partial_j(q^{-j+i}x_j\partial_i - q^{-1}x_i\partial_j) \\ &= q^{-j+i}x_i\partial_j x_j\partial_i + q^{-1}(x_i\partial_j)^2 \\ -q^{j-i}\vartheta_{ji}^{-}x_i\partial_j &= -(q^{j-i}x_j\partial_i - qx_i\partial_j)x_i\partial_j \\ &= -q^{j-i}x_j\partial_i x_i\partial_j + q(x_i\partial_j)^2. \end{aligned}$$

Then with the cancellation of the terms for $(x_i\partial_j)^2$, it turns up to

$$\begin{aligned} &(q-q^{-1})\sum_{i<j} q^{n-i-j+1}(q^{-j+i+1}(x_i\partial_i)(\partial_j x_j) - q^{j-i-1}(x_j\partial_j)(\partial_i x_i)) \\ &= (q-q^{-1})\sum_{i<j} q^{n-2j+2}(x_i\partial_i)(\partial_j x_j) \\ &\quad - \sum_{j<i}(q-q^{-1})q^{n-2j}(x_i\partial_i)(\partial_j x_j). \end{aligned}$$

Let us first add up with respect to $j$ in this double summation, namely compute

$$S_i = (q-q^{-1})\left(\sum_{i<j} q^{n-2j+2}(\partial_j x_j) - \sum_{j<i} q^{n-2j}(\partial_j x_j)\right).$$

Using $(q-q^{-1})\partial_j x_j = q\omega_j - q^{-1}\omega_{j+1}$, we see

$$\begin{aligned} (q-q^{-1})\sum_{j=1}^{i-1} q^{n-2j}\partial_j x_j &= \sum_{j=1}^{i-1} q^{n-2j}(q\omega_j - q^{-1}\omega_{j+1}) \\ &= q^{n-1}\omega_1 - q^{n-2i+1}\omega_i, \end{aligned}$$

$$(q - q^{-1}) \sum_{j=i+1}^{n} q^{n-2j+2} \partial_j x_j = \sum_{j=i+1}^{n} q^{n-2j+2} (q\omega_j - q^{-1}\omega_{j+1})$$
$$= q^{n-2i+1}\omega_{i+1} - q^{-n+1}\omega_{n+1}.$$

Note here $\omega_1 = \gamma$ and $\omega_{n+1} = \gamma^{-1}$. Then the second summation $\sum_i x_i \partial_i S_i$ in the right-hand side of fake Capelli identity becomes

$$\sum_i x_i \partial_i S_i = -\sum_i x_i \partial_i (q^{n-1}\gamma + q^{-n+1}\gamma^{-1})$$
$$+ \sum_i q^{n-2i+1} x_i \partial_i (\omega_i + \omega_{i+1})$$
$$= -\{\gamma\}(q^{n-1}\gamma + q^{-n+1}\gamma^{-1}) + \sum_i q^{n-2i+1} x_i \partial_i (\omega_i + \omega_{i+1}).$$

We have thus obtained the identity from these

$$Q\Delta - \{\gamma\}\{q^n\gamma\} = \sum_{i<j} q^{n-i-j+1} \vartheta_{ji}^{-} \vartheta_{ji}^{+} - \{\gamma\}(q^{n-1}\gamma + q^{-n+1}\gamma^{-1}).$$

An easy calculation shows that $\{q^n\gamma\} - \{q^{n-2}\gamma\} = q^{n-1}\gamma + q^{-n+1}\gamma^{-1}$, which concludes the mock Capelli identity.

## References


[Ab] Abe, E.: *Hopf Algebras*, Cambridge Math. Tracts 74, Cambridge Univ. Press, 1980.

[B] Bergman, G. M.: The diamond lemma for ring theory, *Adv. Math.* 29 (1978) 178–218.

[D] Drinfel'd, V. G.: Quantum groups, in *Proc. Int. Cong. Math. Berkeley*, 1986, pp. 798–820.

[F] Fischer, E.: Über algebraische Modulsysteme und lineare homogene partielle Differentialgleichungen mit konstaten Koeffizienten, *J. Reine Angew. Math.* 140 (1911) 48–81.

[GK] Gavrilik, A. M. and Klimyk, A. U.: $q$-Deformed orthogonal and pseudo-orthogonal algebras and their representations, *Lett. Math. Phys.* 21 (1991) 215–220.

[Ha] Hayashi, T.: $Q$-analogues of Clifford and Weyl algebras – spinor and oscillator representations of quantum enveloping algebras, *Comm. Math. Phys.* 127 (1990) 129–144.

[HiW] Hibi, T. and Wakayama, M.: A $q$-analogue of Capelli's identity for $\mathrm{GL}_q(2)$, *Adv. in Math.* (to appear).

[H1] Howe, R.: Remarks on classical invariant theory, *Trans. Amer. Math. Soc.* 313 (1989) 539–570; Erratum, *Trans. Amer. Math. Soc.* 318 (1990) p. 823.

[H2] Howe, R.: Dual pairs in physics: Harmonic oscillators, photons, electrons, and singletons, in *Lectures in Applied Math.* vol. 21, 1985, pp. 179–207.

[HU] Howe, R. and Umeda, T.: The Capelli identity, the double commutant theorem, and multiplicity-free actions, *Math. Ann.* 290 (1991) 565–619.

[Ja] Jackson, F. H.: On $q$-functions and a certain difference operator, *Trans. Roy. Soc. Edinburgh* 46 (1908) 253–281.

[J1] Jimbo, M.: A $q$-difference analogue of $\mathsf{U}(\mathfrak{g})$ and the Yang–Baxter equation, *Lett. Math. Phys.* 10 (1985) 63–69.

[J2] Jimbo, M.: A $q$-analogue of $\mathsf{U}(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation, *Lett. Math. Phys.* 11 (1986) 247–252.

[Na1] Nazarov, M. L.: Quantum Berezinian and the classical Capelli identity, *Lett. Math. Phys.* 21 (1991) 123–131.

[Na2] Nazarov, M. L.: private communication.

[N1] Noumi, M.: A remark on semisimple elements in $U_q(\mathfrak{sl}(2,\mathbb{C}))$, Combinatorial Aspects in Representation Theory and Geometry, RIMS Kôkyûroku 765 (1991) 71–78.

[N2] Noumi, M.: A realization of Macdonald's symmetric polynomials on quantum homogeneous spaces, in Proceedings of the 21st International Conference on Differential Geometric Methods in Theoretical Physics, Tianjin China, 1992.

[N3] Noumi, M.: Quantum Grassmannians and $q$-hypergeometric series, CWI Quarterly 5 (1992) 293–307.

[N4] Noumi, M.: Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces, to appear, *Adv. Math.*.

[NUW1] Noumi, M., Umeda, T. and Wakayama, M.: A quantum analogue of the Capelli identity and an elementary differential calculas on $\mathrm{GL}_q(n)$ *Duke Math. J.* 76 (1994) 567–594.

[NUW2] Noumi, M., Umeda, T. and Wakayama, M.: A quantum dual pair $(\mathfrak{sl}_2, \mathfrak{o}_n)$ and the associated Capelli identity, *Lett. Math. Phys.* 34 (1995) 1–8.

[O] Olshanski, G. I.: Twisted Yangians and infinite-dimensional classical Lie algebras, in Quantum Groups, *Lecture Notes in Math.* 1510, Springer Verlag (1992), pp. 103–120.

[RTF] Reshetikhin, N. Yu., Takhtadzyan, L. A. and Faddeev. L. D.: Quantization of the Lie groups and Lie algebras, *Leningrad Math. J.* 1 (1990) 193–225.

[U] Umeda, T.: Notes on almost homogeneity, preprint (1993).

[UW] Umeda, T. and Wakayama, M.: Another look at the differential operators on quantum matrix spaces and its applications, in preparation.

[Wy] Weyl, H.: The Classical Groups, their Invariants and Representations, Princeton Univ. Press, 1946.