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Weight multiplicity polynomials for affine Kac–Moody algebras of type $A_r^{(1)}$

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Abstract. For the affine Kac–Moody algebras $X_r^{(1)}$ it has been conjectured by Benkart and Kass that for fixed dominant weights λ, μ , the multiplicity of the weight μ in the irreducible $X_r^{(1)}$ -module $L(\lambda)$ of highest weight λ is a polynomial in r which depends on the type X of the algebra. In this paper we provide a precise conjecture for the degree of that polynomial for the algebras $A_r^{(1)}$. To offer evidence for this conjecture we prove it for all dominant weights λ and all weights μ of depth ≤ 2 by explicitly exhibiting the polynomials as expressions involving Kostka numbers.

Key words: Affine Kac–Moody Lie algebras, weight multiplicity, Kostka numbers

Introduction

The representation theory of affine Kac–Moody algebras has played an increasingly important role in statistical mechanics, in conformal field theory, and in string theory. The characters of the irreducible highest weight representations give interesting combinatorial identities, and the string functions and generalized string functions of these representations are modular functions related to theta functions. In this work we develop yet another connection between the representation theory of affine Kac–Moody algebras and combinatorics.

We consider the untwisted affine Kac–Moody Lie algebras $X_r^{(1)}$ for $X = A, B, C, D$ over the field \mathbb{C} of complex numbers. The algebra $X_r^{(1)}$ can be constructed from the simple finite-dimensional Lie algebra of type X_r by tensoring with the Laurent polynomials in z and extending by a central element c and a degree derivation d :

$$X_r^{(1)} = (X_r \otimes \mathbb{C}[z, z^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

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Let $\mathfrak{h} = (\mathcal{H} \otimes 1) \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the Cartan subalgebra of $X_r^{(1)}$ obtained from a Cartan subalgebra \mathcal{H} of X_r , and let λ denote a dominant integral weight relative to \mathfrak{h} . The character, $\text{ch } L(\lambda) = \sum_{\mu} \dim L(\lambda)_{\mu} e^{\mu}$, records the multiplicity, $\text{mult}(\mu) = \dim L(\lambda)_{\mu}$, of each weight μ in the irreducible $X_r^{(1)}$ -module $L(\lambda)$ with highest weight λ , and the well-known Weyl–Kac character formula ([Kc], p. 173) provides a theoretically useful expression for $\text{ch } L(\lambda)$. However, since the character formula involves sums over the Weyl group of $X_r^{(1)}$ in both its numerator and denominator, it is very difficult and impractical to apply from a computational standpoint. From the character formula Peterson has derived Freudenthal-type weight and root multiplicity formulas, which provide recursive ways of computing weight and root multiplicities (see [Kc], Exercises 11.11, 11.12, 11.14). Frenkel and Kac [FK] have determined weight multiplicities in level one highest weight representations, and Feingold and Lepowsky [FL] have calculated the multiplicities of weights of level one, two, and three representations for the affine Kac Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$. But general information about weight multiplicities is very limited.

It has been conjectured by Benkart and Kass that for fixed λ and μ the multiplicity $\dim L(\lambda)_{\mu}$ of the weight μ in the $X_r^{(1)}$ -module $L(\lambda)$ is a polynomial in r which depends on X for all sufficiently large values of r . The article [BK] discusses evidence for this claim, gives a proof of it in some special cases, makes further conjectures about the degree and coefficients of the polynomials, and uses the polynomials to introduce the notion of a ‘rank-zero’ string function.

In this paper (see Conjectures A and A' of Section 1), we give a precise conjecture concerning the degree of the polynomial for algebras of type A . To offer evidence for the conjecture, we prove it for all dominant weights λ and all weights μ of depth ≤ 2 , that is, for all weights of the form $\mu = \nu, \nu - \delta$, and $\nu - 2\delta$, where δ is the null root and ν is as in (1.27) below. For those weights we explicitly exhibit the polynomials as expressions involving the well-known Kostka numbers (see [M]), which count the number of column-strict tableaux of a given partition shape. Our approach is to apply the root multiplicity formula which Kang [Kn2] derived from the Euler–Poincaré Principle for Kac–Moody algebras to reduce the problem to computations involving the representation theory of $\mathfrak{sl}(r+1, \mathbb{C})$. It is there that the Kostka numbers enter the picture, for they give the multiplicities of the weights in a finite-dimensional irreducible $\mathfrak{sl}(r+1, \mathbb{C})$ -module. Many interesting combinatorial identities involving various Kostka numbers arise in the calculations. We have assembled these identities in the final section of the paper.

1. The affine Lie algebra $A_r^{(1)}$ and the conjecture

1.1. Assume A_r is the simple Lie algebra $\mathfrak{sl}(r+1, \mathbb{C})$ of all $(r+1) \times (r+1)$

complex matrices of trace zero under the commutator product $[x, y] = xy - yx$, and let \mathcal{H} denote the Cartan subalgebra of all diagonal matrices in A_r . Let $\varepsilon_i \in \mathcal{H}^*$ be the projection map which takes a matrix to its (i, i) -entry. Then $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, 2, \dots, r$ are the simple roots, and $\omega_i = \sum_{j=1}^i \varepsilon_j$ for $i = 1, 2, \dots, r$ are the fundamental weights with respect to \mathcal{H} . A dominant weight $\lambda = \sum_{i=1}^\ell a_i \omega_i$, $a_i \in \mathbb{Z}_{\geq 0}$, $a_\ell \neq 0$, determines a partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0\}$ whose i th part is $\lambda_i = \sum_{j=i}^\ell a_j$, and $\lambda = \sum_{i=1}^\ell \lambda_i \varepsilon_i$. The number $\ell(\lambda)$ of nonzero parts, (which is $\ell \leq r$ here) is the *length* of the partition λ . We write $\lambda \vdash m$ to signify that λ is a partition of $m = |\lambda| \stackrel{\text{def}}{=} \sum_{i=1}^\ell \lambda_i$, and ignore the distinction between two partitions that differ only in the number of trailing zeros. Since $\varepsilon_1 + \dots + \varepsilon_{r+1} = 0$ by the trace zero condition, we will not differentiate between λ and $\lambda + \varepsilon_1 + \dots + \varepsilon_{r+1}$ as A_r -weights.

1.2. Let $V(\lambda)$ denote the irreducible $\mathfrak{sl}(r + 1, \mathbb{C})$ -module with highest weight $\lambda = \lambda_1 \varepsilon_1 + \dots + \lambda_{r+1} \varepsilon_{r+1}$, where $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r+1} \geq 0\}$. Suppose that $\nu = \{\nu_1 \geq \nu_2 \geq \dots \geq \nu_{\ell(\nu)} > 0\}$ is a partition, and let $p_i(\nu)$ denote the i th partial sum $p_i(\nu) = \nu_1 + \dots + \nu_i$ of the parts of ν . Then $p_i(\nu) = p_{\ell(\nu)}(\nu) = |\nu|$ for all $i \geq \ell(\nu)$. By $\lambda \succeq \nu$ we mean that $p_i(\lambda) \geq p_i(\nu)$ for all i , and if that is the case, then we say λ *dominates* ν . When $\ell(\nu) \leq r + 1$, then $\nu = \nu_1 \varepsilon_1 + \dots + \nu_{r+1} \varepsilon_{r+1}$ is a weight of $V(\lambda)$ if and only if $\lambda - \nu = \sum_{m=1}^r c_m \alpha_m$, where $\alpha_m = \varepsilon_m - \varepsilon_{m+1}$ and $c_m \in \mathbb{Z}_{\geq 0}$ for all m , which is true if and only if $p_i(\lambda) \geq p_i(\nu)$ for all i , that is, $\lambda \succeq \nu$ in the dominance order (see for example, [BBL], (4.7)).

1.3. The symmetric group S_{r+1} , which is the Weyl group of $\mathfrak{sl}(r + 1, \mathbb{C})$, acts on \mathcal{H}^* by permuting the ε_i so that $\sigma \varepsilon_i = \varepsilon_{\sigma i}$ for all $\sigma \in S_{r+1}$. Every weight ω of $V(\lambda)$ is conjugate under S_{r+1} to a unique dominant weight $\bar{\omega}$ whose coefficients are arranged in descending order, hence form a partition. Since conjugate weights have identical multiplicities, it suffices to determine the dominant weights (which can be done from 1.2) and their multiplicities.

1.4. Associated to the partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0\}$ is its Ferrers diagram or Young frame $F(\lambda)$ having λ_i left-justified boxes in the i th row for $i = 1, \dots, \ell$. A column-strict tableau T of shape λ is obtained by filling in $F(\lambda)$ with numbers from $\{1, 2, \dots, r + 1\}$ so that the entries weakly increase across the rows from left to right and strictly increase from top to bottom down each column. The *column-strict tableaux* of shape λ index a basis for the $\mathfrak{sl}(r + 1, \mathbb{C})$ -module $V(\lambda)$. The *weight* of T is $\omega(T) \stackrel{\text{def}}{=} \sum_{j=1}^{r+1} (\#j\text{'s}) \varepsilon_j$. In particular, if $\lambda = \{3 \geq 3 \geq 1 > 0\} = \{3^2, 1\} \vdash 7$ and

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline 3 & & \\ \hline \end{array},$$

then $\omega(T) = 2\varepsilon_1 + 2\varepsilon_2 + 3\varepsilon_3$ is the weight of T , and $\overline{\omega(T)} = 3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3$ is its unique dominant conjugate. Thus, the multiplicity of the weight $\nu = \nu_1\varepsilon_1 + \nu_2\varepsilon_2 + \dots + \nu_{r+1}\varepsilon_{r+1}$ in $V(\lambda)$ is the number of column-strict tableaux T of shape λ and weight $\omega(T) = \nu$. When ν is a partition (i.e. when ν is dominant) that number is just the *Kostka number* $K_{\lambda,\nu}$. To summarize we have the following well-known result:

PROPOSITION 1.5 *Suppose that ν is a dominant weight of $V(\lambda)$. Then $\dim V(\lambda)_\nu = K_{\lambda,\nu}$, where $K_{\lambda,\nu}$ is the Kostka number which counts the number of column-strict tableaux of shape λ and weight ν . Thus, $K_{\lambda,\nu} \neq 0$ if and only if ν is a weight of $V(\lambda)$ if and only if $\lambda \succeq \nu$.*

1.6. The affine Kac–Moody Lie algebra $A_r^{(1)}$ associated to $A_r = \mathfrak{sl}(r + 1, \mathbb{C})$ can be realized as the Lie algebra $(\mathfrak{sl}(r + 1, \mathbb{C}) \otimes \mathbb{C}[z, z^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$ having Lie brackets given by

$$[x \otimes z^i, y \otimes z^j] = [x, y] \otimes z^{i+j} + i\delta_{i,-j}\kappa(x, y)c,$$

$$[d, x \otimes z^i] = ix \otimes z^i,$$

$$[c, A_r^{(1)}] = (0),$$

where $x, y \in \mathfrak{sl}(r + 1, \mathbb{C})$ and $\kappa(\cdot, \cdot)$ is the Killing form of $\mathfrak{sl}(r + 1, \mathbb{C})$. The matrices $E_i = E_{i,i+1}$, $F_i = E_{i+1,i}$, and $H_i = [E_i, F_i] = E_{i,i} - E_{i+1,i+1}$ for $i = 1, \dots, r$, where $E_{i,j}$ denotes the $(r + 1) \times (r + 1)$ matrix unit having 1 as the (i, j) -entry and 0 everywhere else, generate $\mathfrak{sl}(r + 1, \mathbb{C})$, and E_i (resp. F_i) corresponds to the simple root $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ (resp. $-\alpha_i$) relative to the Cartan subalgebra $\mathcal{H} = \text{span}\{H_1, \dots, H_r\}$. The matrix $E_0 = E_{r+1,1}$, (resp. $F_0 = E_{1,r+1}$), is a basis for the root space of $\mathfrak{sl}(r + 1, \mathbb{C})$ associated to $-\theta$ (resp. θ) where $\theta = \alpha_1 + \dots + \alpha_r$ is the highest root relative to \mathcal{H} , and for $H_0 = [E_0, F_0] = E_{r+1,r+1} - E_{1,1}$ the relations $[H_0, E_0] = 2E_0$ and $[H_0, F_0] = -2F_0$ hold. The affine algebra $A_r^{(1)}$ is generated by the elements $\{e_i, f_i, h_i, d \mid i = 0, 1, \dots, r\}$ where $e_i = E_i \otimes 1$, $f_i = F_i \otimes 1$, $h_i = [e_i, f_i] = H_i \otimes 1$ for $i = 1, \dots, r$, and $e_0 = E_0 \otimes z$, $f_0 = F_0 \otimes z^{-1}$, and $[e_0, f_0] = h_0 = H_0 \otimes 1 + c$. Relative to the Cartan subalgebra $\mathfrak{h} = \text{span}\{h_0, h_1, \dots, h_r, d\}$ of $A_r^{(1)}$, the roots α_i , which correspond to E_i for $i = 0, 1, \dots, r$, are the simple roots. The Cartan matrix $C(A_r^{(1)})$ of $A_r^{(1)}$ is the matrix whose (i, j) -entry $a_{i,j}$ is $\alpha_j(h_i)$ for $i, j \in \{0, \dots, r\}$. If the first row and column are deleted, the resulting matrix is just the Cartan matrix $C(A_r)$ of $A_r = \mathfrak{sl}(r + 1, \mathbb{C})$. Since the Cartan matrix of $A_r^{(1)}$ is singular, there is a nonzero vector, namely $(1, 1, \dots, 1)^t$, annihilated by $C(A_r^{(1)})$. The corresponding root $\delta = \alpha_0 + \alpha_1 + \dots + \alpha_r$ is the *null root* of $A_r^{(1)}$ and $\alpha_0 = \delta - \theta$.

1.7. The fundamental weights $\Lambda_0, \Lambda_1, \dots, \Lambda_r$ are dual to the elements h_0, h_1, \dots, h_r so that $\Lambda_i(h_j) = \delta_{i,j}$. Consequently, the dual space \mathfrak{h}^* of \mathfrak{h} is given by

$$\begin{aligned}\mathfrak{h}^* &= \mathbb{C}\Lambda_0 \oplus \mathbb{C}\Lambda_1 \oplus \cdots \oplus \mathbb{C}\Lambda_r \oplus \mathbb{C}\delta \\ &= \mathbb{C}\Lambda_0 \oplus \mathbb{C}\alpha_0 \oplus \mathbb{C}\alpha_1 \oplus \cdots \oplus \mathbb{C}\alpha_r,\end{aligned}\tag{1.8}$$

where $\alpha_i, \delta, \Lambda_i$ satisfy

$$\begin{aligned}\alpha_i(h_j) &= a_{j,i}, & \alpha_i(d) &= \delta_{0,i}, \\ \Lambda_i(h_j) &= \delta_{i,j}, & \Lambda_i(d) &= 0, \\ \delta(h_j) &= 0, & \delta(d) &= 1, \quad \text{for all } i, j = 0, 1, \dots, r.\end{aligned}\tag{1.9}$$

1.10. The set P^+ of dominant weights consists of those elements $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_j)$ is a nonnegative integer for all $j = 0, 1, \dots, r$. Let $\lambda \in P^+$ and let $L(\lambda)$ denote the integrable irreducible $A_r^{(1)}$ -module with highest weight λ . The central element c acts as the scalar $\lambda(c)$ on $L(\lambda)$, and the value $\lambda(c)$ is termed the *level* of $L(\lambda)$. Any weight μ of $L(\lambda)$ must have the same level as λ . Moreover, since $\mu \leq \lambda$, (that is, $\mu = \lambda - \sum_{j=0}^r c_j \alpha_j$, where $c_j \in \mathbb{Z}_{\geq 0}$ for all j), the weights of $L(\lambda)$ must belong to the same coset as λ in the weight lattice P modulo the root lattice Q . As $\dim L(\lambda)_\mu = \dim L(\lambda)_{\sigma\mu}$ for all σ in the Weyl group of $A_r^{(1)}$, it suffices to determine the multiplicities of the dominant weights μ .

1.11. Fix a positive integer l and suppose $\lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_r\Lambda_r - m\delta$ is a dominant integral weight of level l , where $m \in \mathbb{Z}$ and $a_i \in \mathbb{Z}_{\geq 0}$ for $i = 0, 1, \dots, r$. Since $c = h_0 + h_1 + \cdots + h_r$, we have $\lambda(c) = a_0 + a_1 + \cdots + a_r = l$. Hence, if $r \geq l$, there must be a *gap* in the expression of λ . That is, there exist nonnegative integers s' and t' with $s' + t' \leq r$ such that

$$a_{s'-1} \neq 0, \quad a_{r-t'+1} \neq 0, \quad \text{and} \quad a_{s'} = a_{s'+1} = \cdots = a_{r-t'} = 0.$$

This allows us to adopt the following point of view: From the s' -tuple $\underline{a} = (a_0, a_1, \dots, a_{s'-1})$ and the t' -tuple $\underline{a}' = (a'_{t'-1}, a'_{t'-2}, \dots, a'_0)$ of nonnegative integers with $a_{s'-1} \neq 0$ and $a'_{t'-1} \neq 0$ and the integer m , we construct the dominant weight $\lambda = \sum_{i=0}^r a_i \Lambda_i - m\delta$ with $a_{s'} = a_{s'+1} = \cdots = a_{r-t'} = 0$ and $a_{r-j} = a'_j$ for $j = 0, \dots, t' - 1$. We regard λ as being the *same* for all $r \geq l$. If $\mu = b_0\Lambda_0 + b_1\Lambda_1 + \cdots + b_r\Lambda_r - n\delta$ is a dominant weight of $L(\lambda)$, then $\lambda(c) = \mu(c) = l$. Thus, if $r \geq l$ there must be a gap in the expression for μ . Moreover, if $r \geq l + s' + t'$, then the gap of λ is sufficiently large that there exists a gap of μ which overlaps the gap of λ . As a result, we can associate determining data $\underline{b} = (b_0, b_1, \dots, b_{s''-1})$, $\underline{b}' = (b_{r-t''+1}, b_{r-t''+2}, \dots, b_r)$, and $n \in \mathbb{Z}$ to μ , where $b_{s''-1} \neq 0$, $b_{r-t''+1} \neq 0$, and s'', t'' are nonnegative integers satisfying $s'' + t'' \leq r$, $s' + t'' \leq r$, and $s'' + t' \leq r$. Hence, if we let $s = \max(s', s'')$ and $t = \max(t', t'')$, the weights λ and μ share a *common gap*:

$$\begin{aligned}a_s &= a_{s+1} = \cdots = a_{r-t} = 0, \\ b_s &= b_{s+1} = \cdots = b_{r-t} = 0.\end{aligned}\tag{1.12}$$

1.13. The fact that the levels of λ and μ are the same is the first equation in (1.14) below, and the statement that λ and μ belong to the same coset of the weight lattice modulo the root lattice is the congruence condition, which is the second equation:

$$\begin{aligned} a_0 + a_1 + \cdots + a_r &= b_0 + b_1 + \cdots + b_r \\ N &\stackrel{\text{def}}{=} (b_1 + 2b_2 + \cdots + rb_r) - (a_1 + 2a_2 + \cdots + ra_r) \equiv 0 \\ &\text{modulo } r + 1. \end{aligned} \quad (1.14)$$

Thus, $d_0 + d_1 + \cdots + d_r = 0$ and $N = d_1 + 2d_2 + \cdots + rd_r$, where $d_i = b_i - a_i$ for $i = 0, 1, \dots, r$. Now if $\mu = \lambda - \sum_{j=0}^r c_j \alpha_j$, then $\mu(h_i) = \lambda(h_i) - \sum_{j=0}^r c_j \alpha_j(h_i)$, which gives $d_i = b_i - a_i = -\sum_{j=0}^r a_{i,j} c_j$, where $a_{i,j}$ is the (i, j) -entry of the Cartan matrix $C(A_r^{(1)})$. Note that $a_{i,j} = 2\delta_{i,j}^{\{r\}} - \delta_{i,j-1}^{\{r\}} - \delta_{i,j+1}^{\{r\}}$ where $\delta_{i,j}^{\{r\}} = 1$ if $i \equiv j$ modulo $r + 1$ and is 0 otherwise. Solving the resulting linear system, $d_i = -\sum_{j=0}^r a_{i,j} c_j$ for $i = 0, 1, \dots, r$ and $\mu(d) = -n$ for the c_j 's, we obtain

$$\begin{aligned} c_0 &= n - m, \\ c_i &= n - m + d_{i+1} + 2d_{i+2} + \cdots + (r - i)d_r \\ &\quad - \frac{N}{r + 1}(r - i + 1) \quad \text{for } i = 0, \dots, r - 1, \\ c_r &= n - m - \frac{N}{r + 1}. \end{aligned} \quad (1.15)$$

In particular, the requirement $c_r \in \mathbb{Z}$ is what forces $r + 1$ to divide N as above, and from

$$\begin{aligned} N &= d_1 + 2d_2 + \cdots + rd_r \\ &= (d_1 + 2d_2 + \cdots + (s - 1)d_{s-1}) \\ &\quad - (td_{r-t+1} + (t - 1)d_{r-t+2} + \cdots + 2d_{r-1} + d_r) \\ &\quad - (r + 1)(d_{r-t+1} + d_{r-t+2} + \cdots + d_r) \end{aligned}$$

we see that

$$d_1 + 2d_2 + \cdots + (s - 1)d_{s-1} = td_{r-t+1} + \cdots + 2d_{r-1} + d_r \quad (1.16)$$

must hold if μ is a weight of $L(\lambda)$ for all $r \geq l + s' + t'$.

1.17. We define the *depth of μ with respect to λ* to be

$$\begin{aligned} d_\lambda(\mu) &\stackrel{\text{def}}{=} n - m - (d_1 + 2d_2 + \cdots + (s - 1)d_{s-1}) \\ &= n - m - (td_{r-t+1} + \cdots + 2d_{r-1} + d_r). \end{aligned} \quad (1.18)$$

Then it follows from (1.15) and (1.17) that

$$c_i = \begin{cases} d_\lambda(\mu) + d_{i+1} + 2d_{i+2} + \dots + (s - 1 - i)d_{s-1} \\ d_\lambda(\mu) \\ d_\lambda(\mu) + (i - (r - t + 1))d_{r-t+1} + \dots + 2d_{i-2} + d_{i-1} \end{cases}$$

if $i = 0, 1, \dots, s - 1,$
 if $i = s, s + 1, \dots, r - t + 1,$
 if $i = r - t + 2, \dots, r,$ respectively.

(1.19)

1.20. This brings us to the following conjecture concerning the multiplicity of such a weight μ in $L(\lambda)$. It is a more explicit version of a conjecture first formulated by Benkart and Kass in 1987 (see [BK]):

CONJECTURE A. *Suppose $m, n \in \mathbb{Z}$, and let $\underline{a} = (a_0, a_1, \dots, a_{s'-1}), \underline{a}' = (a'_{s'-1}, a'_{s'-2}, \dots, a'_0), \underline{b} = (b_0, b_1, \dots, b_{s''-1})$ and $\underline{b}' = (b'_{s''-1}, b'_{s''-2}, \dots, b'_0)$ be tuples of nonnegative integers. Set $\lambda = \sum_{i=0}^r a_i \Lambda_i - m\delta$, where $a_{s'} = a_{s'+1} = \dots = a_{r-t'} = 0, a_{r-j} = a'_j$ for $j = 0, \dots, t' - 1$, and $\mu = \sum_{i=0}^r b_i \Lambda_i - n\delta$ where $b_{s''} = b_{s''+1} = \dots = b_{r-t''} = 0$, and $b_{r-j} = b'_j$ for $j = 0, \dots, t'' - 1$. If μ is a weight of $L(\lambda)$ for all $r \geq l + s' + t'$, then its multiplicity $\text{mult}(\mu)$ is a polynomial in r of degree equal to the depth $d_\lambda(\mu) = n - m - (d_1 + 2d_2 + \dots + (s - 1)d_{s-1})$ for all $r \geq l + s' + t'$.*

1.21. Note that for λ and μ as in the conjecture, $\lambda - \mu = \sum_{i=0}^r c_i \alpha_i = \sum_{i=0}^r (c_i - d_\lambda(\mu)) \alpha_i + d_\lambda(\mu) \delta$. Since $c_i = d_\lambda(\mu)$ for all $i = s, s + 1, \dots, r - t + 1$ by (1.19), it follows that the coefficient of those α_i 's in $\lambda - \mu$ are negative whenever $d_\lambda(\mu) < 0$. But since $\lambda - \mu \in \sum_{i=0}^r \mathbb{Z}_{\geq 0} \alpha_i$ must hold for all weights μ of $L(\lambda)$, it must be that the multiplicity of μ is zero whenever $d_\lambda(\mu) < 0$, and that is what is meant in Conjecture A when the depth is negative.

EXAMPLE 1.22 Let $\lambda = 2\Lambda_0$ and $\mu = 2\Lambda_0 - \alpha_0$. Then $\mu(h_1) = 1 = \mu(h_r), \mu(h_i) = 0$ for all other i , and $\mu(d) = -1$. Thus, $\mu = \Lambda_1 + \Lambda_r - \delta$ so that $m = 0$ and $n = 1$. In this example $N = r + 1$ and $d_\lambda(\mu) = 1 - 1 = 0$. Consequently, $\text{mult}(\mu)$ is conjectured to be a constant polynomial in this case, and the tables of [KMPS] confirm this conjecture since $\text{mult}(\mu) = 1$ for all $r \leq 8$ in those tables. Also, the multiplicity of $\Lambda_1 + \Lambda_r$ is given as zero there, which is consistent with the fact that the depth is negative for $\Lambda_1 + \Lambda_r$.

1.23. When λ and μ are as above, the multiplicity of μ in $L(\lambda)$ is the same as the multiplicity of $\mu + k\delta$ in $L(\lambda + k\delta)$ for any $k \in \mathbb{Z}$. Indeed by (9.10.1) of [Kc], the irreducible $A_r^{(1)}$ -module $L(k\delta)$ with highest weight $k\delta$ is one-dimensional. Consequently, $L(\lambda + k\delta) \cong L(\lambda) \otimes L(k\delta)$, and $\dim L(\lambda + k\delta)_{\mu+k\delta} = \dim L(\lambda)_\mu$ for any $k \in \mathbb{Z}$, as claimed. Thus in what follows, we can assume by translating the weights λ and μ by $m\delta$, that $m = 0$ and $\mu = \sum_{i=0}^r b_i \Lambda_i - n\delta$.

1.24. Suppose then that $\lambda = \sum_{i=0}^r a_i \Lambda_i$ and $\mu = \nu - n\delta$, where $\nu = \sum_{i=0}^r b_i \Lambda_i$. If we perform the rotation $r - t + 1 \rightarrow 1$ on λ and ν , we obtain:

$$\begin{aligned} \lambda' &= a_{r-t+1} \Lambda_1 + a_{r-t+2} \Lambda_2 + \cdots + a_r \Lambda_t + a_0 \Lambda_{t+1} + \cdots + a_{s-1} \Lambda_{s+t}, \\ \nu' &= b_{r-t+1} \Lambda_1 + b_{r-t+2} \Lambda_2 + \cdots + b_r \Lambda_t + b_0 \Lambda_{t+1} + \cdots + b_{s-1} \Lambda_{s+t}. \end{aligned} \tag{1.25}$$

We would like to argue that $N' = d_{r-t+1} + 2d_{r-t+2} + \cdots + td_r + (t + 1)d_0 + \cdots + (s + t)d_{s-1} = 0$. Using the expression in (1.16) above we have

$$\begin{aligned} 0 &= -td_{r-t+1} - (t - 1)d_{r-t+2} - \cdots - 2d_{r-1} \\ &\quad -d_r + d_1 + 2d_2 + \cdots + (s - 1)d_{s-1} \end{aligned} \tag{1.26}$$

and adding $0 = (t + 1)(d_0 + d_1 + \cdots + d_r)$ to the right side of (1.26) gives

$$0 = d_{r-t+1} + 2d_{r-t+2} + \cdots + td_r + (t + 1)d_0 + \cdots + (s + t)d_{s-1} \tag{1.27}$$

as desired. Therefore, if $\mu' = \nu' - n\delta$, then the depth of μ' with respect to λ' is

$$\begin{aligned} d_{\lambda'}(\mu') &= n - (d_{r-t+1} + 2d_{r-t+2} + \cdots + td_r \\ &\quad + (t + 1)d_0 + \cdots + (s + t)d_{s-1}) = n \end{aligned}$$

by (1.27). In particular, when $\lambda = \sum_{i=0}^r a_i \Lambda_i$ and $\mu = \sum_{i=0}^r b_i \Lambda_i - n\delta$, then the multiplicity of μ in $L(\lambda)$ is the same as the multiplicity of $\mu' - (d_1 + 2d_2 + \cdots + (s - 1)d_{s-1})\delta$ in the module $L(\lambda')$, because $d_1 + 2d_2 + \cdots + (s - 1)d_{s-1}$ is the difference in the depths. Thus, by replacing λ by λ' and μ by μ' , we may assume that $\lambda = \sum_{i=1}^q a_i \Lambda_i$, $\mu = \sum_{i=1}^q b_i \Lambda_i - n\delta$ for some $q < r$ such that not both a_q and b_q are zero, and $d_\lambda(\mu) = n$. For such pairs of weights we have the following conjecture:

CONJECTURE A'. Assume that $\lambda = \sum_{i=1}^q a_i \Lambda_i$ is a dominant integral weight of $A_r^{(1)}$ and $\mu = \sum_{i=1}^q b_i \Lambda_i - n\delta$ is a dominant weight of the irreducible $A_r^{(1)}$ -module $L(\lambda)$ where not both a_q and b_q are zero. Then the multiplicity $\text{mult}(\mu)$ of μ in $L(\lambda)$ is a polynomial in r of degree $d_\lambda(\mu) = n$ for all $r > q$.

1.28. Let $\nu = \sum_{i=1}^q b_i \Lambda_i$. When such λ and μ are restricted to the Cartan sub-algebra $(\mathcal{H} \otimes 1) \cong \mathcal{H} = \sum_{i=1}^r \mathbb{C}h_i$ of $(\mathfrak{sl}(r + 1, \mathbb{C}) \otimes 1) \cong \mathfrak{sl}(r + 1, \mathbb{C})$, then $\lambda = \sum_{i=1}^q a_i \omega_i = \sum_{i=1}^q \lambda_i \varepsilon_i$ and $\mu = \nu = \sum_{i=1}^q b_i \omega_i = \sum_{i=1}^q \nu_i \varepsilon_i$, where $\lambda_i = \sum_{j=i}^q a_j$ and $\nu_i = \sum_{j=i}^q b_j$ are the parts of the associated partitions, and $\sum_{i=1}^q \lambda_i = a_1 + 2a_2 + \cdots + qa_q = b_1 + 2b_2 + \cdots + qb_q = \sum_{i=1}^q \nu_i$ by (1.25). Thus, λ and ν

can be regarded as partitions of the same number.

1.29. In this paper we establish Conjecture A' for all λ and all $\mu = \nu - n\delta$ with $n = 0, 1, 2$ by explicitly exhibiting $\dim L(\lambda)_{\nu-n\delta}$ as a polynomial in r whose coefficients involve Kostka numbers. To accomplish this we work inside a bigger Kac-Moody Lie algebra $\mathcal{G} = \mathcal{G}(C)$ corresponding to a certain $(r+2) \times (r+2)$ Cartan matrix C . Since the Cartan matrix $C(A_r^{(1)})$ of $A_r^{(1)}$ is symmetric, there is a nondegenerate symmetric bilinear form $(|)$ on the dual space \mathfrak{h}^* of the Cartan subalgebra \mathfrak{h} of $A_r^{(1)}$. The natural pairing between \mathfrak{h}^* and \mathfrak{h} allows us to define a nondegenerate symmetric bilinear form, also denoted by $(|)$ on \mathfrak{h} . Let $-\alpha_{-1} = \lambda \in \mathfrak{h}^*$ and let C be the matrix whose i, j entry is $a_{i,j} = 2(\alpha_i|\alpha_j)/(\alpha_i|\alpha_i)$ for all $i, j = -1, 0, 1, \dots, r$. Then the first column of C consists of the entries $2, 0, -a_1, \dots, -a_q$ followed by $r - q$ zeros, and deleting the first row and column just gives $C(A_r^{(1)})$. The Kac-Moody algebra $\mathcal{G}(C)$ has generators h_i, e_i, f_i , for $i = -1, 0, 1, \dots, r$, where h_i, e_i, f_i , for $i = 0, 1, \dots, r$ together with h_{-1} generate a subalgebra isomorphic to $A_r^{(1)}$. The Cartan subalgebra \mathfrak{h} of $A_r^{(1)}$ is the span of h_{-1}, h_0, \dots, h_r , equivalently, h_0, h_1, \dots, h_r, d , equivalently, h_1, \dots, h_r, c, d . The algebra $\mathcal{G} = \mathcal{G}(C)$ can be realized as the minimal graded Lie algebra with local part $L(\lambda) \oplus A_r^{(1)} \oplus L(\lambda)^*$ where $L(\lambda)^*$ is the finite dual of $L(\lambda)$ (see [BKM], Section 2). The generator f_{-1} can be taken to be the highest weight vector in $L(\lambda)$ and the generator e_{-1} as the lowest weight vector in $L(\lambda)^*$ in a basis dual to a basis of $L(\lambda)$. The weight $\mu = \nu - n\delta$ of $L(\lambda)$ can be regarded as a root of \mathcal{G} , and its multiplicity in \mathcal{G} is the same as in $L(\lambda)$.

1.30. Suppose now that $d_\lambda(\mu) = 0$ so that $\lambda = \sum_{i=1}^q a_i \Lambda_i$ and $\mu = \sum_{i=1}^q b_i \Lambda_i$. Then by (1.19), $\mu = \lambda - \sum_{i=1}^{q-1} c_i \alpha_i$. Hence, the multiplicity of μ in $L(\lambda)$ is the multiplicity of μ in $\mathcal{G}(C)$ which is the number of linearly independent vectors of the form $[f_{i_1}, [f_{i_2}, [\dots, [f_{i_k}, f_{-1}]]]$ in $\mathcal{G}(C)$, where f_j appears c_j times in the expression for each $j \in \{1, 2, \dots, q-1\}$. This is clearly independent of r . Thus, the multiplicity is constant for weights μ of depth zero.

1.31. To establish the conjecture for the cases $n = 1, 2$, in the next section we apply the root multiplicity formula of [Kn2] to the algebra $\mathcal{G}(C)$ to reduce considerations to $\mathfrak{sl}(r+1, \mathbb{C})$ -modules. In Sections 3 and 4 we use the expressions for weight multiplicities in irreducible $\mathfrak{sl}(r+1, \mathbb{C})$ -modules given in terms of Kostka numbers to derive the polynomials.

2. The S-gradation and the root multiplicity formula

2.1. We begin this section with a discussion of gradations of arbitrary symmetrizable Kac-Moody algebras and related root multiplicity formulas, and then specialize the results to the particular algebras described at the end of Section 1.

2.2. Let $C = (a_{i,j})_{i,j \in \mathcal{I}}$ be a symmetrizable generalized Cartan matrix. Assume

$\mathcal{G} = \mathcal{G}(C)$ is the associated Kac–Moody algebra over \mathbb{C} generated by $\{e_i, f_i | i \in \mathcal{I}\}$ and the vector space \mathfrak{h} of dimension $2 | \mathcal{I} | - \text{rank}(C)$ subject to the usual Serre relations (see [Kc]). Let $\{\alpha_i | i \in \mathcal{I}\} \subset \mathfrak{h}^*$ and $\{h_i | i \in \mathcal{I}\} \subset \mathfrak{h}$ denote the simple roots and coroots respectively. The root lattice $Q = \sum_{i \in \mathcal{I}} \mathbb{Z} \alpha_i$ contains the positive cone $Q^+ = \sum_{i \in \mathcal{I}} \mathbb{Z}_{\geq 0} \alpha_i$, and the negative cone $Q^- = -Q^+$. The positive cone provides a partial order on \mathfrak{h}^* in which $\xi \geq \eta$ for $\xi, \eta \in \mathfrak{h}^*$ if and only if $\xi - \eta \in Q^+$. The space \mathfrak{h} is a Cartan subalgebra of \mathcal{G} and relative to its adjoint action, \mathcal{G} decomposes into root spaces, $\mathcal{G} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathcal{G}_\alpha)$, where $\mathcal{G}_\alpha = \{x \in \mathcal{G} | [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. The set $\Delta = \{\alpha \in Q | \mathcal{G}_\alpha \neq (0)\}$ of roots splits into positive and negative roots: $\Delta = \Delta^+ \cup \Delta^-$, where $\Delta^\pm \subseteq Q^\pm$. Then $\alpha \in \Delta^+$ (resp. $\alpha \in \Delta^-$) if and only if $\alpha > 0$ (resp. $\alpha < 0$). The simple reflections $s_i: \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ defined by $s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i = \lambda - \lambda(h_i) \alpha_i$ for $i \in \mathcal{I}$ generate the Weyl group W of \mathcal{G} . Each $\sigma \in W$ is a product of the simple reflections, and the length $l(\sigma)$ of σ is just minimal number of simple reflections giving σ .

2.3. Fix a subset S of \mathcal{I} , and assume $\mathcal{G}_S = \mathcal{G}(C_S)$ is the Kac–Moody Lie algebra associated with the Cartan matrix $C_S = (a_{i,j})_{i,j \in S}$. Let $\Delta_S, \Delta_S^+, \Delta_S^-$, and W_S denote the roots, positive roots, negative roots, and Weyl group of \mathcal{G}_S , respectively. Assume $\Delta^\pm(S) = \Delta^\pm \setminus \Delta_S^\pm$, and let

$$\begin{aligned} W(S) &= \{\sigma \in W | \Phi_\sigma \subset \Delta^+(S)\} \quad \text{where} \\ \Phi_\sigma &= \{\alpha \in \Delta^+ | \sigma^{-1}(\alpha) < 0\}. \end{aligned} \tag{2.4}$$

For doing explicit computations with $W(S)$, the following inductive construction is especially useful:

LEMMA 2.5 (See for example, [Kn1]). *Suppose $\sigma = \sigma' s_j$ and $l(\sigma) = l(\sigma') + 1$. Then $\sigma \in W(S)$ if and only if $\sigma' \in W(S)$ and $\sigma'(\alpha_j) \in \Delta^+(S)$.*

2.6. The *generalized height* of $\alpha = \sum_{i \in \mathcal{I}} k_i \alpha_i \in Q$ with respect to S is defined by $ht^S(\alpha) = \sum_{i \in \mathcal{I} \setminus S} k_i$, and it determines a \mathbb{Z} -gradation $\mathcal{G} = \oplus_{j \in \mathbb{Z}} \mathcal{G}_j^{(S)}$ of \mathcal{G} in which $\mathcal{G}_j^{(S)} = \sum_{\alpha, ht^S(\alpha)=j} \mathcal{G}_\alpha$. The elements $x \in \mathcal{G}_\alpha \subseteq \mathcal{G}_j^{(S)}$ are said to have degree $\text{deg}^S(x) = j$. Then $\mathcal{G}_0^{(S)} = \mathcal{G}_S + \mathfrak{h}$, and all the homogeneous subspaces $\mathcal{G}_j^{(S)}$ are integrable, and hence completely reducible, modules over $\mathcal{G}_0^{(S)}$. The spaces $\mathcal{G}_\pm^{(S)} = \oplus_{j \geq \pm 1} \mathcal{G}_{\pm j}^{(S)}$ give the triangular decomposition $\mathcal{G} = \mathcal{G}_-^{(S)} \oplus \mathcal{G}_0^{(S)} \oplus \mathcal{G}_+^{(S)}$.

2.7. The homology modules $H_k(\mathcal{G}_-^{(S)}) = H_k(\mathcal{G}_-^{(S)}, \mathbb{C})$ with coefficients in the trivial $\mathcal{G}_-^{(S)}$ -module \mathbb{C} inherit the \mathbb{Z} -gradation from $\wedge^k(\mathcal{G}_-^{(S)})$. They have a $\mathcal{G}_0^{(S)}$ -module structure which can be determined by the following formula, often referred to as *Kostant’s formula* because Kostant proved the analogue of this result for finite-dimensional semisimple Lie algebras.

THEOREM 2.8 ([GL], [L]).

$$H_k(\mathcal{G}_-^{(S)}) \cong \sum_{\substack{\sigma \in W(S) \\ l(\sigma)=k}} L_S(\sigma\rho - \rho),$$

where $\rho \in \mathfrak{h}^*$ satisfies $\rho(h_i) = 1$ for all $i \in \mathcal{I}$, and $L_S(\lambda)$ is the irreducible $\mathcal{G}_0^{(S)}$ -module with highest weight λ .

2.9. Consider the formal alternating direct sum of $\mathcal{G}_0^{(S)}$ -modules

$$M = \sum_{k=1}^{\infty} (-1)^{k+1} H_k(\mathcal{G}_-^{(S)}). \tag{2.10}$$

By Kostant’s formula we have,

$$M = \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{\substack{\sigma \in W(S) \\ l(\sigma)=k}} L_S(\sigma\rho - \rho) = \sum_{\substack{\sigma \in W(S) \\ l(\sigma)=k}} (-1)^{l(\sigma)+1} L_S(\sigma\rho - \rho).$$

2.11. Now for $\gamma \in Q^-$ we define the multiplicity of the γ -weight space of M to be the sum of dimensions:

$$\dim M_\gamma = \sum_{\substack{\sigma \in W(S) \\ l(\sigma)=k}} (-1)^{l(\sigma)+1} L_S(\sigma\rho - \rho)_\gamma.$$

It is possible that a particular weight space can have negative weight multiplicity.

2.12. Let $P(M) = \{\gamma \in Q^- \mid \dim M_\gamma \neq (0)\}$. We can totally order the elements of Q^- (for example, by height and within a given height lexicographically). Using this total ordering we can enumerate the elements of $P(M)$, say $P(M) = \{\gamma_i \mid i \geq 1\}$. For $\gamma \in Q^-$ define

$$T(\gamma) = \left\{ (m) = (m_i)_{i \geq 1} \mid m_i \in \mathbb{Z}_{>0}, \sum_i m_i \gamma_i = \gamma \right\} \tag{2.13}$$

and

$$B(\gamma) = \sum_{(m) \in T(\gamma)} \frac{(\sum_i m_i - 1)!}{\prod_i (m_i!)} \prod_i (\dim M_{\gamma_i})^{m_i}. \tag{2.14}$$

Since $T(\gamma)$ is the set of decompositions of γ into a sum of elements $\gamma_i \in P(M)$ for which $\gamma_i \leq \gamma$ in the enumeration, the set $T(\gamma)$ is finite. Using these notions, Kang has derived the following root multiplicity formula:

THEOREM 2.15 [Kn2]. *Let $\alpha \in \Delta^-(S)$. Then the multiplicity of the root α in the Kac–Moody algebra $\mathcal{G} = \mathcal{G}_-^{(S)} \oplus \mathcal{G}_0^{(S)} \oplus \mathcal{G}_+^{(S)}$ is given by*

$$\text{mult}(\alpha) = \dim \mathcal{G}_\alpha = \dim(\mathcal{G}_-^{(S)})_\alpha = \sum_{\gamma|\alpha} \mu\left(\frac{\alpha}{\gamma}\right) \frac{\gamma}{\alpha} B(\gamma), \tag{2.16}$$

where μ is the classical Möbius function, and $\gamma|\alpha$ if $\alpha = k\gamma$ for some positive integer k , in which case $\alpha/\gamma = k$ and $\gamma/\alpha = 1/k$.

2.17. We return now to our setting in (1.28). Assume $\lambda = \sum_{i=1}^q a_i \Lambda_i = \sum_{i=1}^q \lambda_i \varepsilon_i$, ($\lambda_i = \sum_{j=i}^q a_j$) is a dominant integral weight of $A_r^{(1)}$ relative to the Cartan subalgebra \mathfrak{h} spanned by h_0, h_1, \dots, h_r, d , equivalently, by h_1, \dots, h_r, c, d . Let $\alpha_0, \alpha_1, \dots, \alpha_r$ be the simple roots of $A_r^{(1)}$ and set $\alpha_{-1} = -\lambda$. The Kac–Moody algebra $\mathcal{G} = \mathcal{G}(C)$ which we consider has Cartan matrix $C = (a_{i,j})_{i,j=-1}^r$, where $a_{i,j} = 2(\alpha_i|\alpha_j)/(\alpha_i|\alpha_i)$. Thus, $\mathcal{I} = \{-1, 0, 1, \dots, r\}$ in this case, and we take $S = \{1, \dots, r\}$. Then $\mathcal{G}_0^{(S)} = \mathfrak{sl}(r+1, \mathbb{C}) + \sum_{i \in \mathcal{I}} \mathbb{C}h_i = \mathfrak{sl}(r+1, \mathbb{C}) + \mathfrak{h}$. Since c and d are central in $\mathcal{G}_0^{(S)}$, they necessarily act as scalars on any finite-dimensional irreducible $\mathcal{G}_0^{(S)}$ -module by Schur’s lemma. As a consequence, V is an irreducible $\mathcal{G}_0^{(S)}$ -module if and only if V is an irreducible $\mathfrak{sl}(r+1, \mathbb{C})$ -module. In particular, $\mathcal{G}_{-1}^{(S)}$ is a $\mathcal{G}_0^{(S)}$ -module, and $\mathcal{G}_{-1}^{(S)} = V(-\alpha_{-1}) \oplus V(-\alpha_0)$ is its decomposition into irreducible $\mathcal{G}_0^{(S)}$ -modules, hence $\mathfrak{sl}(r+1, \mathbb{C})$ -modules, where $-\alpha_{-1} = \lambda$ and $-\alpha_0 = \Lambda_1 + \Lambda_r = 2\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_r$.

2.18. Let $\alpha \in \Delta^-(S)$. Then $ht^S(\alpha) < 0$. By Theorem 2.8, (2.10), and (2.16), it follows that $V(\sigma\rho - \rho)$, which is $L_S(\sigma\rho - \rho)$ as a $\mathfrak{sl}(r+1, \mathbb{C})$ -module, can contribute to $\text{mult}(\alpha)$ only if $0 > ht^S(\sigma\rho - \rho) \geq ht^S(\alpha)$.

PROPOSITION 2.19 *If $\sigma \in W(S)$ and $l(\sigma) = k$, then $ht^S(\sigma\rho - \rho) \leq -k$.*

Proof. We proceed by induction on $l(\sigma)$. If $l(\sigma) = 1$, then $\sigma = s_{-1}$ or s_0 . Since $s_i\rho - \rho = -\alpha_i$ for each i , the result holds in this case. Now assume the result for all elements in $W(S)$ of length $\leq k$, and suppose that $\sigma = \sigma' s_j \in W(S)$ where $l(\sigma) = l(\sigma') + 1 = k + 1$. By Lemma 2.5, $\sigma' \in W(S)$ and $\sigma\alpha_j \in \Delta^+(S)$. Then

$$\sigma\rho - \rho = \sigma' s_j \rho - \rho = \sigma'(s_j \rho - \rho) + \sigma'\rho - \rho = -\sigma'\alpha_j + \sigma'\rho - \rho,$$

where $ht^S(-\sigma'\alpha_j) \leq -1$ since $\sigma'\alpha_j \in \Delta^+(S)$. We may apply the induction hypothesis to $\sigma'\rho - \rho$ to conclude $ht^S(\sigma'\rho - \rho) \leq -k$. Hence, $ht^S(\sigma\rho - \rho) = ht^S(-\sigma'\alpha_j) + ht^S(\sigma'\rho - \rho) \leq -1 - k = -(1 + k)$ as desired. \square

2.20. We want to compute the multiplicities of roots in $\mathcal{G}(C)$ of the form $\alpha = -\alpha_{-1} - \beta - n\delta$, where $\beta = \sum_{i=1}^r k_i \alpha_i$ and $\delta = \alpha_0 + \alpha_1 + \dots + \alpha_r$. Since $ht^S(\alpha) = -(n+1)$, it follows from Proposition 2.19, that only the weights of the

$\mathfrak{sl}(r + 1, \mathbb{C})$ -modules $V(\sigma\rho - \rho)$ with $\sigma \in W(S)$ and $l(\sigma) \leq n + 1$ can contribute to the calculation of $\text{mult}(\alpha)$ in (2.16). Moreover, in order for $\sigma\rho - \rho = -\sum_{i=-1}^r \ell_i \alpha_i$, $\ell_i \geq 0$, to give a γ in the decomposition in (2.16) with $\gamma|\alpha$, we must have $\ell_{-1} \leq 1$ and $\ell_0 \leq n$. Thus, to compute $\text{mult}(\alpha)$ for $\alpha = -\alpha_{-1} - \beta - \delta$ (i.e. the $n = 1$ case), it suffices to look at $V(-\alpha_0)$, $V(-\alpha_{-1})$ and $V(-\alpha_{-1} - \alpha_0)$ corresponding to $\sigma = s_{-1}, s_0$, and $s_{-1}s_0$ in $W(S)$, respectively. Similarly, to determine $\text{mult}(\alpha)$ for $\alpha = \alpha_{-1} - \beta - 2\delta$ (i.e. the $n = 2$ case), we need only look at the weights of the $\mathfrak{sl}(r + 1, \mathbb{C})$ -modules $V(-\alpha_{-1}), V(-\alpha_0), V(-2\alpha_0 - \alpha_1), V(-2\alpha_0 - \alpha_r), V(-\alpha_{-1} - \alpha_0), V(-\alpha_{-1} - 2\alpha_0 - \alpha_r)$ and $V(-\alpha_{-1} - 2\alpha_0 - \alpha_1)$ when $a_1 = 0$, i.e. when $\lambda_1 = \lambda_2$, which correspond to $s_{-1}, s_0, s_0s_1, s_0s_r, s_{-1}s_0, s_{-1}s_0s_r$, and $s_{-1}s_0s_1$ (when $a_1 = 0$) respectively.

3. The depth one case

3.1. This section is devoted to proving that the multiplicity of a depth one weight in an irreducible $A_r^{(1)}$ -module $L(\lambda)$ is a polynomial of degree one in r . In particular, we obtain an explicit expression for the multiplicity as a polynomial in r with coefficients involving Kostka numbers. Recall that a dominant integral weight $\lambda = a_1\Lambda_1 + \dots + a_q\Lambda_q$ of $A_r^{(1)}$ when restricted to $A_r = \mathfrak{sl}(r + 1, \mathbb{C})$ is given by $\lambda_1\varepsilon_1 + \dots + \lambda_q\varepsilon_q$ where $\lambda_i = a_i + \dots + a_q$. The coefficients determine a partition, $\{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q \geq 0\}$, which we also denote by λ . The multiplicity of a dominant weight ν in the irreducible A_r -module $V(\lambda)$ labelled by λ is the Kostka number $K_{\lambda, \nu}$ which is the number of column-strict tableaux of shape λ and weight ν . When $\xi = \xi_1\varepsilon_1 + \dots + \xi_{r+1}\varepsilon_{r+1}$ is any weight of $\mathfrak{sl}(r + 1, \mathbb{C})$, we set $K_{\lambda, \xi} \stackrel{\text{def}}{=} K_{\lambda, \bar{\xi}}$, where $\bar{\xi}$ is the unique dominant conjugate of ξ obtained by arranging the coefficients of the ε_i 's in descending order. In the calculations in this section and the next we use some identities involving Kostka numbers which we assemble and prove in the final section of the paper.

THEOREM 3.2 *Assume $L(\lambda)$ is the irreducible module for $A_r^{(1)}$ with highest weight $\lambda = a_1\Lambda_1 + \dots + a_q\Lambda_q$. Suppose $\nu - \delta$ is a weight of $L(\lambda)$ of depth 1 where $\nu = b_1\Lambda_1 + \dots + b_q\Lambda_q$. Then for all $r > q$*

$$\begin{aligned} \dim L(\lambda)_{\nu - \delta} &= rK_{\lambda, \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}} + (q - 1)\left(K_{\lambda, \nu} - K_{\lambda, \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}}\right) \\ &\quad + \sum_{\substack{i=1, j=2 \\ i \neq j}}^q K_{\lambda, \overline{\nu + \varepsilon_j - \varepsilon_i}} + (\delta_{\ell(\lambda), q} - 1)K_{\lambda + \zeta_{q-1} + \varepsilon_1, \nu + \zeta_q}, \end{aligned} \tag{3.3}$$

where $\zeta_t \stackrel{\text{def}}{=} \varepsilon_1 + \dots + \varepsilon_t$.

Proof. Since $\nu - \delta$ is a weight of $L(\lambda)$, we have $\text{level}(\lambda) = a_1 + \dots + a_q = b_1 + \dots + b_q = \text{level}(\nu)$, and $a_1 + 2a_2 + \dots + qa_q \equiv b_1 + 2b_2 + \dots + qb_q$ modulo

$r + 1$. Consider the associated partitions, $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q \geq 0\}$, where $\lambda_i = a_i + \dots + a_q$ and $\nu = \{\nu_1 \geq \nu_2 \geq \dots \geq \nu_q \geq 0\}$, where $\nu_i = b_i + \dots + b_q$ for $i = 1, \dots, q$. In particular, $\lambda = \lambda_1 \varepsilon_1 + \dots + \lambda_q \varepsilon_q$ and $\nu = \nu_1 \varepsilon_1 + \dots + \nu_q \varepsilon_q$ when restricted to $\mathfrak{sl}(r + 1, \mathbb{C})$, and as the levels of λ and ν are the same, $\lambda_1 = a_1 + \dots + a_q = b_1 + \dots + b_q = \nu_1$.

As we observed in paragraph 2.20, to compute $\dim L(\lambda)_{\nu-\delta}$, we need only determine the multiplicity of $\alpha = \nu - \delta$ in the irreducible A_r -modules $V(-\alpha_0)$, $V(-\alpha_{-1})$ and $V(-\alpha_{-1} - \alpha_0)$ with $-\alpha_{-1} = \lambda = \lambda_1 \varepsilon_1 + \dots + \lambda_q \varepsilon_q$, $-\alpha_0 = \varepsilon_1 + \zeta_r$, and $-\alpha_{-1} - \alpha_0 = \lambda + \varepsilon_1 + \zeta_r$, where $\zeta_r = \varepsilon_1 + \dots + \varepsilon_r$. The weight $\nu - \delta$ equals $\nu + \zeta_{r+1} = (\nu_1 + 1)\varepsilon_1 + \dots + (\nu_{r+1} + 1)\varepsilon_{r+1}$ when expressed in terms of the ε_i 's. The contribution from $V(-\alpha_{-1} - \alpha_0)$ will be negative since this term corresponds to $s_{-1}s_0$ in 2.11 and the other two will be positive.

First note that

$$\begin{aligned} \dim V(-\alpha_{-1} - \alpha_0)_\alpha &= K_{\lambda+\varepsilon_1+\zeta_r, \nu+\zeta_{r+1}} \\ &= K_{\lambda+\varepsilon_1+\zeta_q, \nu+\zeta_{q+1}} + (r - q)K_{\lambda+\varepsilon_1, \overline{\nu+\varepsilon_{q+1}}} \end{aligned} \tag{3.4}$$

by (5.3) of Section 5.

Now it remains to determine the ways of expressing $\alpha = \nu - \delta = \nu + \zeta_{r+1}$ as a sum

$$\alpha = \nu - \delta = \omega + \phi,$$

where ω is a weight of $V(-\alpha_{-1}) = V(\lambda)$ and ϕ is a weight of $V(-\alpha_0)$. Since $V(-\alpha_0)$ is the adjoint representation of A_r , its weights are the roots of A_r , which have multiplicity one, and 0, which has multiplicity r . Thus, when $\phi = 0 = \varepsilon_1 + \dots + \varepsilon_{r+1}$ and $\omega = \nu$, $\alpha = \nu + (\varepsilon_1 + \dots + \varepsilon_{r+1})$. This will contribute

$$K_{\lambda, \nu r} \tag{3.5}$$

to the weight multiplicity computation. For ϕ a root, $\alpha = \nu + \zeta_{r+1} = (\nu + \zeta_{r+1} - \phi) + (\phi)$. Thus, we get a contribution of

$$K_{\lambda, \overline{\nu-\phi}} \tag{3.6}$$

to the multiplicity computation for each root $\phi = \varepsilon_i - \varepsilon_j$, $1 \leq i \neq j \leq r + 1$. Of course, if $\nu - \phi$ is not a weight of $V(-\alpha_{-1}) = V(\lambda)$, the Kostka number in (3.6) is zero. Looking at $\nu - \phi = \nu + \varepsilon_j - \varepsilon_i$, we see there are q choices of i that can give weights, since weights must be nonnegative combinations of the ε_k 's. For each i there are r choices of j since $j \neq i$. Thus, the total contribution from the various terms in (3.6) is:

$$\sum_{i=1}^q \sum_{\substack{j=1 \\ j \neq i}}^{r+1} K_{\lambda, \overline{\nu+\varepsilon_j-\varepsilon_i}} = \sum_{\substack{i, j=1 \\ j \neq i}}^q K_{\lambda, \overline{\nu+\varepsilon_j-\varepsilon_i}} + (r + 1 - q) \sum_{i=1}^q K_{\lambda, \overline{\nu+\varepsilon_{q+1}-\varepsilon_i}}. \tag{3.7}$$

Consequently, combining (3.4), (3.5), and (3.7) gives:

$$\begin{aligned} \dim L(\lambda)_{\nu-\delta} &= rK_{\lambda,\nu} + (r+1-q) \sum_{i=1}^q K_{\lambda,\overline{\nu+\varepsilon_{q+1}-\varepsilon_i}} + \sum_{\substack{i,j=1 \\ j \neq i}}^q K_{\lambda,\overline{\nu+\varepsilon_j-\varepsilon_i}} \\ &\quad - (r-q)K_{\lambda+\varepsilon_1,\overline{\nu+\varepsilon_{q+1}}} - K_{\lambda+\zeta_q+\varepsilon_1,\nu+\zeta_{q+1}}. \end{aligned} \tag{3.8}$$

The coefficient of r in (3.8) is

$$K_{\lambda,\nu} + \sum_{i=1}^q K_{\lambda,\overline{\nu+\varepsilon_{q+1}-\varepsilon_i}} - K_{\lambda+\varepsilon_1,\overline{\nu+\varepsilon_{q+1}}} = K_{\lambda,\overline{\nu+\varepsilon_{q+1}-\varepsilon_1}}, \tag{3.9}$$

where the last equality results from using

$$K_{\lambda+\varepsilon_1,\overline{\nu+\varepsilon_{q+1}}} = K_{\lambda,\nu} + \sum_{i=2}^q K_{\lambda,\overline{\nu+\varepsilon_{q+1}-\varepsilon_i}} \tag{3.10}$$

from Proposition 5.14 of the last section.

Now let us consider the constant term in (3.8) which is

$$\begin{aligned} (1-q) \sum_{i=1}^q K_{\lambda,\overline{\nu+\varepsilon_{q+1}-\varepsilon_i}} + \sum_{\substack{i,j=1 \\ i \neq j}}^q K_{\lambda,\overline{\nu+\varepsilon_j-\varepsilon_i}} \\ + qK_{\lambda+\varepsilon_1,\overline{\nu+\varepsilon_{q+1}}} - K_{\lambda+\zeta_q+\varepsilon_1,\nu+\zeta_{q+1}}. \end{aligned} \tag{3.11}$$

Then using (3.10), we see that (3.11) simplifies to

$$\begin{aligned} K_{\lambda+\varepsilon_1,\overline{\nu+\varepsilon_{q+1}}} + (q-1) \left(K_{\lambda,\nu} - K_{\lambda,\overline{\nu+\varepsilon_{q+1}-\varepsilon_1}} \right) \\ + \sum_{\substack{i,j=1 \\ i \neq j}}^q K_{\lambda,\overline{\nu+\varepsilon_j-\varepsilon_i}} - K_{\lambda+\varepsilon_1+\zeta_q,\nu+\zeta_{q+1}}. \end{aligned} \tag{3.12}$$

The last term $K_{\lambda+\varepsilon_1+\zeta_q,\nu+\zeta_{q+1}}$ may be reduced using the relation

$$K_{\lambda+\zeta_q+\varepsilon_1,\nu+\zeta_{q+1}} = K_{\lambda+\varepsilon_1,\overline{\nu+\varepsilon_{q+1}}} + (1-\delta_{\ell(\lambda),q})K_{\lambda+\zeta_{q-1}+\varepsilon_1,\nu+\zeta_q} \tag{3.13}$$

which is argued in Proposition 5.6. As a result, equation (3.11) simplifies to

$$\begin{aligned} (q-1) \left(K_{\lambda,\nu} - K_{\lambda,\overline{\nu+\varepsilon_{q+1}-\varepsilon_1}} \right) + \sum_{\substack{i,j=1 \\ i \neq j}}^q K_{\lambda,\overline{\nu+\varepsilon_j-\varepsilon_i}} \\ + (\delta_{\ell(\lambda),q} - 1)K_{\lambda+\zeta_{q-1}+\varepsilon_1,\nu+\zeta_q}. \end{aligned} \tag{3.14}$$

One further minor reduction to note is that when $j = 1$, then $K_{\lambda, \overline{\nu + \varepsilon_j - \varepsilon_i}} = 0$ since $\nu + \varepsilon_1 - \varepsilon_i$ is not dominated by λ . Thus, the constant term is

$$\begin{aligned}
 (q - 1) \left(K_{\lambda, \nu} - K_{\lambda, \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}} \right) &+ \sum_{\substack{i=1, j=2 \\ i \neq j}}^q K_{\lambda, \overline{\nu + \varepsilon_j - \varepsilon_i}} \\
 &+ (\delta_{\ell(\lambda), q} - 1) K_{\lambda + \zeta_{q-1} + \varepsilon_1, \nu + \zeta_q}. \tag{3.15}
 \end{aligned}$$

The expression in (3.3) is now just a consequence of (3.9) and (3.15).

What remains to be shown is that when $\nu - \delta$ is a weight of $L(\lambda)$, the degree of the corresponding polynomial is exactly one, that is, the lead coefficient $K_{\lambda, \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}}$ is nonzero. To accomplish this we prove that if a term $K_{\lambda, \omega}$ appears in the polynomial, then $\omega \succeq \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}$. Then by Proposition 1.5, $K_{\lambda, \omega} \neq 0$ implies $\lambda \succeq \omega \succeq \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}$ so that $K_{\lambda, \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}} \neq 0$. Thus, we would have that if any such term in the polynomial expression (3.3) is nonzero, then the lead coefficient is nonzero. But in Proposition 5.17 of Section 5 we argue that $\overline{\nu + \varepsilon_j - \varepsilon_i} \succeq \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}$ for all $1 \leq i, j \leq q$. In Proposition 5.18 (i) we prove that when $\ell(\lambda) < q$ and $K_{\lambda + \varepsilon_1 + \zeta_{q-1}, \nu + \zeta_q} \neq 0$, then $K_{\lambda, \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}} \neq 0$. Hence, when $\dim L(\lambda)_{\nu - \delta} \neq 0$ the polynomial in (3.3) has degree exactly one. \square

EXAMPLE 3.16 Assume $\lambda = \nu$. Then (3.3) in this special case reads:

$$\begin{aligned}
 \dim L(\lambda)_{\lambda - \delta} &= r K_{\lambda, \overline{\lambda + \varepsilon_{q+1} - \varepsilon_1}} + (q - 1) \left(1 - K_{\lambda, \overline{\lambda + \varepsilon_{q+1} - \varepsilon_1}} \right) \\
 &+ \sum_{1 \leq i < j \leq q} K_{\lambda, \overline{\lambda + \varepsilon_j - \varepsilon_i}} \\
 &= (r - q + 1) K_{\lambda, \overline{\lambda + \varepsilon_{q+1} - \varepsilon_1}} + (q - 1) \\
 &+ \sum_{1 \leq i < j \leq q} K_{\lambda, \overline{\lambda + \varepsilon_j - \varepsilon_i}}. \tag{3.17}
 \end{aligned}$$

We have used the fact that $K_{\lambda, \overline{\lambda + \varepsilon_j - \varepsilon_i}} = 0$ whenever $i > j$ because then λ does not dominate $\lambda + \varepsilon_j - \varepsilon_i$. To consider a particular example, let $\lambda = \Lambda_1 + 2\Lambda_2 = \nu$ so that $\lambda = 3\varepsilon_1 + 2\varepsilon_2 = \nu$ as A_r -weights. Then $\overline{\lambda + \varepsilon_{q+1} - \varepsilon_1} = 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$, and there are two column-strict tableaux of shape λ with that weight. The sum in (3.17) consists of just one term $K_{\lambda, \overline{\lambda + \varepsilon_2 - \varepsilon_1}}$, and since $\overline{\lambda + \varepsilon_2 - \varepsilon_1} = \lambda$, $K_{\lambda, \overline{\lambda + \varepsilon_2 - \varepsilon_1}} = 1$. Thus, for $\lambda = \Lambda_1 + 2\Lambda_2 = \nu$, $\dim L(\lambda)_{\lambda - \delta} = 2(r - 1) + 1 + 1 = 2r$.

EXAMPLE 3.18 Assume $\lambda = m\Lambda_1$ for any $m \geq 1$ and let $\nu = b_1\Lambda_1 + \dots + b_q\Lambda_q$. Then $\lambda = m\varepsilon_1$ and $\nu = \nu_1\varepsilon_1 + \dots + \nu_q\varepsilon_q$, where $\nu_1 = m$. Since $K_{\lambda, \omega} = 0$ unless ω and λ partition the same integer, we see that (3.3) gives 0 for a multiplicity unless

ν is a partition of m also. The condition $\nu_1 = m$ forces $\lambda = m\Lambda_1 = \nu$, so that now (3.17) applies. Thus,

$$\dim L(\lambda)_{\nu-\delta} = rK_{\lambda, \overline{\lambda+\varepsilon_2-\varepsilon_1}} = r.$$

EXAMPLE 3.19 Let $\lambda = \Lambda_0 + \Lambda_1$ and $\nu = \Lambda_r + \Lambda_2$. Since λ involves Λ_0 in its expression and ν involves Λ_r , we rotate the weights. Thus, we will assume $\lambda = \Lambda_2 + \Lambda_3$ and $\nu = \Lambda_1 + \Lambda_4$. As weights of A_r they become $\lambda = 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$ and $\nu = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$. Then $K_{\lambda, \overline{\lambda+\varepsilon_{q+1}-\varepsilon_1}} = K_{\lambda, \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5} = 5$, and $K_{\lambda, \nu} = 2$. Thus,

$$\begin{aligned} \dim L(\lambda)_{\nu-\delta} &= rK_{\lambda, \overline{\lambda+\varepsilon_{q+1}-\varepsilon_1}} + (q-1)\left(K_{\lambda, \nu} - K_{\lambda, \overline{\lambda+\varepsilon_{q+1}-\varepsilon_1}}\right) \\ &\quad + \sum_{\substack{i=1, j=2 \\ i \neq j}}^q K_{\lambda, \overline{\lambda+\varepsilon_j-\varepsilon_i}} + (\delta_{\ell(\lambda), q} - 1)K_{\lambda+\zeta_{q-1}+\varepsilon_1, \nu+\zeta_q} \\ &= 5r + 3(2-5) + \sum_{\substack{i=1, j=2 \\ i \neq j}}^q K_{\lambda, \overline{\lambda+\varepsilon_j-\varepsilon_i}} - K_{\lambda+\zeta_{q-1}+\varepsilon_1, \nu+\zeta_q} \\ &= 5r + 3(2-5) + 3 \cdot 2 + 2 + 2 + 2 - 6 \\ &= 5r - 3, \end{aligned}$$

when $\lambda = \Lambda_0 + \Lambda_1$ and $\nu = \Lambda_r + \Lambda_2$.

4. The depth two case

4.1. As in Section 3 we assume that $\lambda = a_1\Lambda_1 + \dots + a_q\Lambda_q$ is a dominant integral weight of $A_r^{(1)}$. In this section we show that the multiplicity of a depth two weight $\nu - 2\delta$ in the irreducible $A_r^{(1)}$ -module $L(\lambda)$ is a polynomial of degree 2 in r and obtain an explicit, albeit somewhat complicated, expression for that polynomial in terms of Kostka numbers. When restricted to $A_r = \mathfrak{sl}(r+1, \mathbb{C})$, $\lambda = \lambda_1\varepsilon_1 + \dots + \lambda_q\varepsilon_q$, where $\lambda_i = a_i + \dots + a_q$ for each i and where we may identify λ with the partition $\{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q \geq 0\}$. We may assume that $\nu = b_1\Lambda_1 + \dots + b_q\Lambda_q$, and that ν as an A_r -weight is $\nu = \nu_1\varepsilon_1 + \dots + \nu_q\varepsilon_q$ corresponding to the partition $\nu = \{\nu_1 \geq \nu_2 \geq \dots \geq \nu_q \geq 0\}$. Using the set-up of Section 2 we let $\alpha = -\alpha_{-1} - \sum_{i=1}^r k_i\alpha_i - 2\delta = \nu - 2\delta = \nu + 2\zeta_{r+1}$. Because the coefficient of α_{-1} in α is -1 , the multiplicity of α is given by

$$\text{mult}(\alpha) = B(\alpha) = \sum_{(m) \in T(\alpha)} \frac{(\sum_i m_i - 1)!}{\prod_i (m_i!)} \prod_i (\dim M_{\gamma_i})^{m_i}, \tag{4.2}$$

where $T(\alpha) = \{(m) = (m_i)_{i \geq 1} \mid m_i \in \mathbb{Z}_{\geq 0}, \sum_i m_i \gamma_i = \alpha\}$ and M is the alternating sum of the homology modules as defined in (2.10). Since the coefficient of

α_{-1} in α is -1 and the coefficient of α_0 in α is -2 , then as we noted in (2.20), the A_r -modules whose weights appear in the decompositions of α are:

$$V(-\alpha_{-1}), V(-\alpha_0) \subseteq H_1(\mathcal{G}_-^{(S)}),$$

$$V(-2\alpha_0 - \alpha_1), V(-2\alpha_0 - \alpha_r), V(-\alpha_{-1} - \alpha_0) \subseteq H_2(\mathcal{G}_-^{(S)}),$$

$$V(-\alpha_{-1} - 2\alpha_0 - \alpha_r) \subseteq H_3(\mathcal{G}_-^{(S)}), \text{ and}$$

$$V(-\alpha_{-1} - 2\alpha_0 - \alpha_1) \subseteq H_3(\mathcal{G}_-^{(S)}) \text{ when } a_1 = 0, \text{ i.e. when } \lambda_1 = \lambda_2.$$

Thus, the decompositions of α in (4.2) must have one of the following forms:

(Case 1) $\alpha = (-\alpha_{-1}-?) + (-\alpha_0-?) + (-\alpha_0-?),$

(Case 2) $\alpha = (-\alpha_{-1}-?) + (-2\alpha_0 - \alpha_1-?),$

(Case 3) $\alpha = (-\alpha_{-1}-?) + (-2\alpha_0 - \alpha_r-?),$

(Case 4) $\alpha = (-\alpha_0-?) + (-\alpha_{-1} - \alpha_0-?),$

(Case 5) $\alpha = (-\alpha_{-1} - 2\alpha_0 - \alpha_r-?),$

(Case 6) $\alpha = (-\alpha_{-1} - 2\alpha_0 - \alpha_1-?)$ (when $\lambda_1 = \lambda_2$),

where the ?'s stand for nonnegative linear combinations of the simple roots $\alpha_1, \alpha_2, \dots, \alpha_r$. Observe that the signs of the corresponding terms in (4.2) are + for Cases 1,5,6, and - for Cases 2,3,4.

Case 1. Suppose that

$$\alpha = (-\alpha_{-1}-?) + (-\alpha_0-?) + (-\alpha_0-?) = \omega + \theta + \eta, \tag{4.3}$$

where ω is a weight of $V(-\alpha_{-1})$ and θ, η are weights of $V(-\alpha_0)$ (the adjoint representation of A_r). The weights of $V(-\alpha_0)$ are of the form $\zeta_{r+1} = \varepsilon_1 + \dots + \varepsilon_{r+1} = 0$ and $\varepsilon_i - \varepsilon_j + \zeta_{r+1}$ ($1 \leq i \neq j \leq r + 1$), whose multiplicities are r and 1 , respectively. If $\theta = \eta = \zeta_{r+1}$, then $\omega = \nu$, and the contribution of the decomposition in (4.3) is:

$$\frac{2!}{2!} r^2 K_{\lambda, \nu} = r^2 K_{\lambda, \nu}. \tag{4.4}$$

If $\theta = \zeta_{r+1}$ and $\eta = \varepsilon_i - \varepsilon_j + \zeta_{r+1}$ ($1 \leq i \neq j \leq r + 1$), then $\omega = \nu + \varepsilon_j - \varepsilon_i$. In order for ω to be a weight of $V(-\alpha_{-1})$, we must have $1 \leq i \leq q, 1 \leq j \leq r + 1, j \neq i$. Therefore the decompositions involving $\theta = \zeta_{r+1}$ and such η 's contribute the following total to (4.2):

$$2!r \sum_{i=1}^q \sum_{\substack{j=1 \\ j \neq i}}^{r+1} K_{\lambda, \nu + \varepsilon_j - \varepsilon_i}. \tag{4.5}$$

Now suppose that $\theta = \varepsilon_i - \varepsilon_j + \zeta_{r+1}$, $\eta = \varepsilon_k - \varepsilon_\ell + \zeta_{r+1}$, and $\omega = \nu + (\varepsilon_j + \varepsilon_\ell) - (\varepsilon_i + \varepsilon_k)$, where we may assume $1 \leq i, k \leq q$, $1 \leq j, \ell \leq r + 1$, $i \neq j$, and $k \neq \ell$. It is convenient at this stage to separate considerations according to the relationships between i, j, k, ℓ . In particular, we obtain for the contribution to (4.2):

$$(i = \ell, j = k) \quad 2! \frac{1}{2} r(r + 1) K_{\lambda, \nu}, \tag{4.6}$$

$$\begin{aligned} (i = \ell, j \neq k) \quad & 2! \sum_{k=1}^q \sum_{\substack{j=1 \\ j \neq k}}^{r+1} \sum_{\substack{i=1 \\ i \neq j, k}}^{r+1} K_{\lambda, \nu + \varepsilon_j - \varepsilon_k} \\ & = 2(r - 1) \sum_{k=1}^q \sum_{\substack{j=1 \\ j \neq k}}^{r+1} K_{\lambda, \nu + \varepsilon_j - \varepsilon_k}, \end{aligned}$$

$(i \neq \ell, j \neq k)$ and :

$$(i = k, j = \ell) \quad \frac{2!}{2!} \sum_{i=1}^q \sum_{\substack{j=1 \\ j \neq i}}^{r+1} K_{\lambda, \nu + 2\varepsilon_j - 2\varepsilon_i},$$

$$(i = k, j \neq \ell) \quad 2! \sum_{i=1}^q \sum_{\substack{1 \leq j < \ell \leq r+1 \\ i \neq j, \ell}} K_{\lambda, \nu + \varepsilon_j + \varepsilon_\ell - 2\varepsilon_i},$$

$$(i \neq k) \quad 2! \sum_{1 \leq i < k \leq q} \sum_{\substack{j, \ell=1 \\ j \neq i, k, \ell \neq i, k}}^{r+1} K_{\lambda, \nu + \varepsilon_j + \varepsilon_\ell - \varepsilon_i - \varepsilon_k}.$$

Case 2. Suppose that

$$\alpha = (-\alpha_{-1} - ?) + (-2\alpha_0 - \alpha_1 - ?) = \omega + \theta, \tag{4.7}$$

where ω is a weight of $V(-\alpha_{-1})$ and θ is a weight of $V(-2\alpha_0 - \alpha_1)$. Now $-2\alpha_0 - \alpha_1 = 2(\varepsilon_1 + \zeta_r) - (\varepsilon_1 - \varepsilon_2) = \varepsilon_1 + \varepsilon_2 + 2\zeta_r = \varepsilon_1 + \varepsilon_2 - 2\varepsilon_{r+1} + 2\zeta_{r+1}$. It is known that the weights of $V(-2\alpha_0 - \alpha_1)$ are of the form:

- (i) $\varepsilon_i + \varepsilon_j - \varepsilon_k - \varepsilon_\ell + 2\zeta_{r+1}$ ($1 \leq i < j \leq r + 1, 1 \leq k \leq \ell \leq r + 1, k \neq i, j$, and $\ell \neq i, j$);
- (ii) $\varepsilon_i - \varepsilon_j + 2\zeta_{r+1}$ ($1 \leq i \neq j \leq r + 1$);
- (iii) $2\zeta_{r+1}$,

whose multiplicities are 1, $r - 1$, and $\frac{1}{2}r(r - 1)$, respectively. Hence the terms coming from Case 2 will contribute the following to (4.2):

$$\begin{aligned}
 & - \sum_{\substack{i,j=1 \\ i < j}}^q \sum_{\substack{1 \leq k \leq \ell \leq r+1 \\ k \neq i, j \quad \ell \neq i, j}} K_{\lambda, \overline{\nu + \varepsilon_k + \varepsilon_\ell - \varepsilon_i - \varepsilon_j}}, \\
 & - (r - 1) \sum_{i=1}^q \sum_{\substack{j=1 \\ j \neq i}}^{r+1} K_{\lambda, \overline{\nu + \varepsilon_j - \varepsilon_i}} - \frac{1}{2}r(r - 1)K_{\lambda, \nu}
 \end{aligned} \tag{4.8}$$

Case 3. Suppose that

$$\alpha = (-\alpha_{-1} - ?) + (-2\alpha_0 - \alpha_r - ?) = \omega + \theta, \tag{4.9}$$

where ω is a weight of $V(-\alpha_{-1})$ and θ is a weight of $V(-2\alpha_0 - \alpha_r)$. It is known that the weights of $V(-2\alpha_0 - \alpha_r)$ are of the form:

- (i) $\varepsilon_i + \varepsilon_j - \varepsilon_k - \varepsilon_\ell + 2\zeta_{r+1}$ ($1 \leq i \leq j \leq r + 1, 1 \leq k < \ell \leq r + 1, i \neq k, \ell, j \neq k, \ell$);
- (ii) $\varepsilon_i - \varepsilon_j + 2\zeta_{r+1}$ ($1 \leq i \neq j \leq r + 1$);
- (iii) $2\zeta_{r+1}$,

whose multiplicities are 1, $r - 1$, and $\frac{1}{2}r(r - 1)$, respectively. Decompositions of α using these weights add the following to (4.2):

$$\begin{aligned}
 & - \sum_{1 \leq i \leq j \leq q} \sum_{\substack{1 \leq k < \ell \leq r+1 \\ i \neq k, \ell, j \neq k, \ell}} K_{\lambda, \overline{\nu + \varepsilon_k + \varepsilon_\ell - \varepsilon_i - \varepsilon_j}}, \\
 & - (r - 1) \sum_{i=1}^q \sum_{\substack{j=1 \\ j \neq i}}^{r+1} K_{\lambda, \overline{\nu + \varepsilon_j - \varepsilon_i}} - \frac{1}{2}r(r - 1)K_{\lambda, \nu}.
 \end{aligned} \tag{4.10}$$

Case 4. Suppose that

$$\alpha = (-\alpha_0 - ?) + (-\alpha_{-1} - \alpha_0 - ?) = \theta + \omega, \tag{4.11}$$

where θ is a weight of $V(-\alpha_0)$ and ω is a weight of $V(-\alpha_{-1} - \alpha_0)$. Recall that the weights of $V(-\alpha_0)$ are of the form ζ_{r+1} and $\varepsilon_i - \varepsilon_j + \zeta_{r+1}$ with multiplicity r and 1, respectively. In Case 4 the contribution to (4.2) then is:

$$-rK_{\lambda + \varepsilon_1 + \zeta_r, \nu + \zeta_{r+1}} - \sum_{\substack{i,j=1 \\ i \neq j}}^{r+1} K_{\lambda + \varepsilon_1 + \zeta_r, \overline{\nu + \varepsilon_j - \varepsilon_i + \zeta_{r+1}}}. \tag{4.12}$$

Case 5. Suppose that

$$\alpha = (-\alpha_{-1} - 2\alpha_0 - \alpha_r - ?) = \nu + 2\zeta_{r+1}, \tag{4.13}$$

a weight of $V(-\alpha_{-1} - 2\alpha_0 - \alpha_r)$, where $-\alpha_{-1} - 2\alpha_0 - \alpha_r = \lambda + 2\varepsilon_1 + \zeta_{r-1} + \zeta_{r+1}$. What is added to (4.2) here is:

$$K_{\lambda+2\varepsilon_1+\zeta_{r-1}+\zeta_{r+1},\nu+2\zeta_{r+1}} = K_{\lambda+2\varepsilon_1+\zeta_{r-1},\nu+\zeta_{r+1}}. \tag{4.14}$$

Case 6. In this final case, which occurs only when $\lambda_1 = \lambda_2$

$$\alpha = (-\alpha_{-1} - 2\alpha_0 - \alpha_1 - ?) = \nu + 2\zeta_{r+1} \tag{4.15}$$

is a weight of the irreducible module $V(-\alpha_{-1} - 2\alpha_0 - \alpha_1)$ having highest weight $-\alpha_{-1} - 2\alpha_0 - \alpha_r = \lambda + \varepsilon_1 + \varepsilon_2 + 2\zeta_r$. The contribution to (4.2) is accordingly:

$$\delta_{\lambda_1,\lambda_2} K_{\lambda+\varepsilon_1+\varepsilon_2+2\zeta_r,\nu+2\zeta_{r+1}}. \tag{4.16}$$

Combining (4.4)–(4.6), (4.8), (4.10), (4.12), (4.14), (4.16) allows us to conclude:

LEMMA 4.17 *The multiplicity of a depth two weight $\nu - 2\delta$ in the irreducible $A_r^{(1)}$ -module $L(\lambda)$ is*

$$\begin{aligned} \text{mult}(\alpha) &= (r^2 + 2r)K_{\lambda,\nu} + 2r \sum_{i=1}^q \sum_{\substack{j=1 \\ j \neq i}}^{r+1} K_{\lambda,\overline{\nu+\varepsilon_j-\varepsilon_i}} \\ &+ \sum_{i=1}^q \sum_{\substack{1 \leq k \leq \ell \leq r+1 \\ i \neq k,\ell}} K_{\lambda,\overline{\nu+\varepsilon_k+\varepsilon_\ell-2\varepsilon_i}} \\ &+ \sum_{1 \leq i < j \leq q} \sum_{\substack{k,\ell=1 \\ k \neq i,j \ \ell \neq i,j}}^{r+1} K_{\lambda,\overline{\nu+\varepsilon_k+\varepsilon_\ell-\varepsilon_i-\varepsilon_j}} \\ &- \sum_{\substack{i,j=1 \\ i \neq j}}^{r+1} K_{\lambda+\varepsilon_1+\zeta_r,\overline{\nu+\varepsilon_j-\varepsilon_i+\zeta_{r+1}}} \\ &- rK_{\lambda+\varepsilon_1+\zeta_r,\nu+\zeta_{r+1}} + K_{\lambda+2\varepsilon_1+\zeta_{r-1},\nu+\zeta_{r+1}} \\ &+ \delta_{\lambda_1,\lambda_2} K_{\lambda+\varepsilon_1+\varepsilon_2+2\zeta_r,\nu+2\zeta_{r+1}}. \end{aligned}$$

EXAMPLE 4.18 Suppose that $\lambda = m\Lambda_1 = \nu$ for some $m \geq 1$. Then as an A_r -weight $\lambda = m\varepsilon_1 = \nu$, and

$$\text{mult}(m\Lambda_1 - 2\delta)$$

$$\begin{aligned} &= r^2 + 2r + 2r \sum_{j=2}^{r+1} K_{\lambda, \overline{\lambda + \varepsilon_j - \varepsilon_1}} + \sum_{2 \leq k \leq \ell \leq r+1} K_{\lambda, \overline{\lambda + \varepsilon_k + \varepsilon_\ell - 2\varepsilon_1}} \\ &\quad - \sum_{\substack{i, j=1 \\ i \neq j}}^{r+1} K_{\lambda + \varepsilon_1 + \zeta_r, \overline{\lambda + \varepsilon_j - \varepsilon_i + \zeta_{r+1}}} - rK_{\lambda + \varepsilon_1 + \zeta_r, \lambda + \zeta_{r+1}} + K_{\lambda + 2\varepsilon_1 + \zeta_{r-1}, \lambda + \zeta_{r+1}}. \end{aligned}$$

Since λ corresponds to a partition with just one part (of size m), the Kostka number $K_{\lambda, \eta} = 1$ for all partitions η of m . When $i = 1$, we have $K_{\lambda + \varepsilon_1 + \zeta_r, \overline{\lambda + \varepsilon_j - \varepsilon_i + \zeta_{r+1}}} = r$ for $j = 2, \dots, r + 1$, since each column-strict tableau of shape $\lambda + \varepsilon_1 + \zeta_r$ and weight $\lambda + \varepsilon_j - \varepsilon_1 + \zeta_{r+1}$ necessarily has m ones, j , and some other integer k , say, in its first row, and there are r choices for k . When $i > 1$, then for all $j = 1, \dots, r + 1$ with $j \neq i$ there is exactly one column-strict tableau of shape $\lambda + \varepsilon_1 + \zeta_r$ and weight $\lambda + \varepsilon_j - \varepsilon_i + \zeta_{r+1} = \lambda + \varepsilon_j + \zeta_r$. Therefore, using these results, we see the above reduces in the following way:

$$\begin{aligned} \text{mult}(m\Lambda_1 - 2\delta) &= r^2 + 2r + 2r^2 + \frac{1}{2}r(r + 1) - r^2 - r^2 - r^2 \\ &\quad + \frac{1}{2}r(r - 1) = r^2 + 2r. \end{aligned}$$

When $m = 1$, then $\sum_{2 \leq k \leq \ell \leq r+1} K_{\lambda, \overline{\lambda + \varepsilon_k + \varepsilon_\ell - 2\varepsilon_1}}$ is 0, and the term $(1/2)r(r + 1)$ should be omitted. Thus, $\text{mult}(\Lambda_1 - 2\delta) = (1/2)(r^2 + 3r)$. (We are grateful to Ronald King for this observation and for his interest in our work.)

In order to see the polynomial behavior of the multiplicity expression in Lemma 4.17 we will split the terms according to whether the indices $k, \ell = 1, \dots, r + 1$ lie in the range $1, \dots, q$ or the range $q + 1, \dots, r + 1$. It follows from the fact that λ dominates μ , that $\ell(\mu) \geq \ell(\lambda) = \ell(\lambda + 2\varepsilon_1)$. Thus, we can apply (5.8), (5.10), and (5.13) from the next section to re-express the last four summands and obtain:

$$\begin{aligned} \text{(i)} \quad &2r \sum_{i=1}^q \sum_{\substack{j=1 \\ j \neq i}}^{r+1} K_{\lambda, \overline{\nu + \varepsilon_j - \varepsilon_i}} \\ &= 2r \sum_{\substack{i, j=1 \\ j \neq i}}^q K_{\lambda, \overline{\nu + \varepsilon_j - \varepsilon_i}} + 2r(r + 1 - q) \sum_{i=1}^q K_{\lambda, \overline{\nu + \varepsilon_{q+1} - \varepsilon_i}}, \end{aligned} \tag{4.19}$$

$$\begin{aligned}
\text{(ii)} \quad & \sum_{i=1}^q \sum_{\substack{1 \leq k \leq \ell \leq r+1 \\ i \neq k, \ell}} K_{\lambda, \overline{\nu + \varepsilon_k + \varepsilon_\ell - 2\varepsilon_i}} \\
&= \sum_{\substack{i, k, \ell=1 \\ k \leq \ell, i \neq k, \ell}}^q K_{\lambda, \overline{\nu + \varepsilon_k + \varepsilon_\ell - 2\varepsilon_i}} + (r+1-q) \sum_{\substack{i, k=1 \\ k \neq i}}^q K_{\lambda, \overline{\nu + \varepsilon_k + \varepsilon_{q+1} - 2\varepsilon_i}} \\
&\quad + (r+1-q) \sum_{i=1}^q K_{\lambda, \overline{\nu + 2\varepsilon_{q+1} - 2\varepsilon_i}} \\
&\quad + \frac{1}{2}(r+1-q)(r-q) \sum_{i=1}^q K_{\lambda, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_i}},
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad & \sum_{1 \leq i < j \leq q} \sum_{\substack{k, \ell=1 \\ k \neq i, j, \ell \neq i, j}}^{r+1} K_{\lambda, \overline{\nu + \varepsilon_k + \varepsilon_\ell - \varepsilon_i - \varepsilon_j}} \\
&= \sum_{\substack{i, j, k, \ell=1 \\ i < j, k \neq i, j, \ell \neq i, j}}^q K_{\lambda, \overline{\nu + \varepsilon_k + \varepsilon_\ell - \varepsilon_i - \varepsilon_j}} \\
&\quad + 2(r+1-q) \sum_{\substack{i, j, k=1 \\ i < j, k \neq i, j}}^q K_{\lambda, \overline{\nu + \varepsilon_k + \varepsilon_{q+1} - \varepsilon_i - \varepsilon_j}} \\
&\quad + (r+1-q) \sum_{1 \leq i < j \leq q} K_{\lambda, \overline{\nu + 2\varepsilon_{q+1} - \varepsilon_i - \varepsilon_j}} \\
&\quad + (r+1-q)(r-q) \sum_{1 \leq i < j \leq q} K_{\lambda, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j}},
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad & - \sum_{\substack{i, j=1 \\ i \neq j}}^{r+1} K_{\lambda + \varepsilon_1 + \zeta_r, \overline{\nu + \varepsilon_j - \varepsilon_i + \zeta_{r+1}}} \\
&= - \sum_{\substack{i, j=1 \\ i \neq j}}^q K_{\lambda + \varepsilon_1 + \zeta_q, \overline{\nu + \varepsilon_j - \varepsilon_i + \zeta_{q+1}}} - (r-q) \sum_{\substack{i, j=1 \\ i \neq j}}^q K_{\lambda + \varepsilon_1, \overline{\nu + \varepsilon_j - \varepsilon_i + \varepsilon_{q+1}}} \\
&\quad - (r+1-q) \sum_{i=1}^q K_{\lambda + \zeta_{q+1} + \varepsilon_1, \overline{\nu + \varepsilon_{q+1} - \varepsilon_i + \zeta_{q+2}}}
\end{aligned}$$

$$\begin{aligned}
 & -(r + 1 - q)(r - 1 - q) \sum_{i=1}^q K_{\lambda+\varepsilon_1, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_i}} \\
 & -(r + 1 - q) \sum_{j=1}^q K_{\lambda+\varepsilon_1, \overline{\nu+\varepsilon_j}} - (r + 1 - q)(r - q) K_{\lambda+\varepsilon_1, \overline{\nu+\varepsilon_{q+1}}},
 \end{aligned}$$

(v) $-r K_{\lambda+\varepsilon_1+\zeta_r, \nu+\zeta_{r+1}} = -r K_{\lambda+\varepsilon_1+\zeta_q, \nu+\zeta_{q+1}} - r(r - q) K_{\lambda+\varepsilon_1, \overline{\nu+\varepsilon_{q+1}}},$

(vi) $K_{\lambda+2\varepsilon_1+\zeta_{r-1}, \nu+\zeta_{r+1}} = (r - q) K_{\lambda+2\varepsilon_1+\zeta_q, \nu+\zeta_{q+2}}$
 $- (1 - \delta_{\ell(\lambda), q})(r - 1 - q) K_{\lambda+2\varepsilon_1+\zeta_{q-1}, \nu+\zeta_{q+1}}$
 $+ \frac{1}{2}(r - q)(r - 1 - q) K_{\lambda+2\varepsilon_1, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}}},$

(vii) $\delta_{\lambda_1, \lambda_2} K_{\lambda+\varepsilon_1+\varepsilon_2+2\zeta_r, \nu+2\zeta_{r+1}}$
 $= \delta_{\lambda_1, \lambda_2} K_{\lambda+\varepsilon_1+\varepsilon_2+2\zeta_q, \nu+2\zeta_{q+1}} + (r - q) \delta_{\lambda_1, \lambda_2} K_{\lambda+\varepsilon_1+\varepsilon_2+\zeta_{q+1}, \nu+\zeta_{q+1}+\zeta_{q+2}}$
 $+ \frac{1}{2}(r - q)(r - 1 - q) \delta_{\lambda_1, \lambda_2} K_{\lambda+\varepsilon_1+\varepsilon_2, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}}}.$

From these equations it is clear that the expression for $\text{mult}(\alpha)$ in Lemma 4.17 is a polynomial in r of degree at most two. Hence, this brings us to the main result of the section:

THEOREM 4.20 *Assume $L(\lambda)$ is the irreducible module for $A_r^{(1)}$ with highest weight $\lambda = a_1\Lambda_1 + \dots + a_q\Lambda_q$ which is dominant integral. Suppose $\nu - 2\delta$ is a weight of $L(\lambda)$ of depth 2 where $\nu = b_1\Lambda_1 + \dots + b_q\Lambda_q$. Then for all $r \geq q$,*

$$\begin{aligned}
 \dim L(\lambda)_{\nu-2\delta} &= Ar^2 + Br + C \quad \text{where} \\
 A &= \begin{cases} \frac{1}{2} \sum_{i,j=2, i>j}^{q+2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_i-\varepsilon_j}} + \frac{1}{2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-2\varepsilon_1}} & \text{when } \lambda_1 > \lambda_2, \\ \frac{1}{2} \sum_{j=2}^{q+2} \widehat{K}_{\lambda, \nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1-\varepsilon_j} + \frac{1}{2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-2\varepsilon_1}} & \text{when } \lambda_1 = \lambda_2, \end{cases}
 \end{aligned}
 \tag{4.21}$$

where $\widehat{K}_{\lambda, \nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1-\varepsilon_j}$ is the number of column-strict tableaux of shape λ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j$ whose $(1, \lambda_1)$ -entry is $> j$. In particular $A \neq 0$, and this polynomial has degree exactly two.

Proof. It is a consequence of Lemma 4.17 and (4.19) that the multiplicity is a polynomial in r of degree at most 2. What remains to be shown is that the lead

coefficient can be calculated by (4.21) and that it is nonzero. By (4.19) the lead coefficient is the sum of the following terms:

- (a) $K_{\lambda, \nu}$
- (b) $2 \sum_{i=1}^q K_{\lambda, \overline{\nu + \varepsilon_{q+1} - \varepsilon_i}}$
- (c) $\frac{1}{2} \sum_{i=1}^q K_{\lambda, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_i}}$
- (d) $\sum_{i,j=1, i < j}^q K_{\lambda, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j}}$
- (e) $-2K_{\lambda + \varepsilon_1, \overline{\nu + \varepsilon_{q+1}}}$
- (f) $-\sum_{i=1}^q K_{\lambda + \varepsilon_1, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i}}$
- (g) $\frac{1}{2} K_{\lambda + 2\varepsilon_1, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2}}}$
- (h) $\frac{1}{2} \delta_{\lambda_1, \lambda_2} K_{\lambda + \varepsilon_1 + \varepsilon_2, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2}}}$

To reduce this sum we will make use of the following observations:

$$K_{\lambda, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j}} = \begin{cases} K_{\lambda, \overline{\nu + \varepsilon_{q+1} - \varepsilon_i}} & \text{if } 1 \leq i \leq q \text{ and } j = q+1, q+2, \\ K_{\lambda, \nu} & \text{if } i = q+1 \text{ and } j = q+2; \end{cases}$$

$$K_{\lambda, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_i}} = 0 \quad \text{if } i = q+1, q+2.$$

Thus, we see that

$$(a) + (b) + (c) + (d) = \frac{1}{2} \sum_{i,j=1}^{q+2} K_{\lambda, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j}}.$$

Now

$$(e) + (f) = -\sum_{i=1}^{q+2} K_{\lambda + \varepsilon_1, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i}}$$

and

$$(g) = \frac{1}{2} \sum_{i,j=2, i \leq j}^{q+2} K_{\lambda, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j}}.$$

Since for $i \neq 1$,

$$K_{\lambda + \varepsilon_1, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i}} = \sum_{j=2}^{q+2} K_{\lambda, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j}},$$

we see that

$$(e) + (f) = -K_{\lambda+\varepsilon_1, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1}} - \sum_{i,j=2}^{q+2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_i-\varepsilon_j}}.$$

Hence, the lead coefficient is given by the following expression:

$$\begin{aligned} (a) - (h) &= -\frac{1}{2} \sum_{i,j=2, i>j}^{q+2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_i-\varepsilon_j}} + \frac{1}{2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-2\varepsilon_1}} \\ &\quad + \sum_{j=2}^{q+2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1-\varepsilon_j}} - K_{\lambda+\varepsilon_1, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1}} \\ &\quad + \frac{1}{2} \delta_{\lambda_1, \lambda_2} K_{\lambda+\varepsilon_1+\varepsilon_2, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}}}. \end{aligned}$$

Suppose first that $\lambda_1 > \lambda_2$. Then since

$$K_{\lambda+\varepsilon_1, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1}} = \sum_{i,j=2, j \geq i}^{q+2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_i-\varepsilon_j}}$$

and

$$\sum_{j=2}^{q+2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1-\varepsilon_j}} = \sum_{i,j=2}^{q+2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_i-\varepsilon_j}}$$

in this case (compare Proposition 5.18 (ii) below), we have:

$$\begin{aligned} (a) - (g) &= \frac{1}{2} \sum_{i,j=2, i>j}^{q+2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_i-\varepsilon_j}} \\ &\quad + \frac{1}{2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-2\varepsilon_1}} \quad \text{when } \lambda_1 > \lambda_2. \end{aligned}$$

Now consider the $\lambda_1 = \lambda_2$ case. The column-strict tableaux of shape $\lambda + \varepsilon_1$ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1$ are in one-to-one correspondence with the tableaux of shape λ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j$ for $j = 2, \dots, q+2$, whose $(1, \lambda_1)$ -entry is $\leq j$. (This can be seen by deleting the last box in the first row of such a tableau.) Therefore:

$$\begin{aligned} &\sum_{j=2}^{q+2} K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1-\varepsilon_j}} - K_{\lambda+\varepsilon_1, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1}} \\ &= \sum_{j=2}^{q+2} \widehat{K}_{\lambda, \nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1-\varepsilon_j}, \end{aligned} \tag{4.22}$$

where $\widehat{K}_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j}$ is the number of column-strict tableaux of shape λ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j$ whose $(1, \lambda_1)$ -entry is $> j$.

Consider the column-strict tableaux of shape λ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j$ for $2 \leq j < i \leq q + 2$. If the $(2, \lambda_2)$ entry is less than or equal to i , we can adjoin a box to the first row with j in it and a box to the second row with i in it and get a column-strict tableau of shape $\lambda + \varepsilon_1 + \varepsilon_2$ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2}$. If on the other hand the $(2, \lambda_2)$ entry is greater than i , then we can remove the entry in the $(1, \lambda_1)$ box, which is necessarily a ‘1’ and replace it with an i to produce a column-strict tableau of shape λ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j$ whose $(1, \lambda_1)$ entry is $> j$. As both of these processes reverse, we see that

$$\begin{aligned} & \sum_{i,j=2, i>j}^{q+2} K_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j} \\ &= K_{\lambda + \varepsilon_1 + \varepsilon_2, \nu + \varepsilon_{q+1} + \varepsilon_{q+2}} + \sum_{j=2}^{q+2} \widehat{K}_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j}, \end{aligned} \tag{4.23}$$

where $\widehat{K}_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j}$ is as before. Thus, combining these expressions with the expression for (a)–(h) gives:

$$\begin{aligned} \text{(a) - (h)} &= \frac{1}{2} \sum_{j=2}^{q+2} \widehat{K}_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j} \\ &+ \frac{1}{2} K_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_1} \quad \text{when } \lambda_1 = \lambda_2, \end{aligned}$$

where $\widehat{K}_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j}$ is the number of column-strict tableaux of shape λ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j$ whose $(1, \lambda_1)$ -entry is $> j$.

It remains to argue that the degree is exactly two. Since we are assuming that $\nu - 2\delta$ is a weight of $L(\lambda)$, its multiplicity, which is given by Lemma 4.17, is nonzero. We show that if any of the Kostka numbers appearing in the multiplicity expression in Lemma 4.17 is nonzero, then the lead coefficient A in (4.21) must be nonzero. If a Kostka number appearing in parts (i), (ii), or (iii) of (4.19) is nonzero, then by Proposition 5.17 either $K_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_1} \neq 0$ or $K_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j} \neq 0$ for some $1 \leq i \neq j \leq \ell(\nu)$. Now $K_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j} \neq 0$ implies $K_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j} \neq 0$ for some $2 \leq i \leq q+2$ by Proposition 5.18 (ii). If $i = j$, then $K_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_1} \neq 0$ by Proposition 5.17 (i). If a Kostka number appearing in (4.19) (iv)–(vii) is nonzero, then once again either $K_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_1} \neq 0$ or $K_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j} \neq 0$ for some $2 \leq i \neq j \leq q + 2$ by Proposition 5.18 (iii). This allows us to conclude when $\lambda_1 > \lambda_2$ that $A \neq 0$.

Now suppose that $\lambda_1 = \lambda_2$. Arguing as in the previous case, we may assume either $K_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_1} \neq 0$ or $K_{\lambda, \nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j} \neq 0$ for some $1 \leq j < i \leq \ell(\nu)$. In the first situation $A \neq 0$. In the second, when $j \geq 2$,

we may assume by (4.23) that $K_{\lambda+\varepsilon_1+\varepsilon_2, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}}} \neq 0$ (else $A \neq 0$). Then $K_{\lambda+\varepsilon_1, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_k}} \neq 0$ for some $k \geq 2$, which can be seen by removing the last box from the second row of a column-strict tableau of shape $\lambda + \varepsilon_1 + \varepsilon_2$ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2}$. Since $\overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_k} = \overline{\nu + \varepsilon_{\ell(\nu)+1} + \varepsilon_{\ell(\nu)+2} - \varepsilon_k} \succeq \overline{\nu + \varepsilon_{\ell(\nu)+1} + \varepsilon_{\ell(\nu)+2} - \varepsilon_1} = \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1}$ by Proposition 5.17 (i), we may suppose that $K_{\lambda+\varepsilon_1, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1}} \neq 0$. Then by (4.22), it follows that $K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1-\varepsilon_\ell}} \neq 0$ for some $\ell \geq 2$. Since $\overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_\ell} \succeq \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_2} \neq 0$ by Proposition 5.17 (iii), we may suppose that $K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1-\varepsilon_2}} \neq 0$. If the $(1, \lambda_1)$ entry of a column-strict tableau of shape λ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_2$ is greater than 2, then $\widehat{K}_{\lambda, \nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1-\varepsilon_2} \neq 0$ and $A \neq 0$ as desired. Otherwise, the $(1, \lambda_1)$ entry must be 2, and replacing it with a 1 gives a column-strict tableau of shape λ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_2$. Since $K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-2\varepsilon_2}} \neq 0$ implies $K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-2\varepsilon_1}} \neq 0$ by Proposition 5.17 (ii), we have in this final case that $A \neq 0$. This completes the proof. \square

5. Kostka number identities

5.1. In this section we establish the Kostka number identities required for the multiplicity computations in Sections 3 and 4. Throughout $\pi = \sum_{i=1}^p \pi_i \varepsilon_i = \{\pi_1 \geq \pi_2 \geq \dots \geq \pi_p \geq 0\}$ and $\eta = \sum_{i=1}^p \eta_i \varepsilon_i = \{\eta_1 \geq \eta_2 \geq \dots \geq \eta_p \geq 0\}$ are partitions, and the lengths $\ell(\pi), \ell(\eta)$ of the partitions π, η are less than or equal to p and at least one of them is p . We assume η partitions m and π partitions $m + n$ for some $n \geq 0$ which we specify in the various results. As before, $\zeta_t = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_t$ so that ζ_t corresponds to the partition of t having all parts of size 1. We use the following facts: (i) $K_{\lambda, \nu}$ represents the number of column-strict tableaux of shape λ and weight ν as in (1.1); (ii) $K_{\lambda, \nu} = 0$ if λ does not dominate ν in the dominance order, see (1.3); (iii) If $\omega = \omega_1 \varepsilon_1 + \omega_2 \varepsilon_2 + \dots + \omega_{r+1} \varepsilon_{r+1}$ where the ω_i are nonnegative integers summing to m , and if σ is a permutation, then $\sigma\omega = \omega_1 \varepsilon_{\sigma 1} + \omega_2 \varepsilon_{\sigma 2} + \dots + \omega_{r+1} \varepsilon_{\sigma(r+1)} = \omega_{\sigma^{-1}1} \varepsilon_1 + \omega_{\sigma^{-1}2} \varepsilon_2 + \dots + \omega_{\sigma^{-1}(r+1)} \varepsilon_{r+1}$. In particular, the ω_i 's can be rearranged to be in descending order, and the resulting partition of m is $\overline{\omega}$ as in (1.4). By definition $K_{\lambda, \omega} = K_{\lambda, \sigma\omega} = K_{\lambda, \overline{\omega}}$ for all permutations σ .

PROPOSITION 5.2 Assume $r \geq p$ and π is a partition of $m + 1$. Then

$$K_{\pi+\zeta_r, \eta+\zeta_{r+1}} = (r - p)K_{\pi, \overline{\eta+\varepsilon_{p+1}}} + K_{\pi+\zeta_p, \eta+\zeta_{p+1}}. \tag{5.3}$$

Proof. The result is clear if $r = p$, so assume $r > p$. A column-strict tableau of shape $\pi + \zeta_r$ and weight $\eta + \zeta_{r+1}$ contains exactly one entry equal to $r + 1$. If the box containing $r + 1$ is in the r th row of the tableau, then deleting it gives a column-strict tableau of shape $\pi + \zeta_{r-1}$ and weight $\eta + \zeta_r$. Otherwise, removing

the box with $r + 1$ in it yields a tableau of shape $\kappa + \zeta_r$ and weight $\eta + \zeta_r$ for some partition $\kappa \subset \pi$ with $|\pi/\kappa| = 1$. Thus,

$$K_{\pi+\zeta_r, \eta+\zeta_{r+1}} = K_{\pi+\zeta_{r-1}, \eta+\zeta_r} + \sum_{\substack{\kappa \subset \pi \\ |\pi/\kappa|=1}} K_{\kappa+\zeta_r, \eta+\zeta_r}. \tag{5.4}$$

Since a column-strict tableau of shape $\kappa + \zeta_r$ and weight $\eta + \zeta_r$ must have $1, \dots, r$ down its first column, $K_{\kappa+\zeta_r, \eta+\zeta_r} = K_{\kappa, \eta}$. As a result,

$$\begin{aligned} K_{\pi+\zeta_r, \eta+\zeta_{r+1}} &= K_{\pi+\zeta_{r-1}, \eta+\zeta_r} + \sum_{\substack{\kappa \subset \pi \\ |\pi/\kappa|=1}} K_{\kappa, \eta} \\ &= K_{\pi+\zeta_{r-1}, \eta+\zeta_r} + K_{\pi, \overline{\eta+\varepsilon_{p+1}}}, \end{aligned} \tag{5.5}$$

where the last equality can be seen by considering column-strict tableaux of shape π and weight $\eta + \varepsilon_{p+1}$ and removing the box which contains $p + 1$. Repeated applications of this argument gives (5.3). \square

The next result in conjunction with Proposition 5.2 allows for further reduction, especially in the case that $\ell(\pi) = p$.

PROPOSITION 5.6 *Assume π is a partition of $m + 1$ and $r \geq p$. Then*

$$K_{\pi+\zeta_p, \overline{\eta+\zeta_{p+1}}} = K_{\pi, \overline{\eta+\varepsilon_{p+1}}} + (1 - \delta_{\ell(\pi), p}) K_{\pi+\zeta_{p-1}, \eta+\zeta_p}, \tag{5.7}$$

$$K_{\pi+\zeta_r, \eta+\zeta_{r+1}} = (r - p + 1) K_{\pi, \overline{\eta+\varepsilon_{p+1}}} + (1 - \delta_{\ell(\pi), p}) K_{\pi+\zeta_{p-1}, \eta+\zeta_p}. \tag{5.8}$$

Proof. The column-strict tableaux of shape $\pi + \zeta_p$ and weight $\eta + \zeta_{p+1}$ which have $1, \dots, p$ down their first column are in one-to-one correspondence with the column-strict tableaux of shape π and weight $\eta + \varepsilon_{p+1}$. If a tableau of shape $\pi + \zeta_p$ and weight $\eta + \zeta_{p+1}$ does not have $1, \dots, p$ down its first column, then $p + 1$ must occur in the last box of the first column, and necessarily π must have length less than p . Such tableaux, when they occur, are in one-to-one correspondence with the column-strict tableaux of shape $\pi + \zeta_{p-1}$ and weight $\eta + \zeta_p$, which can be seen by removing the box with $p + 1$ in it from the first column. Thus (5.7) holds, and (5.8) then follows from Proposition 5.2. \square

The ‘quadratic version’ of Proposition 5.2 is the following:

PROPOSITION 5.9 *Let π be a partition of $m + 2$ and η be a partition of m with $\ell(\eta) = p$. Then for $r \geq p$ we have*

$$\begin{aligned} K_{\pi+2\zeta_r, \eta+2\zeta_{r+1}} &= \frac{1}{2}(r - p + 1)(r - p) K_{\pi, \eta+\varepsilon_{p+1}+\varepsilon_{p+2}} \\ &\quad + (r - p) K_{\pi+\zeta_p, \eta+\varepsilon_{p+1}+\zeta_{p+1}} + K_{\pi+2\zeta_p, \eta+2\zeta_{p+1}}. \end{aligned} \tag{5.10}$$

Proof. We may assume $r > p$ as otherwise the result is trivial. For a column-strict tableau of shape $\pi + 2\zeta_r$ and weight $\eta + 2\zeta_{r+1}$ there are two possibilities: either (i) it has $1, \dots, r$ down its first column or (ii) the last entry in both of the first two columns is $r + 1$. In Case (i) the tableaux are in one-to-one correspondence with the tableaux of shape $\pi + \zeta_r$ and weight $\eta + \varepsilon_{r+1} + \zeta_{r+1}$. As $\eta + \varepsilon_{r+1} + \zeta_{r+1}$ can be brought to $\eta + \varepsilon_{p+1} + \zeta_{r+1}$ by permutation, the number of these tableaux is $K_{\pi+\zeta_r, \eta+\varepsilon_{p+1}+\zeta_{r+1}}$. By Proposition 5.2,

$$\begin{aligned} K_{\pi+\zeta_r, \eta+\varepsilon_{p+1}+\zeta_{r+1}} &= (r - (p + 1))K_{\pi, \eta+\varepsilon_{p+1}+\varepsilon_{p+2}} \\ &\quad + K_{\pi+\zeta_{p+1}, \eta+\varepsilon_{p+1}+\zeta_{p+2}} \\ &= (r - p)K_{\pi, \eta+\varepsilon_{p+1}+\varepsilon_{p+2}} + K_{\pi+\zeta_p, \eta+\varepsilon_{p+1}+\zeta_{p+1}}. \end{aligned} \tag{5.11}$$

The last equality in (5.11) follows because a tableau of shape $\pi + \zeta_{p+1}$ and weight $\eta + \varepsilon_{p+1} + \zeta_{p+2}$ either has $1, \dots, p + 1$ down its first column or the last entry in column 1 is $p + 2$, and so

$$K_{\pi+\zeta_{p+1}, \eta+\varepsilon_{p+1}+\zeta_{p+2}} = K_{\pi, \eta+\varepsilon_{p+1}+\varepsilon_{p+2}} + K_{\pi+\zeta_p, \eta+\varepsilon_{p+1}+\zeta_{p+1}}.$$

Since the number of column-strict tableaux of shape $\pi + 2\zeta_r$ and weight $\eta + 2\zeta_{r+1}$ having $r + 1$ as the last entry in both the first and second columns (Case (ii)) is $K_{\pi+2\zeta_{r-1}, \eta+2\zeta_r}$, we have

$$\begin{aligned} K_{\pi+2\zeta_r, \eta+2\zeta_{r+1}} &= K_{\pi+2\zeta_{r-1}, \eta+2\zeta_r} + (r - p)K_{\pi, \eta+\varepsilon_{p+1}+\varepsilon_{p+2}} \\ &\quad + K_{\pi+\zeta_p, \eta+\varepsilon_{p+1}+\zeta_{p+1}} \end{aligned}$$

and we may repeat this argument $r - p$ times to obtain (5.10) as claimed. □

PROPOSITION 5.12 *Let π be a partition of $m + 2$ and η be a partition of m with $p = \ell(\eta) \geq \ell(\pi)$. Then for all r with $r - 1 \geq p$,*

$$\begin{aligned} K_{\pi+\zeta_{r-1}, \eta+\zeta_{r+1}} &= \frac{1}{2}(r - p)(r - p - 1)K_{\pi, \eta+\varepsilon_{p+1}+\varepsilon_{p+2}} \\ &\quad + (r - p)K_{\pi+\zeta_p, \eta+\zeta_{p+2}} \\ &\quad - (1 - \delta_{\ell(\pi), p})(r - 1 - p)K_{\pi+\zeta_{p-1}, \eta+\zeta_{p+1}}. \end{aligned} \tag{5.13}$$

Proof. A column-strict tableau of shape $\pi + \zeta_{r-1}$ and weight $\eta + \zeta_{r+1}$ has exactly one entry equal to $r + 1$. If this entry occurs at the end of the first column, then necessarily $r - 1 > \ell(\pi)$ and deleting the last box in the first column produces a column-strict tableau of shape $\pi + \zeta_{r-2}$ and weight $\eta + \zeta_r$. Otherwise, removing the $r + 1$ entry produces a tableau of shape $\kappa + \zeta_{r-1}$ and weight $\eta + \zeta_r$. Thus, we have

$$\begin{aligned}
 K_{\pi+\zeta_{r-1},\eta+\zeta_{r+1}} &= K_{\pi+\zeta_{r-2},\eta+\zeta_r} + \sum_{\substack{\kappa \subset \pi \\ |\pi/\kappa|=1}} K_{\kappa+\zeta_{r-1},\eta+\zeta_r} \\
 &= K_{\pi+\zeta_{r-2},\eta+\zeta_r} + (r-1-p) \sum_{\substack{\kappa \subset \pi \\ |\pi/\kappa|=1}} K_{\kappa,\eta+\varepsilon_{p+1}} \\
 &\quad + \sum_{\substack{\kappa \subset \pi \\ |\pi/\kappa|=1}} K_{\kappa+\zeta_p,\eta+\zeta_{p+1}} \quad \text{by (5.3)} \\
 &= K_{\pi+\zeta_{r-2},\eta+\zeta_r} + (r-1-p)K_{\pi,\eta+\varepsilon_{p+1}+\varepsilon_{p+2}} \\
 &\quad + K_{\pi+\zeta_p,\eta+\zeta_{p+2}} - K_{\pi+\zeta_{p-1},\eta+\zeta_{p+1}}
 \end{aligned}$$

by the first line, where the last term occurs only if $p > \ell(\pi)$. Iterating this process and summing the coefficients of $K_{\pi,\eta+\varepsilon_{p+1}+\varepsilon_{p+2}}$ yields (5.13). \square

PROPOSITION 5.14 *Assume π and η are partitions of m and their first parts are equal ($\pi_1 = \eta_1$). Then*

$$K_{\pi+\varepsilon_1,\eta+\varepsilon_{p+1}} = K_{\pi,\eta} + \sum_{i=2}^p K_{\pi,\eta+\varepsilon_{p+1}-\varepsilon_i}. \tag{5.15}$$

Proof. Take a column-strict tableau of shape $\pi + \varepsilon_1$ and weight $\eta + \varepsilon_{p+1}$, and let i be the last entry in the first row. Since $\pi_1 = \eta_1$ we have $i \geq 2$. Now if $i = p + 1$, then removing the last box in row one gives a column-strict tableau of shape π and weight η . If $2 \leq i \leq p$, then removing the last box in the first row gives a column-strict tableau of shape π and weight $\eta + \varepsilon_{p+1} - \varepsilon_i$. As both of these processes reverse, (5.15) must hold. \square

5.16. To show that the multiplicity polynomial has degree exactly one in the depth one case and degree exactly two in the depth two case, we required that certain partitions dominate others, and that certain Kostka numbers being nonzero imply others are nonzero. The particular facts we need are contained in the next two results. Here $\lambda = \lambda_1\varepsilon_1 + \dots + \lambda_q\varepsilon_q$ where $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q \geq 0\}$, $\nu = \nu_1\varepsilon_1 + \dots + \nu_q\varepsilon_q$ where $\nu = \{\nu_1 \geq \nu_2 \geq \dots \geq \nu_q \geq 0\}$, $\lambda_1 = \nu_1$, and either λ_q or ν_q is nonzero. Recall from (1.2) that $\lambda \succeq \nu$ for partitions λ and ν of the same number if and only if $\lambda - \nu = \sum_{m=1}^r c_m \alpha_m$ where $\alpha_m = \varepsilon_m - \varepsilon_{m+1}$ and $c_m \in \mathbb{Z}_{\geq 0}$ for all m . This is the case if and only if $K_{\lambda,\nu} \neq 0$. Since the column-strict tableau of shape λ and weight $\omega = \omega_1\varepsilon_1 + \dots + \omega_{r+1}\varepsilon_{r+1}$ index a basis for the ω weight space of the irreducible $\mathfrak{sl}(r+1, \mathbb{C})$ -module $V(\lambda)$, and since weights in a Weyl group orbit have the same multiplicity, $K_{\lambda,\bar{\omega}} \neq 0$ if and only if there are column-strict tableaux of shape λ and weight ω .

PROPOSITION 5.17

- (i) $\overline{\nu + \varepsilon_j - \varepsilon_i} \succeq \overline{\nu + \varepsilon_{q+1} - \varepsilon_1} \succeq \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_1}$ for all $1 \leq i \leq \ell(\nu)$ and all $1 \leq j \leq r + 1$.
- (ii) $\overline{\nu + \varepsilon_j + \varepsilon_k - 2\varepsilon_i} \succeq \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_i} \succeq \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_1}$ for all $1 \leq i \leq \ell(\nu)$ and all $1 \leq j \leq k \leq r + 1$ with $i \neq j, k$.
- (iii) $\overline{\nu + \varepsilon_k + \varepsilon_\ell - \varepsilon_i - \varepsilon_j} \succeq \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_h}$ for all $1 \leq i < h \leq j \leq \ell(\nu)$ and $1 \leq k, \ell \leq r + 1$ with $k \neq i, j$ and $\ell \neq i, j$.

Proof. Suppose $\nu = \{\nu_1 \geq \dots \geq \nu_p > 0\}$. For $1 \leq k \leq p$ assume $k', k'' \in \{1, \dots, p\}$ are such that

$$\nu_{k'-1} > \nu_{k'} = \dots = \nu_k = \dots = \nu_{k''} > \nu_{k''+1}.$$

Then for $1 \leq i \leq p = \ell(\nu)$ and $1 \leq j \leq r + 1$ with $j \neq i$ we have

$$\overline{\nu + \varepsilon_j - \varepsilon_i} = \begin{cases} \nu & \text{if } j \leq p \text{ and } \nu_{j'} = \nu_{i''} - 1, \\ \nu + \varepsilon_{p+1} - \varepsilon_{i''} & \text{if } j \geq p + 1, \text{ and either } i'' \neq p \text{ or } \nu_p \neq 1, \\ \nu & \text{if } j \geq p + 1, i'' = p, \text{ and } \nu_p = 1, \\ \nu + \varepsilon_{j'} - \varepsilon_{i''} & \text{otherwise.} \end{cases}$$

In particular,

$$\overline{\nu + \varepsilon_{q+1} - \varepsilon_1} = \begin{cases} \text{(a) } \nu & \text{if } \nu_1 = \dots = \nu_p = 1, \\ \text{(b) } \nu + \varepsilon_{p+1} - \varepsilon_{1''} & \text{otherwise.} \end{cases}$$

To establish part (i) observe that there are values s and t ($s = t$ allowed) with $1'' \leq s \leq p$ and $1 \leq t \leq p + 1$ so that

$$\begin{aligned} \overline{\nu + \varepsilon_j - \varepsilon_i} &= \nu + \varepsilon_t - \varepsilon_s = \nu + \varepsilon_{p+1} - \varepsilon_{1''} + (\varepsilon_t - \varepsilon_{p+1}) + (\varepsilon_{1''} - \varepsilon_s) \\ &= \overline{\nu + \varepsilon_{q+1} - \varepsilon_1} + (\varepsilon_t - \varepsilon_{p+1}) + (\varepsilon_{1''} - \varepsilon_s) \end{aligned}$$

when (b) above holds. Since $\varepsilon_{1''} - \varepsilon_s$ and $\varepsilon_t - \varepsilon_{p+1}$ are positive roots of A_r , hence nonnegative integer combinations of the simple roots α_m , it is clear that $\nu \succeq \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}$ when (b) holds. Now in case (a) we have that $\overline{\nu + \varepsilon_j - \varepsilon_i}$ equals $2\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{p-1}$ or ν , both of which are clearly $\succeq \nu = \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}$. For the second inequality in (i) let $\omega = \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}$ and $\omega' = \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_1}$ and note in treating such a weight as ω' we always assume $\nu_1 \geq 2$. Then for some s with $1'' \leq s \leq p$ we have

$$\begin{aligned} \omega &= \{\nu_2 = \dots = \nu_{1''} > \nu_1 - 1 \geq \nu_{1''+1} \geq \dots \geq \nu_{s-1} \geq \nu_s \geq \dots \geq \nu_p \geq 1\} \\ \omega' &= \{\nu_2 = \dots = \nu_{1''} > \nu_{1''+1} \geq \nu_{1''+2} \geq \dots \geq \nu_{s-1} \geq \nu_1 - 2 \\ &\geq \dots \geq \nu_p \geq 1\}. \end{aligned}$$

When the length of ω' is $p - 1$, the last two 1's are interchanged with ν_p which is 0. It is apparent the partial sums of the parts of ω are all greater than or equal to the partial sums of ω' so that $\omega \succeq \omega'$ as desired.

(ii) First let us show that $\overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_i} \succeq \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_1}$. There exist $1 \leq s \leq t \leq p$ such that $\overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_i} = \overline{\nu + \varepsilon_{m+1} + \varepsilon_{m+2} - 2\varepsilon_t}$ and $\overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_1} = \overline{\nu + \varepsilon_{n+1} + \varepsilon_{n+2} - 2\varepsilon_s}$ where m is the length of $\nu - 2\varepsilon_i$ and n is the length of $\nu - 2\varepsilon_1$. Then since $m \leq n$,

$$\begin{aligned} \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_i} &= \overline{\nu + \varepsilon_{m+1} + \varepsilon_{m+2} - 2\varepsilon_t} \\ &= \overline{\nu + \varepsilon_{n+1} + \varepsilon_{n+2} - 2\varepsilon_s + 2(\varepsilon_s - \varepsilon_t)} \\ &\quad + (\varepsilon_{m+1} - \varepsilon_{n+1}) + (\varepsilon_{m+2} - \varepsilon_{n+2}) \end{aligned}$$

which gives $\overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_i} \succeq \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_1}$.

Now there exist s, t, u, v such that $\overline{\nu + \varepsilon_j + \varepsilon_k - 2\varepsilon_i} = \overline{\nu + \varepsilon_s + \varepsilon_t - 2\varepsilon_u}$ and $\overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_i} = \overline{\nu + \varepsilon_{m+1} + \varepsilon_{m+2} - 2\varepsilon_v}$ where m is the length of $\nu - 2\varepsilon_i$ and $v \leq u$. But then

$$\begin{aligned} \overline{\nu + \varepsilon_j + \varepsilon_k - 2\varepsilon_i} &= \overline{\nu + \varepsilon_s + \varepsilon_t - 2\varepsilon_u} \\ &= \overline{\nu + \varepsilon_{m+1} + \varepsilon_{m+2} - 2\varepsilon_v + (\varepsilon_s - \varepsilon_{m+1})} \\ &\quad + (\varepsilon_t - \varepsilon_{m+2}) + 2(\varepsilon_v - \varepsilon_u) \end{aligned}$$

from which it follows that $\overline{\nu + \varepsilon_j + \varepsilon_k - 2\varepsilon_i} \succeq \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - 2\varepsilon_i}$.

(iii) If m is the length of $\nu - \varepsilon_i - \varepsilon_h$ and n is the length of $\nu - \varepsilon_i - \varepsilon_j$, then $n \leq m$. Suppose $\omega = \overline{\nu + \varepsilon_k + \varepsilon_\ell - \varepsilon_i - \varepsilon_j}$ and $\omega' = \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_h}$. Then there exist $a, b, s, t \in \{1, \dots, p = \ell(\nu)\}$, $1 \leq u \leq n + 1 \leq m + 1$, and $1 \leq v \leq n + 2 \leq m + 2$ such that $a < b, s < t, u \leq v, \omega = \overline{\nu + \varepsilon_u + \varepsilon_v - \varepsilon_s - \varepsilon_t}$, and $\omega' = \overline{\nu + \varepsilon_{m+1} + \varepsilon_{m+2} - \varepsilon_a - \varepsilon_b}$. Then $a \leq s, b \leq t$ hold. Consequently,

$$\begin{aligned} \omega &= \overline{\nu + \varepsilon_u + \varepsilon_v - \varepsilon_s - \varepsilon_t} \\ &= \overline{\nu + \varepsilon_{m+1} + \varepsilon_{m+2} - \varepsilon_a - \varepsilon_b + (\varepsilon_u - \varepsilon_{m+1})} \\ &\quad + (\varepsilon_v - \varepsilon_{m+2}) + (\varepsilon_a - \varepsilon_s) + (\varepsilon_b - \varepsilon_t) \\ &= \omega' + (\varepsilon_u - \varepsilon_{m+1}) + (\varepsilon_v - \varepsilon_{m+2}) + (\varepsilon_a - \varepsilon_s) + (\varepsilon_b - \varepsilon_t). \end{aligned}$$

Thus, $\overline{\nu + \varepsilon_k + \varepsilon_\ell - \varepsilon_i - \varepsilon_j} \succeq \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_h}$, as claimed. □

PROPOSITION 5.18

- (i) If $\ell(\lambda) < q$ and $K_{\lambda + \varepsilon_1 + \zeta_{q-1}, \nu + \zeta_q} \neq 0$, then $K_{\lambda, \overline{\nu + \varepsilon_{q+1} - \varepsilon_1}} \neq 0$.
- (ii) If $K_{\lambda, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j}} \neq 0$ for some $2 \leq j \leq \ell(\nu)$, then $K_{\lambda, \overline{\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j}} \neq 0$ for some i (possibly equal to j) with $2 \leq i \leq q + 2$.

(iii) If (a) $K_{\lambda+\varepsilon_1+\zeta_r, \overline{\nu+\varepsilon_k-\varepsilon_i+\zeta_{r+1}}} \neq 0$ for some i, k ($i = k$ allowed), or

(b) $K_{\lambda+2\varepsilon_1+\zeta_{r-1}, \nu+\zeta_{r+1}} \neq 0$, or

(c) $K_{\lambda+\varepsilon_1+\varepsilon_2+2\zeta_r, \nu+2\zeta_{r+1}} \neq 0$,

then either $K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-2\varepsilon_1}} \neq 0$ or $K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_i-\varepsilon_j}} \neq 0$ for some $2 \leq i \neq j \leq q+2$.

Proof. (i) If $\ell(\lambda) < q$ and $K_{\lambda+\varepsilon_1+\zeta_{q-1}, \nu+\zeta_q} \neq 0$, then there are column-strict tableaux of shape $\lambda + \varepsilon_1 + \zeta_{q-1}$ and weight $\nu + \zeta_q$. The first column of such a tableau has $1, 2, \dots, q$ with some integer $j \geq 2$ omitted. Removing that column produces a column-strict tableau of shape $\lambda + \varepsilon_1$ and weight $\nu + \varepsilon_j$. Such a tableau must have $\lambda_1 = \nu_1$ entries equal to 1 across its first row followed by some integer i with $2 \leq i \leq q$. Deleting the last entry in the first row produces a column-strict tableau of shape λ and weight $\nu + \varepsilon_j - \varepsilon_i$. Therefore, $K_{\lambda, \nu+\varepsilon_j-\varepsilon_i}$ must be nonzero for some i, j with $2 \leq i, j \leq q$. If $i = j$ then $K_{\lambda, \nu}$ must be nonzero; otherwise, there exist i, j with $i \neq j$ such that $K_{\lambda, \nu+\varepsilon_j-\varepsilon_i} \neq 0$. Since both of these being nonzero imply $K_{\lambda, \overline{\nu+\varepsilon_{q+1}-\varepsilon_1}} \neq 0$ by Proposition 5.17, we have the desired conclusion.

(ii) If $K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_1-\varepsilon_j}} \neq 0$, then there are column-strict tableaux of shape λ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_1 - \varepsilon_j$. Such a tableau has $\lambda_1 - 1 = \nu_1 - 1$ entries equal to 1 across its first row followed by some entry $i \geq 2$. We may change that last entry to a 1 to produce a column-strict tableau of shape λ and weight $\nu + \varepsilon_{q+1} + \varepsilon_{q+2} - \varepsilon_i - \varepsilon_j$. It follows that $K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_i-\varepsilon_j}} \neq 0$ for some $i \in \{2, \dots, q+2\}$.

(iii) (a) Removing the first column and deleting the last box from the first row of a column-strict tableau of shape $\lambda + \varepsilon_1 + \zeta_r$ and weight $\nu + \varepsilon_k - \varepsilon_i + \zeta_{r+1}$ produces a column-strict tableau of shape λ and weight $\nu + \varepsilon_k + \varepsilon_\ell - \varepsilon_i - \varepsilon_j$ for some $2 \leq j, \ell \leq r+1$. (b) Analogously, by deleting the first column of length $r-1$ and the last two boxes in the first row of a column-strict tableau of shape $\lambda + 2\varepsilon_1 + \zeta_{r-1}$ and weight $\nu + \zeta_{r+1}$, we obtain a column-strict tableau of shape λ and weight $\nu + \varepsilon_k + \varepsilon_\ell - \varepsilon_i - \varepsilon_j$ for some $2 \leq i \leq j \leq r+1$ and $2 \leq k < \ell \leq r+1$. (c) By removing the first two columns and the last box in the first two rows of column-strict tableau of shape $\lambda + \varepsilon_1 + \varepsilon_2 + 2\zeta_r$ and weight $\nu + 2\zeta_{r+1}$ we get a column-strict tableau of shape λ and weight $\nu + \varepsilon_k + \varepsilon_\ell - \varepsilon_i - \varepsilon_j$ where $2 \leq k \leq \ell \leq r+1$ and $2 \leq i < j \leq r+1$. Thus, in all the cases (a),(b),(c) there is a column-strict tableau of shape λ and weight $\nu + \varepsilon_k + \varepsilon_\ell - \varepsilon_i - \varepsilon_j$. When $\{i, j\} = \{k, \ell\}$, such a tableau has weight ν , so that $K_{\lambda, \nu} \neq 0$. When $|\{i, j\} \cap \{k, \ell\}| = 1$, then the tableau has weight $\nu + \varepsilon_s - \varepsilon_t$ for some $s \neq t$ with $t \leq \ell(\nu)$. Consequently, $K_{\lambda, \overline{\nu+\varepsilon_s-\varepsilon_t}} \neq 0$. Finally, when $\{i, j\} \cap \{k, \ell\} = \emptyset$, then $K_{\lambda, \overline{\nu+\varepsilon_k+\varepsilon_\ell-\varepsilon_i-\varepsilon_j}} \neq 0$. Applying Proposition 5.17, we see that either $K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-2\varepsilon_1}} \neq 0$ or $K_{\lambda, \overline{\nu+\varepsilon_{q+1}+\varepsilon_{q+2}-\varepsilon_i-\varepsilon_j}} \neq 0$ for some $2 \leq i \neq j \leq \ell(\nu)$.

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