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Automorphisms and local rigidity of regular varieties

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0. Introduction

Let X be a projective algebraic variety over an algebraically closed field of characteristic zero. Suppose that an affine algebraic group G acts transitively on X; such varieties are called flag spaces. Let $\mathcal{T}_X$ denote its tangent bundle, with symmetric algebra $S^\bullet(\mathcal{T}_X)$. Then, the cohomology groups $H^i(X, \mathcal{T}_X)$ vanish for $i \geqslant 1$. The complex analytic version of this result was proved by R. Bott in 1957, using Kodaira's vanishing theorem, [Bott] Theorem VII. By Serre's GAGA principle, the algebraic version follows immediately. In the late seventies, R. Elkik showed that $H^i(X, S^\bullet(\mathcal{T}_X))$ vanishes for all $i \geqslant 1$, using Grauert-Riemenschneider's vanishing theorem. This implies readily Bott's result.

By Kodaira–Spencer theory, the vanishing of $H^1(X, \mathcal{T}_X)$ implies that complex flag varieties admit no local deformation of their complex structures. In other words, for any continuous family of complex varieties X_t parametrized by a complex manifold T, with X_t topologically isomorphic to X for all t, and X_0 analytically isomorphic to X, then X_t is analytically isomorphic to X in a neighborhood of $0 \in T$.

Another application of this vanishing theorem is to show that every regular function on the cotangent bundle of X is the symbol of a differential operator on X with regular coefficients. This result is basic in the theory of $\mathcal{D}$-modules on flag varieties. It is easy to find examples of varieties where the vanishing theorem fails, for instance the plane minus a point. It is harder to find algebraic varieties where the correspondence between symbols and differential operators ceases to be well-behaved. However, moduli spaces of vector bundles over curves provide such examples, see [BK].

One can look for a more general vanishing theorem, namely find conditions on a line bundle L over a flag variety X such that $H^i(X, L \otimes S^\bullet(\mathcal{T}_X)) = 0$ for all $i \geqslant 1$. This property holds if L is globally generated. B. Broer has given a characterization of all line bundles, for which this vanishing result holds, [Bro] Theorem 2.11. This generalized vanishing theorem has further applications in representation theory, see [Bry].

With these motivations in mind, we consider here a class of G-varieties X called *regular varieties*; see 2.1. Regular varieties were defined separately in [BDP] and [Gin]. In this article, we show that both definitions coincide, see 2.5. Regular varieties need not be homogeneous for a group, but they must be smooth and contain an open G-orbit, say Ω. In fact, complete regular varieties must contain an open B-orbit for a Borel subgroup B of G; such varieties are called *spherical.* The wonderful compactifications of symmetric spaces and all smooth toric varieties are examples of regular G-varieties.

The divisor $D := X \setminus \Omega$ is called the *boundary* of X. Since X is regular, D has only normal crossings. The *action sheaf* $\mathcal{S}_X$ is the subsheaf of $\mathcal{T}_X$ made of all vector fields tangent to D, i.e. preserving the defining ideal of D. The regularity of X implies that $\mathcal{S}_X$ is locally free, and coincides with $\mathcal{T}_X$ on Ω.

When Ω is proper over an affine variety, we show that $H^i(X, L\otimes S^\bullet(\mathcal{S}_X))$ vanish for all $i \geqslant 1$ and for all globally generated line bundles L on X, Theorem 3.2. Our hypothesis is satisfied if the isotropy groups of Ω are reductive. Hence, our result holds for smooth toric varieties, and for wonderful compactifications of symmetric spaces as well. For smooth toric varieties, it reduces to a known vanishing result on the cohomology of globally generated line bundles. Our hypothesis is also verified if X is a flag variety, and hence Theorem 3.2 implies also Bott's and Elkik's vanishing theorems for flag varieties, and their twists by line bundles, see 3.7.

Our vanishing theorem combined with subsequent work of F. Knop [K3] has several useful applications for regular varieties:

(o) Local rigidity of the complex structure of X relatively to the class of the divisor D.

(i) A general characterization of regular G-varieties X which are homogeneous under their automorphism groups. The criterion is that every irreducible G-stable divisor is numerically effective, see 4.1. A striking example is that the wonderful compactifications of symmetric spaces of rank one are homogeneous varieties for an overgroup.

(ii) If a projective regular variety X is Fano, then $H^i(X, \mathcal{T}_X) = 0$ for all $i \geqslant 0$; in particular X is locally rigid, see 4.2.

(iii) A good correspondence between symbols and differential operators coming from the group action, see 4.3.

Our vanishing result does not always hold if we replace $\mathcal{S}_X$ by the whole tangent sheaf to X. Indeed in 4.4, we give examples of varieties X, regular for $G = \mathrm{SO}(2n+1)$, made of only two G-orbits and for which $H^1(X, \mathcal{T}_X) = k^{2n+1}$ and $H^2(X, \mathcal{T}_X) = 0$. Hence, these varieties admit many local deformations of their complex structures, yet they do not admit deformations preserving the class of their boundary divisor.

Our main motivation to prove the vanishing Theorem 3.2 was to study the category of modules $\mathcal{M}$ over the sheaf $\mathcal{D}_X$ of differential operators on X. In particular, we wanted to find appropriate conditions on X and $\mathcal{M}$, to ensure that $\mathcal{M}$ has no higher cohomology and is generated by global sections, as in the work of Beilinson–Bernstein for flag varieties, see [Bien]. We have found conditions that are still unsatisfactory to our liking, so we did not include them in this article.

We proved the vanishing Theorem 3.2 in the autumn 1991. At that time, we had evidence, and conjectured, that our assumption that Ω is proper over an affine, was not necessary to prove the vanishing theorem for regular G-varieties. Inspired by our result, F. Knop proved in 1992 a generalized version of this vanishing theorem which applies in a suitable sense to all G-varieties, see [K3]. His version relies on a careful study of the group action, while our version follows a more geometric approach.

1. Preliminaries

All our geometric objects are defined over an algebraically closed field k of characteristic zero. All varieties and subvarieties are irreducible. Let us recall first a fundamental vanishing theorem.

KODAIRA VANISHING THEOREM 1.1. *Let X be a smooth projective variety, with canonical bundle ω_X, and let L be an ample line bundle on X. Then $H^i(X, L\otimes \omega_X) = 0$ for $i \geqslant 1$.*

Varieties arising from homogeneous spaces of linear algebraic groups tend to have a negative canonical bundle. They are often Fano, i.e. their anti-canonical line bundle ω_X^{-1} is ample. Hence in the above theorem, the tensor product with ω_X helps to obtain a vanishing result for line bundles that fail to be ample.

To state the Kawamata–Viehweg vanishing theorem, we present a few definitions. Further details can be found in [KMM]. Let X be a normal variety; consider a Cartier divisor D on X, and a reduced, irreducible complete curve $C \subset X$. We define the intersection number $(D \cdot C)$ as follows: denote by $\pi\colon \tilde{C} \to C$ the normalization, then $(D \cdot C)$ is the degree of the line bundle $\pi^*(\mathcal{O}_X(D)|C)$ on $\tilde{C}$.

DEFINITION 1.2. A Cartier divisor D in X is *numerically effective*, or *nef* for short, if $(D \cdot C) \geqslant 0$ for any reduced, irreducible curve $C \subset X$.

Suppose now that X is a complete, normal variety. For any Cartier divisor D on X, the *Iitaka dimension* of D is by definition the integer k such that there exists $a, b > 0$ and $c \in \mathbb{N}_{>0}$ for which the following inequalities hold for all $m \in \mathbb{N}$ sufficiently large:

$$a\, m^k \leqslant \dim \Gamma(X, \mathcal{O}_X(mcD)) \leqslant b\, m^k.$$

This integer k is denoted by $\kappa(D)$.

DEFINITION 1.3. A divisor $D \in \mathrm{Div} X$ is called *big* if $\kappa(D) = \dim X$.

A nef divisor is big if and only if its self-intersection number (D^n) is positive, where $n = \dim X$, see e.g. [Vie] 3.2. Clearly, any ample divisor is both nef and big. The converse does not hold in general; for instance, let X be the blowup of $\mathbb{P}^n$ at one point, and let D be the pull-back of the hyperplane line bundle on $\mathbb{P}^n$, then D is big and nef, but not ample.

A $\mathbb{Q}$-divisor D on a normal variety X is a formal finite sum $D = \Sigma\, a_i\, D_i$ where a_i are rational numbers, and D_i are irreducible divisors. For any rational number a, we denote by $[a]$ the largest integer n such that $n \leqslant a$ (the integral part of a). We set: $\langle a \rangle = a - [a]$ (the fractional part of a) and $\lceil a \rceil = -[-a]$ (the round-up of a). We then set: $\langle D \rangle = \Sigma \langle a_i \rangle D_i$ (the fractional part of D) and $\lceil D \rceil = \Sigma \lceil a_i \rceil D_i$ (the round-up of D). There are obvious notions of nef and big $\mathbb{Q}$-Cartier divisors.

KAWAMATA–VIEHWEG VANISHING THEOREM 1.4 ([KMM] 1-2-3, [Viel] Theorem 1). *Let X be a projective, nonsingular variety. Let D be a $\mathbb{Q}$-divisor on X satisfying:*

(i) *D is nef and big,*
(ii) *the fractional part $\langle D \rangle$ has support with only normal crossings.*

Then $H^i(X, \mathcal{O}_X(\lceil D \rceil) \otimes \omega_X) = 0$ for $i \geqslant 1$.

Since ample line bundles are nef and big, this theorem implies the Kodaira vanishing theorem.

2. Regular G-varieties

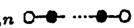
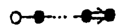
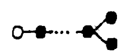
2.1. DEFINITIONS

We continue with the previous notation. Let X be an algebraic variety on which a connected affine algebraic group G acts with an open dense orbit Ω. Then X is called *regular* if it verifies the following three conditions (see [BDP]):

(i) The closure of every G-orbit is smooth.
(ii) For any orbit closure $Y \neq X$, Y is the transversal intersection of the orbit closures of codimension one containing Y.
(iii) The isotropy group of any point $p \in X$ has a dense orbit in the normal space to the orbit $G \cdot p$ in X.

It follows from (i) that X is smooth, and from (ii) that G has only a finite number of orbits in X. Denote by Ω the open orbit. Let H be the isotropy group in G of a point of Ω; then $\Omega \simeq G/H$. The set $D := X \setminus \Omega$ is called the boundary divisor of X; let $X_1, \ldots, X_l$ be its irreducible components.

Here are some examples. $\mathbb{P}^n$, the n-dimensional projective space with the standard action of the torus $G = (\mathbb{G}_m)^n$ is a regular variety; so is every smooth toric variety. The table below contains a list of examples for which G is almost simple, X contains exactly two G-orbits, and the open orbit Ω is affine; see D. Ahiezer, [A1] Table 2. Note that the list consists precisely of completions of rank one symmetric spaces, plus two exceptions, up to covering.

X	G	H	P	
$\mathbb{P}^n \times (\mathbb{P}^n)^*$	PGL$(n+1)$	GL(n)	$P_{1,n}$	
$Q(n)$	SO(n)	SO$(n-1)$	P_1	
$\mathbb{P}^{n-1}$	SO(n)/center	S(O(1) × O$(n-1)$)/center	P_1	 (O O for n=4)
Gr$(2, 2n)$	SP$(2n)$	SP$(2n-2)$ × SP(2)/center	P_2	
 E_6/P_1	F_4	Spin(9)	P_4	
$Q(7)$	G_2	SL(3)	P_1	
$\mathbb{P}^6$	G_2	Norm(SL(3))	P_1	
$Q(8)$	Spin(7)	G_2	P_3	
$\mathbb{P}^7$	SO(7)	G_2	P_3	

Here Gr(d, m) denotes the Grassmannian of d-planes in k^m. $\mathbb{P}^n$ =Gr$(1, n+1)$, and $(\mathbb{P}^n)^*$ =Gr$(n, n+1)$ denotes the same projective space with contragredient action of PGL$(n+1)$. $Q(n)$ denotes the quadric $z_0^2 = \Sigma_{i=1}^n z_i^2$ in $\mathbb{P}^n$. SO(n) acts on $Q(n)$ by $[z_0 \colon z'] \mapsto [z_0 \colon gz']$ for $z' \in k^n$ and $g \in$ SO(n). The right hand column refers to the parabolic subgroup of G which stabilizes a point of the closed orbit. P_i, resp. $P_{i,j}$, denotes the parabolic subgroup obtained by adjoining all simple root subgroups to a Borel subgroup, except for the root subgroups corresponding to the simple roots i, resp. i and j. The order on the Dynkin diagram is taken to be the standard one, with arrows placed on the right hand side and pointing to the right, except for SP$(2n)$ for which they point left. Note that P is a maximal parabolic subgroup, except in the first case, i.e. $\mathbb{P}^n \times (\mathbb{P}^n)^*$. Note that $Q(4)$ is the regular completion of SL(2) $\simeq$ SO(4)/SO(3).

Finally two examples of non-regular varieties:

$X = \mathbb{P}^1$ with the standard action of the additive group $\mathbb{G}_a$ is not a regular variety, because $\mathbb{G}_a$ acts trivially in the tangent space to the fixed point. Note that the symmetry group $\mathbb{G}_a$ is unipotent, hence this case would not have been included in our study, anyway.

$X = \mathbb{A}^2$ with the standard action of SL(2) is also not a regular variety. Indeed $\mathbb{A}^2$ contains two orbits: $\{0\}$ and its complement $\mathbb{A}^2 \setminus \{0\}$. Since there is no orbit of codimension one, X cannot be regular. On the other hand, the blow up of $\mathbb{A}^2$ at the origin is a regular variety for SL(2).

2.2. Relations with spherical varieties

A G-variety is called *spherical* if a Borel subgroup B of G has an open dense orbit in X. The second author and E. Vinberg have shown that spherical varieties contain only a finite number of G-orbits, and even a finite number of B-orbits.

A *color* in a G-variety X is an irreducible B-stable divisor containing a G-orbit, but which is not itself G-stable. For example, take $G = \mathrm{SL}(2)$ with the natural action on $X = \mathbb{A}^2$, then the point 0 is a closed G-orbit. Let B be the subgroup of upper triangular matrices; it preserves the x-axis which contains the origin. Yet, the x-axis is not G-stable, hence X has one color. A G-variety is said to be *without color* if every irreducible B-stable divisor containing a G-orbit is itself G-stable.

In order to describe regular varieties, let us review a result on the local structure of spherical varieties. Let X be spherical, and let B be a Borel subgroup of G. Let Ω_0 be the open orbit of B in Ω. Put

$$\begin{aligned} P &= \{s \in G \mid s \cdot \Omega_0 = \Omega_0\}, \\ X_0 &= \{x \in X \mid B \cdot x \text{ is open in } G \cdot x\}. \end{aligned}$$

Then P is a parabolic subgroup of G containing B. Let N denote the unipotent radical of P. According to [BP] Proposition 3.4, if X has no color, then X_0 is open in X and stable by P. Moreover, there exists a closed subvariety Z in X_0, and a Levi subgroup L in P such that:

(a) Z is stable by L, and the derived subgroup (L, L) acts trivially on Z,
(b) The natural morphism $N \times Z \to X_0$, given by the group action, is an isomorphism.

The torus $L/(L, L)$ acts in Z with an open dense orbit. Let A be the image of this torus in the automorphism group of Z; then $Z = \overline{A}$ is a toric variety for A. Since X has no color, every G-orbit in X meets Z in a single A-orbit.

The rank of X, denoted by $\mathrm{rk}(X)$, is by definition $\dim(A) = \dim(Z)$. If X contains a projective G-orbit Y, then $\mathrm{rk}(X)$ equals the codimension of Y in X.

PROPOSITION 2.2.1. *Let X be a complete and smooth G-variety. Then X is regular if and only if it is spherical and without color.*

Proof. $\Leftarrow$ Suppose X is spherical and without color. Consider the open affine chart described above: $N \times Z \hookrightarrow X$. Then Z is a smooth affine variety, toric for $L/(L, L)$, with a fixed point o. It follows that Z is isomorphic to affine r-space ($r = \mathrm{rk}(X)$) with the standard action of the r-dimensional torus. Clearly, Z is regular for A, or L. Because any G-orbit in X meets Z transversally along a single L-orbit, X is regular for G.

$\Rightarrow$ Suppose X is regular, complete and smooth. Theorem 1.4 in [BLV] asserts that if X is a normal G-variety and $z \in X$ is any point lying in a projective G-orbit in X, then there is an open affine neighborhood of z isomorphic to $N \times Z$, where

N is the unipotent radical of a parabolic subgroup P opposite to the stabilizer G_z, and Z is an affine subvariety of X, containing z and stable by the Levi subgroup $L = P \cap G_z$. If X is smooth, then so is Z. Moreover, we can identify the normal space to the orbit $G \cdot z$ at z with the tangent space $T_z Z$.

In our case, by condition (ii), the representation of G_z in this normal space decomposes as a direct sum of one-dimensional representations. Moreover, (iii) implies that the corresponding characters of G_z are linearly independent. Therefore, the Levi subgroup L of G_z acts on $T_z Z$ with a dense orbit, and its derived subgroup (L, L) acts trivially. Hence a very special case of Luna's slice theorem implies that Z is L-isomorphic to $T_z Z$, and hence that Z is a toric variety for $L/(L, L)$. It follows that X is spherical. Moreover, there are no colors since every L-stable divisor in Z is the intersection of Z with a G-stable divisor in X.

COROLLARY 2.2.2. *A homogeneous space for G admits a regular completion if and only if it is spherical.*

Proof. If X is a regular completion of a homogeneous space, then Proposition 2.2.1 asserts that X is spherical. Therefore the underlying homogeneous space is also spherical.

Conversely, given a spherical homogeneous space Ω, its completions without color correspond to subdivisions of its cone of valuations, [BP]. There exists a regular subdivision, i.e. a subdivision such that every cone is generated by part of a basis of the lattice, see [KKMS] Chap. I, Sect. 2, Theorem 11. The corresponding toric variety is smooth, therefore so is the completion of Ω, according to the above result of local structure.

2.3. ACTION SHEAF OF A REGULAR VARIETY

From now on, we suppose that X is regular and complete, with a normal crossing boundary divisor $D = X_1 \cup X_2 \cup \cdots \cup X_l$.

Let $\mathcal{T}_X$ denotes the tangent sheaf to X, and let $\mathcal{I}$ be the ideal sheaf of D. The *action sheaf* $\mathcal{S}_X$ of X is the subsheaf of $\mathcal{T}_X$ made of vector fields tangent to D, i.e.

$$\mathcal{S}_X = \{v \in \mathcal{T}_X \mid v \cdot \mathcal{I}_Y \subseteq \mathcal{I}_Y\}.$$

Another notation for the action sheaf $\mathcal{S}_X$ is $\mathcal{T}_X(-\log D)$, and it is then called the logarithmic tangent bundle. Choose local coordinates $x_1, \ldots, x_n$ on X, $n = \dim(X)$, such that D is defined locally by the equation $x_1 \cdot x_2 \cdot \cdots \cdot x_l = 0$. Set $\partial_i = \partial/\partial x_i$. Then, the vector fields $x_1\partial_1, \ldots, x_l\partial_l, \ \partial_{l+1}, \ldots, \partial_n$ form a local $\mathcal{O}_X$-basis of $\mathcal{S}_X$. Therefore, $\mathcal{S}_X$ is a locally free $\mathcal{O}_X$-module of rank d. The Lie bracket of vector fields induces a Lie algebra structure on $\mathcal{S}_X$.

Let $\mathfrak{g}$ be the Lie algebra of G. The G-action on X gives a canonical morphism

$$op\colon \mathcal{O}_X \otimes \mathfrak{g} \to \mathcal{T}_X.$$

Observe that G preserves each component of the boundary of X, since such a component is the closure of a G-orbit. Consequently, the image of op lies in the action sheaf $\mathcal{S}_X$. Recall from Section 2.2 that X_0 denotes the union of all B-orbits open in the corresponding G-orbits. We have $X_0 \simeq N \times Z$, where Z is a smooth toric variety for the torus A.

PROPOSITION 2.3.1.

(i) *There is an exact sequence of* $(\mathcal{O}_{X_0}, P)$*-modules, where* P *acts trivially on* $\mathfrak{a}$*:*

$$0 \to \mathcal{O}_{X_0} \otimes \mathfrak{n} \to \mathcal{S}_X|_{X_0} \to \mathcal{O}_{X_0} \otimes \mathfrak{a} \to 0.$$

(ii) *The sheaf* $\mathcal{S}_X$ *is locally free, and generated by* $op(\mathfrak{g})$. *In particular,* $\mathcal{S}_X$ *is generated by global sections.*

Proof. The quotient of X_0 by N exists and is isomorphic to Z. The quotient morphism $q\colon X_0 \to Z$ is equivariant for P, and N acts trivially on Z. We have an exact sequence

$$0 \to \mathcal{T}_{X_0/Z} \to \mathcal{T}_{X_0} \to q^*\mathcal{T}_Z \to 0.$$

Moreover, the canonical map $\mathcal{O}_{X_0} \otimes \mathfrak{n} \to \mathcal{T}_{X_0/Z}$ is an isomorphism. Hence, we obtain an exact sequence

$$0 \to \mathcal{O}_{X_0} \otimes \mathfrak{n} \to \mathcal{S}_X|_{X_0} \to q^*\mathcal{S}_Z \to 0.$$

Since Z is a smooth toric variety for A, the A-action yields an isomorphism

$$op_Z\colon \mathcal{O}_Z \otimes \mathfrak{a} \to \mathcal{S}_Z,$$

cf. [Oda] Proposition 3.1.

This proves (i) and (ii).

PROPOSITION 2.3.2. *There is an exact sequence*

$$0 \to \mathcal{S}_X \to \mathcal{T}_X \to \bigoplus_{i=1}^{l} \mathcal{O}_X(X_i) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i} \to 0.$$

REMARK. In local coordinates $x_1, \ldots, x_n$, we can describe the map

$$\mathcal{T}_X \to \bigoplus_{i=1}^{l} \mathcal{O}_X(X_i) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i}\colon \sum_i a_i\partial_i \mapsto \sum_i \left(\frac{a_i}{x_i} \operatorname{mod} x_i\right).$$

Proof. Let Ω_X denote the sheaf of regular differential 1-forms on X, and $\Omega_X(\log D)$ the sheaf of differential 1-forms on X with at most logarithmic singularities along D, see [Dan] Section 15. Consider the exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \Omega_X \to \Omega_X(\log D) \to \bigoplus_{i=1}^{l} \mathcal{O}_{X_i} \to 0,$$

where the map on the right is given by residues, see e.g. [Dan] 15.7. Applying the duality functor $\mathrm{Hom}_{\mathcal{O}_X}(.\,,\mathcal{O}_X)$, we obtain:

$$0 \to 0 \to \mathcal{T}_X(-\log D) \to \mathcal{T}_X \to \bigoplus_{i=1}^{l} \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_{X_i},\mathcal{O}_X) \to 0.$$

On the other hand, the $\mathcal{O}_X$-dual of the exact sequence $0 \to \mathcal{O}_X(-X_i) \to \mathcal{O}_X \to \mathcal{O}_{X_i} \to 0$ is:

$$0 \to 0 \to \mathcal{O}_X \to \mathcal{O}_X(X_i) \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_{X_i},\mathcal{O}_X) \to 0.$$

Thus, $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_{X_i},\mathcal{O}_X)$ is isomorphic to $\mathcal{O}_X(X_i) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i}$. Using the identity $\mathcal{S}_X = \mathcal{T}_X(-\log D)$, we obtain the proposition.

2.4. Restriction to a regular subvariety

Let Y be a subvariety of X defined by the intersection of irreducible components of the normal crossing divisor $D = X_1 \cup \cdots \cup X_l$. Re-ordering these components if necessary, we have $Y = X_1 \cap \cdots \cap X_c$, where c is the codimension of Y in X. Let $\mathcal{I}_Y$ be the ideal sheaf of Y.

The normal bundle of Y decomposes as a direct sum of line bundles

$$\mathcal{N}_Y = \bigoplus_{i=1}^{c} \mathcal{O}_X(X_i)|_Y.$$

Consider the total space of this normal bundle

$$N = \mathrm{Spec}_{\mathcal{O}_Y}\left(\bigoplus_{m=0}^{\infty} \mathcal{I}_Y^m/\mathcal{I}_Y^{m+1}\right).$$

Then, we have

$$\mathcal{O}_N = S^\bullet_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2) = S^\bullet_{\mathcal{O}_Y}\left(\bigoplus_{i=1}^{c} \mathcal{O}_X(-X_i)|_Y\right)$$

and $\mathcal{O}_N$ is naturally endowed with a gradation by $\mathbb{N}^c$ induced by the decomposition of $\mathcal{N}_Y$ as a sum of c line bundles.

Define $\mathcal{S}_N$ to be the sheaf of all derivations δ of $\mathcal{O}_Y$ such that δ preserves the gradation by $\mathbb{N}^c$, and that $\delta|_{\mathcal{O}_Y}$ preserves the ideal sheaves of $Y \cap X_i$, for $c+1 \leqslant i \leqslant l$. In more geometric terms, $\mathcal{S}_N$ is the sheaf on Y made of all vector fields on N which commute with the action of $(k^*)^c$ defined by the gradation by $\mathbb{N}^c$, and whose restriction to Y stabilizes the subvarieties $Y \cap X_i$ for $c+1 \leqslant i \leqslant l$.

There is a natural map

$$\mathcal{S}_X/\mathcal{I}_Y\mathcal{S}_X = \mathcal{S}_X|_Y \to \mathcal{S}_N$$

given as follows. Every $\delta \in \mathcal{S}_X$ acts on the line bundles $\mathcal{O}_X(-X_i)$ for $i = 1, \ldots, l$. Hence, δ maps $\mathcal{I}_Y$ into itself, and similarly for all the powers of $\mathcal{I}_Y$, because δ is a derivation. Therefore, δ induces a derivation of $\mathcal{O}_N$, which preserves the gradation and the subvarieties $Y \cap X_i$. If moreover $\delta \in \mathcal{I}_Y\mathcal{S}_X$, then $\delta(\mathcal{O}_X) \subset \mathcal{I}_Y$ and $\delta(\mathcal{I}_Y) \subset \mathcal{I}_Y^2$. Therefore, δ acts trivially on $\mathcal{O}_N$.

PROPOSITION 2.4.1. *We have an isomorphism $\mathcal{S}_X|_Y \simeq \mathcal{S}_N$ as $\mathcal{O}_Y$-modules.*

Proof. Choose local coordinates $x_1, \ldots, x_n$ on X, $n = \dim(X)$, such that D is defined locally by the equation $x_1 \cdot x_2 \cdot \cdots \cdot x_l = 0$. Then, the vector fields $x_1\partial_1, \ldots, x_l\partial_l$, $\partial_{l+1}, \ldots, \partial_n$ form a local $\mathcal{O}_X$-basis of $\mathcal{S}_X$. The kernel of $\mathcal{S}_X|_Y \to \mathcal{S}_N$ is the image in $\mathcal{S}_X|_Y = \mathcal{S}_X/\mathcal{I}_Y\mathcal{S}_X$ of the derivations $\delta \in \operatorname{Der}\mathcal{O}_X$ such that $\delta(\mathcal{I}_D) \subseteq \mathcal{I}_D, \delta(\mathcal{O}_X) \subseteq \mathcal{I}_Y$ and $\delta(\mathcal{I}_Y) \subseteq \mathcal{I}_Y^2$. Indeed, the last two conditions are equivalent to $\delta(\mathcal{I}_Y^m) \subseteq \mathcal{I}_Y^{m+1}, \forall m \in \mathbb{N}$. Since the local equation for Y is $x_1 \cdot \cdots \cdot x_c = 0$, we see that the kernel of $\mathcal{S}_X \to \mathcal{S}_N$ is generated locally by $x_i x_j \partial_j$ for $1 \leqslant i, j \leqslant c$ and $x_i\partial_j$ for $1 \leqslant i \leqslant c, c+1 \leqslant j \leqslant l$. Thus, this kernel is exactly $\mathcal{I}_Y\mathcal{S}_X$.

The surjectivity of the map $\mathcal{S}_X|_Y \to \mathcal{S}_N$ follows by the same token: the images of $x_1\partial_1, \ldots, x_c\partial_c$ generate the subsheaf of $\mathcal{S}_N$ made of $\mathcal{O}_Y$-linear derivations, while the images of $x_{c+1}\partial_{c+1}, \ldots, x_l\partial_l, \partial_{l+1}, \ldots, \partial_n$ generate the subsheaf of $\mathcal{S}_N$ made of derivations induced by $\mathcal{O}_Y$.

PROPOSITION 2.4.2. *We have an exact sequence:*

$$0 \to \mathcal{O}_Y^c \to \mathcal{S}_X|_Y \to \mathcal{S}_Y \to 0.$$

Proof. By the above proposition, $\mathcal{S}_X|_Y \simeq \mathcal{S}_N$. Every derivation $\delta \in \mathcal{S}_N$ preserves $\mathcal{O}_Y \subseteq \mathcal{O}_N$, and induces on it a element of $\mathcal{S}_Y$. Moreover, every derivation of $\mathcal{O}_Y$ can be trivially extended to $\mathcal{O}_N$. Hence, we have an exact sequence

$$0 \to \mathrm{Der.Gr}_{\mathcal{O}_Y}(\mathcal{O}_N) \to \mathcal{S}_N \to \mathcal{S}_Y \to 0,$$

where $\mathrm{Der.Gr}_{\mathcal{O}_Y}(\mathcal{O}_N)$ consists of all derivations of $\mathcal{O}_N$ over $\mathcal{O}_Y$ which preserve the gradation.

Now, $\operatorname{Der}_{\mathcal{O}_Y}(\mathcal{O}_N) = \operatorname{Der}_{\mathcal{O}_Y}(S^\bullet_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2)) = \operatorname{Der}_{\mathcal{O}_Y} S^\bullet_{\mathcal{O}_Y} \bigoplus_{i=1}^c \mathcal{O}_Y(-X_i)$. The subalgebra of $\operatorname{Der}_{\mathcal{O}_Y}(\mathcal{O}_N)$ which stabilizes the gradation can be identified with $\bigoplus_{i=1}^l \operatorname{Hom}(\mathcal{O}_Y(-X_i), \mathcal{O}_Y(-X_i)) \simeq \mathcal{O}_Y^c$.

2.5. EQUIVALENT CHARACTERIZATION OF REGULAR ACTION

We continue with the notation of 2.4, and bring in the group action, with associated morphism:

$$op\colon \mathcal{O}_X \otimes \mathfrak{g} \to \mathcal{T}_X.$$

PROPOSITION 2.5. *X is a regular G-variety with boundary D if and only if the image of op is $\mathcal{S}_X$.*

Proof. Suppose $\mathrm{Im}(op) = \mathcal{S}_X$; we must show that X is a regular G-variety. Since $\mathcal{S}_X$ and $\mathcal{T}_X$ coincide on $X \setminus D$, the tangent sheaf to $X \setminus D$ is generated by $\mathfrak{g}$. Therefore, $X \setminus D$ is a G-orbit. Let $Y \subset X$ be a subvariety defined as in 2.4.1; then Y is G-stable. The composite morphism

$$\mathcal{O}_X \otimes \mathfrak{g} \to \mathcal{S}_X|_Y \to \mathcal{S}_N$$

is surjective, hence so is $\mathcal{O}_Y \otimes \mathfrak{g} \to \mathcal{S}_N$. In particular, $\mathcal{O}_Y \otimes \mathfrak{g} \to \mathcal{S}_Y$ is surjective, thus G has a dense orbit in Y; let us denote this orbit by Ω'. Moreover, the following map is also surjective:

$$\mathcal{O}_{\Omega'} \otimes \mathfrak{g} \to \mathrm{Der.Gr}\, \mathcal{S}^\bullet_{\mathcal{O}_{\Omega'}}(\oplus_{i=1}^c \mathcal{O}_{\Omega'}(-X_i))$$

where Der.Gr denotes as before the derivation which preserve the gradation. It follows that G acts transitively in the normal bundle to Ω' minus its boundary divisors. Hence, the isotropy group of any point $y \in \Omega'$ has a dense orbit in the normal space to Ω' at y.

The converse implication follows by reversing the arguments in the previous discussion.

Given a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X, the fiber of $\mathcal{F}$ at $x \in X$ is the vector space $\mathcal{F}(x) := \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$, where $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ at x, and $\mathfrak{m}_x$ is the maximal ideal in the local ring $\mathcal{O}_{X,x}$. By Nakayama's lemma, *op* is surjective if and only if the induced fiber map $\mathfrak{g} \to \mathcal{S}_X(x)$ is surjective, for all $x \in X$. The above proposition shows that Definition 4.2.1 in [Gin] of regular G-action is equivalent to Definition 1.1.

2.6. IITAKA DIMENSION OF $\mathcal{S}_X$

For any locally free sheaf $\mathcal{E}$ on a complete algebraic variety Z, we define the *Iitaka dimension* $\kappa(\mathcal{E})$ to be the Iitaka dimension of the tautological line bundle on the projective bundle

$$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}_{\mathcal{O}_Z}\, S^\bullet(\mathcal{E}).$$

Then the dimension of $\Gamma(Z, S^m\mathcal{E})$ grows like $m^{\kappa(\mathcal{E})}$.

The following result will play a key role in the proof of our vanishing result. Recall that $\mathrm{rk}(X)$ denotes the rank of X defined in 2.2.

PROPOSITION 2.6.1. *For any G-stable subvariety Y of X, we have:*

$$\kappa(\mathcal{S}_X|_Y) = 2\dim(X) - \mathrm{rk}(X) - 1.$$

Note that the right hand side of this equality does not depend on Y.

Proof. First observe that for any exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{E} \to \mathcal{F} \to 0$$

with $\mathcal{E}$ and $\mathcal{F}$ locally free sheaves on Y, and for any integer $m \geqslant 1$, we have an exact sequence

$$0 \to S^{m-1}\mathcal{E} \to S^m\mathcal{E} \to S^m\mathcal{F} \to 0.$$

It follows that $\kappa(\mathcal{E}) = \kappa(\mathcal{F}) + 1$. Applying this observation to the exact sequence

$$0 \to \mathcal{O}_Y^c \to \mathcal{S}_X|_Y \to \mathcal{S}_Y \to 0$$

in 2.4.2, we obtain $\kappa(\mathcal{S}_X|_Y) = \kappa(\mathcal{S}_Y) + c$. Now, the rank of Y equals $\mathrm{rk}(X) - c$, while $\dim(Y) = \dim(X) - c$. Therefore, the equality in the proposition is equivalent to:

$$\kappa(\mathcal{S}_Y) = 2\dim(Y) - \mathrm{r}(Y) - 1.$$

In other words, we can suppose that $Y = X$.

Let S^*X be the spectrum of the symmetric algebra of $\mathcal{S}_X$. There is a 'moment map'

$$\mu \colon S^*X \to \mathfrak{g}^*$$

induced by the action map $op \colon \mathfrak{g} \to H^0(X, \mathcal{S}_X)$. Recall that $\mathcal{S}_X$ is generated by global sections, see Proposition 2.3.1 above. Thus, μ is proper, $\mathcal{S}_X$ is numerically effective, and $\kappa(\mathcal{S}_X) + 1$ is the dimension of the image of μ. On Ω, S^*X coincides with the total space of the cotangent bundle of X. Thus, our statement follows from [K1] Satz 7.1, which asserts in particular that the dimension of the image of μ equals $2\dim(X) - \mathrm{rk}(X)$ for spherical X.

A locally free sheaf $\mathcal{E}$ on Z is called *big* if $\kappa(\mathcal{E})$ is maximal, i.e. $\kappa(\mathcal{E}) = \dim(Z) + \mathrm{rk}(\mathcal{E}) - 1$. It is equivalent to say that the tautological fiber bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}\, S^\bullet(\mathcal{E})$ is big.

COROLLARY 2.6.2. *For every closed G-orbit Y in X, the sheaf $\mathcal{S}_X|_Y$ is big.*

2.7. Independence of the Regular Completion

We show that the cohomology of the symmetric algebra $S^\bullet(\mathcal{S}_X)$ depends only on the open G-orbit Ω and not on the particular regular completion X.

Let X' be another complete regular G-variety, endowed with a birational G-equivariant morphism $\pi \colon X' \to X$.

PROPOSITION 2.7.1. *For any integer $i \geqslant 1$, we have: $R^i\pi_*S^\bullet(\mathcal{S}_{X'}) = 0$. Moreover, $\pi_*S^\bullet(\mathcal{S}_{X'}) = S^\bullet(\mathcal{S}_X)$.*

Proof. For every closed G-orbit Y, let X_Y be the set of $x \in X$ such that Y is contained in the closure of $G \cdot x$. Then, X_Y is a G-stable open subset of X, containing Y as its unique closed G-orbit; we will say that X_Y is *simple*. By replacing X, resp. X', by X_Y, resp. $\pi^{-1}(X_Y)$, we can assume that X is simple, with $\pi : X' \to X$ birational and proper. However, X and X' need no longer be complete.

Then, Z (defined in 2.2) is a simple toric variety, hence affine, and X_0 is also affine. Since the translates $g \cdot X_0$ ($g \in G$) cover X, it suffices to verify that:

$$H^0(\pi^{-1}(X_0), S^\bullet(\mathcal{S}_{X'})) = H^0(X_0, S^\bullet(\mathcal{S}_X)),$$

and

$$H^i(\pi^{-1}(X_0), S^\bullet(\mathcal{S}_{X'})) = 0 \quad \text{for all } i \geqslant 1.$$

Now, $\pi^{-1}(X_0)$ equals X'_0. Namely, the complement of X'_0 is the union of the B-stable irreducible divisors in X' which meet the open G-orbit; and the same holds for the complement of $\pi^{-1}(X_0)$, because π is an isomorphism on this open orbit. Furthermore, the restriction of $S^\bullet(\mathcal{S}_{X'})$ to X'_0 is isomorphic as $\mathcal{O}_{X'_0}$-module to $\mathcal{O}_{X'_0} \otimes S^\bullet(\mathfrak{n} \oplus \mathfrak{a})$. Since π is birational and proper, and X, X' are smooth, we have: $\pi_*\mathcal{O}_{X'} = \mathcal{O}_X$ by Zariski's main theorem, and $R^i\pi_*\mathcal{O}_{X'} = 0$ for all $i \geqslant 1$, e.g. by [B2]. This implies the result.

COROLLARY 2.7.2. *For any line bundle L over X, and for any integer $i \geqslant 0$, we have an isomorphism*

$$H^i(X, L \otimes S^\bullet(\mathcal{S}_X)) \simeq H^i(X', (\pi^*L) \otimes S^\bullet(\mathcal{S}_{X'})).$$

Proof. This follows readily from the Leray spectral sequence and the projection formula.

That the cohomology of $S^\bullet(\mathcal{S}_X)$ depends only on Ω, and not on a particular regular completion, follows now from Corollary 2.7.2 and the fact that any two regular completions of Ω are dominated by a third one.

3. Cohomology of the action sheaf

3.1. A SEPARATION CONDITION FOR X

We continue with the previous notation. In particular, X denotes a complete, regular G-variety, with open orbit Ω (a spherical homogeneous space). To Ω are associated several combinatorial objects, whose definition will be recalled below. We will characterize those Ω which are proper over an affine, in terms of these combinatorial data.

Given a Borel subgroup B of G, let Ω_0 be the open orbit of B in Ω. Note that Ω_0 is also open in X, and hence that every B-invariant rational function on X is constant. Since B is solvable, Ω_0 is an affine variety, and its complement in Ω is pure of codimension one. Let $\mathcal{D}$ be the set of irreducible components of $\Omega \setminus \Omega_0$. Then $\mathcal{D}$ is the set of 'possible colors' for Ω.

Each $Z \in \mathcal{D}$ is a B-stable divisor; let v_Z be the corresponding valuation of $k(X)$. Let $k(X)^{(B)}$ be the set of eigenvectors of B in $k(X)$, and let Λ be the set of eigencharacters, also called weights, of B in $k(X)$. Then, Λ is a free abelian group of finite rank.

Note that every $f \in k(X)^{(B)}$ is determined uniquely, up to a scalar multiple, by its weight $\chi_f \in \Lambda$, because every B-invariant function in $k(X)$ is constant. Hence, for any $Z \in \mathcal{D}$, we can define $\rho(Z) \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ by $\rho(Z)(\chi_f) = v_Z(f)$.

Let $\mathcal{V}$ be the set of G-invariant discrete valuations of $k(X)$ over k. To each valuation $v \in \mathcal{V}$, we can associate, as above, $\rho(v) \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q}) =: \mathcal{Q}$. It is known that the map ρ identifies $\mathcal{V}$ with a finitely generated convex cone in $\mathcal{Q}$, with non-empty interior (cf. [K2] Corollary 6.3).

The following result is an immediate consequence of [K2] 5.2, 5.4 and 7.7.

LEMMA 3.1. *The following conditions are equivalent:*

(i) *Ω is proper over an affine variety.*
(ii) *There is a weight $f \in \Lambda$ such that $f < 0$ on $\rho(\mathcal{D}) \setminus \{0\}$ and $f \geqslant 0$ on $\mathcal{V}$.*

For the isotropy subgroup H of a point of Ω, Condition 3.1 (i) means that H is a parabolic subgroup of some reductive subgroup of G. In particular, this condition is satisfied if X is a smooth toric variety, or a wonderful compactification of a symmetric space, or a flag variety.

3.2. The Vanishing Theorem

THEOREM 3.2. *If Ω is proper over an affine, then:*

$$H^i(X, L \otimes S^m(\mathcal{S}_X)) = 0$$

for every integers $i \geqslant 1$, $m \geqslant 0$ and line bundle L on X generated by global sections.

Note that the special case $m = 0$ asserts that $H^i(X, L) = 0$ if L is generated by global sections. This result was proved by the second author [B2] Corollary 1 for any complete, spherical variety. F. Knop proved in [K3] a generalization of Theorem 3.2 which avoids condition 3.1 (ii), but does not allow twisting by line bundles L. We will give several applications of this vanishing result in Part 4. For the convenience of the reader, we give here a synopsis of the proof of our theorem.

(1) Condition 3.1 (ii) implies (for projective X) that the cone of nef and effective $\mathbb{Q}$-divisors with support in D has a non-empty interior in the linear space of $\mathbb{Q}$-divisors with support in D.
(2) From Step 1, we deduce the existence of an effective $\mathbb{Q}$-divisor E supported on the whole D, such that $\mathcal{O}_{\mathbb{P}(S^*X)}(1) \otimes \pi^*\mathcal{O}_X(E)$ is nef and big, where $\pi\colon \mathbb{P}(S^*X) \to X$.

(3) The canonical bundle of $\mathbb{P}(S^*X)$ is

$$\omega_{\mathbb{P}(S^*X)} = \mathcal{O}_{\mathbb{P}(S^*X)}(-n) \otimes \pi^*\mathcal{O}_X(-D),$$

where $n = \dim(X)$.

(4) Let $E = \Sigma_{i=1}^{\ell} N_i X_i$. Let $N = \max\{N_1, \ldots, N_\ell\}$. For $m \geqslant 0$, define:

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}(S^*X)}(m+n) \otimes \pi^*\mathcal{O}_X(E/N).$$

Applying the Kawamata–Viehweg vanishing Theorem 1.4 to $\mathcal{L}$ and using Steps 2 and 3, we get:

$$H^i(X, S^m \mathcal{S}_X) = H^i(\mathbb{P}(S^*X), \mathcal{O}_{\mathbb{P}(S^*X)}(m)) = 0 \quad \text{for } i \geqslant 1.$$

3.3. A TECHNICAL LEMMA

LEMMA 3.3. *Consider a projective, regular variety X such that Ω is proper over an affine. Then the cone of nef and effective $\mathbb{Q}$-divisors with support in D has a non-empty interior in the linear space of $\mathbb{Q}$-divisors with support in D.*

Proof. Recall that $X_1, \ldots, X_\ell$ denote the boundary divisors of X; let $v_1, \ldots, v_\ell \in \mathcal{V}$ be the associated valuations. Furthermore, denote by $(C_i)_{i \in I}$ the subdivision of $\mathcal{V}$ into convex cones which is associated to X, see e.g. [BP]. Then each cone C_i is generated by some subset of $\{v_1, \ldots, v_\ell\}$. Now any $\mathbb{Q}$-divisor E with support in D defines a piecewise linear function Φ_E on the fan $(C_i)_{i \in I}$, as follows: The value of Φ_E at v_j is the coefficient of D_j in E. This sets-up a bijective correspondence $E \to \Phi_E$ between $\mathbb{Q}$-divisors with support in D, and $\mathbb{Q}$-valued piecewise linear functions on $\mathcal{V}$. We denote its inverse by $\Phi \to E_\Phi$. By [B1] 3.3, the divisor E is nef if and only if Φ_E is convex and moreover $v_D(\Phi_{E,i}) \leqslant 0$ for all $D \in \mathcal{D}$, where $\Phi_{E,i}$ denotes the linear function $\Phi_E|_{C_i}$.

By assumption, there exists a linear function f such that $f \geqslant 0$ on $\mathcal{V}$ and that $f < 0$ on $\rho(\mathcal{D}) \setminus \{0\}$. Moreover, because X is projective, there exists a piecewise linear function g on $\mathcal{V}$, such that g is strictly convex (this follows from [B1] 3.3). Furthermore, we may assume that $g > 0$ on $\mathcal{V} \setminus \{0\}$. Namely, g is the support function of some convex polyhedron P. Translating P (which amounts to adding a linear function to g), we may assume that P contains the origin in its relative interior, and then g is positive outside of the origin.

Now set $\Phi = f + \varepsilon g$ with $\varepsilon > 0$ small enough. Then Φ is strictly convex, $\Phi > 0$ on $\mathcal{V} \setminus \{0\}$ and $v_D(\Phi_i) < 0$ for all $i \in I$ and all $D \in \mathcal{D}$ such that $\rho(v_D) \neq 0$. Therefore, E_Φ is in the interior of the cone of nef, effective $\mathbb{Q}$-divisors with support in D.

3.4. KEY LEMMA

We denote by $\pi\colon \mathbb{P} \to X$ the projective bundle associated to $\mathcal{S}_X$, i.e. $\mathbb{P} = \operatorname{Proj} S^\bullet(\mathcal{S}_X)$.

LEMMA 3.4. *If Ω is proper over an affine and if X is projective, then there exists an effective, nef divisor E, supported on the whole boundary D, such that the invertible sheaf $\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{O}_X(E)$ is big.*

Proof. Assume that for all effective, nef divisors E supported on the whole D, the invertible sheaf $\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{O}_X(E)$ is not big. Then the intersection number

$$(c_1(\mathcal{O}_{\mathbb{P}}(1)) + \pi^*E)^{\dim(\mathbb{P})}$$

vanishes, see 1.3. On the other hand, since $\mathcal{O}_{\mathbb{P}}(1)$ and $\pi^*\mathcal{O}_X(E)$ are generated by global sections, we have

$$c_1(\mathcal{O}_{\mathbb{P}}(1))^u \cdot (\pi^*E)^v \geqslant 0,$$

for $u + v = \dim(\mathbb{P})$. It follows that all these intersection numbers vanish. But any divisor with support in D is the difference of two effective, nef divisors with support on the whole D, see 3.3. Therefore, we have

$$(c_1(\mathcal{O}_{\mathbb{P}}(1))^u \cdot \pi^*D_1 \ldots \pi^*D_v = 0$$

for all divisors $D_1, \ldots, D_v$ with support in D. Choose a closed G-orbit Y in X, then Y is the transversal intersection of r irreducible G-stable divisors where $r = \mathrm{rk}(X)$. Therefore,

$$(c_1(\mathcal{O}_{\mathbb{P}}(1)|_{\pi^{-1}(Y)}))^{\dim(\mathbb{P})-r} = 0,$$

that is: $\kappa(\mathcal{S}_X|_Y) < \dim(\mathbb{P}) - r = 2\dim(X) - 1 - r$, and we have a contradiction with Proposition 2.6.1.

3.5. Canonical bundle of $\mathbb{P}$

Recall that $\mathbb{P}$ denotes the projective bundle associated to $\mathcal{S}_X$.

LEMMA 3.5. *The canonical sheaf of $\mathbb{P}$ is given by*

$$\omega_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-n) \otimes \pi^*\mathcal{O}_X(-D),$$

where $n = \dim(X) = \mathrm{rk}\, \mathcal{S}_X$.

Proof. Recall that

$$\omega_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-n) \otimes \pi^*((\wedge^n \mathcal{S}_X) \otimes \omega_X),$$

(see for instance [Sch] p. 139). Now, from the exact sequence in Proposition 2.3.2:

$$0 \to \mathcal{S}_X \to \mathcal{T}_X \to \bigoplus_{i=1}^{\ell} \mathcal{O}_X(X_i) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i} \to 0,$$

we obtain the isomorphism

$$\wedge^n \mathcal{S}_X \simeq (\wedge^n \mathcal{T}_X) \otimes \mathcal{O}_X(-X_1 - \cdots - X_\ell),$$

whence

$$(\wedge^n \mathcal{S}_X) \otimes \omega_X \simeq \mathcal{O}_X(-D).$$

3.6. End of the proof of the vanishing theorem

Using Corollary 2.7.2 and the fact that any complete regular variety is dominated by a projective, regular variety, we may assume that X is projective. Let E be as in Lemma 3.4. Write

$$E = \sum_{i=1}^{\ell} N_i D_i,$$

with $N_i > 0$ for all i. Set $N = \max(N_1, \ldots, N_\ell)$, $n = \dim(X)$ and

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}}(m+n) \otimes \pi^* \left(L \otimes \mathcal{O}_X \left(\frac{E}{N} \right) \right),$$

where m is a non-negative integer. Since $\mathcal{O}_{\mathbb{P}}(1)$ and L are nef, it follows from Lemma 3.4 above and [Sch] Lemma 1.3, 1.4, that $\mathcal{L}$ is nef and big. Furthermore, $\lceil E/N \rceil = D$ by choice of N. Therefore, applying the Kawamata–Viehweg vanishing Theorem 1.4 together with Lemma 3.5, we get:

$$H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m) \otimes \pi^* L) = 0 \quad \text{for all } i \geqslant 1.$$

We conclude the proof using the fact that:

$$H^i(\mathbb{P}, \pi^* L \otimes \mathcal{O}_{\mathbb{P}}(m)) \simeq H^i(X, L \otimes S^m(\mathcal{S}_X)).$$

3.7. Simplifications in the case of flag varieties

In the case where $X = G/P$ is a homogeneous space, i.e. a flag space, then the action sheaf $\mathcal{S}_X$ coincides with the tangent sheaf $\mathcal{T}_X$. Furthermore D is empty, and $\omega_{\mathbb{P}}^{-1} = \mathcal{O}_{\mathbb{P}}(n)$ is nef and big. Indeed, $\mathcal{O}_{\mathbb{P}}(1)$ is generated by its global sections. To show that it is big, observe that the associated morphism is the projectivization of the moment map

$$\mu \colon T^*X \to \mathfrak{g}^*,$$

and this map is known to be generically finite, see e.g. [Bien] IV.1.1. Note that $\omega_{\mathbb{P}}^{-1}$ is not ample in general, because the projectivization of the moment map is rarely finite.

Therefore, for any line bundle L generated by global sections, $\pi^* L \otimes \mathcal{O}_{\mathbb{P}}(1)$ is also nef and big. Applying the Kawamata–Viehweg vanishing Theorem 1.4, we have

$$H^i(X, L \otimes S^m(\mathcal{S}_X)) \simeq H^i(\mathbb{P}, \pi^* L \otimes \mathcal{O}_{\mathbb{P}}(m+n) \otimes \omega_{\mathbb{P}}) = 0 \quad \text{for } i \geqslant 1.$$

This provides a direct proof of Theorem 3.2 in the case of flag spaces, and of Bott's theorem mentioned in the introduction.

4. Applications

4.1. AUTOMORPHISM GROUP OF A REGULAR G-VARIETY

We continue with the previous notation; in particular, X denotes a complete, regular G-variety. Let Aut(X) denote the automorphism group of X considered as an algebraic variety. Since X is complete, smooth and unirational, (in fact it is even rational like all spherical varieties), the identity component of Aut(X) is an affine algebraic group, and its Lie algebra consists of the space of global vector fields $H^0(X, \mathcal{T}_X)$ (cf. [MO] Theorems 3.6, 3.7 or [Ram] Corollary 1). Using a vanishing theorem of F. Knop ([K3] Theorem 4.1), we describe the G-module $H^0(X, \mathcal{T}_X)$.

For $1 \leqslant i \leqslant l$, let V_i be the quotient of $H^0(X, \mathcal{O}_X(X_i))$ modulo the line of constant sections. Since X is spherical, the G-module V_i is multiplicity free.

PROPOSITION 4.1.1. *We have an exact sequence of G-modules*

$$0 \to H^0(X, \mathcal{S}_X) \to H^0(X, \mathcal{T}_X) \to \bigoplus_{i=1}^{l} V_i \to 0.$$

Proof. By Proposition 2.3.2 and the vanishing of $H^1(X, \mathcal{S}_X)$ (see [K3] Theorem 4.1), we have an exact sequence:

$$0 \to H^0(X, \mathcal{S}_X) \to H^0(X, \mathcal{T}_X) \to \bigoplus_{i=1}^{l} H^0(X_i, \mathcal{O}_X(X_i)|_{X_i}) \to 0.$$

Now, consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(X_i) \to \mathcal{O}_X(X_i)|_{X_i} \to 0.$$

Observe that $H^1(X, \mathcal{O}_X)$ is zero since X is smooth, complete and unirational. Hence, $H^0(X_i, \mathcal{O}_X(X_i)|_{X_i})$ is isomorphic to V_i.

Remark. The image of $\mathfrak{g}$ in $H^0(X, \mathcal{T}_X)$ is contained in $H^0(X, \mathcal{S}_X)$, and it may happen that this inclusion be strict. For instance, if G is a symplectic group of rank m, and X is a projective space of dimension $2m-1$. Then G acts transitively in X, hence X is a regular G-variety. Since the boundary D is empty, $H^0(X, \mathcal{S}_X) = H^0(X, \mathcal{T}_X)$, but this latter space can be identified with the space of all $2m \times 2m$ matrices of trace zero, which is strictly larger than $\mathfrak{g}$ for $m \geqslant 2$. Namely, $H^0(X, \mathcal{T}_X)$ is the Lie algebra of the automorphism group of X, i.e. of $\mathrm{PGL}_{2m}(k)$.

THEOREM 4.1.2. *For a complete regular variety X, the following conditions are equivalent:*

(i) *Aut(X) is transitive*
(ii) *For $1 \leqslant i \leqslant l$, the invertible sheaf $\mathcal{O}_X(X_i)$ is generated by global sections*
(iii) *For $1 \leqslant i \leqslant l$, the invertible sheaf $\mathcal{O}_X(X_i)$ is numerically effective.*

Proof. (i) $\Rightarrow$ (ii) If the group of automorphisms of X is transitive, then $\mathcal{T}_X$ is generated by global sections. Therefore, by Proposition 2.3.2, the sheaf $\mathcal{O}_X(X_i)|_{X_i}$ is also generated by global sections. Now the vanishing of $H^1(X, \mathcal{O}_X)$ implies that the map

$$H^0(X, \mathcal{O}_X(X_i)) \to H^0(X_i, \mathcal{O}_X(X_i)|_{X_i})$$

is surjective, hence $\mathcal{O}_X(X_i)$ is generated by global sections.

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(ii) follows from the fact that in a complete spherical variety, every numerically effective Cartier divisor is generated by its global sections. Indeed, it follows from [B1] Section 3, that in such a variety X, numerical equivalence coincides with rational equivalence. Moreover, in the space $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, the closure of the cone of ample divisor classes is the set of divisor classes generated by global sections.

(ii)$\Rightarrow$(i) By [K3] Theorem 4.1, we have $H^i(X, S^\bullet \mathcal{S}_X) = 0$ for all $i \geqslant 1$. In particular, $H^1(X, \mathcal{S}_X) = 0$. Therefore, every global section of $\mathcal{O}_X(X_i)$ can be lifted to a global section of $\mathcal{T}_X$. By proposition 2.3.2, we conclude that $\mathcal{T}_X$ is generated by global sections.

For complete regular varieties containing only two orbits, we recover the following result due to D. Ahiezer [A2] whose proof relied on his classification of regular varieties with only two orbits in [A1].

COROLLARY 4.1.3. *Let X be a complete regular G-variety, containing exactly two G-orbits. Then the following conditions are equivalent:*

(i) *There exist regular non-constant functions on Ω.*
(ii) *The normal bundle of the closed G-orbit Y in X has non-zero global sections.*
(iii) *The automorphism group of X is transitive.*

Proof. (i)$\Rightarrow$(ii) Let f be a non-constant regular function on Ω. Then f extends uniquely to a rational function on X with a pole of order $n > 0$ on Y. Thus, f defines a global section of $\mathcal{O}_X(nY)$, which generates this sheaf on a open dense subset of Y. Since Y is homogeneous, the sheaf $\mathcal{O}_X(nY)$ is generated by global sections; hence so is $\mathcal{N}_Y^n$, where $\mathcal{N}_Y$ denotes the normal (line) bundle of Y. Because Y is a flag variety, it follows that $\mathcal{N}_Y$ is also generated by global sections.

(ii)$\Rightarrow$(iii) follows from the above theorem and the surjectivity of the map

$$H^0(X, \mathcal{O}_X(Y)) \to H^0(Y, \mathcal{N}_Y)$$

(a consequence of the vanishing of $H^1(X, \mathcal{O}_X)$).

(iii)$\Rightarrow$(i) is also implied by the above theorem, and the fact that every non-constant global section of $\mathcal{O}_X(Y)$ defines a non-constant regular function on Ω.

This corollary implies for instance that all the varieties in the table of Section 2.1 are homogeneous for an overgroup of G. It would be interesting to have a classification of homogeneous regular varieties, or equivalently a classification of flag varieties that are regular completions of spherical spaces.

4.2. Rigidity of spherical Fano varieties

The vanishing of $H^1(X, \mathcal{T}_X)$ implies by Kodaira–Spencer theory that X is locally rigid, i.e. that the complex structure of X is locally unique, see [MK]. In the relative case (X, D), the vanishing of $H^1(X, \mathcal{S}_X)$ implies that the inclusion of the divisor $D \hookrightarrow X$ is locally rigid, see [Bin] for the theory of deformations of diagrams. Hence regular varieties are locally rigid modulo their boundary, because of the vanishing Theorem 4.1 in [K3].

In this section, we show that if X is regular and Fano, then it is locally rigid even without group action, i.e. $H^1(X, \mathcal{T}_X) = 0$. Recall that a variety is called Fano if its anti-canonical line bundle ω_X^{-1} is ample. All flag spaces are Fano. The complete symmetric varieties of De Concini–Procesi are Fano as well. For a characterization of smooth toric Fano varieties, see [Oda] Lemma 2.20.

PROPOSITION 4.2. *Let X be a regular projective Fano G-variety. Then*

(i) $H^i(X, \mathcal{T}_X) = 0$ *for any* $i \geqslant 1$.
(ii) *If* Ω *is proper over an affine, then* $H^i(X, L \otimes \mathcal{T}_X) = 0$ *for all* $i \geqslant 1$ *and line bundle* L *generated by its global sections.*

Proof. (ii) For any irreducible divisor Y in X, let $\mathcal{N}_Y$ be its normal line bundle: $\mathcal{N}_Y = \mathcal{O}_X(Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. Let $X_1, \dots, X_\ell$ be the components of the boundary of X. Put $\mathcal{N}_j = \mathcal{N}_{X_j}$. Recall from 2.3.2, the exact sequence:

$$0 \to L \otimes \mathcal{S}_X \to L \otimes \mathcal{T}_X \to \bigoplus_{j=1}^{l} L \otimes \mathcal{N}_j \to 0.$$

Because Ω is assumed to be proper over an affine, we have $H^i(X, L \otimes \mathcal{S}_X) = 0$ for $i \geqslant 1$, by Theorem 3.2. To prove the proposition, it suffices to show that $H^i(X_j, L \otimes \mathcal{N}_j) = 0$ for $i \geqslant 1$. But

$$L \otimes \mathcal{N}_j = L \otimes \omega_X^{-1} \otimes \omega_X \otimes \mathcal{N}_j = L \otimes \omega_X^{-1} \otimes \omega_{X_j}.$$

Now L is nef and ω_X^{-1} is ample, therefore $L \otimes \omega_X^{-1}$ is ample. Then, by Kodaira vanishing Theorem 1.1, we have

$$H^i(X_j, L \otimes \mathcal{N}_j) = 0 \quad \text{for } i \geqslant 1.$$

The proof of (i) is similar, using [K3] Theorem 4.1.

4.3. Differential operators and their symbols

Let $\mathcal{D}_X$ be the sheaf of algebraic differential operators on the complete, regular variety X, with boundary D. Denote by $\mathcal{C}_X$ the subsheaf of $\mathcal{D}_X$ made of operators strongly tangent to the boundary D

$$\begin{aligned}\mathcal{C}_X = \{P \in \mathcal{D}_X \mid & \text{ for every } G\text{-orbit } Y \text{ with ideal sheaf } \mathcal{I}_Y, \\ & P \cdot \mathcal{I}_Y^m \subseteq \mathcal{I}_Y^m, \quad \forall m \in \mathbf{N}\}.\end{aligned}$$

Since X is regular, $\mathcal{C}_X$ is the $\mathcal{O}_X$-subalgebra of $\mathcal{D}_X$ generated by the action sheaf $\mathcal{S}_X$. Recall that $\mathcal{S}_X$ is a locally free $\mathcal{O}_X$-module of rank $n = \dim(X)$. The Lie bracket of vector fields induces a Lie algebra structure on $\mathcal{S}_X$.

The order filtration on $\mathcal{D}_X$ induces a filtration on $\mathcal{C}_X$, whose associated graded algebra is the symmetric algebra $S^\bullet \mathcal{S}_X$ of the action bundle $\mathcal{S}_X$. This filtration induces also a filtration on $H^0(X, \mathcal{C}_X)$. To every differential operator $p \in \mathcal{C}_X$, we can attach a symbol $\sigma(p) \in S^\bullet \mathcal{S}_X$ which is simply the image of p in the associated graded algebra. Locally, it is clear that every symbol comes from a differential operator, because we can study this question in an affine open set and the result is known to hold for affine space with a union of hyperplanes as boundary. But at the global level, we loose all explicit description in coordinates, and one can ask whether every symbol in $H^0(X, S^\bullet(\mathcal{S}_X))$ comes from an operator in $H^0(X, \mathcal{C}_X)$, in other words whether we can commute taking the cohomology of $\mathcal{C}_X$ with taking its associated graded. The answer is affirmative by the following result.

PROPOSITION 4.3.1. *For any complete, regular variety X, we have*

$$\mathrm{gr} H^0(X, \mathcal{C}_X) \simeq H^0(X, S^\bullet(\mathcal{S}_X)),$$

$$H^i(X, \mathcal{C}_X^m) = 0 \quad \textit{for } i \geqslant 1, m \geqslant 0,$$

and

$$H^i(X, \mathcal{C}_X) = 0 \quad \textit{for } i \geqslant 1.$$

Proof. For every integer $m \geqslant 0$, we have an exact sequence

$$0 \to \mathcal{C}_X^m \to \mathcal{C}_X^{m+1} \to S^{m+1}(\mathcal{S}_X) \to 0.$$

Therefore, we get a long exact sequence of cohomology spaces

$$\begin{aligned}0 &\to H^0(X, \mathcal{C}_X^m) \to H^0(X, \mathcal{C}_X^{m+1}) \to H^0(X, S^{m+1}(\mathcal{S}_X)) \\ &\to H^1(X, \mathcal{C}_X^m) \to H^1(X, \mathcal{C}_X^{m+1}) \to 0,\end{aligned}$$

and an isomorphism

$$H^i(X, \mathcal{C}_X^m) \simeq H^i(X, \mathcal{C}_X^{m+1}) \quad \text{for } i \geqslant 2.$$

Namely, the sheaf $S^{m+1}(\mathcal{S}_X)$ has no higher cohomology by [K3] Theorem 4.1.

In particular, the map

$$H^i(X, \mathcal{C}_X^m) \to H^i(X, \mathcal{C}_X^{m+1})$$

is onto for $i \geqslant 1$. Now, $\mathcal{C}_X^0$ is isomorphic to $\mathcal{O}_X$, hence we have: $H^i(X, \mathcal{C}_X^0) = 0$. Therefore, $H^i(X, \mathcal{C}_X^m) = 0$ by induction on m, and the proposition follows.

COROLLARY 4.3.2. *On any complete, regular variety X, the symbol map is surjective.*

4.4. REGULAR G-VARIETIES CAN HAVE DEFORMATIONS

In this section, we show that a regular G-variety X can admit deformations even if it does not have any deformation keeping the divisor class of D fixed. In other words, we shall construct regular G-varieties X such that $H^1(X, \mathcal{S}_X) = 0$ but $H^1(X, \mathcal{T}_X) \neq 0$. To establish the existence of deformations of their complex structures, one needs to check that there are no obstructions, i.e. $H^2(X, \mathcal{T}_X) = 0$, see [MK] p. 155.

Let V be a k-vector space of odd dimension $2n+1$. Let q be a non-degenerate quadratic form on q, and choose a maximal isotropic subspace $U \subset V$, i.e. a subspace of dimension n, such that $q|_U = 0$. Take $G = \mathrm{SO}(V, q) = \mathrm{SO}(2n+1)$, and P to be the stabilizer of U in G. Then P is a maximal parabolic subgroup of G with Levi factor $L \simeq \mathrm{GL}(n)$, and we have a surjective morphism $f\colon P \to \mathrm{GL}(U)$. Let $\mathbb{P}(U)$, resp. $\mathbb{P}(U^*)$, be the projective space of U, resp. of the dual of U. Then P acts diagonally in $F = \mathbb{P}(U) \times \mathbb{P}(U^*)$ by f and its contragredient. Let X be the fiber product $G \times_P F$, with projection $p\colon X \to G/P$.

PROPOSITION 4.4.

(i) *X is a projective regular G-variety, with only two orbits.*
(ii) *The G-module $H^1(X, \mathcal{T}_X)$ is isomorphic to $V = k^{2n+1}$.*
(iii) $H^i(X, \mathcal{T}_X) = 0$ *for all* $i \geqslant 2$.

Therefore, X admits a $2n+1$-dimensional family of complex deformations.

Proof. (i) F is projective and smooth, hence the same holds for X. P operates in F via its Levi factor L. F contains one open dense L-orbit consisting of the pairs $(d, H) \in \mathbb{P}(U) \times \mathbb{P}(U^*)$ such that the line d is not in the hyperplane H. The complement of this orbit: $\{(d, H) \in \mathbb{P}(U) \times \mathbb{P}(U^*) | d \in H\}$ is homogeneous for L, and has codimension one. Thus, F is a spherical variety for the Levi factor L, with only two orbits. Because the closed orbit in F is a divisor, F has no color for L and hence F is a regular L-variety. Consequently, X is a regular G-variety.

(ii) and (iii) Consider the exact sequence of $\mathcal{O}_X$-G-modules

$$0 \to \mathcal{R} \to \mathcal{T}_X \to p^*\mathcal{T}_{G/P} \to 0,$$

where $\mathcal{R}$ denotes the relative tangent bundle. Note that

$$p_*(p^*\mathcal{O}_{G/P}) = p_*\mathcal{O}_X = \mathcal{O}_{G/P} \quad \text{because } H^0(F,\mathcal{O}_F) = k,$$
$$R^ip_*(p^*\mathcal{O}_{G/P}) = R^ip_*\mathcal{O}_X = 0 \quad \text{for } i \geqslant 1, \quad \text{because } H^i(F,\mathcal{O}_F) = 0.$$

By the projection formula, we have

$$p_*(p^*\mathcal{T}_{G/P}) = \mathcal{T}_{G/P} \quad \text{and } R^jp_*(p^*\mathcal{T}_{G/P}) = 0 \quad \text{for all } j \geqslant 1.$$

Moreover, $H^i(G/P,\mathcal{T}_{G/P}) = 0$ for all $i \geqslant 1$ by [Bott], whence

$$H^i(X,p^*\mathcal{T}_{G/P}) = 0 \quad \text{for } i \geqslant 1 \quad \text{and}$$
$$H^0(X,p^*\mathcal{T}_{G/P}) = H^0(G/P,\mathcal{T}_{G/P}).$$

Thus, we get an exact sequence

$$H^0(X,\mathcal{T}_X) \to H^0(G/P,\mathcal{T}_{G/P}) \to H^1(X,\mathcal{R}) \to H^1(X,\mathcal{T}_X) \to 0$$

and isomorphisms

$$H^i(X,\mathcal{R}) \simeq H^i(X,\mathcal{T}_X)$$

for $i \geqslant 2$. The restriction of $\mathcal{R}$ to F coincides with the tangent bundle of $F = \mathbb{P}(U) \times \mathbb{P}(U^*)$. Therefore, $H^0(F,\mathcal{R}|F) \simeq \mathfrak{sl}(U) \oplus \mathfrak{sl}(U)$ as GL(U)-modules, while $H^i(F,\mathcal{R}|_F) = 0$ for all $i \geqslant 1$, by the vanishing Theorem 3.2 for flag varieties, see 3.7. This implies that the locally free G-sheaf $p_*\mathcal{R}$ on G/P is associated to the semisimple P-module $\mathfrak{sl}(U) \oplus \mathfrak{sl}(U)$, and that $R^ip_*\mathcal{R} = 0$ for $i \geqslant 1$.

Using the Borel–Weil–Bott theorem [Bott], we check that $H^1(G/P,p_*\mathcal{R})$ is isomorphic to $V \oplus V$ and that $H^i(G/P,p_*\mathcal{R}) = 0$ for any $i \neq 1$. Let $e_1,\dots,e_n$ be unit vectors in $\mathfrak{t}^* \simeq k^n$, the dual of a Cartan subalgebra of $\mathfrak{g}$, and choose $e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_n$ as simple roots. Then $\rho = (n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{1}{2})$ is the half sum of the positive roots. The highest weight of the $L = \mathrm{GL}(n)$-module $\mathfrak{sl}(U) \simeq \mathfrak{sl}(n)$ is $\lambda = (1,0,\dots,0,-1)$. Consider the dominant weights in $\mathfrak{t}^*$ of the form $w(\lambda + \rho) - \rho$, where w is an element of the Weyl group W of $\mathfrak{g}$. If s is the simple reflection in the nth simple root e_n, then $s(\lambda+\rho)-\rho = (1,0,\dots,0)$, which is dominant and is the highest weight of V as G-module. For all other choices of $w \in W$, the weights $w(\lambda + \rho) - \rho$ are not dominant. This proves our assertion. It follows that $H^1(X,\mathcal{R}) \simeq V \oplus V$, and $H^i(X,\mathcal{R}) = 0$ for $i \neq 1$, whence (iii). In particular, $H^0(X,\mathcal{R}) = 0$, i.e. there are no global vector fields tangent to the fibers of p, and $H^0(X,\mathcal{T}_X) = \mathfrak{g}$ injects into $H^0(G/P,\mathcal{T}_{G/P})$.

Now, the automorphism group of G/P is actually $\mathrm{SO}(2n+2) \supseteq \mathrm{SO}(2n+1) = G$. Hence, as G-module, $H^0(G/P,\mathcal{T}_{G/P}) = \mathfrak{so}(2n+2) \simeq \mathfrak{so}(2n+1) \oplus V = \mathfrak{g} \oplus V$.

The action map $\mathfrak{g} \to H^0(X,\mathcal{T}_X)$ is an isomorphism. Indeed, one can check using Theorem 4.1.2 that the automorphism group of X is exactly G. The above exact sequence in cohomology translates into an exact sequence of G-modules:

$$\mathfrak{g} \to \mathfrak{g} \oplus V \to V \oplus V \to H^1(X,\mathcal{T}_X) \to 0.$$

The first map is injective, for it is induced by the map from $\mathfrak{g}$ to $H^0(G/P, \mathcal{T}_{G/P})$ defined by the G-action on G/P. Hence, V is isomorphic to $H^1(X, \mathcal{T}_X)$.

The last sentence of the theorem follow by the Kodaira–Nirenberg–Spencer Theorem, [MK] p. 155.

Note that in this example, we also have $H^1(X, \mathcal{D}_X^1) \neq 0$. Indeed, we have a short exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{D}_X^1 \to \mathcal{T}_X \to 0,$$

which splits canonically by evaluating elements of $\mathcal{D}_X^1$ at the constant function 1. In particular, the symbol map is always surjective for differential operators of order $\leqslant 1$. Thus, $H^1(X, \mathcal{D}_X^1)$ is isomorphic to $H^1(X, \mathcal{T}_X) \simeq V$.

To complete our study of these varieties X, one can show that they are locally rigid modulo their boundary divisors, i.e. $H^i(X, \mathcal{S}_X) = 0$ for $i \geqslant 1$. We cannot use Theorem 3.2, because X does not satisfy the Condition 3.1 (ii). Indeed, using the notation of 3.1, the cone $\mathcal{V}$ consists of a half-line, and $\mathcal{V}_0 = 0$. The set $\mathcal{D}$ of possible colors for the open G-orbit in X contains three elements. Under the map ρ, one is mapped inside $\mathcal{V}$, the two others are mapped on the opposite half-line. Thus, there is no weight $f \in \Lambda$ that can separate $\rho(\mathcal{D})$ and $\mathcal{V}$, and Condition 3.1 (ii) fails. However, the method of proof of Theorem 4.4 applies – with a lengthy computation – or one can use the results of [K3] to show that $H^i(X, \mathcal{S}_X) = 0$ for $i \geqslant 1$.

4.5. Remarks on global rigidity

The results proved in this paper address only the local uniqueness of complex structures on regular varieties. The question of global uniqueness of complex structures is quite different, and in fact, uniqueness does not hold in general. Except for special cases such as projective spaces, flag varieties admit many homogeneous complex structures. The number of these is, however, finite.

Let k be the field of complex numbers, and let K be a maximal compact subgroup of G. Borel–Hirzebruch have shown that the K-homogeneous complex structures on varieties of complete flags are in bijection with elements of the Weyl group of K, see [BH] 13.8.

Let us explain how to construct these complex structures. Let M be a Cartan subgroup of K, let W be the Weyl group of K, and let B be any Borel subgroup of G containing M. Then, the flag variety of G can be written as $X = K/M \simeq G/B$. This presentation equips X with the complex structure of a complex homogeneous space for G.

There are exactly $|W|$ such Borel subgroups: $\{wBw^{-1} | w \in W\}$, hence one could think that the complex varieties $G/(wBw^{-1})$ are the ones of Borel–Hirzebruch's result. However, all the varieties $G/(wBw^{-1})$ are isomorphic to

$X = G/B$. Indeed, G acts on them, and B has a fixed point by Borel's fixed point theorem. Therefore, they are isomorphic to G/B.

To obtain a K-equivariant almost complex structure on K/M, it suffices to have an M-equivariant linear transformation J of order 4, of the complexified tangent space $T_{eM}(K/M)_{\mathbb{C}}$ such that the only eigenvalues of J are i and $-i$, and the corresponding eigenspaces T^+ and T^- (the holomorphic and anti-holomorphic tangent spaces) are of equal dimension. Now,

$$T_{eM}(K/M)_{\mathbb{C}} = T^+ \oplus T^- = \mathfrak{g}/\mathfrak{t} = \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{t}$ is the Lie algebra of T, the complexification of M, and Φ is the root system of T in G. Moreover, this complex structure is integrable if and only if $\mathfrak{t} \oplus T^+$ is a Lie subalgebra of $\mathfrak{g}$.

J commute with the action of M if and only if T^+ and T^- are direct sums of root spaces, because all the roots of M are distinct. We can then choose $\mathfrak{t} \oplus T^+$ to be the Lie algebra of any Borel subalgebra containing $\mathfrak{t}$, namely $Ad(w)\mathfrak{b}$, for $w \in W$. Using the group K to spread this choice of holomorphic tangent space all around K/M, this yields a K-equivariant complex structure on K/M.

The $|W|$ complex structures on K/M mentioned above are obtained this way. One checks that these complex structures are also Kähler. They are pairwise distinct, since they have distinct Chern classes in $H^2(X, \mathbb{Z})$, see [BH] Sect. 10. Nevertheless, these complex structures are equivalent under the group of diffeomorphisms of X induced by the right action of W on K/M. W acts transitively on the set of Borel subgroups containing T.

For varieties of partial flags, this equivalence might fail due to the fact that parabolic subgroups having a given Levi subgroup may not be conjugate by G.

It is worth observing that only one of the complex structures constructed above admits G as a group of complex automorphisms, namely the complex structure corresponding to the choice of $\mathfrak{t} + T^+ = \mathfrak{b}$ at the origin of G/B. Indeed, all other choices for T^+ are not B-stable, hence they do not yield a G-equivariant complex structure.

The structure defined by $\mathfrak{t} + T^- = \mathfrak{b}^-$ corresponds to the complex conjugate variety. It admits G as group of anti-holomorphic automorphisms.

It would be very interesting to have further finiteness results on the global moduli spaces of regular varieties.

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