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The overconvergence of morphisms of étale φ - ∇ -spaces on a local field

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1. Introduction

Let X be a smooth curve over a perfect field k of positive characteristic p and let \overline{X} be the smooth compactification of X . The category of unit-root F -isocrystals over X is equivalent to the category of p -adic representations of $\pi_1(X, *)$. And it is conjectured that p -adic representations with finite monodromy around $D = \overline{X} - X$, which means that the inertia group of the Galois group around D acts trivially after a finite extension, correspond to overconvergent unit-root F -isocrystals. So the natural functor

$$\left(\begin{array}{c} \text{overconvergent unit-root} \\ F\text{-isocrystals on } X \text{ around } D \end{array} \right) \longrightarrow \left(\begin{array}{c} \text{unit-root} \\ F\text{-isocrystals on } X \end{array} \right)$$

is expected to be fully faithful. Crew showed finite monodromy theorem in rank one case and he proved full faithfulness for rank one objects. (See [1])

In this paper we will show full faithfulness of the above functor for arbitrary rank. To do this we study the local version of the above problem. In [3] Fontaine defined étale φ -modules on $\text{Spec } k((t))$, a spectrum of the field of formal Laurent series over k , whose category is equivalent to the category of p -adic representations of $\text{Gal}(k((t))^{\text{sep}}/k((t)))$. We will define a subcategory of overconvergent étale φ - ∇ -modules on $\text{Spec } k((t))$. (∇ means a connection) It is conjectured that the category of overconvergent étale φ - ∇ -modules corresponds to that of p -adic representations of $\text{Gal}(k((t))^{\text{sep}}/k((t)))$ with finite monodromy. We will show that the category of overconvergent étale φ -modules, considering only Frobenius structures and forgetting connections, is a full subcategory of étale φ -modules. As a corollary of this result we obtain the full faithfulness for the local version.

It is quite natural to expect that the natural functor, in general, from the category of overconvergent F -isocrystals, admitting arbitrary slopes, to the category of F -isocrystals or its local version is fully faithful. In this case the connection, however, plays an essential role and our method is useless.

2. Preliminaries

(2.1) Let p be a prime number and let Λ be a discrete valuation ring which is finite over the ring \mathbb{Z}_p of p -adic integers. We put $q = p^f$ to be the number of elements of the residue field of Λ . We denote by Λ_0 the maximum unramified subring of Λ .

Let k be a perfect field of characteristic p . We assume that k contains the field \mathbb{F}_q of q elements. Let $W = W(k)$, R and K be the ring of Witt vectors with k -coefficients, $\Lambda \otimes_{\Lambda_0} W$ and $R[p^{-1}]$ respectively. We define an endomorphism σ on $K = \Lambda[p^{-1}] \otimes_{\Lambda_0} W$, which we call frobenius, by $1 \otimes \text{frob}^f$. Here frob is the usual frobenius on W which is induced by the map $x \mapsto x^p$ on k . We denote by $||$ an absolute value of K .

We consider the ring

$$O_{\mathcal{E}} = \varprojlim ((R/p^n R)[[T]][[T^{-1}]]).$$

Then $O_{\mathcal{E}}$ is a complete discrete valuation ring whose residue field is identified with $k((t))$ by the map $T \mapsto t$. We denote by $\mathcal{E} = O_{\mathcal{E}}[p^{-1}]$ the fraction field of $O_{\mathcal{E}}$. An element of \mathcal{E} is uniquely expressed by formal Laurent series

$$\sum_{n=-\infty}^{n=\infty} a_n T^n, \quad a_n \in K, \quad |a_n| \text{ is bounded and } |a_n| \rightarrow 0 \text{ (} n \rightarrow -\infty \text{)}.$$

The absolute value $||_g$ of \mathcal{E} , which is called the Gauss norm, is given by

$$\left| \sum a_n T^n \right|_g = \sup\{|a_n| \mid n \in \mathbb{Z}\}.$$

A series $\sum a_n T^n \in \mathcal{E}$ is called overconvergent if and only if there exist real numbers $C > 0$ and $0 < \eta < 1$ such that $|a_n| \leq C\eta^{-n}$ for all negative integer n . We denote by \mathcal{E}^\dagger (resp. $O_{\mathcal{E}}^\dagger$) the subfield of \mathcal{E} which consists of overconvergent series (resp. the subring $\mathcal{E}^\dagger \cap O_{\mathcal{E}}$ of $O_{\mathcal{E}}$). \mathcal{E}^\dagger is also a discrete valuation field with the absolute value $||_g$ whose integer ring is $O_{\mathcal{E}}^\dagger$ and whose residue field is also $k((t))$.

We fix a frobenius endomorphism σ on \mathcal{E} , which extends the frobenius on K and is also denoted by σ . We assume that the endomorphism σ is a lifting of q th power map on $k((t))$ such that \mathcal{E}^\dagger is stable under σ . It is easy to see that σ is a continuous map on \mathcal{E} and that the condition $\sigma(\mathcal{E}^\dagger) \subset \mathcal{E}^\dagger$ is equivalent to the condition

$$T^\sigma \in \mathcal{E}^\dagger.$$

One can easily see that the fixed part of the frobenius σ in \mathcal{E} is $\Lambda[p^{-1}]$.

We define a derivation D on \mathcal{E} by

$$D \left(\sum a_n T^n \right) = \sum n a_n T^{n-1}.$$

The derivation D is a continuous one on \mathcal{E} and one sees easily that $D(x) \in \mathcal{E}^\dagger$ if and only if $x \in \mathcal{E}^\dagger$.

We have the relation $D \circ \sigma = \frac{D(T^\sigma)}{T^\sigma} \sigma \circ D$. If, for example, we choose the frobenius on \mathcal{E} by $T^\sigma = T^q$, then we have $D \circ \sigma = q\sigma \circ D$.

(2.2) We define a system of invariants $\alpha_N : \mathcal{E} \rightarrow \mathbb{R}_{\geq 0}$ for integers N by

$$\alpha_N \left(\sum a_n T^n \right) = \sup\{|a_n|; n \leq N\}.$$

It is well-defined since $|a_n| \rightarrow 0$ ($n \rightarrow -\infty$). For any element x in \mathcal{E} , one has $\alpha_N(x) \leq |x|_g$ for all N and equality holds for sufficient large N . By definition, $N \leq M$ implies $\alpha_N(x) \leq \alpha_M(x)$. From the properties of non-archimedean absolute value we have the following inequalities:

$$\begin{aligned} \alpha_N(x + y) &\leq \max\{\alpha_N(x), \alpha_N(y)\} \\ \alpha_N(xy) &\leq \sup\{\alpha_L(x)\alpha_M(y) \mid L + M = N\} \end{aligned} \tag{2.2.1}$$

for all $x, y \in \mathcal{E}$. By definition, x is contained in \mathcal{E}^\dagger if and only if there exist real numbers $C > 0$ and $0 < \eta < 1$ such that $\alpha_N(x) \leq C\eta^{-N}$ for all negative integers N .

Now we show a stability theorem for the frobenius σ which is the key proposition in our arguments.

(2.2.2) PROPOSITION. *Let r be a positive integer and let a_1, a_2, \dots, a_r be elements in \mathcal{E}^\dagger such that $|a_i|_g \leq 1$ for all i . If an element x in \mathcal{E} satisfies the equality:*

$$x^{\sigma^r} + a_1 x^{\sigma^{r-1}} + \dots + a_{r-1} x^\sigma + a_r x = 0, \tag{2.2.3}$$

then x is contained in \mathcal{E}^\dagger .

First we prove (2.2.2) in the case where $T^\sigma = T^q$. This case avoids the confusion arising from σ and helps to understand the case of general σ .

As all a_i 's are contained in \mathcal{E}^\dagger , there exist real numbers $C > 0$ and $0 < \eta < 1$ such that $\alpha_N(a_i) \leq C\eta^{-N}$ for negative integer N and for all i . Assume that x is not contained in \mathcal{E}^\dagger . We may assume that $|x|_g \leq 1$. Then $\alpha_N(x) \leq 1$ for all N and there exists a negative integer L such that $\alpha_L(x) > C\eta^{-L}$ and $\alpha_{L-1}(x) < \alpha_L(x)$. Now we calculate $\alpha_{q^r L}$ of each term in the left-hand side of (2.2.3) using (2.2.1). Since $T^\sigma = T^q$, we have

$$\alpha_{q^r L}(x^{\sigma^r}) = \alpha_L(x)$$

and

$$\alpha_{q^r L}(a_i x^{\sigma^{r-i}}) \leq \sup\{\alpha_M(a_i)\alpha_N(x); M + q^{r-i}N = q^r L\} \quad (1 \leq i \leq r).$$

If $M \leq (q^r - q^{r-i})L$ in the second inequality, then $\alpha_M(a_i)\alpha_N(x) \leq C\eta^{-(q^r - q^{r-i})L} \leq C\eta^{-L} < \alpha_L(x)$. If $M > (q^r - q^{r-i})L$ in the second inequality, then $N < L$ and $\alpha_M(a_i)\alpha_N(x) < \alpha_L(x)$ since $|a_i|_g \leq 1$. So we have

$$\alpha_{q^r L}(a_i x^{\sigma^{r-i}}) < \alpha_L(x)$$

for all $1 \leq i \leq r$. This contradicts (2.2.3). □

To prove (2.2.2) for general σ , we prepare the following lemma.

(2.2.4) LEMMA. *If x is not contained in \mathcal{E}^\dagger , then, for any real number $0 < \eta < 1$, there exist infinitely many negative integers L such that*

$$\alpha_L(x)\eta^{L-N} > \alpha_N(x)$$

for all integer $L < N < 0$.

Proof. If there exists no integer $L \leq -2$ which satisfies this property, then, for any negative integer N , there exists an integer $N < M < 0$ such that $\alpha_N(x)\eta^{N-M} \leq \alpha_M(x)$. Applying this finitely many time, we have $\alpha_N(x) \leq \alpha_{-1}(x)\eta^{-N-1}$ for all negative integers N . We have a contradiction. If there exists only finitely many such negative integers L , then we choose η close enough to 1 and we get a contradiction similar to the first part. □

Proof of (2.2.2). We define $u \in \mathcal{E}^\dagger$ by $T^\sigma = T^q u$. Then $|u - 1|_g < 1$. As all a_i 's are contained in \mathcal{E}^\dagger , there exist real numbers $C > 0$ and $0 < \eta < 1$ such that $\alpha_N(u) \leq \eta^{-N}$ and $\alpha_N(a_i) \leq C\eta^{-N}$ for all negative integers N for all i . We may assume that $|x|_g \leq 1$, equivalently that $\alpha_N(x) \leq 1$ for all integers N . Assume that x is not contained in \mathcal{E}^\dagger . Then there exists a negative integer L such that $\alpha_L(x) > \max\{C, 1\}\eta^{-L}$, $\alpha_{L-1}(x) < \alpha_L(x)$ and $\alpha_L(x)\eta^{L-N} > \alpha_N(x)$ for all $L < N < 0$ by (2.2.4). First we will show

$$\begin{cases} \alpha_{q^j L}(x^{\sigma^j}) = \alpha_L(x) > \max\{C, 1\}\eta^{-L}; \\ \alpha_N(x^{\sigma^j}) < \alpha_{q^j L}(x^{\sigma^j}) & \text{if } N < q^j L; \\ \alpha_{q^j L}(x^{\sigma^j})\eta^{q^j L - N} > \alpha_N(x^{\sigma^j}) & \text{if } q^j L < N < 0 \end{cases} \quad (2.2.5)$$

for all $0 \leq j \leq r$ inductively. If $j = 0$, we have nothing to prove by the assumption on L . We put $y = \sum y_n T^n = x^{\sigma^{j-1}}$. We observe that $|y|_g = |x|_g \leq 1$. For any integer N we have

$$\alpha_N(y_n^\sigma T^{qn} u^n) = \alpha_{N-qn}(y_n u^n) \begin{cases} < \alpha_n(y) & \text{if } N > qn; \\ \leq \alpha_n(y) & \text{if } N = qn; \\ \leq \alpha_n(y)\eta^{-N+qn} & \text{if } N < qn \end{cases} \quad (2.2.6)$$

by the assumption on u and (2.2.1) for each n . In the case $N = q^j L$ we get

$$\alpha_{q^j L}(y_{q^j-1L}^\sigma T^{q^j L} u^{q^{j-1}L}) = \alpha_0(y_{q^j-1L} u^{q^{j-1}L}) = \alpha_{q^j-1L}(y)$$

and

$$\alpha_{q^j L}(y_n^\sigma T^{qn} u^n) < \begin{cases} \alpha_{q^j-1L}(y) & \text{if } n < q^{j-1}L; \\ \alpha_{q^j-1L}(y)\eta^{(q^{j-1}L-n)-(q^jL-qn)} & \text{if } q^{j-1}L < n < 0; \\ \eta^{-q^jL} & \text{if } n > 0 \end{cases}$$

by (2.2.6) and the assumption on the induction. Thus we obtain

$$\alpha_{q^j L}(y^\sigma) = \alpha_{q^j-1L}(y) = \alpha_L(x).$$

from (2.2.1). In the case $N < q^j L$ we get

$$\alpha_N(y_n^\sigma T^{qn} u^n) < \begin{cases} \alpha_{q^j-1L}(y) & \text{if } qn \leq N; \\ \alpha_{q^j-1L}(y) & \text{if } N < qn \leq q^j L; \\ \alpha_{q^j-1L}(y)\eta^{(q^{j-1}L-n)-(N-qn)} & \text{if } q^j L < qn < 0; \\ \eta^{-N} < \eta^{-q^j L} & \text{if } qn \geq 0 \end{cases}$$

by (2.2.6) and the assumption on the induction. Since

$$(q^{j-1}L - n) - (N - qn) > (q - 1)(n - q^{j-1}L) > 0$$

for $q^j L < qn < 0$, we have

$$\alpha_N(y^\sigma) < \alpha_{q^j L}(y^\sigma)$$

from (2.2.1). In the case $q^j L < N < 0$ we get

$$\alpha_N(y_n^\sigma T^{qn} u^n) < \begin{cases} \alpha_{q^j-1L}(y) & \text{if } qn \leq q^j L; \\ \alpha_{q^j-1L}(y)\eta^{q^{j-1}L-n} & \text{if } q^j L < qn \leq N; \\ \alpha_{q^j-1L}(y)\eta^{q^{j-1}L-n-(N-qn)} & \text{if } N < qn < 0; \\ \eta^{(q^j-1)L-N} & \text{if } qn \geq 0. \end{cases}$$

by (2.2.6) and the assumption on the induction. Since $q^j L - N \leq q^j L - qn < q^{j-1}L - n < 0$ for $q^j L < qn \leq N$ and $q^j - N < (q^{j-1}L - n) - (N - qn)$ for $N < qn < 0$, we have

$$\alpha_{q^j L}(y^\sigma)\eta^{q^j L-N} > \alpha_N(y^\sigma)$$

by (2.2.1).

Now we calculate $\alpha_{q^r L}$ of each term in the left-hand side of (2.2.3) using (2.2.5). We have

$$\alpha_{q^r L}(x^{\sigma^r}) = \alpha_L(x)$$

and

$$\alpha_{q^r L}(a_i x^{\sigma^{r-i}}) \leq \sup\{\alpha_M(a_i)\alpha_N(x^{\sigma^{r-i}}) \mid M + N = q^r L\} \quad \text{for } 1 \leq i \leq r.$$

If $M \leq (q^r - q^{r-i})L$ in the last inequality, then $\alpha_M(a_i)\alpha_N(x^{\sigma^{r-i}}) \leq C\eta^{-(q^r - q^{r-i})L} \leq C\eta^{-L} < \alpha_L(x)$ for $|x^{\sigma^{r-i}}|_g = |x|_g \leq 1$. If $M > (q^r - q^{r-i})L$ in the last inequality, then $N < q^{r-i}L$ and $\alpha_M(a_i)\alpha_N(x^{\sigma^{r-i}}) < \alpha_L(x)$ for $|a_i|_g \leq 1$. So we obtain

$$\alpha_{q^r L}(a_i x^{\sigma^{r-i}}) < \alpha_L(x)$$

for all $1 \leq i \leq r$. This contradicts (2.2.3). □

(2.2.7) REMARK. The assertion of (2.2.2) does not always hold if $|a_i|_g > 1$ for some i . For example, we consider the case that $T^\sigma = T^q$. Consider the element

$$x = \sum_{i=0}^{\infty} \pi^i T^{-q^i} \notin \mathcal{E}^\dagger.$$

Then x satisfies the relation

$$\pi x^{\sigma^2} - (1 + \pi T^{1-q})x^\sigma + T^{1-q}x = 0.$$

3. Overconvergent étale φ - ∇ -spaces on $\text{Spec } k((t))$

We keep the notation as in Section 2. In this section we define overconvergent étale Λ - φ - ∇ -spaces on $\text{Spec } k((t))$ and show some properties of them which we will use.

(3.1) We fix a lifting $O_{\mathcal{E}}$ of $k((t))$ in characteristic 0 and a lifting σ on \mathcal{E} of q th power map on $k((t))$ as in the previous section. Now we define some notions.

DEFINITION (1) A Λ - φ -module (resp. an overconvergent Λ - φ -module) on $\text{Spec } k((t))$ is a free $O_{\mathcal{E}}$ (resp. $O_{\mathcal{E}}^\dagger$)-module M of finite rank with a σ -linear endomorphism $\varphi : M \rightarrow M$, which we call Frobenius, satisfying the following condition;

$$\begin{aligned} \varphi(M) \text{ spans } \mathcal{E} \otimes_{O_{\mathcal{E}}} M \text{ (resp. } \mathcal{E}^\dagger \otimes_{O_{\mathcal{E}}^\dagger} M) \text{ as an} \\ \mathcal{E} \text{ (resp. an } \mathcal{E}^\dagger) \text{-vector space.} \end{aligned} \tag{3.1.1}$$

(2) A Λ - φ - ∇ -module (resp. an overconvergent Λ - φ - ∇ -module) on $\text{Spec } k((t))$ is a Λ - φ -module (M, φ) (resp. an overconvergent Λ - φ -module (M, φ)) with an additive endomorphism $\nabla : M \rightarrow M$, which we call a connection, satisfying the following conditions;

$$\nabla(ax) = D(a)x + a\nabla(x) \text{ for all } a \in O_{\mathcal{E}} \text{ (resp. } a \in O_{\mathcal{E}}^{\dagger}) \text{ and } x \in M; \quad (3.1.2)$$

$$\nabla \circ \varphi = \frac{D(T^{\sigma})}{T^{\sigma}} \varphi \circ \nabla. \quad (3.1.3)$$

(3) A Λ - φ -space (resp. an overconvergent Λ - φ -space, a Λ - φ - ∇ -space, an overconvergent Λ - φ - ∇ -space) on $\text{Spec } k((t))$ is an \mathcal{E} (resp. $\mathcal{E}^{\dagger}, \mathcal{E}, \mathcal{E}^{\dagger}$)-vector space E of finite dimension with φ (resp. φ, φ and ∇, φ and ∇) which satisfy the above conditions (3.1.1)–(3.1.3) when we replace M and $O_{\mathcal{E}}$ with E and \mathcal{E} (resp. $\mathcal{E}^{\dagger}, \mathcal{E}, \mathcal{E}^{\dagger}$), respectively.

(4) A morphism of Λ - φ -modules (resp. \dots) is an $O_{\mathcal{E}}$ (resp. \dots)-linear homomorphism which commutes with all additional structures.

In the rest of this paper we use the terminology φ -module (resp. \dots) instead of Λ - φ -module (resp. \dots) for simplicity.

REMARK. (1) In [3] Fontaine defined φ -modules and our φ -modules (resp. overconvergent φ -modules) are $O_{\mathcal{E}}$ - φ -modules (resp. $O_{\mathcal{E}}^{\dagger}$ - φ -modules) of his definition. In the etale case Fontaine’s φ -module has a natural connection and the category of etale φ -modules on $\text{Spec } k((t))$ which is defined by Fontaine coincides with our category of etale φ - ∇ -modules. (See (3.2) and (3.3))

(2) We omit the conditions of convergence for a connection in the definition of φ -modules (resp. \dots). In the etale case the connection satisfies the condition of topological quasi-nilpotence from (3.1.3), so the category of unit-root F -isocrystals on $\text{Spec } k((t))$ (see [5] and [1]) is equivalent to the category of etale φ - ∇ -spaces.

We define tensor products and duals of φ -modules (resp. \dots) as follows.

Let $(E_1, \varphi_1, \nabla_1)$ and $(E_2, \varphi_2, \nabla_2)$ be φ - ∇ -spaces. We define their tensor product (E, φ, ∇) by $E = E_1 \otimes_{\mathcal{E}} E_2, \varphi = \varphi_1 \otimes \varphi_2$ and $\nabla = \nabla_1 \otimes 1 + 1 \otimes \nabla_2$.

For a φ - ∇ -space (E, φ, ∇) , we define the dual E^{\vee} of E by $E^{\vee} = \text{Hom}_{\mathcal{E}}(E, \mathcal{E})$, $\varphi^{\vee}(f) = (1 \otimes \sigma) \circ \sigma^* f \circ (1 \otimes \varphi)^{-1}$ and $\nabla^{\vee}(f)(x) = D(f(x)) - f(\nabla(x))$ for $f \in E^{\vee}$ and $x \in E$, where $1 \otimes \varphi : \sigma^* E \rightarrow E$ is the isomorphism which is induced by φ . Here $\sigma^* E$ is the scalar extension of E by $\sigma : \mathcal{E} \rightarrow \mathcal{E}$.

The other cases are same as in the above definition.

There exists a natural functor ν^* from the category of overconvergent φ -modules (resp. \dots) to that of φ -modules (resp. \dots) which is defined by the scalar extension $O_{\mathcal{E}}^{\dagger} \rightarrow O_{\mathcal{E}}$ (resp. \dots). We can easily see that the functor ν^* commutes with taking a tensor product and dual.

We show that there exists a cyclic vector in a φ -space (resp. in an overconvergent φ -space). (c.f. [7](3.3))

(3.1.4) LEMMA. *Let E be an \mathcal{E} (resp. an \mathcal{E}^\dagger)-vector space of finite dimension with a σ -linear endomorphism $\varphi : E \rightarrow E$ such that $\varphi(E)$ spans E as an \mathcal{E} (resp. an \mathcal{E}^\dagger)-vector space. Then there exists a cyclic vector $e \in E$, that is, there exists an element $e \in E$ such that $e, \varphi(e), \dots, \varphi^{r-1}(e)$ is a basis of E where $r = \dim E$.*

Proof. We will show the case of \mathcal{E} -vector spaces. Set

$$s = \max\{s(x) \mid x, \varphi(x), \dots, \varphi^{s(x)-1}(x) \text{ are linearly independent over } \mathcal{E} \text{ in } E\}.$$

If $s = r$, then there is nothing to prove. Assume $s < r$ and $x, \varphi(x), \dots, \varphi^{s-1}(x)$ are linearly independent over \mathcal{E} in E . As $\varphi(E)$ spans E over \mathcal{E} , there exists an element $y \in E$ such that $x, \varphi(x), \dots, \varphi^{s-1}(x)$ and $\varphi^s(y)$ are linearly independent over \mathcal{E} . By the assumption on s , we have

$$(x + ay) \wedge \varphi(x + ay) \wedge \dots \wedge \varphi^s(x + ay) = 0 \text{ in } \bigwedge^{s+1} E$$

for all $a \in \mathcal{E}$. Since there exist sufficiently many elements in \mathcal{E} (for example we may choose $1, T, \dots$, for a), we have

$$x \wedge \varphi(x) \wedge \dots \wedge \varphi^{s-1}(x) \wedge \varphi^s(y) = 0.$$

This contradicts the choice of y . The case of \mathcal{E}^\dagger -vector spaces is similar. □

(3.2) Let $k((t))^{\text{alg}}$ be an algebraic closure of $k((t))$ and let $W(k((t))^{\text{alg}})$ be the ring of Witt vectors with $k((t))^{\text{alg}}$ -coefficients. We can embed $O_{\mathcal{E}}$ into $\Lambda \otimes_{\Lambda_0} W(k((t))^{\text{alg}})$ such that the frobenius σ on $O_{\mathcal{E}}$ commutes with the endomorphism $1 \otimes \text{frob}^f$ on $\Lambda \otimes_{\Lambda_0} W(k((t))^{\text{alg}})$, where frob is the usual frobenius on $W(k((t))^{\text{alg}})$. We denote by $\tilde{\mathcal{E}}$ the fraction field of $\Lambda \otimes_{\Lambda_0} W(k((t))^{\text{alg}})$ and we regard \mathcal{E} as a subfield of $\tilde{\mathcal{E}}$.

Let E be a φ -space (resp. an overconvergent φ -space) on $\text{Spec } k((t))$. Then $\tilde{\mathcal{E}} \otimes E$ is naturally an F -space on $\text{Spec } k((t))^{\text{alg}}$. By the classification theorem of F -spaces on algebraically closed field, the F -space $\tilde{\mathcal{E}} \otimes_{\mathcal{E}} E$ is determined by its slopes. (See [2] Chapter IV, for example, and we can generalize the classification theorem for our Λ - F -spaces on algebraically closed field) We define slopes of a φ -space E (resp. an overconvergent φ -space) on $\text{Spec } k((t))$ by the slopes of $\tilde{\mathcal{E}} \otimes E$ as an F -space on $\text{Spec } k((t))^{\text{alg}}$.

Now we define an etale φ -space (resp. an overconvergent φ -space). A φ -space (resp. an overconvergent φ -space) on $\text{Spec } k((t))$ is etale if and only if all its slopes are 0. A φ -module (resp. an overconvergent φ -module) is etale if and only if all its slopes are 0 after inverting p . Equivalently, a φ -module M (resp. an overconvergent

φ -module M) is etale if and only if the morphism $1 \otimes \varphi : \sigma^*M \rightarrow M$, which is induced by φ , is an isomorphism. By the theory of slopes, tensor products and dual of etale objects are also etale.

According to the classification theorem of F -spaces on algebraically closed field, we characterize etale φ -spaces (resp. overconvergent etale φ -spaces) using a cyclic vector;

(3.2.1) PROPOSITION. *Let E be a φ -space (resp. an overconvergent φ -space) on $\text{Spec } k((t))$ and let e be a cyclic vector of E which satisfies the relation*

$$\varphi^r(e) + a_1\varphi^{r-1}(e) + \dots + a_r e = 0 \quad a_i \in \mathcal{E} \text{ (resp. } \mathcal{E}^\dagger). \tag{3.2.2}$$

Then E is etale if and only if $|a_i|_g \leq 1$ for all $1 \leq i \leq r - 1$ and $|a_r|_g = 1$.

(3.2.3) REMARK. The Newton polygone of (3.2.2) gives slopes of φ -spaces E . Therefore all slopes of E are greater than or equal to 0 if and only if $|a_i|_g \leq 1$ for all i .

(3.3) Now we discuss the relation between the Frobenius and the connection.

In the case of an etale φ - ∇ -space E (resp. an overconvergent etale φ - ∇ -space E) over $\text{Spec } k((t))$ the Frobenius φ determines the connection ∇ and there exists a φ - ∇ -lattice of E , that is, an $O_{\mathcal{E}}$ (resp. $O_{\mathcal{E}}^\dagger$)-submodule which spans E over \mathcal{E} (resp. \mathcal{E}^\dagger) and which is stable under the Frobenius φ and the connection ∇ . By (3.1.4) and (3.2.1) there exists a basis e_1, e_2, \dots, e_r ($r = \dim E$) and a matrix $A \in \text{GL}_r(O_{\mathcal{E}})$ (resp. a matrix $A \in \text{GL}_r(O_{\mathcal{E}}^\dagger)$) such that

$$\varphi(e_1, \dots, e_r) = (e_1, \dots, e_r)A.$$

We set a matrix C of degree r with \mathcal{E} (resp. \mathcal{E}^\dagger)-coefficients by

$$\nabla(e_1, \dots, e_r) = (e_1, \dots, e_r)C.$$

From the condition (3.1.3) we have the following relation;

$$A^D + CA = \frac{D(T^\sigma)}{T^\sigma} AC^\sigma,$$

where $A^D = (D(a_{ij}))$ and $C^\sigma = (c_{ij}^\sigma)$ for $A = (a_{ij})$ and $C = (c_{ij})$. For a matrix $X \in \text{M}_r(\mathcal{E})$ we define $\psi(X) = \frac{D(T^\sigma)}{T^\sigma} AX^\sigma A^{-1}$. As $|\frac{D(T^\sigma)}{T^\sigma}|_g < 1$, ψ is a contraction operator on $\text{M}_r(\mathcal{E})$ for p -adic topology and we have

$$C = -(1 - \psi)^{-1}(A^D A^{-1}). \tag{3.3.1}$$

So C is uniquely determined by A and all the coefficients of C are contained in $O_{\mathcal{E}}$ (resp. $O_{\mathcal{E}}^\dagger$).

(3.3.2) THEOREM. *The category of etale φ - ∇ -modules (resp. overconvergent etale φ - ∇ -modules, etale φ - ∇ -spaces, overconvergent etale φ - ∇ -spaces) is naturally a full subcategory of that of etale φ -modules (resp. overconvergent etale φ -modules, etale φ -spaces, overconvergent etale φ -spaces) by the forgetful functor. Moreover the category of etale φ - ∇ -modules (resp. etale φ - ∇ -spaces) on $\text{Spec } k((t))$ is equivalent to that of etale φ -modules (resp. etale φ -spaces).*

In the overconvergent case, on the contrary, the category of overconvergent φ -spaces is not equivalent to that of overconvergent φ - ∇ -spaces. Because all coefficients of C are not always contained in \mathcal{E}^\dagger when we determine the connection by the relation (3.3.1).

EXAMPLE. Fix a frobenius σ by $T^\sigma = T^p$. Let $a = 1 - \frac{\pi}{T} \in O_{\mathcal{E}}^\dagger$. Then we have $c = -(1 - p\sigma)^{-1}(a^{-1}D(a)) \notin O_{\mathcal{E}}^\dagger$. Define a φ - ∇ -space E on $\text{Spec } k((t))$ of rank one by $\varphi(e) = ae$ and $\nabla(e) = ce$, where e is a basis of E . So the coefficients of the frobenius structure of E is an overconvergent, but the coefficients of the connection are not contained in \mathcal{E}^\dagger . Moreover the p -adic representation of $\text{Gal}(k((t))^{\text{sep}}/k((t)))$ which corresponds to E in the sense of Fontaine [3] is not of finite monodromy. [1]

REMARK. In general, the connection of φ - ∇ -spaces is not determined by its Frobenius structures. Because the operator ψ is not a contraction in this case.

4. Full faithfulness

We keep the same notation as in the previous section. In this section we will show the local version that the natural functors

$$\begin{aligned} & \left(\begin{array}{c} \text{overconvergent etale} \\ \varphi\text{-spaces on } \text{Spec } k((t)) \end{array} \right) \xrightarrow{\nu^*} \left(\begin{array}{c} \text{etale} \\ \varphi\text{-spaces on } \text{Spec } k((t)) \end{array} \right) \\ & \left(\begin{array}{c} \text{overconvergent etale} \\ \varphi\text{-}\nabla\text{-spaces on } \text{Spec } k((t)) \end{array} \right) \xrightarrow{\nu^*} \left(\begin{array}{c} \text{etale} \\ \varphi\text{-}\nabla\text{-spaces on } \text{Spec } k((t)) \end{array} \right) \end{aligned}$$

defined in (3.1) are fully faithful.

(4.1) First we consider only frobenius structures.

Let r be a positive integer and let A be an $r \times r$ invertible matrix with $\tilde{\mathcal{E}}$ -coefficients. We define a Λ - φ -space E_A on $\text{Spec } k((t))^{\text{alg}}$ associated to A by its frobenius linear morphism

$$\varphi((e_1, e_2, \dots, e_r)) = (e_1, e_2, \dots, e_r)A,$$

where e_1, e_2, \dots, e_r is a basis of E_A on $\tilde{\mathcal{E}}$. We say that A is etale if and only if all slopes of E_A are 0.

(4.1.1) PROPOSITION. *Let A be an invertible matrix of degree r with \mathcal{E}^\dagger -coefficients such that A is etale. Assume that $\mathbf{x} = (x_1, x_2, \dots, x_r) \in \mathcal{E}^r$ satisfies the relation*

$$A^t \mathbf{x}^\sigma = {}^t \mathbf{x}.$$

Then \mathbf{x} is contained in $(\mathcal{E}^\dagger)^r$.

Proof. Let L be an \mathcal{E}^\dagger -subvector space of \mathcal{E} which is generated by x_1, x_2, \dots, x_r . Then L is stable under the Frobenius σ on \mathcal{E} . So we can regard L as an overconvergent φ -space. By the construction, there is a natural surjection from E_A^\vee to L as overconvergent φ -spaces. So all slopes of L are 0 by the slope theory of φ -spaces. From (3.1.4) and (3.2.1), there exists a cyclic vector $y \in L$ which satisfies the relation

$$y^{\sigma^s} + a_1 y^{\sigma^{s-1}} + \dots + a_{s-1} y^\sigma + a_s y = 0$$

such that $|a_i|_g \leq 1$ for all $1 \leq i \leq s - 1$ and $|a_s|_g = 1$. Here $s = \dim_{\mathcal{E}^\dagger} L$. So y contains in \mathcal{E}^\dagger and $s = 1$ from (2.2.2). Therefore x_i 's are included in \mathcal{E}^\dagger . □

(4.1.2) REMARK. The assertion of (4.1.1) is also true if all slopes of E_A are less than or equal to 0 by the remark (3.2.3).

(4.1.3) THEOREM. *The natural functor ν^* from the category of overconvergent etale φ -spaces on $\text{Spec } k((t))$ to that of etale φ -spaces on $\text{Spec } k((t))$ is fully faithful.*

Proof. Let \mathcal{C} be either the category of overconvergent etale φ -spaces or the category of etale φ -spaces. Define

$$H_{\mathcal{C}}^0(E) = \{y \in E \mid \varphi(y) = y\}$$

for any object E of \mathcal{C} . Then one can easily see that

$$\text{Hom}_{\mathcal{C}}(E_1, E_2) = H_{\mathcal{C}}^0(E_1^\vee \otimes E_2).$$

So we have to show the natural map

$$H_{(\text{over.}\varphi\text{-sp})}^0(E) \rightarrow H_{(\varphi\text{-sp})}^0(\nu^* E)$$

is an isomorphism for any overconvergent etale φ -space E . The injectivity is trivial. We show the surjectivity. Let e_1, e_2, \dots, e_r be a basis of E on \mathcal{E}^\dagger and define $A \in \text{GL}_r(\mathcal{E}^\dagger)$ by $\varphi(e) = eA$ where $e = (e_1, e_2, \dots, e_r)$. If $y = e^t \mathbf{x} = e^t(x_1, x_2, \dots, x_r) \in H_{(\varphi\text{-sp})}^0(\nu^* E)$, then \mathbf{x} satisfies the relation

$$A^t \mathbf{x}^\sigma = {}^t \mathbf{x}.$$

From (4.1.1) \mathbf{x} is contained in $(\mathcal{E}^\dagger)^r$. Hence $y \in H_{(\text{over.}\varphi\text{-sp})}^0(E)$. □

REMARK. In general case, not restricting to etale case, the full faithfulness of the natural functor from the category of overconvergent φ -spaces to that of φ -spaces does not hold by (2.2.7).

(4.2) Since the category of etale φ - ∇ -spaces (resp. overconvergent etale φ - ∇ -spaces) on $\text{Spec } k((t))$ is a full subcategory of etale φ -spaces (resp. overconvergent etale φ -spaces) on $\text{Spec } k((t))$ (3.3.2), the theorem below follows from (4.1.3).

(4.2.1) THEOREM. *The natural functor ν^* from the category of overconvergent etale φ - ∇ -spaces on $\text{Spec } k((t))$ to that of etale φ - ∇ -spaces on $\text{Spec } k((t))$ is fully faithful.*

(4.3) Now we consider lattices. As $M = (\mathcal{E}^\dagger \otimes_{O_\mathcal{E}^\dagger} M) \cap (O_\mathcal{E} \otimes_{O_\mathcal{E}^\dagger} M)$ for a free $O_\mathcal{E}^\dagger$ -module M , we have

(4.3.1) THEOREM. *The natural functors*

$$\left(\begin{array}{c} \text{overconvergent etale} \\ \varphi\text{-modules on } \text{Spec } k((t)) \end{array} \right) \longrightarrow \left(\begin{array}{c} \text{etale} \\ \varphi\text{-modules on } \text{Spec } k((t)) \end{array} \right)$$

$$\left(\begin{array}{c} \text{overconvergent etale} \\ \varphi\text{-}\nabla\text{-modules on } \text{Spec } k((t)) \end{array} \right) \longrightarrow \left(\begin{array}{c} \text{etale} \\ \varphi\text{-}\nabla\text{-modules on } \text{Spec } k((t)) \end{array} \right)$$

are fully faithful.

5. Application to the case of curves

(5.1) Let X be a smooth curve over k , where k is a perfect field of positive characteristic. Let \overline{X} and D be the smooth compactification of X and $\overline{X} - X$, respectively. We apply (4.2.1) and (4.3.1) to the unit-root F -isocrystals on X . (See [1])

(5.1.1) THEOREM. *The natural functors*

$$\left(\begin{array}{c} \text{overconvergent unit-root} \\ \Lambda\text{-}F\text{-isocrystals on } X \text{ around } D \end{array} \right) \longrightarrow \left(\begin{array}{c} \text{unit-root} \\ \Lambda\text{-}F\text{-isocrystals on } X \end{array} \right)$$

$$\left(\begin{array}{c} \text{overconvergent unit-root} \\ \Lambda\text{-}F\text{-crystals on } X \text{ around } D \end{array} \right) \longrightarrow \left(\begin{array}{c} \text{unit-root} \\ \Lambda\text{-}F\text{-crystals on } X \end{array} \right)$$

are fully faithful.

Proof. By the argument of Crew in [1](4.6)–(4.10) we can reduce (5.1.1) to the local case. Therefore (5.1.1) follows from (4.2.1) and (4.3.1). \square

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