

COMPOSITIO MATHEMATICA

SOO TECK LEE

Degenerate principal series representations of $Sp(2n, \mathbf{R})$

Compositio Mathematica, tome 103, n° 2 (1996), p. 123-151

<http://www.numdam.org/item?id=CM_1996__103_2_123_0>

© Foundation Compositio Mathematica, 1996, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

Degenerate principal series representations of $\mathrm{Sp}(2n, \mathbf{R})$

SOO TECK LEE

Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511, Republic of Singapore.

Received 13 July 1994; accepted 9 May 1995

Abstract. We study the infinitesimal action of $\mathfrak{sp}(2n, \mathbf{R})$ on the degenerate principal series representations of $\mathrm{Sp}(2n, \mathbf{R})$ associated with a maximal parabolic subgroup. We then deduce the module structure and unitarity of these representations.

Key words: Complementary series, degenerate principal series representations, socle series, unitary representations.

1. Introduction

Bargmann's calculation of the infinitesimal action of $\mathfrak{sl}(2, \mathbf{R})$ on the principal series representations of $\mathrm{SL}(2, \mathbf{R})$ ([2]) is probably the most straightforward example in the study of infinite dimensional representations of semisimple Lie groups. However his ideas have not been extended to more complicated groups until recently when Howe and Tan ([8]) apply them to study some degenerate principal series representations of $\mathrm{O}(p, q)$, $\mathrm{U}(p, q)$ and $\mathrm{Sp}(p, q)$. Bargmann's ideas are also applicable to the study of another degenerate series of $\mathrm{U}(n, n)$. This has been done in the author's dissertation ([11]). The calculations involved in these examples are elementary and the results show that the enveloping algebra transforms the K -types according to some very simple scalar expressions involving the parameters of the representation spaces. With this information the module structure and unitarity of the representations become transparent. Moreover these calculations require no special technology. It is therefore desirable to extend this technique as widely as possible. In this paper, we shall use this method to study a degenerate series of $\mathrm{Sp}(2n, \mathbf{R})$. The degenerate series of $\mathrm{GL}(n, \mathbf{R})$ and $\mathrm{GL}(n, \mathbf{C})$ will be studied in an upcoming joint paper with R. Howe ([7]).

We shall study the following degenerate series representations of $\mathrm{Sp}(2n, \mathbf{R})$. Let P be the maximal parabolic subgroup of $\mathrm{Sp}(2n, \mathbf{R})$ with a Levi decomposition $P = MN$ where

$$M = \left\{ \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix} : a \in \mathrm{GL}(n, \mathbf{R}) \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in M(n, \mathbf{R}), b = b^t \right\}.$$

For convenience, we shall denote the elements $\begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ in P by m_a and n_b respectively. For each $\sigma \in \mathbf{C}$, we let $\chi_\sigma^\pm : P \rightarrow \mathbf{C}^\times$ be the characters given by

$$\chi_\sigma^\pm(m_a n_b) = \begin{cases} (\det a)^\sigma & \text{if } \det a > 0, \\ \pm |\det a|^\sigma & \text{if } \det a < 0. \end{cases}$$

We shall study the corresponding induced representations $I^\pm(\sigma)$. The representation spaces for $I^\pm(\sigma)$ are respectively

$$\{f \in C^\infty(\mathrm{Sp}(2n, \mathbf{R})) : f(pg) = \delta(p)^{1/2} \chi_\sigma^\pm(p) f(g), g \in \mathrm{Sp}(2n, \mathbf{R}), p \in P\}$$

(δ is the modular function of P) and on which $\mathrm{Sp}(2n, \mathbf{R})$ acts by right translation, i.e.

$$(g.f)(h) = f(hg) \quad (g, h \in \mathrm{Sp}(2n, \mathbf{R})).$$

We shall briefly describe our methods. We first identify the representations $I^\pm(\sigma)$ with some function spaces $\mathcal{S}^{\alpha, \pm}(X^\sigma)$. Let $K \cong \mathrm{U}(n)$ be a maximal compact subgroup of $\mathrm{Sp}(2n, \mathbf{R})$ and let \mathfrak{k} be its Lie algebra. Let $\mathfrak{sp}(2n, \mathbf{R}) = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. For a K -type V_μ of $\mathcal{S}^{\alpha, \pm}(X^\sigma)$, we shall calculate explicitly the images of the K highest weight vectors in $V_\mu \otimes \mathfrak{p}_\mathbf{C}$ under the map $v \otimes p \rightarrow p.v$ ($p \in \mathfrak{p}_\mathbf{C}, v \in V_\mu$). With this information, we are able to determine (1) the reducibility of $I^\pm(\sigma)$, (2) the complementary series, (3) all the irreducible constituents of $I^\pm(\sigma)$ when it is reducible and determine which of them are unitarizable, and (4) the socle series and the module diagrams of $I^\pm(\sigma)$.

THEOREM 4.3. *If n is even and $-\frac{1}{2} < \sigma < \frac{1}{2}$, then $I^\pm(\sigma)$ is unitarizable.*

The irreducible constituents and the socle series of $I^+(\sigma)$ at the points of reducibility are described in Theorems 5.2, 5.4, 5.5 and 5.6. The module diagrams of $I^+(\sigma)$ for several typical cases are given in Fig. 6, 7, 12 and 13. These diagrams greatly enhance our understanding of the general results.

We are hardly the first to study these representations and some of our results are already in the literature. Kudla and Rallis have studied the degenerate series of the metaplectic group in [10] in relation to local theta correspondence. These representations are also among the examples studied by Johnson ([9]), Sahi ([12],[13]) and Zhang ([16]). Our methods are elementary and are considerably different from all of the above. In particular both Sahi and Zhang use Jordan algebra techniques, but our methods do not depend on the Jordan algebra structure associated to such spaces. They are thus potentially more general. For instance, they are applicable to the degenerate series of $\mathrm{GL}(n, \mathbf{R})$ and $\mathrm{GL}(n, \mathbf{C})$ (c.f. [7]).

This paper is arranged as follows. In section 2, we shall identify each of these representations with a function space $\mathcal{S}^{\alpha,+}(X^o)$ or $\mathcal{S}^{\alpha,-}(X^o)$. We then give an explicit description for the highest weight vectors in the K -types of these modules. In section 3, we shall determine how the enveloping algebra of $\mathfrak{sp}(2n, \mathbf{C})$ transforms these highest weight vectors in the K -types. In section 4, we discuss the reducibility of $I^\pm(\sigma)$ and determine the complementary series. Finally in section 5, we describe the subquotients of $I^+(\sigma)$ and determine which of them are unitarizable. Module diagrams for several typical cases are also given.

2. The modules $\mathcal{S}^{\alpha,\pm}(X^o)$ and their K -types

In this section we shall first identify the representations $I^\pm(\sigma)$ with the function spaces $\mathcal{S}^{\alpha,\pm}(X^o)$ in which the action by the Lie algebra of $\text{Sp}(2n, \mathbf{R})$ can be explicitly described. We then decompose $\mathcal{S}^{\alpha,\pm}(X^o)$ into a sum of K -types and give an explicit description of a highest weight vector in each K -type. As $\text{Sp}(2, \mathbf{R})$ is just $\text{SL}(2, \mathbf{R})$, we shall assume that $n \geq 2$ throughout this paper.

Let $M_{2n,n}(\mathbf{R})$ be the space of all $2n \times n$ real matrices. We consider the action of $\text{Sp}(2n, \mathbf{R})$ on $M_{2n,n}(\mathbf{R})$ given by

$$\tau(g)x = (g^{-1})^t x, \quad g \in \text{Sp}(2n, \mathbf{R}), x \in M_{2n,n}(\mathbf{R}). \tag{2.1}$$

Let 0_n and I_n be the $n \times n$ zero matrix and $n \times n$ identity matrix respectively. Let $x_o = \begin{pmatrix} 0_n \\ I_n \end{pmatrix}$ and let X^o be the $\text{Sp}(2n, \mathbf{R})$ -orbit of x_o in $M_{2n,n}(\mathbf{R})$. If we regard x_o as a map from \mathbf{R}^n to \mathbf{R}^{2n} , then its range is totally isotropic with respect to the standard symplectic form defining $\text{Sp}(2n, \mathbf{R})$. Hence each element of X^o has the property that its range is totally isotropic with respect to this form.

For each $\alpha \in \mathbf{C}$, we consider the function spaces

$$\begin{aligned} \mathcal{S}^{\alpha,\pm}(X^o) &= \{f \in C^\infty(X^o) : \\ f(xa) &= \begin{cases} (\det a)^\alpha f(x) & \text{if } \det a > 0, \\ \pm |\det a|^\alpha f(x) & \text{if } \det a < 0, \end{cases} \quad \forall x \in X^o, a \in \text{GL}(n, \mathbf{R}) \} \end{aligned}$$

and let $\text{Sp}(2n, \mathbf{R})$ act on them by

$$(g.f)(x) = f[\tau(g^{-1})x] = f(g^t x), \quad x \in X^o, g \in \text{Sp}(2n, \mathbf{R}).$$

Let $\rho_n = \frac{n+1}{2}$. Then the modular function of P is given by

$$\delta(p) = |\det a|^{2\rho_n} = \chi_{2\rho_n}^+(p), \quad (p = m_a n_b \in P).$$

Now as $\text{Sp}(2n, \mathbf{R})$ -modules, we have

$$\mathcal{S}^{\alpha,\pm}(X^o) \cong I^\pm(-\alpha - \rho_n). \tag{2.2}$$

In fact, for each $f \in \mathcal{S}^{\alpha, \pm}(X^o)$, we set $\tilde{f} : \mathrm{Sp}(2n, \mathbf{R}) \rightarrow \mathbf{C}$, $\tilde{f}(g) = f(\tau(g^{-1})x_o)$. Then since $\tau(m_a)x_o = x_o a$ for $a \in \mathrm{GL}(n, \mathbf{R})$, one easily verifies that $\tilde{f} \in I^{\pm}(-\alpha - \rho_n)$. Hence the map $f \rightarrow \tilde{f}$ is an isomorphism for $\mathcal{S}^{\alpha, \pm}(X^o) \cong I^{\pm}(-\alpha - \rho_n)$.

We now let $\theta(g) = (\overline{g^{-1}})^t$, $g \in \mathrm{Sp}(2n, \mathbf{R})$. Then θ is a Cartan involution on $\mathrm{Sp}(2n, \mathbf{R})$ and

$$K = \mathrm{Sp}(2n, \mathbf{R})^\theta = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathrm{GL}(n, \mathbf{R}), ab^t = b^t a, aa^t + bb^t = I_n \right\}$$

is a maximal compact subgroup of $\mathrm{Sp}(2n, \mathbf{R})$. It is isomorphic to $U(n)$. In fact, the map $\phi : K \rightarrow U(n)$ given by

$$\phi \left[\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right] = a + ib$$

is an isomorphism. We also note that

$$P \cap K = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in O(n) \right\} \cong O(n)$$

and $\chi_{-\alpha}^+|_{P \cap K} = 1$, and that $\chi_{-\alpha}^-|_{P \cap K} = \det_{O(n)}$. Here 1 denotes the trivial character and $\det_{O(n)}$ is the determinant character of $O(n)$. It follows that as representations of K ,

$$\begin{aligned} \mathcal{S}^{\alpha, +}(X^o)|_K &\cong \mathrm{Ind}_{P \cap K}^K \chi_{-\alpha}^+|_{P \cap K} \cong \mathrm{Ind}_{O(n)}^{U(n)} 1, \\ \mathcal{S}^{\alpha, -}(X^o)|_K &\cong \mathrm{Ind}_{P \cap K}^K \chi_{-\alpha}^-|_{P \cap K} \cong \mathrm{Ind}_{O(n)}^{U(n)} \det_{O(n)}. \end{aligned}$$

Here ‘Ind’ denotes unnormalized induction. Now it is easy to deduce the K -structure of $\mathcal{S}^{\alpha, \pm}(X^o)$. Let Λ^+ denote the set of all dominant weights of $U(n)$. Λ^+ can be identified with the set of all n -tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ of integers satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ (see [14]). For $1 \leq j \leq n$, we let

$$e_j = \overbrace{(0, \dots, 0, 1, 0, \dots, 0)}^j. \tag{2.3}$$

Thus for $k \in \mathbf{Z}$ and $\lambda \in \Lambda^+$, $\lambda + ke_j = (\lambda_1, \dots, \lambda_{j-1}, \lambda_j + k, \lambda_{j+1}, \dots, \lambda_n)$. We shall also write $2\lambda = (2\lambda_1, \dots, 2\lambda_n)$ and $2\lambda + \mathbf{1} = (2\lambda_1 + 1, \dots, 2\lambda_n + 1)$. For $\lambda \in \Lambda^+$, V_λ shall denote a copy of the irreducible representation of $U(n)$ with highest weight λ . It is well known that ([3])

$$\mathrm{Ind}_{O(n)}^{U(n)} 1 \cong \sum_{\lambda \in \Lambda} V_{2\lambda}.$$

On the other hand, since $\mathrm{Ind}_{O(n)}^{U(n)} \det_{O(n)} \cong \det_{U(n)} \otimes \mathrm{Ind}_{O(n)}^{U(n)} 1$, we have

$$\mathrm{Ind}_{O(n)}^{U(n)} \det_{O(n)} \cong \sum_{\lambda \in \Lambda} V_{2\lambda + \mathbf{1}}.$$

We shall give an explicit description of the K highest weight vectors in the K -types later.

The Lie algebra of $\mathrm{Sp}(2n, \mathbf{R})$ is

$$\mathfrak{sp}(2n, \mathbf{R}) = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1^t \end{pmatrix} : x_1, x_2, x_3 \in \mathfrak{gl}(n, \mathbf{R}), x_2 = x_2^t, x_3 = x_3^t \right\}.$$

It has a Cartan decomposition

$$\mathfrak{sp}(2n, \mathbf{R}) = \mathfrak{p} + \mathfrak{k}$$

where

$$\mathfrak{p} = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} : \alpha, \beta \in \mathfrak{gl}_n(\mathbf{R}), \alpha = \alpha^t, \beta = \beta^t \right\}$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathfrak{gl}_n(\mathbf{R}), \alpha = -\alpha^t, \beta = \beta^t \right\}.$$

\mathfrak{k} is the Lie algebra of K . The map $\phi: \mathfrak{k} \rightarrow \mathfrak{u}(n)$ given by

$$\phi \left[\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \right] = \alpha + i\beta \quad (2.4)$$

is an isomorphism of Lie algebras and it extends naturally to an isomorphism $\phi: \mathfrak{k}_{\mathbf{C}} \rightarrow \mathfrak{gl}_n(\mathbf{C})$ for the complexified Lie algebras. For $1 \leq k, l \leq n$, let e_{kl} be the element of $\mathfrak{gl}_n(\mathbf{C})$ with 1 at its (k, l) position and 0 elsewhere. Then one can check that

$$\phi^{-1}(e_{kl}) = \begin{pmatrix} \frac{1}{2}(e_{kl} - e_{lk}) & -\frac{i}{2}(e_{kl} + e_{lk}) \\ \frac{i}{2}(e_{kl} + e_{lk}) & \frac{1}{2}(e_{kl} - e_{lk}) \end{pmatrix}.$$

Now \mathfrak{p} is invariant under the adjoint action by K . Thus its complexification

$$\mathfrak{p}_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{p} \cong \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} : \alpha, \beta \in \mathfrak{gl}_n(\mathbf{C}), \alpha = \alpha^t, \beta = \beta^t \right\}$$

is a K -module. Let

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} a & ia \\ ia & -a \end{pmatrix} : a \in \mathfrak{gl}_n(\mathbf{C}), a = a^t \right\},$$

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} a & -ia \\ -ia & -a \end{pmatrix} : a \in \mathfrak{gl}_n(\mathbf{C}), a = a^t \right\}.$$

Then \mathfrak{p}^+ and \mathfrak{p}^- are submodules of $\mathfrak{p}_{\mathbf{C}}$. Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the standard basis of \mathbf{C}^n and $\{\varepsilon_1^*, \dots, \varepsilon_n^*\}$ be the corresponding dual basis in \mathbf{C}^{n*} . For $1 \leq k, j \leq n$, let ε_{kj} and ε_{kj}^* denote the images of $\varepsilon_k \otimes \varepsilon_j$ and $\varepsilon_k^* \otimes \varepsilon_j^*$ under the canonical projections $\mathbf{C}^n \otimes \mathbf{C}^n \rightarrow S^2(\mathbf{C}^n)$ and $\mathbf{C}^{n*} \otimes \mathbf{C}^{n*} \rightarrow S^2(\mathbf{C}^{n*})$ respectively. We now observe

that as K -modules, $\mathfrak{p}^+ \cong S^2(\mathbf{C}^n)$ and $\mathfrak{p}^- \cong S^2(\mathbf{C}^{n*})$. In fact, the linear maps $\psi^+ : \mathfrak{p}^+ \rightarrow S^2(\mathbf{C}^n)$ and $\psi^- : \mathfrak{p}^- \rightarrow S^2(\mathbf{C}^{n*})$ given by

$$\begin{aligned} \psi^+ \left(\begin{array}{cc} \frac{1}{2}(e_{pl} + e_{lp}) & \frac{i}{2}(e_{pl} + e_{lp}) \\ \frac{i}{2}(e_{pl} + e_{lp}) & -\frac{1}{2}(e_{pl} + e_{lp}) \end{array} \right) &= \varepsilon_{pl}, \\ \psi^- \left(\begin{array}{cc} \frac{1}{2}(e_{pl} + e_{lp}) & -\frac{i}{2}(e_{pl} + e_{lp}) \\ -\frac{i}{2}(e_{pl} + e_{lp}) & -\frac{1}{2}(e_{pl} + e_{lp}) \end{array} \right) &= \varepsilon_{pl}^*, \end{aligned}$$

are K -module isomorphisms.

For $1 \leq p \leq 2n$ and $1 \leq q \leq n$, E_{pq} shall denote the matrix in $M_{2n,n}(\mathbf{R})$ with 1 at its (p, q) th entry and 0 elsewhere. For $1 \leq k, j \leq n$, let x_{kj} and y_{kj} be linear functional on $M_{2n,n}(\mathbf{R})$ specified by

$$x_{kj}(E_{pq}) = \begin{cases} 1 & \text{if } p = k, q = j \\ 0 & \text{otherwise,} \end{cases} \quad y_{kj}(E_{pq}) = \begin{cases} 1 & \text{if } p = n + k, q = j \\ 0 & \text{otherwise.} \end{cases}$$

We then set $z_{k,j} = x_{kj} + iy_{kj}$ and $\bar{z}_{k,j} = x_{kj} - iy_{kj}$. Hence we can identify a point $p \in M_{2n,n}(\mathbf{R})$ with a point $z = (z_{kj}) \in M_{n,n}(\mathbf{C})$ where $z_{jk} = x_{kj}(p) + iy_{kj}(p)$. In particular we can regard X^o as a subset of $M_{n,n}(\mathbf{C})$. We shall frequently make this identification without comment.

Now the action (2.1) induces an action of $\text{Sp}(2n, \mathbf{R})$ on the polynomial algebra $P(M_{2n,n}(\mathbf{C})) \cong P(\mathbf{C} \otimes M_{2n,n}(\mathbf{R}))$. Direct calculations show that $\varphi^{-1}(e_{kl})$, $(\psi^+)^{-1}(\varepsilon_{k,l})$ and $(\psi^-)^{-1}(\varepsilon_{k,l}^*)$ act on $P(M_{2n,n}(\mathbf{C}))$ by the following differential operators

$$\begin{aligned} \varphi^{-1}(e_{kl}) &= \sum_{j=1}^n \left(z_{kj} \frac{\partial}{\partial z_{lj}} - \bar{z}_{lj} \frac{\partial}{\partial \bar{z}_{kj}} \right), \\ (\psi^+)^{-1}(\varepsilon_{kl}) &= \sum_{j=1}^n \left(z_{kj} \frac{\partial}{\partial \bar{z}_{lj}} + z_{lj} \frac{\partial}{\partial \bar{z}_{kj}} \right), \\ (\psi^-)^{-1}(\varepsilon_{kl}^*) &= \sum_{j=1}^n \left(\bar{z}_{kj} \frac{\partial}{\partial z_{lj}} + \bar{z}_{lj} \frac{\partial}{\partial z_{kj}} \right). \end{aligned} \tag{2.5}$$

From now on we shall abuse notations and simply write e_{kl} , ε_{kl} and ε_{kl}^* for $\varphi^{-1}(e_{kl})$, $(\psi^+)^{-1}(\varepsilon_{kl})$ and $(\psi^-)^{-1}(\varepsilon_{kl}^*)$ respectively.

For each $1 \leq j \leq n - 1$, let

$$\gamma_j = \begin{vmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ z_{j1} & z_{j2} & \cdots & z_{jn} \\ \bar{z}_{(j+1)1} & \bar{z}_{(j+1)2} & \cdots & \bar{z}_{(j+1)n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \bar{z}_{n1} & \bar{z}_{n2} & \cdots & \bar{z}_{nn} \end{vmatrix}. \tag{2.6}$$

For convenience, we write $z_l = (z_{l1}, \dots, z_{ln})$, and $\bar{z}_l = (\bar{z}_{l1}, \dots, \bar{z}_{ln})$, so that $\gamma_j = \det(z_1, \dots, z_j, \bar{z}_{j+1}, \dots, \bar{z}_n)$. We also set $\gamma_0 = \det(\bar{z}_1, \dots, \bar{z}_n)$ and $\gamma_n = \det(z_1, \dots, z_n)$. It is clear from eq. (2.5) that each γ_j is a highest weight vector in $P(M_{2n,n}(\mathbf{C}))$ for K and it has weight

$$\left(\overbrace{1, \dots, 1}^j, \overbrace{-1, \dots, -1}^{n-j} \right).$$

Moreover each γ_j is an eigenvector for $\text{GL}(n, \mathbf{R})$: for $z \in M_{2n,n}(\mathbf{R})$ and $a \in \text{GL}(n, \mathbf{R})$ we have $\gamma_j(za) = (\det a)\gamma_j(z)$. Now the elements of $P(M_{n,n}(\mathbf{C}))$ can be regarded as complex functions on $M_{2n,n}(\mathbf{R})$. The assignment $f \rightarrow f|_{X^o}$ is clearly a K -module map from $P(M_{n,n}(\mathbf{C})) \rightarrow C^\infty(X^o)$. For $0 \leq j \leq n$, one can check that $\gamma_j|_{X^o} \neq 0$. Consequently, the restriction of each γ_j to X^o is also a highest weight vector in $C^\infty(X^o)$. For convenience, we shall also denote $\gamma_j|_{X^o}$ by γ_j .

The following lemma gives highest weight vectors in the K -types. We omit its proof as it can be verified directly. Note that the function $(\det z)(\det \bar{z})$ on X^o is an invariant for K .

LEMMA 2.1. *For each $\lambda \in \Lambda$, the functions on X^o*

$$\begin{aligned} \xi_{2\lambda} &= \gamma_1^{\lambda_1 - \lambda_2} \gamma_2^{\lambda_2 - \lambda_3} \dots \gamma_{n-1}^{\lambda_{n-1} - \lambda_n} (\det z)^{\lambda_n + \frac{\alpha}{2}} (\det \bar{z})^{\frac{\alpha}{2} - \lambda_1}, \\ \xi_{2\lambda+1} &= \gamma_1^{\lambda_1 - \lambda_2} \gamma_2^{\lambda_2 - \lambda_3} \dots \gamma_{n-1}^{\lambda_{n-1} - \lambda_n} (\det z)^{\lambda_n + \frac{\alpha+1}{2}} (\det \bar{z})^{\frac{\alpha-1}{2} - \lambda_1}, \end{aligned}$$

are highest weight vectors in $V_{2\lambda}$ and $V_{2\lambda+1}$ respectively.

3. Transition of K -types

In this section, we shall first derive explicit formulas for the highest weight vectors for $K \cong U(n)$ in the tensor products $V_\lambda \otimes S^2(\mathbf{C}^n)$ and $V_\lambda \otimes S^2(\mathbf{C}^{n*})$. We then use these results to compute the action of $\mathfrak{p}_{\mathbf{C}}$ on the K -types.

We now recall some notations used in [11]. For $1 \leq a, b \leq n$, let $h_{ab} = e_{aa} - e_{bb}$. For $1 \leq m < j \leq n$, F_{mj} and S_{mj} are elements in $\mathcal{U}(\mathfrak{gl}_n(\mathbf{C}))$ given by (c.f. eqs. (3.1) and (3.8) of [11])

$$\begin{aligned} F_{mj} &= \sum_I (-1)^{j-m-l+1} e_{j,i_l} e_{i_l, i_{l-1}} \dots e_{i_2, i_1} e_{i_1, m} \prod_{\substack{a \notin I \\ m+1 \leq a \leq j-1}} (h_{aj} + j - a), \\ S_{mj} &= \sum_I e_{i_1, m} e_{i_2, i_1} \dots e_{i_l, i_{l-1}} e_{j, i_l} \prod_{\substack{a \notin I \\ m+1 \leq a \leq j-1}} (h_{ma} + a - m), \end{aligned}$$

where $I = \{i_1 < i_2 < \dots < i_l\}$ in the sums runs over all subsets of $\{m+1, \dots, j-1\}$. For $1 \leq p < q \leq n$, we let (c.f. eq. (3.2) of [11])

$$H(j; p, q) = \prod_{a=p}^q (h_{a,j} + j - a), \quad H^-(j; p, q) = \prod_{a=p}^q (h_{a,j} + j - a - 1).$$

PROPOSITION 3.1. *Let $\lambda \in \Lambda^+$ and V_λ be an irreducible $U(n)$ module of highest weight λ . Let u be a highest weight vector in V_λ .*

(a) *If $\lambda_{j-1} \geq \lambda_j + 2$, then $V_\lambda \otimes S^2(\mathbf{C}^n)$ has a highest weight vector of weight $\lambda + 2e_j$ given by*

$$\nu_{\lambda+2e_j} = \sum_{m=1}^j \sum_{t=1}^j (-1)^{m+t} F_{mj} \\ \times \{ [F_{tj} H^-(j; 1, m-1) H^-(j; 1, t-1).u] \otimes \varepsilon_{tm} \}.$$

(b) *If $\lambda_j \geq \lambda_{j+1} + 2$, then $V_\lambda \otimes S^2(\mathbf{C}^{n*})$ has a highest weight vector of weight $\lambda - 2e_j$ given by*

$$\omega_{\lambda-2e_j} = \sum_{m=j}^n \sum_{t=j}^n S_{jm} \left\{ \left[S_{jt} \prod_{a=m+1}^n (h_{ja} + a - j - 1) \right. \right. \\ \left. \left. \times \prod_{b=t+1}^n (h_{jb} + b - j - 1).u \right] \otimes \varepsilon_{tm}^* \right\}.$$

Proof. By the second formula of Proposition 3.6 of [11],

$$X_j = \sum_{t=1}^j (-1)^{t+1} [F_{tj} H^-(j; 1, t-1).u] \otimes \varepsilon_t,$$

is a highest weight vector in $V_\lambda \otimes \mathbf{C}^n$ of weight $\lambda + e_j$. Let the module generated by X_j be W . Then by the first formula of Proposition 3.6 of [11],

$$Y_j = \sum_{m=1}^j (-1)^{m+1} F_{mj} [H(j; 1, m-1) X_j \otimes \varepsilon_m] \\ = \sum_{m=1}^j \sum_{t=1}^j (-1)^{m+t} F_{mj} \\ \times \{ [F_{tj} H^-(j; 1, m-1) H^-(j; 1, t-1).u] \otimes \varepsilon_t \otimes \varepsilon_m \},$$

is a highest weight vector of weight $\lambda + 2e_j$ in $W \otimes \mathbf{C}^n \hookrightarrow V_\lambda \otimes \mathbf{C}^n \otimes \mathbf{C}^n$. If $\pi: V_\lambda \otimes \mathbf{C}^n \otimes \mathbf{C}^n \rightarrow V_\lambda \otimes S^2(\mathbf{C}^n)$ is the canonical projection, then

$$\nu_{\lambda+2e_j} = \pi(Y_j) = \sum_{m=1}^j \sum_{t=1}^j (-1)^{m+t} F_{mj} \\ \times \{ [F_{tj} H^-(j; 1, m-1) H^-(j; 1, t-1).u] \otimes \varepsilon_{tm} \},$$

is a highest weight vector in $V_\lambda \otimes S^2(\mathbf{C}^n)$ of weight $\lambda + 2e_j$. The proof for (b) is similar (use Proposition 3.10 of [11]). \square

We shall denote the space of K -finite vector in $S^{\alpha, \pm}(X^o)$ by $S^{\alpha, \pm}(X^o)_K$. For each $\lambda \in \Lambda^+$, let $\mu = 2\lambda$ or $2\lambda + \mathbf{1}$. We consider the K -module map $m_\mu : V_\mu \otimes \mathfrak{p}_\mathbf{C} \rightarrow S^{\alpha, \pm}(X^o)_K$ given by

$$m_\mu(v \otimes p) = p.v, \quad p \in \mathfrak{p}_\mathbf{C}, v \in V_\mu.$$

Since $\mathfrak{p}_\mathbf{C} = \mathfrak{p}^+ \oplus \mathfrak{p}^- \cong S^2(\mathbf{C}^n) \oplus S^2(\mathbf{C}^{n*})$ as a K -module, we can obtain expressions for highest weight vectors in $V_\mu \otimes \mathfrak{p}_\mathbf{C}$ using Proposition 3.1.

We shall first study the transition from V_μ to $V_{\mu+2e_j}$. By Proposition 3.1,

$$\nu_{\mu+2e_j} = \sum_{m=1}^n \sum_{t=1}^n (-1)^{m+t} F_{mj} \{ [F_{tj} H^-(j; 1, m-1) H^-(j; 1, t-1) \cdot \xi_\mu] \otimes \varepsilon_{tm} \}$$

is a highest weight vector of weight $\mu + 2e_j$ in $V_\mu \otimes \mathfrak{p}^+$. Thus

$$m_\mu(\nu_{\mu+2e_j}) = \sum_{m=1}^n \sum_{t=1}^n (-1)^{m+t} F_{mj} \varepsilon_{tm} F_{tj} H^-(j; 1, m-1) H^-(j; 1, t-1) \cdot \xi_\mu.$$

We know that $m_\mu(\nu_{\mu+2e_j})$ is a multiple of $\xi_{\mu+2e_j}$. In the remaining of this section we shall compute explicitly this multiple. We shall now establish several preliminary results.

LEMMA 3.2. *Let $z = (z_{ab}) \in X^o$. Then for $1 \leq k, r \leq n$ with $k \neq r$, we have*

$$\sum_{j=1}^n \begin{vmatrix} z_{jk} & z_{jr} \\ \bar{z}_{jk} & \bar{z}_{jr} \end{vmatrix} = 0.$$

Proof. z defines a linear map $T : \mathbf{R}^n \rightarrow \mathbf{R}^{2n}$. Since $z \in X^o$, the range of T is totally isotropic with respect to the standard symplectic form on $\langle \cdot, \cdot \rangle$ on \mathbf{R}^{2n} . This space is spanned by the columns $z^{(j)}$ ($1 \leq j \leq n$) of z . Hence

$$-2i \langle z^{(k)}, z^{(r)} \rangle = \sum_{j=1}^n \begin{vmatrix} z_{jk} & z_{jr} \\ \bar{z}_{jk} & \bar{z}_{jr} \end{vmatrix} = 0. \quad \square$$

LEMMA 3.3. (i) *For $j \geq p > q$ or $p > q > j$, $e_{p,q} \gamma_j = 0$.*

(ii) *For $p > j \geq q$, we have*

$$e_{p,q} \gamma_j = 2 \det(z_1, \dots, z_{q-1}, z_p, z_{q+1}, \dots, z_j, \bar{z}_{j+1}, \dots, \bar{z}_n).$$

Proof. (i) is clear from eq. (2.5). We now prove (ii). For $p > j \geq q$, we have

$$e_{p,q} \gamma_j = \det(z_1, \dots, z_{q-1}, z_p, z_{q+1}, \dots, z_j, \bar{z}_{j+1}, \dots, \bar{z}_n) - \det(z_1, \dots, z_j, \bar{z}_{j+1}, \dots, \bar{z}_{p-1}, \bar{z}_q, \bar{z}_{p+1}, \dots, \bar{z}_n).$$

We let

$$d_1 = \det(z_p, \bar{z}_p, z_1, \dots, z_{q-1}, z_{q+1}, \dots, z_j, \bar{z}_{j+1}, \dots, \bar{z}_{p-1}, \bar{z}_{p+1}, \dots, \bar{z}_n),$$

$$d_2 = \det(z_q, \bar{z}_q, z_1, \dots, z_{q-1}, z_{q+1}, \dots, z_j, \bar{z}_{j+1}, \dots, \bar{z}_{p-1}, \bar{z}_{p+1}, \dots, \bar{z}_n).$$

Then it suffices to prove $d_1 + d_2 = 0$.

For $1 \leq r < k \leq n$, we let $D(r, k)$ be the complementary minor of the minor $\begin{vmatrix} z_{pr} & z_{pk} \\ \bar{z}_{pr} & \bar{z}_{pk} \end{vmatrix}$ of d_1 . Note that $D(r, k)$ is also the complementary minor of the minor $\begin{vmatrix} z_{qr} & z_{qk} \\ \bar{z}_{qr} & \bar{z}_{qk} \end{vmatrix}$ of d_2 . Using Laplace's expansion, we have

$$\begin{aligned} d_1 + d_2 &= \sum_{1 \leq r < k \leq n} (-1)^{r+k+1} \left(\begin{vmatrix} z_{pr} & z_{pk} \\ \bar{z}_{pr} & \bar{z}_{pk} \end{vmatrix} + \begin{vmatrix} z_{qr} & z_{qk} \\ \bar{z}_{qr} & \bar{z}_{qk} \end{vmatrix} \right) D(r, k) \\ &= - \sum_{1 \leq r < k \leq n} (-1)^{r+k+1} \sum_{l \neq p, q} \begin{vmatrix} z_{lr} & z_{lk} \\ \bar{z}_{lr} & \bar{z}_{lk} \end{vmatrix} D(r, k) \quad (\text{by Lemma 3.2}) \\ &= - \sum_{l \neq p, q} \det(z_l, \bar{z}_l, z_1, \dots, z_{q-1}, z_{q+1}, \dots, z_j, \bar{z}_{j+1}, \dots, \bar{z}_{p-1}, \bar{z}_{p+1}, \dots, \bar{z}_n) \\ &= 0. \end{aligned} \quad \square$$

LEMMA 3.4. For $m \leq k < j \leq n$, we have

$$(2\gamma_{j-1})(e_{jm} \cdot \gamma_k) - \sum_{t=k+1}^{j-1} (e_{jt} \cdot \gamma_{j-1})(e_{tm} \cdot \gamma_k) = (e_{jm} \cdot \gamma_{j-1})(2\gamma_k).$$

Proof. We shall omit the details of the proof as it is similar to that of Lemma 4.5 in [11]. For $z \in X^o$ with $(e_{jm} \cdot \gamma_{j-1})(z) \neq 0$, we apply Cramer's rule to the system of linear equations in the unknowns $x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_j, y_j, \dots, y_n$ given by

$$\begin{aligned} x_1 z_1 + x_2 z_2 + \dots + x_{m-1} z_{m-1} + x_j z_j + x_{m+1} z_{m+1} + \dots \\ + x_{j-1} z_{j-1} + y_{j+1} \bar{z}_{j+1} + \dots + y_n \bar{z}_n = z_m \end{aligned}$$

and use Lemma 3.3 to describe its solutions. □

We shall now introduce a more convenient notation for the highest weight vectors $\xi_{2\lambda}$ and $\xi_{2\lambda+1}$ in the K -types. We shall fix a complex number α in the rest of this section. Let $\lambda \in \Lambda^+$. For $\mu = 2\lambda$ or $2\lambda + \mathbf{1}$, we define $\mathbf{l}(\mu) = (l_0, l_1, \dots, l_n)$ by

$$l_0 = \begin{cases} \frac{\alpha}{2} - \lambda_1 & \text{if } \mu = 2\lambda \\ \frac{\alpha - 1}{2} - \lambda_1 & \text{if } \mu = 2\lambda + \mathbf{1}, \end{cases} \quad l_n = \begin{cases} \lambda_n + \frac{\alpha}{2} & \text{if } \mu = 2\lambda \\ \lambda_n + \frac{\alpha + 1}{2} & \text{if } \mu = 2\lambda + \mathbf{1}, \end{cases}$$

and

$$l_j = \lambda_j - \lambda_{j+1} \quad (1 \leq j \leq n - 1).$$

We now set

$$\gamma^{l(\mu)} = \gamma_0^{l_0} \gamma_1^{l_1} \cdots \gamma_n^{l_n}. \tag{3.7}$$

Then one sees that $\gamma^{l(\mu)} = \xi_\mu$ for $\mu = 2\lambda$ or $2\lambda + 1$. For $0 \leq j \leq n$, $\mathbf{1} \pm \varepsilon_j$ shall denote the $n + 1$ tuple of numbers with its j -th coordinate $l_j \pm 1$ and other coordinates the same as that of $\mathbf{1}$. For $1 \leq j \leq n$, $q \leq p, q \leq n$ and $\mu \in \Lambda^+$, we let (c.f. eq. (4.6) of [11])

$$\mu(j; p, q) = \prod_{a=p}^q (\mu_a - \mu_j + j - a - 1).$$

Thus if v is a highest weight vector for $\mathfrak{gl}_n(\mathbf{C})$ with weight μ , then $H^-(j; p, q).v = \mu(j; p, q)v$. We also note that $(2\lambda)(j; p, q) = (2\lambda + \mathbf{1})(j; p, q)$.

PROPOSITION 3.5. *Let $\lambda \in \Lambda^+$. For $\mu = 2\lambda$ or $2\lambda + \mathbf{1}$ and $1 \leq m \leq j - 1$, we have*

$$F_{mj}.\gamma^{l(\mu)} = \frac{1}{2}(-1)^{m+j+1} \mu(j; m, j - 1)(e_{jm}.\gamma_{j-1})\gamma^{l-\varepsilon_{j-1}}.$$

Proof. The proof is a calculation using Lemma 4.4 of [11] and Lemma 3.4 above. Again we omit the details as it is similar to Proposition 4.8 of [11]. \square

With reasoning similar to Lemma 3.3, we also have for $n \geq p > q > k \geq 1$ and $z \in X^o$,

$$\begin{aligned} \det(z_1, \dots, z_k, \bar{z}_{k+1}, \dots, \bar{z}_{q-1}, z_p, \bar{z}_{q+1}, \dots, \bar{z}_n) \\ = \det(z_1, \dots, z_k, \bar{z}_{k+1}, \dots, \bar{z}_{p-1}, z_q, \bar{z}_{p+1}, \dots, \bar{z}_n). \end{aligned}$$

This implies

$$\varepsilon_{p,q}.\gamma_k = 2 \det(z_1, \dots, z_k, \bar{z}_{k+1}, \dots, \bar{z}_{q-1}, z_p, \bar{z}_{q+1}, \dots, \bar{z}_p, \dots, \bar{z}_n). \tag{3.8}$$

Eq. (3.8) and Cramer's rule now imply the following identity.

LEMMA 3.6. *For $1 \leq k < m \leq j \leq n$, we have*

$$\begin{aligned} (2\gamma_{j-1})(\varepsilon_{jm}.\gamma_k) - \sum_{t=k+1}^{j-1} (e_{jt}.\gamma_{j-1})(\varepsilon_{tm}.\gamma_k) &= (\varepsilon_{jm}.\gamma_{j-1})(2\gamma_k) \\ &= \begin{cases} 4\gamma_j\gamma_k & \text{if } m = j, \\ 0 & \text{if } m \neq j. \end{cases} \end{aligned}$$

PROPOSITION 3.7. *For $\mu = 2\lambda$ or $2\lambda + \mathbf{1}$ where $\lambda \in \Lambda^+$, we have*

$$m_\mu(\nu_{\mu+2\varepsilon_j}) = (\alpha - \mu_j + j - 1)[\mu(j; 1, j - 1)]^2 \xi_{\mu+2\varepsilon_j}.$$

Proof. We recall that

$$m_\mu(\nu_{\mu+2e_j}) = \sum_{m=1}^j \sum_{t=1}^j (-1)^{m+t} F_{mj} \varepsilon_{tm} F_{tj} H^-(j; 1, m-1) H^-(j; 1, t-1) \cdot \xi_\mu$$

and that ξ_μ is of the form $\gamma^{\mathbf{l}}$ where $\mathbf{l} = \mathbf{l}(\mu)$ (c.f. eq. (3.7)).

By Proposition 3.5, we have

$$\begin{aligned} & \sum_{t=1}^j (-1)^t \varepsilon_{tj} F_{tj} H^-(j; 1, t-1) \cdot \xi_\mu \\ &= \sum_{t=1}^{j-1} (-1)^t \mu(j; 1, t-1) \frac{1}{2} (-1)^{j+t+1} \mu(j; t, j-1) \varepsilon_{tj} \{ (e_{jt} \cdot \gamma_{j-1}) \gamma^{1-\varepsilon_{j-1}} \} \\ & \quad + (-1)^j \mu(j; 1, j-1) \varepsilon_{jj} \cdot \gamma^{\mathbf{l}} \\ &= \frac{1}{2} (-1)^j \mu(j; 1, j-1) \left\{ - \sum_{t=1}^{j-1} [(\varepsilon_{tj} e_{jt} \cdot \gamma_{j-1}) \gamma^{1-\varepsilon_{j-1}} \right. \\ & \quad \left. + \sum_{k=0}^{t-1} l_k (e_{jt} \cdot \gamma_{j-1}) (\varepsilon_{tj} \cdot \gamma_k) \gamma^{1-\varepsilon_k - \varepsilon_{j-1}}] + \sum_{k=0}^{j-1} 2l_k (\varepsilon_{jj} \cdot \gamma_k) \gamma^{1-\varepsilon_k} \right\} \\ &= \frac{1}{2} (-1)^j \mu(j; 1, j-1) \left\{ - \sum_{t=1}^{j-1} (-2\gamma_j) \gamma^{1-\varepsilon_{j-1}} \right. \\ & \quad - \sum_{k=0}^{j-2} l_k \sum_{t=k+1}^{j-1} (e_{jt} \cdot \gamma_{j-1}) (\varepsilon_{tj} \cdot \gamma_k) \gamma^{1-\varepsilon_k - \varepsilon_{j-1}} \\ & \quad \left. + \sum_{k=0}^{j-2} 2l_k (\varepsilon_{jj} \cdot \gamma_k) \gamma^{1-\varepsilon_k} + 2l_{j-1} (2\gamma_j) \gamma^{1-\varepsilon_{j-1}} \right\} \\ & \quad (\text{since } \varepsilon_{tj} e_{jt} \cdot \gamma_{j-1} = -2\gamma_j, \varepsilon_{jj} \cdot \gamma_{j-1} = 2\gamma_j) \\ &= \frac{1}{2} (-1)^j \mu(j; 1, j-1) \left\{ 2(j-1) \xi_{\mu+2e_j} + \sum_{k=0}^{j-2} l_k [(\varepsilon_{jj} \cdot \gamma_k) (2\gamma_{j-1}) \right. \\ & \quad \left. - \sum_{t=k+1}^{j-1} (e_{jt} \cdot \gamma_{j-1}) (\varepsilon_{tj} \cdot \gamma_k)] \gamma^{1-\varepsilon_k - \varepsilon_{j-1}} + 4l_{j-1} \xi_{\mu+2e_j} \right\} \\ &= \frac{1}{2} (-1)^j \mu(j; 1, j-1) \left\{ 2(2l_{j-1} + j-1) \xi_{\mu+2e_j} \right. \\ & \quad \left. + \sum_{k=0}^{j-2} l_k (4\gamma_j \gamma_k) \gamma^{1-\varepsilon_k - \varepsilon_{j-1}} \right\} \end{aligned}$$

$$\begin{aligned}
 &= (-1)^j \left(\sum_{k=0}^{j-2} 2l_k + 2l_{j-1} + j - 1 \right) \mu(j; 1, j - 1) \xi_{\mu+2e_j} \\
 &= (-1)^j (\alpha - \mu_j + j - 1) \mu(j; 1, j - 1) \xi_{\mu+2e_j}.
 \end{aligned}$$

Next for $1 \leq m \leq j - 1$,

$$\begin{aligned}
 &\sum_{t=1}^j (-1)^t \varepsilon_{tm} F_{tj} H^-(j; 1, t - 1) \cdot \xi_{\mu} \\
 &= \sum_{t=1}^{j-1} (-1)^t \mu(j; 1, t - 1) \frac{1}{2} (-1)^{j+t+1} \mu(j; t, j - 1) \varepsilon_{tm} \{ (e_{jt} \cdot \gamma_{j-1}) \gamma^{1-\varepsilon_{j-1}} \} \\
 &\quad + (-1)^j \mu(j; 1, j - 1) \varepsilon_{jm} \cdot \gamma^1 \\
 &= \frac{1}{2} (-1)^{j+1} \mu(j; 1, j - 1) \left\{ \sum_{t=1}^{j-1} \left[(\varepsilon_{tm} e_{jt} \cdot \gamma_{j-1}) \gamma^{1-\varepsilon_{j-1}} \right. \right. \\
 &\quad \left. \left. + \sum_{k=0}^{\min(t-1, m-1)} l_k (e_{jt} \cdot \gamma_{j-1}) (\varepsilon_{tm} \cdot \gamma_k) \gamma^{1-\varepsilon_k - \varepsilon_{j-1}} \right] - 2 \sum_{k=0}^{m-1} l_k (\varepsilon_{jm} \cdot \gamma_k) \gamma^{1-\varepsilon_k} \right\} \\
 &= \frac{1}{2} (-1)^j \mu(j; 1, j - 1) \left\{ - \sum_{k=0}^{m-1} l_k \sum_{t=k+1}^{j-1} (e_{jt} \cdot \gamma_{j-1}) (\varepsilon_{tm} \cdot \gamma_k) \gamma^{1-\varepsilon_k - \varepsilon_{j-1}} \right. \\
 &\quad \left. + \sum_{k=0}^{m-1} l_k (\varepsilon_{jm} \cdot \gamma_k) \gamma^{1-\varepsilon_k} \right\} \quad (\text{since } \varepsilon_{tm} e_{jt} \cdot \gamma_{j-1} = 0, \forall 1 \leq t \leq j - 1) \\
 &= \frac{1}{2} (-1)^j \mu(j; 1, j - 1) \sum_{k=0}^{m-1} l_k \left[(2\gamma_{j-1}) (\varepsilon_{jm} \cdot \gamma_k) \right. \\
 &\quad \left. - \sum_{t=k+1}^{j-1} (e_{jt} \cdot \gamma_{j-1}) (\varepsilon_{tm} \cdot \gamma_k) \right] \gamma^{1-\varepsilon_k - \varepsilon_{j-1}} \\
 &= 0, \quad \text{by Lemma 3.6.}
 \end{aligned}$$

Hence

$$\begin{aligned}
 m_{\mu}(\nu_{\mu+2e_j}) &= (-1)^j \mu(j; 1, j - 1) \\
 &\quad \times \{ (-1)^j \mu(j; 1, j - 1) (\alpha - \mu_j + j - 1) \xi_{\mu+2e_j} \} \\
 &= (\alpha - \mu_j + j - 1) [\mu(j; 1, j - 1)]^2 \xi_{\mu+2e_j}. \quad \square
 \end{aligned}$$

Next we consider the transition from V_{μ} to $V_{\mu-2e_j}$. By part (b) of Proposition 3.1, the vector in $V_{\mu} \otimes S^2(\mathbf{C}^{n*})$ given by

$$\omega_{\mu-2e_j} = \sum_{m=j}^n \sum_{t=j}^n S_{jm} \left\{ \left[S_{jt} \prod_{a=m+1}^n (h_{ja} + a - j - 1) \right. \right. \\ \left. \left. \times \prod_{b=t+1}^n (h_{jb} + b - j - 1) \cdot \xi_{\mu} \right] \otimes \varepsilon_{tm}^* \right\}$$

is a highest weight vector of weight $\mu - 2e_j$. Hence we need to compute

$$m_{\mu}(\omega_{\mu-2e_j}) = \sum_{m=j}^n \sum_{t=j}^n S_{jm} \varepsilon_{tm}^* S_{jt} \\ \times \prod_{a=m+1}^n (h_{ja} + a - j - 1) \prod_{b=t+1}^n (h_{jb} + b - j - 1) \cdot \xi_{\mu}.$$

We first observe that the operator S_{jm} which appears in the expression for $\omega_{\mu-2e_j}$ now plays the role of F_{mj} in $\nu_{\mu+2e_j}$. Using arguments similar to Lemma 4.4 of [11], one can show that

$$S_{jm} = \sum_{r=j}^{m-1} e_{mr} S_{j,r} \prod_{a=r+1}^{t-1} (h_{j,a} + a - j - 1).$$

Using this identity we can then carry out a parallel analysis on the transition from V_{μ} to $V_{\mu-2e_j}$. We shall omit the details and shall only give the final result.

PROPOSITION 3.8

$$m_{\mu}(\omega_{\mu-2e_j}) = (\alpha + \mu_j + n - j) \prod_{a=j+1}^n (\mu_j - \mu_a + a - j - 1)^2 \xi_{\mu-2e_j}.$$

Alternatively, one can also obtain transition coefficients for the ‘downward transition’ from V_{μ} to $V_{\mu-2e_j}$ by considering the ‘upward transition’ from V_{μ} to $V_{\mu+2e_j}$ in the Hermitian dual.

4. Reducibility and complementary series

In this section we shall discuss the reducibility of $I^{\pm}(\sigma)$ and determine the complementary series.

The reducibility of $I^{\pm}(\sigma)$ is first determined by Kudla and Rallis in [10]. They study the action by the enveloping algebra and determine the obstructions to transition between an arbitrary K -type and a scalar K -type, and from which they prove:

THEOREM 4.1. ([10]) $I^\pm(\sigma)$ is irreducible if and only if $\sigma + \rho_n \notin \mathbf{Z}$.

We can also deduce Theorem 4.1 from our results. In Propositions 3.7 and 3.8, we have determined the obstructions to transition between arbitrary K -types. We shall call the scalar expressions $\alpha - \mu_j + j - 1$ and $\alpha + \mu_j + n - j$ ‘transition coefficients’. It is clear that $\mathcal{S}^{\alpha, \pm}(X^o)$ is irreducible if and only if all the transition coefficients are non zero, and this occurs precisely when $\alpha \notin \mathbf{Z}$. The theorem then follows from the isomorphism (2.2) $\mathcal{S}^{\alpha, \pm}(X^o) \cong I^\pm(\sigma)$ where $\sigma = -\alpha - \rho_n$. In the next section, we shall study in detail the module structure of $I^\pm(\sigma)$ at the points of reducibility.

Next we shall determine which of the modules $I^\pm(\sigma)$ define unitary representations. We know from the theory of unitary induction that the module $I^\pm(\sigma)$ can be given a $\mathrm{Sp}(2n, \mathbf{R})$ invariant inner product (given by integration over K) when $\mathrm{Re}(\sigma) = 0$. We call the set of σ such that $\mathrm{Re}(\sigma) = 0$ the *unitary axis*. On the other hand, there exists σ not on the unitary axis such that $I^\pm(\sigma)$ still can be given a $\mathrm{Sp}(2n, \mathbf{R})$ invariant inner product. We shall call this family of unitary representations the *complementary series*. We shall now determine the complementary series.

Let

$$\Lambda_e^+ = \{2\lambda : \lambda \in \Lambda^+\}, \quad \Lambda_o^- = \{2\lambda + \mathbf{1} : \lambda \in \Lambda^+\}.$$

Then Λ_e^+ and Λ_o^- are the highest weights occurring in $\mathcal{S}^{\alpha, +}(X^o)$ and $\mathcal{S}^{\alpha, -}(X^o)$ respectively. Now each K -type V_λ of $\mathcal{S}^{\alpha, \pm}(X^o)$ has a K -invariant inner product given by

$$\langle f_1, f_2 \rangle_\lambda = \int_K f_1(kx_o) \overline{f_2(kx_o)} dk.$$

Since V_λ is an irreducible K module, any K -invariant inner product on V_λ is a multiple of $\langle \cdot, \cdot \rangle_\lambda$. Thus if $\langle \cdot, \cdot \rangle$ is a $\mathrm{Sp}(2n, \mathbf{R})$ invariant inner product on $\mathcal{S}^{\alpha, +}(X^o)$ (respectively $\mathcal{S}^{\alpha, -}(X^o)$), then there exists positive constants $\{c_\lambda\}_{\lambda \in \Lambda_e^+}$ (respectively $\{c_\lambda\}_{\lambda \in \Lambda_o^-}$) such that

$$\langle f_1, f_2 \rangle = c_\lambda \langle f_1, f_2 \rangle_\lambda, \quad \forall f_1, f_2 \in V_\lambda.$$

Since the K -types of $\mathcal{S}^{\alpha, \pm}(X^o)$ are mutually orthogonal with respect $\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle$ is completely determined by the constants $\{c_\lambda\}$. Using similar arguments as the $U(n, n)$ case (see section 9 of [11]), we obtain the following:

LEMMA 4.2. *The inner product on $\mathcal{S}^{\alpha, +}(X^o)$ (respectively $\mathcal{S}^{\alpha, -}(X^o)$) defined by the constants $\{c_\lambda\}_{\lambda \in \Lambda_e^+}$ (respectively $\{c_\lambda\}_{\lambda \in \Lambda_o^-}$) is $\mathrm{Sp}(2n, \mathbf{R})$ invariant if and only if*

$$(\alpha - \lambda_j + j - 1)c_{\lambda+2e_j} + (\bar{\alpha} + \lambda_j + n - j + 2)c_\lambda = 0$$

for all $\lambda \in \Lambda_e^+$ (respectively $\lambda \in \Lambda_o^-$) and all $1 \leq j \leq n$.

We let

$$N_{\lambda,j} = \frac{\alpha - \lambda_j + j - 1}{\bar{\alpha} + \lambda_j + n - j + 2} = -\frac{c_\lambda}{c_{\lambda+2e_j}}. \tag{4.9}$$

Then $\mathcal{S}^{\alpha,+}(X^o)$ (respectively $\mathcal{S}^{\alpha,-}(X^o)$) is unitarizable if and only if $N_{\lambda,j} < 0$ for all $\lambda \in \Lambda_e^+$ (respectively for all $\lambda \in \Lambda_o^+$) and for all j .

We let $\tilde{\alpha} = \alpha + \frac{n+1}{2}$. Then

$$N_{\lambda,j} = \frac{\tilde{\alpha} - (\frac{n+1}{2} + \lambda_j - j + 1)}{\bar{\tilde{\alpha}} + (\frac{n+1}{2} + \lambda_j - j + 1)}.$$

We write $m = \frac{n+1}{2} + \lambda_j - j + 1$. Then $N_{\lambda,j} = (\tilde{\alpha} - m)/(\bar{\tilde{\alpha}} + m)$. Thus $N_{\lambda,j}$ is real for all λ and for j if and only if either $\text{Re}(\tilde{\alpha}) = 0$ or $\tilde{\alpha}$ is real. The case $\text{Re}(\tilde{\alpha}) = 0$ corresponds to the unitary axis. If $\tilde{\alpha}$ is real, then

$$N_{\lambda,j} = \frac{\tilde{\alpha} - m}{\bar{\tilde{\alpha}} + m} < 0 \iff (\tilde{\alpha} - m)(\tilde{\alpha} + m) < 0 \iff \tilde{\alpha}^2 < m^2.$$

If n is odd, then $m \in \mathbf{Z}$. In particular for $m = 0$ there is no solution for $\tilde{\alpha}$. On the other hand, if n is even then the minimum value of m^2 is $(\frac{1}{2})^2$ so that

$$N_{\lambda,j} < 0, \forall \lambda, j \iff \tilde{\alpha}^2 < \frac{1}{4} \iff |\tilde{\alpha}| < \frac{1}{2} \iff -\frac{1}{2}n - 1 < \alpha < -\frac{1}{2}n.$$

These α 's give the complementary series. We now recall that $\mathcal{S}^{\alpha,\pm}(X^o) \cong I^\pm(\sigma)$ where $\sigma = -\alpha - \rho_n$. Thus we have proved:

THEOREM 4.3. *If n is even and $-\frac{1}{2} < \sigma < \frac{1}{2}$, then $I^\pm(\sigma)$ is unitarizable.*

5. Subquotients of $I^+(\sigma)$ and their unitarity

In this section, we shall give a detailed description of the module structure of $I^+(\sigma)$ when it is reducible. We shall describe all the irreducible constituents of $I^+(\sigma)$ and determine which of them are unitarizable, i.e., possess a $\text{Sp}(2n, \mathbf{R})$ invariant positive definite inner product. We also describe the socle series and module diagram of $I^+(\sigma)$. The structure of $I^-(\sigma)$ is very similar and will be left to the readers.

We recall that (c.f. eq. (2.2)) $I^+(\sigma) \cong \mathcal{S}^{\alpha,+}(X^o)$ where $\alpha = -\sigma - \rho_n$, and throughout this section we shall always assume that $I^+(\sigma)$ is reducible (i.e., $\sigma + \rho_n \in \mathbf{Z}$). We find it more convenient to derive intermediate results in the model $\mathcal{S}^{\alpha,+}(X^o)$, but shall state the main theorems in the more standard model $I^+(\sigma)$. We now identify each of the K -types V_μ of $\mathcal{S}^{\alpha,+}(X^o)$ with the integral point $\mu = (\mu_1, \dots, \mu_n)$ in \mathbf{R}^n . Let x_1, \dots, x_n be the standard coordinates of \mathbf{R}^n . The transition formulas in section 3 tell us that we should consider the hyperplanes:

$$\ell_j^+ : x_j = \alpha + j - 1, \quad \ell_j^- : x_j = -(\alpha + n - j) \quad (1 \leq j \leq n). \tag{5.10}$$

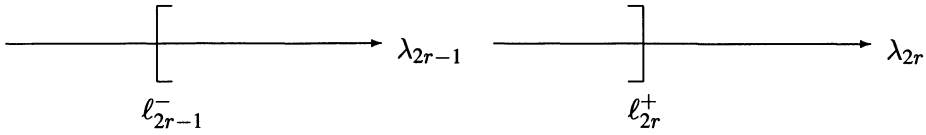


Fig. 1.

These are the ‘potential barriers’ to the transition of K -types in the sense that if a K -type V_μ lies on say ℓ_j^+ then the enveloping algebra $\mathcal{U}(\mathfrak{sp}(n, \mathbf{C}))$ is unable to transform vectors in V_μ to vectors in $V_{\mu+2e_j}$. However since the highest weights of the K -types V_μ in $\mathcal{S}^{\alpha,+}(X^o)$ are the form $\mu = 2\lambda$ where $\lambda \in \Lambda^+$, not all the hyperplanes given in (5.10) affect transition. We shall consider two cases, i.e., n even and n odd. By Theorem 4.1, if n is even, then $I^+(\sigma)$ is reducible if and only if $\sigma \in \frac{1}{2} + \mathbf{Z}$; and if n is odd, then $I^+(\sigma)$ is reducible if and only if $\sigma \in \mathbf{Z}$.

Case: $n = 2m$ even. We shall first assume that α is odd. Since n is even, for odd j , only ℓ_j^- affects the transition of the K -types, and for even j , only ℓ_j^+ affects the transitions of the K -types. Thus there is only one ‘barrier’ along each coordinate axis which is effective. This situation can be visualized as in Fig. 1.

The symbol $[$ means that transition of K -types from left to right is permissible but K -types at the right side of the barrier can not move across the barrier to reach the left side of the barrier. The symbol $]$ is interpreted similarly.

For $1 \leq r \leq m$, we define

$$\begin{aligned} X_1^r &= \{\lambda \in \Lambda_e^+ : \lambda_{2r-1} < -(\alpha + n - 2r + 1)\}, \\ X_2^r &= \{\lambda \in \Lambda_e^+ : \lambda_{2r-1} \geq -(\alpha + n - 2r + 1)\}, \\ Y_1^r &= \{\lambda \in \Lambda_e^+ : \lambda_{2r} \leq \alpha + 2r - 1\}, \\ Y_2^r &= \{\lambda \in \Lambda_e^+ : \lambda_{2r} > \alpha + 2r - 1\}. \end{aligned}$$

For $1 \leq p, q \leq m + 1$, we set

$$\begin{aligned} L_{pq} &= (X_2^1 \cap \dots \cap X_2^{p-1} \cap X_1^p \cap \dots \cap X_1^m) \\ &\quad \cap (Y_2^1 \cap \dots \cap Y_2^{q-1} \cap Y_1^q \cap \dots \cap Y_1^m). \end{aligned}$$

Observe that because of the dominance condition $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ on Λ^+ , intersections of the X_j^r 's and the Y_k^m 's other than those of the form L_{pq} are empty. The set of nonempty L_{pq} forms a partition for Λ^+ . If $L_{pq} \neq \emptyset$, we call the subspace $\sum_{\lambda \in L_{pq}} V_\lambda$ a ‘constituent’ of $\mathcal{S}^{\alpha,+}(X^o)$. For convenience we shall also denote this subspace by L_{pq} .

LEMMA 5.1. (i) For $p < q$, $L_{pq} \neq \emptyset$ if and only if $p - q \geq \alpha + m + 1$. In this case, L_{pq} is unitarizable if and only if $p - q = \alpha + m + 1$.

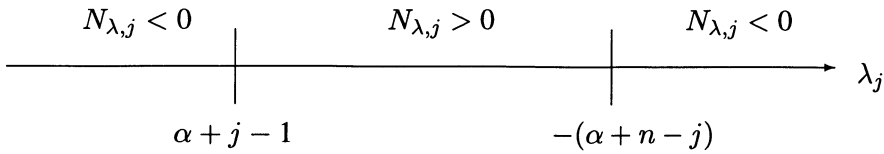


Fig. 2.

- (ii) For each $1 \leq p \leq m + 1$, $L_{p,p}$ is always nonempty. $L_{p,p}$ is unitarizable if and only if $\alpha \geq -m - 1$.
- (iii) For each $1 \leq q \leq m$, $L_{q+1,q}$ is always nonempty. $L_{q+1,q}$ is unitarizable if and only if $\alpha \leq -m$.
- (iv) For $p > q + 1$, $L_{p,q} \neq \emptyset$ if and only if $p - q \leq \alpha + m + 1$ and it is unitarizable if and only if $p - q = \alpha + m + 1$.

Proof. As the proofs are elementary, we shall only prove (i).

If $\lambda \in L_{pq}$, then in particular $\lambda \in X_1^p$ and $\lambda \in Y_2^{q-1}$. Consequently, $\lambda_{2p-1} \leq -(\alpha + n - 2p + 1) - 2 = -\alpha - n + 2p - 3$ and $\lambda_{2q-2} \geq (\alpha + 2q - 2 - 1) + 2 = \alpha + 2q - 1$. Since $\lambda_{2p-1} \geq \lambda_{2q-2}$, we have $-\alpha - n + 2p - 3 \geq \alpha + 2q - 1$ which gives $p - q \geq \alpha + m + 1$.

Conversely suppose that $p - q \geq \alpha + m + 1$. We let $\lambda = (\lambda_1, \dots, \lambda_n)$ be given by

$$\lambda_j = \begin{cases} -\alpha - n + 2p - 3 & \text{if } 1 \leq j \leq 2q - 1, \\ \alpha + 2q - 1 & \text{if } 2q \leq j \leq n. \end{cases}$$

Then $\lambda \in L_{pq}$ so that $L_{pq} \neq \emptyset$.

For unitarity, we recall the definition of $N_{\lambda, j}$ given in (4.9) and that $S^{\alpha,+}(X^o)$ is unitarizable if and only if all $N_{\lambda, j} < 0$. Similarly L_{pq} is unitarizable if and only if $N_{\lambda, j} < 0$ for all $\lambda \in L_{p,q}$ and $1 \leq j \leq n$ such that $\lambda, \lambda + e_j \in L_{pq}$. We now divide each axis λ_j into 3 portions as shown in Fig. 2.

If $\lambda \in L_{pq}$, then for $2p - 1 \leq j \leq 2q - 2$, λ_j is confined to the range

$$\alpha + 2q - 1 \leq \lambda_j \leq -\alpha - n + 2p - 3.$$

This is to ensure that the dominance condition is met. Now $N_{\lambda, j}$ is positive only if it corresponds to a transition in the middle portion. Hence L_{pq} is unitarizable if and only if $\alpha + 2q - 1 = -\alpha - n + 2p - 3$ because in this case there is no transition within the middle portion of the axes λ_j for $2p - 1 \leq j \leq 2q - 2$ (see Fig. 3). This occurs when

$$\alpha + 2q - 1 = -\alpha - n + 2p - 3 \iff q - p = -(\alpha + m + 1). \quad \square$$

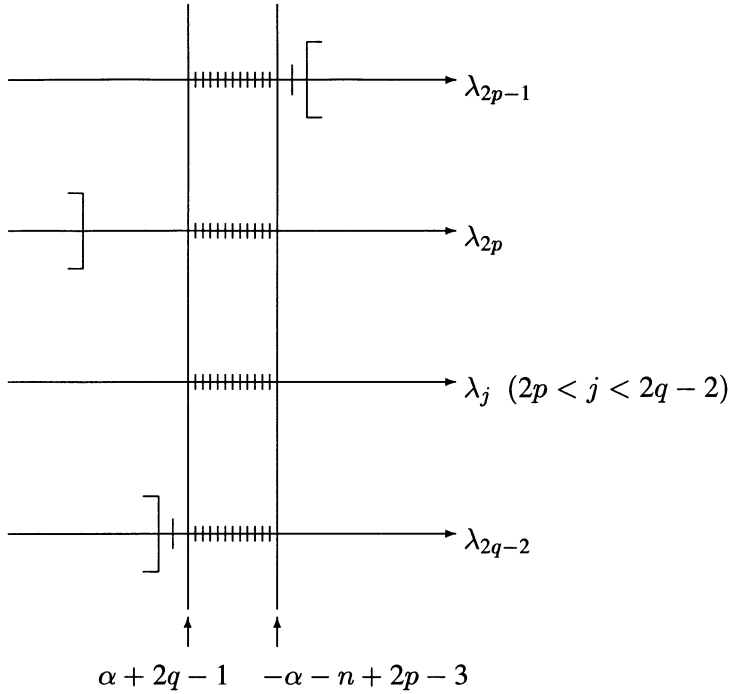


Fig. 3.

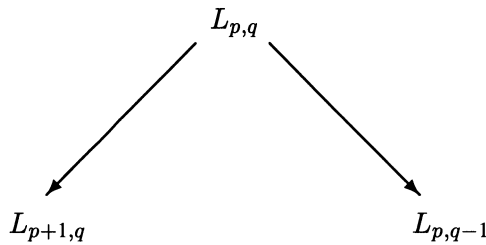


Fig. 4.

The definition of $L_{p,q}$ leads to the transition relation as shown in Fig. 4. Loosely speaking, it means that vectors in $L_{p,q}$ can be transformed by the enveloping algebra to $L_{p+1,q}$ and $L_{p,q-1}$ but the converse is not. We can now arrange all the $(m + 1)^2$ possible irreducible constituents of $S^{\alpha,+}(X^o)$ into a ‘square’ in such a way which is consistent with the above transition relation (see Fig. 5).

For a given α , we can use Lemma 5.1 to determine which of the constituents L_{pq} are nonempty. If we remove those empty L_{pq} from the ‘square’ in Fig. 5, then the remaining configuration is the module diagram (see [1] or section 7 of [11] for a precise definition) of $S^{\alpha,+}(X^o)$. We can then easily read off the socle series (and

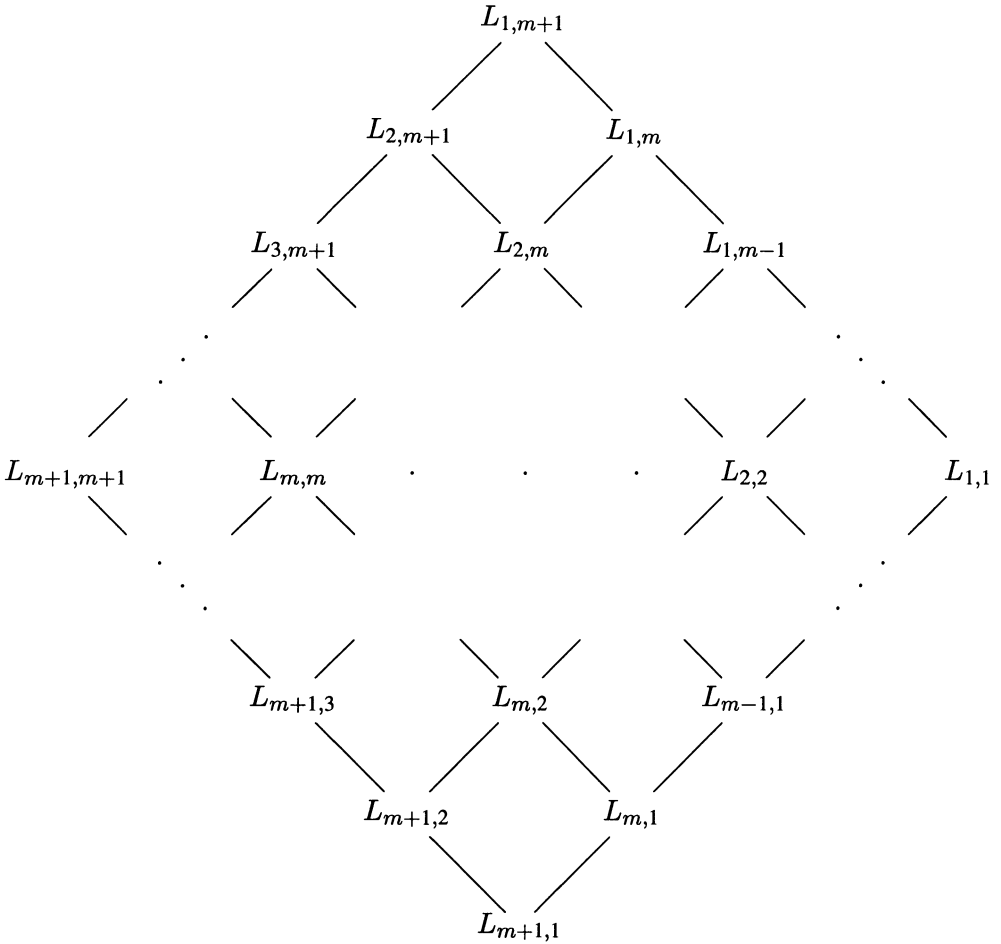


Fig. 5.

hence a composition series) of $S^{\alpha,+}(X^o)$ (and hence of $I^+(\sigma)$). Recall that (c.f. [6]) the socle of a module M is the sum of all irreducible submodules of M , and it is denoted by $\text{Soc}(M)$. The socle series of $I^+(\sigma)$ is the ascending chain

$$\text{Soc}^0(I^+(\sigma)) \subseteq \text{Soc}^1(I^+(\sigma)) \subseteq \text{Soc}^2(I^+(\sigma)) \subseteq \dots$$

of submodules of $I^+(\sigma)$ defined inductively by setting $\text{Soc}^0(I^+(\sigma)) = 0$ and

$$\text{Soc}^{r+1}(I^+(\sigma))/\text{Soc}^r(I^+(\sigma)) = \text{Soc}(I^+(\sigma)/\text{Soc}^r(I^+(\sigma)))$$

for any nonnegative integer r . We now observe that α is odd if and only if $[\sigma] \equiv m \pmod{2}$. Here $[\sigma]$ denote the greatest integer less than or equal to σ . Thus we obtain the following theorem.

THEOREM 5.2. *Let $n = 2m$ be an positive even integer, and let $\sigma \in \frac{1}{2} + \mathbf{Z}$ be such that $[\sigma] \equiv m \pmod{2}$.*

(i) *If $\sigma \leq \frac{1}{2}$, then*

$$I^+(\sigma) = \oplus\{L_{p,q} : 0 \leq p - q \leq r_1\},$$

where $r_1 = \min([\sigma] + 1, m)$, and the socle series of $I^+(\sigma)$ is given by

$$\text{Soc}^j(I^+(\sigma)) = \begin{cases} \oplus_{r_1-j+1 \leq p-q \leq r_1} L_{p,q} & 1 \leq j \leq r_1, \\ I^+(\sigma) & j \geq r_1 + 1. \end{cases}$$

Moreover, a constituent $L_{p,q}$ of $I^+(\sigma)$ is unitarizable if and only if $p = q$ or if $-m + \frac{1}{2} \leq \sigma \leq \frac{1}{2}$ and $p - q = r_1$.

(ii) *If $\sigma \geq \frac{3}{2}$, then*

$$I^+(\sigma) = \oplus\{L_{pq} : -1 \leq q - p \leq r_2\},$$

where $r_2 = \min([\sigma], m)$, and the socle series of $I^+(\sigma)$ is given by

$$\text{Soc}^j(I^+(\sigma)) = \begin{cases} \oplus_{-1 \leq q-p \leq j-2} L_{pq} & 1 \leq j \leq r_2 + 1, \\ I^+(\sigma) & j \geq r_2 + 2. \end{cases}$$

Moreover, a constituent L_{pq} of $I^+(\sigma)$ is unitarizable if and only if $p = q + 1$ or $\frac{3}{2} \leq \sigma \leq m + \frac{1}{2}$ and $q - p = r_2$.

Next we shall consider the case when α is even. We note that

$$\alpha \text{ is even} \iff [\sigma] \equiv m + 1 \pmod{2}.$$

It is easy to check that if σ is such that $[\sigma] \equiv m + 1 \pmod{2}$, then $[-\sigma] \equiv m \pmod{2}$. Now the module structure of $I^+(-\sigma)$ can be obtained from Theorem 5.2. On the other hand, the underlying $(\mathfrak{sp}(2n, \mathbf{C}), K)$ module structure of $I^+(\sigma)$ is contragredient to that of $I^+(-\sigma)$. Hence the module structure of $I^+(\sigma)$ can be derived from the module structure of $I^+(-\sigma)$. In particular, a composition series of $I^+(\sigma)$ can be obtained by ‘reversing’ a composition series of $I^+(-\sigma)$.

We shall now use Theorem 5.2 and the above observation to construct the module diagrams of $I^+(\sigma)$ for some specific cases. We recall that when n is even, $I^+(0)$ is irreducible, and the complementary series occurs in the range $-\frac{1}{2} < \sigma < \frac{1}{2}$. The module $I^+(\sigma)$ is reducible at the end points of the complementary series (i.e. $\sigma = \pm \frac{1}{2}$). The diagrams for the degenerate series of $\text{Sp}(8, \mathbf{R})$ and of $\text{Sp}(12, \mathbf{R})$ at the reducibility points are given in Fig. 6 and Fig. 7, respectively. We have used a blackened circle to denote a unitary constituent, and a unblackened circle to denote a non-unitary constituent. The module diagrams for other cases can be worked out similarly, by using Theorem 5.2.

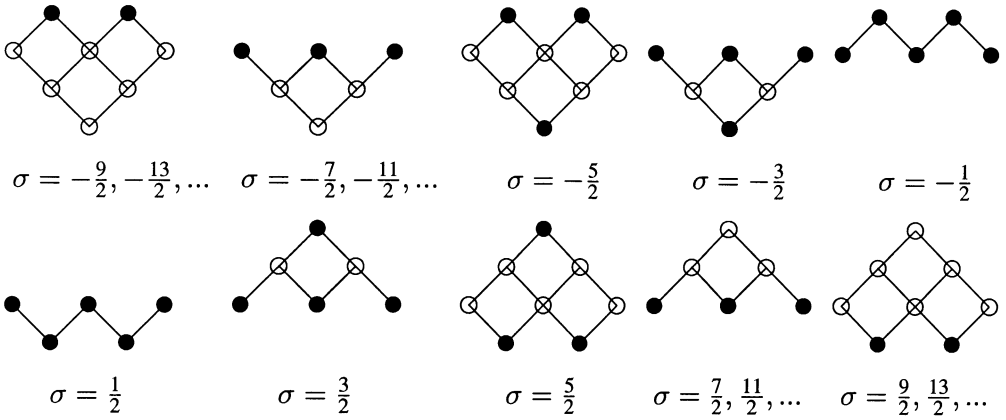


Fig. 6.

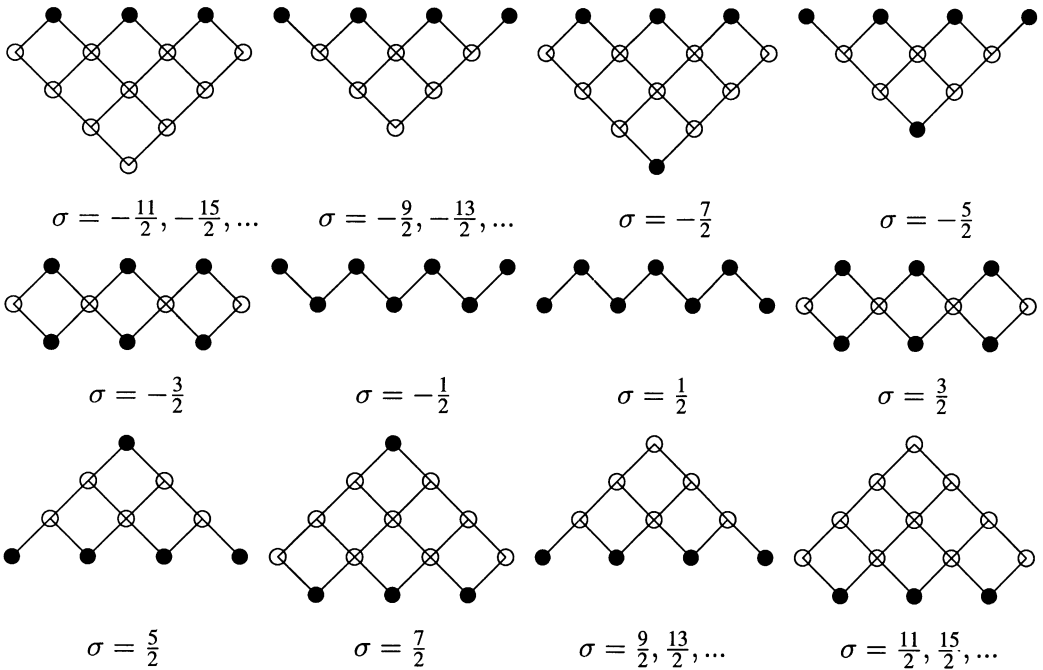


Fig. 7.

Case: $n = 2m + 1$ odd. Our analysis in this case will be similar to the $U(n, n)$ case ([11]). We need to consider two subcases: α odd and α even.

Subcase: α odd. Recall that the potential barriers ℓ_j^+ and ℓ_j^- are defined by the equations $x_j = \alpha + j - 1$ and $x_j = -(\alpha + n - j)$ respectively. Since both α and

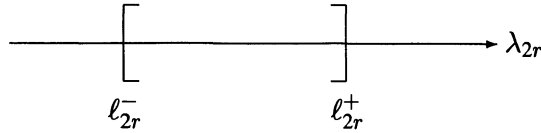


Fig. 8.

n are odd, $\alpha + j - 1$ and $-(\alpha + n - j)$ are odd or even depends on whether j is odd or even. For odd j , both ℓ_j^+ and ℓ_j^- do not contain any K -types of $S^{\alpha,+}(X^o)$. Hence they play no role in the transition of the K -types. On the other hand, for even j , both ℓ_j^+ and ℓ_j^- affect the transition of K -types.

Let

$$G(\alpha) = (\alpha + j - 1) - [-(\alpha + n - j)] = 2\alpha + n - 1. \tag{5.11}$$

We can think of $G(\alpha)$ as the gap between the two barriers along a axis. Note that in this case it is always even. We shall first consider the case when $G(\alpha) \geq 0$. This occurs when $\alpha \geq -m$. The barriers along the even axes can be visualized as in Fig. 8.

For each $1 \leq r \leq m$, we let

$$\begin{aligned} A_1^r &= \{\lambda \in \Lambda_e^+ : \lambda_{2r} < -(\alpha + n - 2r)\}, \\ A_2^r &= \{\lambda \in \Lambda_e^+ : -(\alpha + n - 2r) \leq \lambda_{2r} \leq \alpha + 2r - 1\}, \\ A_3^r &= \{\lambda \in \Lambda_e^+ : \lambda_{2r} > \alpha + 2r - 1\}. \end{aligned}$$

For a n -tuple $a = (a_1, \dots, a_n)$ of integers with $a_j = 1, 2$ or 3 , we set

$$R_a = A_{a_1}^1 \cap \dots \cap A_{a_m}^m.$$

Let $D = \{a : a_j = 1, 2$ or $3; a_1 \geq a_2 \geq \dots \geq a_m\}$. Then the dominance condition on Λ_e^+ forces $R_a = \emptyset$ for all $a \notin D$. Let s and t be nonnegative integers such that $s + t \leq n$. Let $a(s, t)$ be the n -tuple of integers given by

$$a(s, t) = (\overbrace{3, \dots, 3}^s, 2, \dots, 2, \overbrace{1, \dots, 1}^t).$$

Thus D is the set of all such $a(s, t)$. Now the definition of R_a leads to the transition relation as shown in Fig. 9. Thus to understand the module structure of $I^+(\sigma)$ in this case, it remains to determine which irreducible constituents are nonempty.

For $a \in D$, let $\ell_2(a)$ denote the number of entries of a which are equal to 2. Elementary arguments similar to the proof of Lemma 6.8 of [11] shows the following.

LEMMA 5.3. *Let α be an odd integer such that $\alpha \geq -m$.*

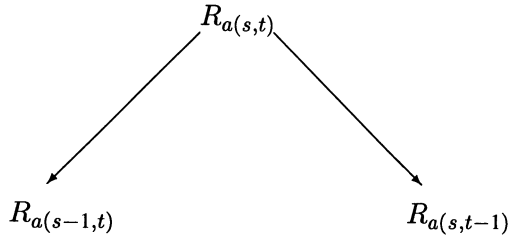


Fig. 9.

- (a) $R_a \neq \emptyset$ if and only if $\ell_2(a) \leq \alpha + m + 1$.
- (b) R_a is unitarizable if and only if $\ell_2(a) = \alpha + m + 1$.

In particular, if $\alpha \geq 1$, then $S^{\alpha,+}(X^o)$ has no unitary constituent.

We note that α is odd if and only if $\sigma \equiv m \pmod{2}$. Thus Lemma 5.3 leads to the following theorem:

THEOREM 5.4. *Let $n = 2m + 1$ be an odd positive integer and let σ be an negative integer such that $\sigma \equiv m \pmod{2}$. Then*

$$I^+(\sigma) = \oplus \{R_a : \ell_2(a) \leq r\},$$

where $r = \min(|\sigma|, m)$, and the socle series of $I^+(\sigma)$ is given by

$$\text{Soc}^j(I^+(\sigma)) = \begin{cases} \oplus \{R_a : r - j + 1 \leq \ell_2(a) \leq r\} & 1 \leq j \leq r, \\ I^+(\sigma) & j \geq r + 1. \end{cases}$$

Moreover, a constituent R_a of $I^+(\sigma)$ is unitarizable if and only if $-m \leq \sigma \leq -1$ and $\ell_2(a) = |\sigma|$.

Next we consider the case when $G(\alpha) = -2$. This occurs when $\alpha = -m - 1$, or equivalently $\sigma = 0$. Thus this is the reducibility point on the unitary axis. In this case, the barriers along the even axes can be visualized as in Fig. 10. Note that ℓ_{2r}^+ and ℓ_{2r}^- are at a distance of 2 units apart.

For $0 \leq p \leq m$, let

$$U_p = \{\mu \in \Lambda_e^+ : \mu_{2p+2} \leq -m + 2p \leq \mu_{2p}\}.$$

It is clear from Fig. 10 that each U_p is an irreducible submodule of $S^{-m-1,+}(X^o)$, and

$$S^{-m-1,+}(X^o) = \bigoplus_{p=0}^m U_p. \tag{5.12}$$

Hence we have proved:

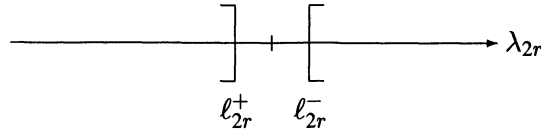


Fig. 10.

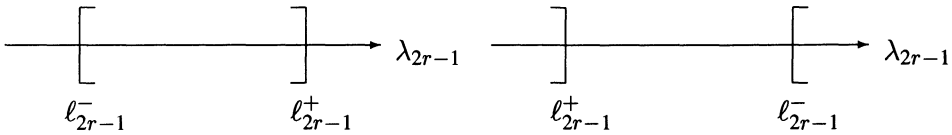


Fig. 11.

THEOREM 5.5. *If $n = 2m + 1$ where m is an even integer, then*

$$I^+(0) = \bigoplus_{p=0}^m U_p$$

is a direct sum of $m + 1$ irreducible submodules.

If σ is a positive integer such that $\sigma \equiv m \pmod{2}$, then module structure of $I^+(\sigma)$ can be deduced from that of $I^+(-\sigma)$, which is given in Theorem 5.4.

Subcase: α even. This case is very similar to the case α odd and n odd, so we shall only state the final results. Note that in this case, the effective barriers occur along the odd axes instead of the even axes. This can be visualized as in Fig. 11.

If $G(\alpha) \geq 0$ (i.e., when $\alpha \geq -m$), we define for each $1 \leq r \leq m + 1$,

$$\begin{aligned} E_1^r &= \{\lambda \in \Lambda_e^+ : \lambda_{2r-1} < -(\alpha + n - 2r + 1)\}, \\ E_2^r &= \{\lambda \in \Lambda_e^+ : -(\alpha + n - 2r + 1) \leq \lambda_{2r-1} \leq \alpha + 2r - 2\}, \\ E_3^r &= \{\lambda \in \Lambda_e^+ : \lambda_{2r-1} > \alpha + 2r - 2\}. \end{aligned}$$

For a $(m + 1)$ -tuple $a = (a_1, \dots, a_{m+1})$ such that $a_j = 1, 2, 3$ for all j and such that $a_1 \geq a_1 \geq \dots \geq a_{m+1}$, we let

$$W_a = E_{a_1}^1 \cap \dots \cap E_{a_{m+1}}^{m+1}.$$

As before, we let $\ell_2(a)$ be the number of entries of a which are equal to 2. We also note that α is even if and only if $\sigma \equiv m + 1 \pmod{2}$.

THEOREM 5.6. *Let $n = 2m + 1$ be a positive odd integer.*

(i) If σ is a negative integer such that $\sigma \equiv m + 1 \pmod{2}$, then

$$I^+(\sigma) = \oplus\{W_a : \ell_2(a) \leq r\},$$

where $r = \min(|\sigma|, m + 1)$, and the socle series of $I^+(\sigma)$ is given by

$$\text{Soc}^j(I^+(\sigma)) = \begin{cases} \oplus\{W_a : r - j + 1 \leq \ell_2(a) \leq r\} & 1 \leq j \leq r, \\ I^+(\sigma) & j \geq r + 1. \end{cases}$$

Moreover, a constituent W_a of $I^+(\sigma)$ is unitarizable if and only if either $a = (3, 3, \dots, 3)$ or $(1, 1, \dots, 1)$ or if $-m - 1 \leq \sigma \leq -1$ and $\ell_2(a) = |\sigma|$.

(ii) If m is odd, then for $0 \leq p \leq m + 1$, the subspace

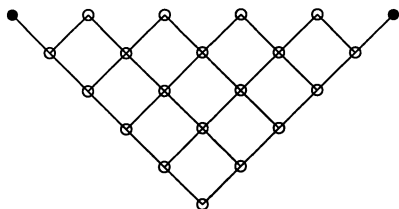
$$F_p = \oplus\{V_\mu : \mu_{2p+1} \leq -m + 2p - 1 \leq \mu_{2p-1}\}$$

is an irreducible submodule of $I^+(0)$, and

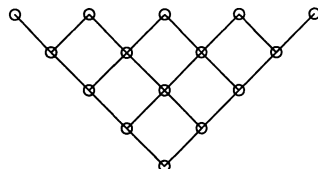
$$I^+(0) = \bigoplus_{p=0}^{m+1} F_p.$$

As before, if σ is a positive integer such that $\sigma \equiv m + 1 \pmod{2}$, then the module structure of $I^+(\sigma)$ can be deduced from the structure of $I^+(-\sigma)$, which is given in Theorem 5.6.

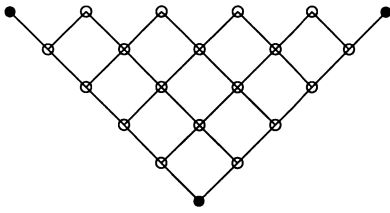
We shall now use Theorem 5.4, 5.5 and 5.6 to construct the module diagram of $I^+(\sigma)$ for two typical cases. The following are the diagrams for the $\text{Sp}(18, \mathbf{R})$ modules $I^+(\sigma)$ with $\sigma = 0, -1, -2, \dots$. If σ' is an positive integer, then the diagram for $I^+(\sigma')$ can be obtained by inverting the diagram of $I^+(-\sigma')$. Recall that a blackened circle denote a unitary constituent and a unblackened circle denote a non-unitary constituent.



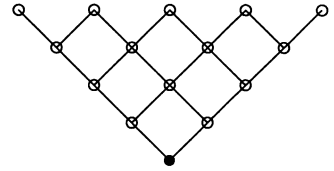
$\sigma = -7, -9, \dots$



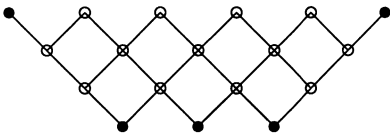
$\sigma = -6, -8, \dots$



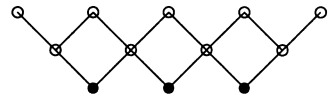
$\sigma = -5$



$\sigma = -4$



$\sigma = -3$



$\sigma = -2$



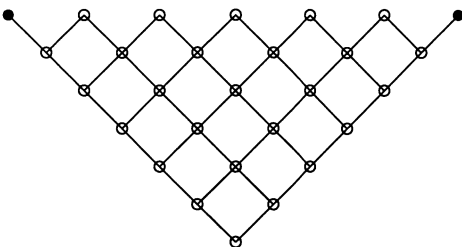
$\sigma = -1$



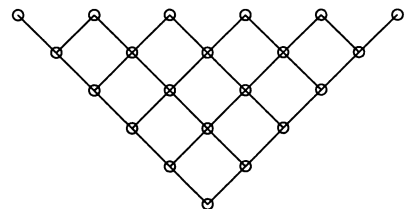
$\sigma = 0$

Fig. 12.

Finally we construct the diagrams for the $Sp(22, \mathbf{R})$ modules $I^+(\sigma)$ for $\sigma = 0, -1, -2, \dots$



$\sigma = -8, -10, \dots$



$\sigma = -7, -9, \dots$

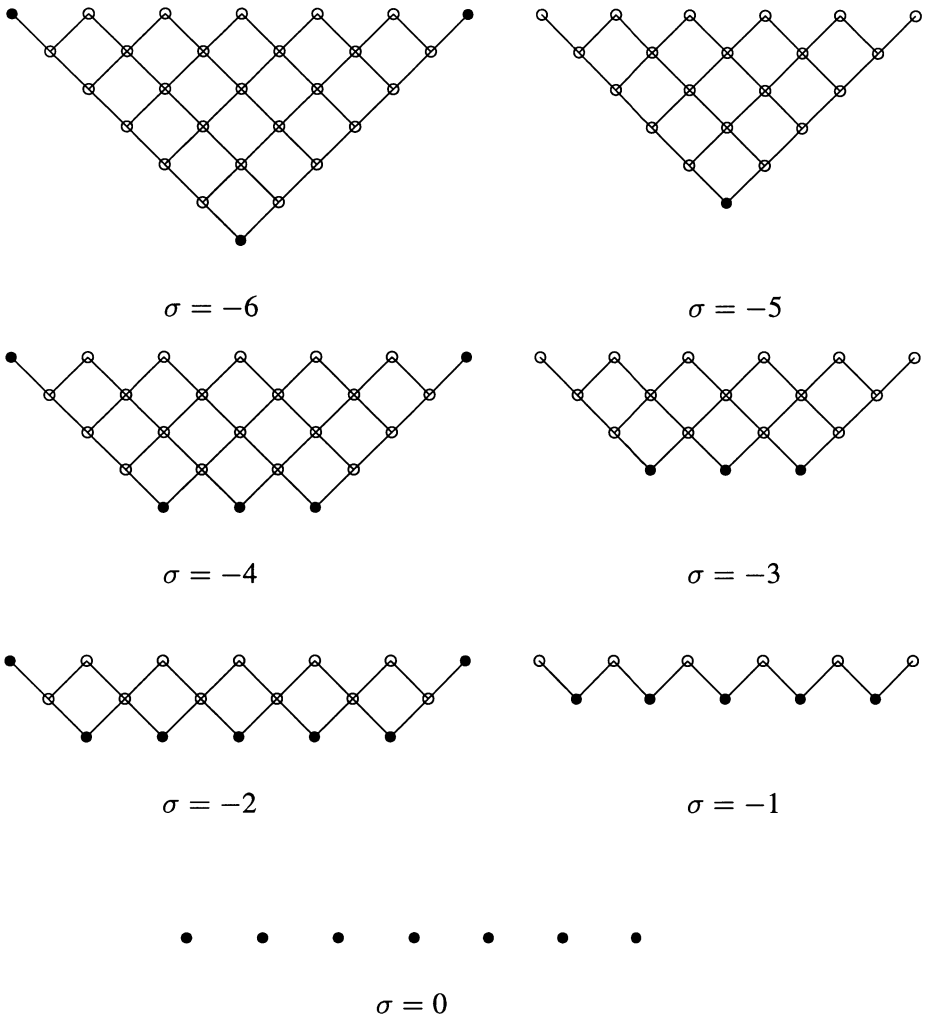


Fig. 13.

Acknowledgements

The author would like to thank the referee for many helpful suggestions.

References

1. Alperin, J.: Diagrams for modules, *J. Pure Appl. Algebra* 16 (1980) 111–119.
2. Bargmann, V.: Irreducible unitary representations of the Lorentz group, *Ann. of Math.* 48 (1947) 568–640.
3. Bernstein, I. N., Gelfand, I. M. and Gelfand, S. I.: Model of representations of Lie groups, *Sel. Math. Sov.* 1 (1981) 121–142.

4. Carter, R.: Raising and lowering operators for \mathfrak{sl}_n , with applications to orthogonal bases of \mathfrak{sl}_n -modules, in The Arcata Conference on Representations of Finite Groups, *Proc. Sympos. Pure Math.* 47, Part 2, 351–366, Amer. Math. Soc., Providence, 1987.
5. Carter, R. and Lusztig, G.: On the modular representations of the general linear and symmetric groups, *Math. Z.* 136 (1974) 193–242.
6. Goodearl, K. and Warfield, R.: *An Introduction to Noncommutative Rings*, London Mathematical Society student Texts 16, Cambridge University Press, Cambridge, 1989.
7. Howe, R. and Lee, S.: Degenerate principal series representations of $\mathrm{GL}(n, \mathbf{C})$ and $\mathrm{GL}(n, \mathbf{R})$, in preparation.
8. Howe, R. and Tan, E.: Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations, *Bull. Amer. Math. Soc.* 28 (1993) 1–74.
9. Johnson, K.: Degenerate principal series on tube type domains, *Contemp. Math.* 138 (1992) 175–187.
10. Kudla, S. and Rallis, S.: Degenerate principal series and invariant distributions, *Israel J. Math.* 69 (1990) 25–45.
11. Lee, S.: On some degenerate principal series representations of $\mathrm{U}(n, n)$, *J. of Funct. Anal.* 126 (1994) 305–366.
12. Sahi, S.: Unitary representations on the Shilov boundary of a symmetric tube domain, in Representations of Groups and Algebras, *Contemp. Math.* 145 (1993) 275–286, Amer. Math. Soc., Providence.
13. Sahi, S.: Jordan algebras and degenerate principal series, *preprint*.
14. Varadarajan, V.: *An Introduction to Harmonic Analysis on Semisimple Lie Groups*, Cambridge Studies in Advanced Mathematics, Vol 16, Cambridge Univ. Press, Cambridge, 1989.
15. Wallach, N. R.: *Real Reductive Groups I*, Academic Press, 1988.
16. Zhang, G.: Jordan algebras and generalized principal series representation, *preprint*.