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Lang's conjecture in characteristic p: an explicit bound

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Let K be a function field in one variable with constant field k and denote by K_a, K_s its algebraic and separable closures, respectively. Let X/K be an algebraic curve of genus at least two. The function field analogue of Mordell's conjecture states that $X(K)$ is finite unless X is K_a-isomorphic to a curve defined over k, in which case X is called isotrivial. This was first proved by Manin [Man] in characteristic zero and, shortly after, another proof was given by Grauert [Gra] and this proof was then adapted by Samuel [Sa] to positive characteristic. Since then several different proofs were given for Mordell's conjecture over function fields. In particular Szpiro [Sz] was the first to prove an effective version of Mordell's conjecture in characteristic p.

Mordell's conjecture was generalized by Lang [L] (and proved through work of Raynaud [R] and Faltings [F]). An analogue of Lang's conjecture over function fields of characteristic p was proved by the second author and Abramovich [V] [AV]. The aim of the present paper is to prove an effective version of Lang's conjecture in characteristic p. Our approach consists of combining the approaches in [B1] and [V] [AV] which in their turn were motivated by Manin's work [Man]. Here is our main result:

THEOREM. *Let K be a function field in one variable and characteristic $p > 0$. Let X be a smooth projective curve of genus $g \geqslant 2$ over K embedded into its Jacobian J. Assume X has non-zero Kodaira-Spencer class (equivalently, X is not defined over K^p). If Γ is a subgroup of $J(K_s)$ such that $\Gamma/p\Gamma$ is finite, then:*

$$\sharp(X \cap \Gamma) \leqslant \sharp(\Gamma/p\Gamma) \cdot p^g \cdot 3^g \cdot (8g-2) \cdot g!$$

The equivalence of the vanishing of the Kodaira-Spencer class of X and X being defined over K^p is proved in [V], Lemma 1.

We stress the fact that we do not assume Γ is finitely generated, which is the main feature in Lang's conjecture that distinguishes it from the Mordell conjecture.

A similar result in characteristic zero was obtained by the first author [B4] but the bound there is huge as compared to the bound here. This is a reflection of the fact that the characteristic p case is in some sense 'easier' than the characteristic zero case.

The question of the existence of this type of bounds for Lang's conjecture was raised by Mazur in [Maz] and is quite different from what one understands by 'effective Mordell'. In particular, even in the special case when Γ in our Theorem is finitely generated, our bound is not a consequence of Szpiro's [Sz]. Indeed, assuming we are in the hypothesis of the Theorem above with Γ finitely generated, let $K_1 \subset K_s$ be the field generated over K by the coordinates of the points in Γ. What Szpiro's 'effective Mordell' yields is a bound for the height of the points in $X(K_1)$ that depends on g = genus of X, p = characteristic of k, q_1 = genus of K_1 and s_1 = number of points of bad reduction of a semistable model of $X \otimes K_1/K_1$. It follows that $\sharp(X \cap \Gamma)$ is bounded by a constant that depends on g, p, q_1, s_1. But of course q_1, s_1 are not bounded by a constant that depends on $\sharp(\Gamma/p\Gamma)$ only; we may always keep $\sharp(\Gamma/p\Gamma)$ constant and vary Γ so that both q_1 and s_1 go to infinity.

In order to prove the Theorem let us start by recalling a construction from [B1]. Assume we have fixed a derivation $\delta = \partial/\partial t$ of K where $t \in K$ is a separable transcendence basis of K/k. Then for any K-scheme X one defines the 'first jet scheme along δ' by the formula

$$X^1 := \mathrm{Spec}(S(\Omega_{X/k})/I),$$

where I is the ideal generated by sections of the form $df - \delta f$ ($f \in \mathcal{O}_X$). This object was analysed in [B2], [B3] where the characteristic zero case only was considered. But many of the facts proved there extend, with identical proofs, in positive characteristic. In particular the following hold. Assume X above is a smooth variety over K. Then exactly as in [B1], p. 1396, X^1 identifies with the torsor for the tangent bundle $TX := \mathrm{Spec}(S(\Omega_{X/K}))$ corresponding to the Kodaira-Spencer class

$$\rho(\delta) \in H^1(X, T_{X/K})$$

(where $\rho \colon \mathrm{Der}_k K \to H^1(X, T_{X/K})$ is the Kodaira-Spencer map; this map played various roles in virtually all approaches to the Mordell and Lang conjectures over function fields.) So exactly as in [B2], section 1, we may write X^1 as the complement of a divisor in a projective bundle:

$$X^1 = \mathbf{P}(E) \backslash \mathbf{P}(\Omega_{X/K}),$$

where E is the vector bundle defined by the extension

$$0 \to \mathcal{O}_X \to E \to \Omega_{X/K} \to 0$$

corresponding to $\rho(\delta) \in H^1(X, T_{X/K}) \simeq \mathrm{Ext}^1(\Omega_{X/K}, \mathcal{O}_X)$.

If X/K is a smooth group scheme then so is X^1/K.

Also, since δ lifts to a derivation of K_s, there is an obvious 'lifting map'

$$\nabla \colon X(K_s) \to X^1(K_s)$$

which in case X/K is a group is a homomorphism.

The following is the characteristic p analogue of a fact from [B3], (2.2):

LEMMA. *If X/K is a smooth projective curve of genus $\geqslant 2$ with non zero-Kodaira-Spencer class then X^1 is an affine surface.*

Proof. By the discussion preceding the Lemma it is enough to check that the divisor $\mathbf{P}(\Omega_{X/K})$ is ample in $\mathbf{P}(E)$, equivalently that E is ample, which is the same as E_a being ample (where E_a is the pull back of E on $X_a := X \otimes_K K_a$). Let $F: X_a \to X_a$ be the absolute Frobenius (viewed as a scheme morphism over the integers). Assume E_a is not ample and seek a contradiction. By the characteristic p analogue of 'Gieseker's Theorem' [Gie] due to Martin-Deschamps [MD] it follows that there exists a power $F^m: X_a \to X_a$ of F such that the pull back of the sequence

$$(*) \quad 0 \to \mathcal{O}_{X_a} \to E \to \Omega_{X_a/K_a} \to 0$$

via F^m splits. Now, since Ω_{X_a/K_a} has degree $2g-2 > (2g-2)/p$, a result of Tango [T] Theorem 15 p. 73 implies that the sequence $(*)$ itself must be split, which contradicts the fact that the Kodaira-Spencer class of X/K is non zero. This completes the proof of the Lemma.

Proof of the Theorem. The closed immersion $X \subset J$ induces a closed immersion $X^1 \subset J^1$. For any point $P \in X(K_s) \cap pJ(K_s)$ we have $\nabla(P) \in X^1(K_s) \cap pJ^1(K_s)$. Since J^1 is an extension of J by a vector group (same argument as in [B2] (2.2)) the algebraic group $B = pJ^1$ coincides with the maximum abelian subvariety of J^1 and the projection $B \to J$ is an isogeny. Moreover by [Ro], p. 704, Lemma 2, the natural isogeny (the Verschiebung) $J^{(p)} \to J$ factors through $B \to J$. Since Verschiebung is of degree p^g, $B \to J$ has degree at most p^g.

In order to prove the Theorem it is obviously enough to prove that, over K_a, $X^1 \cap B$ is finite, of cardinality at most $p^g \cdot 3^g \cdot (8g-2) \cdot g!$ Finiteness follows trivially from our Lemma above: X^1 is affine and B is complete and both are closed in J^1 so their intersection is closed in both X^1 and B, so $X^1 \cap B$ is both affine and complete, hence it is finite over K. To estimate its cardinality we use Bézout's theorem in Fulton's form, along the lines of [B4] (except that here we do not need any 'iteration' and we do not have to take multiplicities into account!).

Recall that X^1 and J^1 are Zariski locally trivial principally homogeneous spaces for the tangent bundles of X and J respectively, corresponding to the Kodaira-Spencer class. Let

$$0 \to \mathcal{O}_X \to E_X \to \Omega_{X/K} \to 0$$

$$0 \to \mathcal{O}_J \to E_J \to \Omega_{J/K} \to 0$$

be the corresponding extensions. Consider the divisors $D_X = \mathbf{P}(\Omega_{X/K}) \subset \mathbf{P}(E_X)$ and $D_J = \mathbf{P}(\Omega_{J/K}) \subset \mathbf{P}(E_J)$. Since $\Omega_{J/K} \simeq \mathcal{O}_J^g$ we have $D_J \simeq J \times \mathbf{P}^{g-1}$.

Recall also that these divisors belong to the linear systems associated to $\mathcal{O}_{\mathbf{P}(E_X)}(1)$ and $\mathcal{O}_{\mathbf{P}(E_J)}(1)$ respectively and that we have identifications $X^1 \simeq \mathbf{P}(E_X)\backslash D_X$ and $J^1 \simeq \mathbf{P}(E_J)\backslash D_J$. Let $\alpha: X \to J$ be the inclusion. There is a natural restriction homomorphism $\alpha^* E_J \to E_X$ prolonging the natural homomorphism $\alpha^*\Omega_{J/K} \to \Omega_{X/K}$ since $E_X = \Omega_{X/K^p} = \Omega_{X/k}$ and similarly for E_J. The homomorphism $\alpha^* E_J \to E_X$ is clearly surjective so it induces a closed embedding $\mathbf{P}(E_X) \subset \mathbf{P}(E_J)$ prolonging the embedding $X^1 \subset J^1$. By abuse we shall still denote by π_X, π_J the projections $\mathbf{P}(E_X) \to X, \mathbf{P}(E_J) \to J$.

Claim. The line bundle $\mathcal{H} := \pi_J^*\mathcal{O}_J(3\Theta) \otimes \mathcal{O}_{\mathbf{P}(E_J)}(1)$ is very ample on $\mathbf{P}(E_J)$. (Here Θ is the theta divisor on J.)

To check the Claim, note first that the trace of the linear system $|\mathcal{H}|$ on D_J is very ample. Indeed

$$\mathcal{H} \otimes \mathcal{O}_{D_J} = \mathcal{H} \otimes \mathcal{O}_{\mathbf{P}(\Omega_{J/K})} = p_1^*\mathcal{O}_J(3\Theta) \otimes p_2^*\mathcal{O}_{\mathbf{P}^{g-1}}(1),$$

where p_1, p_2 are the two projections of $D_J = J \times \mathbf{P}^{g-1}$ onto the factors. So $\mathcal{H} \otimes \mathcal{O}_{D_J}$ is very ample on D_J, cf. [Mum] p. 163. Furthermore we have an exact sequence

$$H^0(\mathbf{P}(E_J), \mathcal{H}) \to H^0(D_J, \mathcal{H} \otimes \mathcal{O}_{D_J}) \to H^1(\mathbf{P}(E_J), \pi_J^*\mathcal{O}_J(3\Theta)).$$

But the H^1 above is zero (use the Leray spectral sequence and the vanishing theorem in [Mum] p. 150) so the trace of $|\mathcal{H}|$ on D_J is a complete linear system and hence is very ample. In particular $|\mathcal{H}|$ separates points of D_J and 'vectors tangent to D_J'. Since $|\mathcal{H}|$ has no base points outside D_J either, it follows that $|\mathcal{H}|$ is base point free on $\mathbf{P}(E_J)$. Hence $|\mathcal{H}|$ restricted to the fibres of π_J is base point free. Since any base point free linear subsystem of $|\mathcal{O}_{\mathbf{P}^g}(1)|$ equals actually the whole of $|\mathcal{O}_{\mathbf{P}^g}(1)|$ it follows that $|\mathcal{H}|$ separates points in each fibre of π_J and separates 'vectors tangent to each fibre'. All these imply that $|\mathcal{H}|$ separate points and tangent vectors on the whole of $\mathbf{P}(E_J)$ and our Claim is proved.

Our last step is to compute the degrees $\deg_{\mathcal{H}}\mathbf{P}(E_X)$ and $\deg_{\mathcal{H}} B$ of $\mathbf{P}(E_X)$ and B respectively, as subvarieties of $\mathbf{P}(E_J)$ with respect to the embedding defined by $\mathcal{H}$. Note that

$$\mathcal{H} \otimes \mathcal{O}_{\mathbf{P}(E_X)} = \pi_X^*\mathcal{O}_X(3\Theta) \otimes \mathcal{O}_{\mathbf{P}(E_X)}(1).$$

We may compute the selfintersection

$$(\mathcal{O}_{\mathbf{P}(E_X)}(1) \cdot \mathcal{O}_{\mathbf{P}(E_X)}(1))_{\mathbf{P}(E_X)} = \deg \Omega_{X/K} = 2g - 2$$

and since $(\Theta \cdot X)_J = g$ we get

$$\deg_{\mathcal{H}}\mathbf{P}(E_X) = 2g - 2 + 6g = 8g - 2.$$

On the other hand we have

$$\mathcal{H} \otimes \mathcal{O}_B \simeq \pi^* \mathcal{O}_J(3\Theta),$$

where $\pi: B \to J$ is the projection which we already know has degree at most p^g. So we get, using $(\Theta^g)_J = g!$, that

$$\deg_{\mathcal{H}} B = p^g \cdot 3^g \cdot g!$$

Now Bezout's theorem in Fulton's form [Fu] p. 148, says that the number of irreducible components in the intersection of two projective varieties of degrees d_1, d_2 cannot exceed $d_1 d_2$. In particular

$$\sharp(X^1 \cap B) \leqslant \deg_{\mathcal{H}} \mathbf{P}(E_X) \cdot \deg_{\mathcal{H}} B \leqslant (8g-2) \cdot p^g \cdot 3^g \cdot g!$$

and our Theorem is proved.

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References

[AV] Abramovich, D., Voloch, J. F.: Toward a proof of the Mordell-Lang conjecture in characteristic p, *Duke Math. J.* **5** (1992) 103–115.

[B1] Buium, A.: Intersections in jet spaces and a conjecture of Lang, S. *Annals of Math.* **136** (1992) 557–567.

[B2] Buium, A.: Geometry of differential polynomial functions I: algebraic groups, *Amer. J. Math.* 115, **6** (1993) 1385–1444.

[B3] Buium, A.: Geometry of differential polynomial functions II: algebraic curves, *Amer. J. Math.*, to appear.

[B4] Buium, A.: Effective bound for the geometric Lang conjecture, *Duke Math. J.* 71, **2** (1993) 475–499.

[F] Faltings, G.: Endlichkeitsatze für abelsche varietäten über Zahlkorpern, *Invent: Math.* **73** (1983) 349–366.

[Fu] Fulton, W.: Intersection Theory, Springer 1984.

[Gie] Gieseker, D.: P-ample bundles and their Chern classes, *Nagoya Math. J.* **43** (1971) 91–116.

[Gra] Grauert, H.: Mordell's Vermutung über rationale Punkte auf algebraische Kurven und Functionenkorper, *Publ. Math. IHES* **25** (1965).

[L] Lang, S.: Division points on curves, *Ann. Mat. Pura Appl.* (4) **70** (1965) 229–234.

[Man] Manin, Yu.: Rational points on algebraic curves over function fields, Izvestija Akad Nauk SSSR, *Mat. Ser. t.* **27** (1963) 1395–1440.

[MD] Martin-Deschamps, M.: Propriétés de descente des variétés a fibre cotangent ample, *Ann. Inst. Fourier*, 34, **3** (1984) 39–64.

[Maz] Mazur, B.: Arithmetic on curves, *Bull. Amer. Math. Soc.* 14, **2** (1986) 207–259.

[Mum] Mumford, D.: Abelian varieties, Oxford Univ. Press 1974.

[R] Raynaud, M.: Courbes sur une variété abélienne et points de torsion, *Invent. Math.* **71** (1983) 207–235.

[Ro] Rosenlicht, M.: Extensions of vector groups by abelian varieties, *Am. J. of Math.* **80** (1958) 685–714.

[Sa] Samuel, P.: Compléments a un article de Hans Grauert sur la conjecture de Mordell, *Publ. Math. IHES* **29** (1966) 55–62.

[Sz] Szpiro, L.: Propriétés numériques du faisceau dualisant relatif, Astérisque **86** (1981) 44–78.

[T] Tango, H.: On the behaviour of extensions of vector bundles under the Frobenius map, *Nagoya Math. J.* **48** (1972) 73–89.

[V] Voloch, J. F.: On the conjectures of Mordell and Lang in positive characteristic, *Invent. Math.* **104** (1991) 643–646.