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A Tate sequence for global units

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Let K/k be a finite Galois extension of number fields with Galois group G, and let S be a finite G-invariant set of primes of K which contains all the archimedean primes. This paper is concerned with deriving an exact sequence

$$0 \to E \to A \to B \to \nabla \to 0$$

of finitely generated $\mathbb{Z}G$-modules, in which E is the group of S-units of K, A is cohomologically trivial, B projective, and ∇ is uniquely determined, in a sense stronger than isomorphism, by K and S.

Such an exact sequence has been constructed by Tate [TN,TS], but only under the assumption that S is *large*, in the sense that the S-class number of K is 1 and that all ramified primes of K/k are in S. In this case ∇ has a very simple description: writing $\mathbb{Z}S$ for the permutation G-module with $\mathbb{Z}$-basis S, then ∇ is the kernel

$$\Delta S = \ker(\mathbb{Z}S \to \mathbb{Z})$$

of the augmentation map $\mathbb{Z}S \to \mathbb{Z}$, which sends every element of S to 1.

Our interest in Tate sequences comes from the observation that they enable the cohomological methods of class field theory to be used in the study of the G-module structure of E. In [GW], the Tate sequence is applied to discuss invariants of the $\mathbb{Z}G$-genus of E. If S is large, then more precise invariants of the location of E in its genus have been considered [RW] assuming, e.g., that K is totally real and G has odd order.

When S is not large, then ∇ turns out to be considerably more complicated than ΔS. It is given by an exact sequence of G-modules

$$0 \to cl \to \nabla \to \overline{\nabla} \to 0,$$

where cl is the S-class group of K and where $\overline{\nabla}$ is a $\mathbb{Z}G$-lattice rather like ΔS but incorporating also some ramification-theoretic information about the set S^{ram} of ramified primes of K/k which are *not* in S. Finally the extension class of the

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above exact sequence for ∇ is uniquely determined. In fact, it can be explicitly described in terms of the exact sequence

$$G(\tilde{K}/K) \hookrightarrow G(\tilde{K}/k) \twoheadrightarrow G$$

of Galois groups, where $\tilde{K}$ is the Hilbert S-class field of K, hence $G(\tilde{K}/K) \simeq cl$. Surprisingly, the proof of this makes essential use of global Weil groups.

1. Constructions and results

We fix the notation of the introduction. In particular, since the set S will not vary, we continue to not display it in our notation (so the K in C_K is to suggest independence from S). The purpose of this section is to sketch the main construction, to formulate some of the problems it raises and to outline our results on these problems. The actual proofs start in Section 4, with Section 2, Section 3 providing some necessary background.

We fix, for now, a choice $*$ of a representative for each orbit of the action of G on the primes of K. Many of the objects we consider will depend on $*$ but, for the sake of eventual clarity, we delay the discussion of this dependence to the end of this section. Let S_*, for a set S of primes of K, denote the intersection of S and $*$.

We begin the construction by choosing a finite G-invariant set of primes S', containing S and *larger* in the sense that:

(i) the S'-class number of K is 1
(ii) S' contains all primes which ramify in K/k
(iii) $\bigcup_{\mathfrak{p}\in S'} G_\mathfrak{p} = G$, where $G_\mathfrak{p}$ is the decomposition group of $\mathfrak{p}$ in K/k.

Such a set S' exists, by the Tchebotarev density theorem, and we will need to discuss the independence of our results from its choice, for which reason S' is explicitly kept in our notation.

The augmentation sequence $0\to\Delta G\to\mathbb{Z}G\to\mathbb{Z}\to 0$ induces an isomorphism $\delta': H^1(G,\mathrm{Hom}(\Delta G, C_K))\to H^2(G,\mathrm{Hom}(\mathbb{Z},C_K)) = H^2(G,C_K)$, since $\mathrm{Hom}(\mathbb{Z}G, C_K)$ is cohomologically trivial. So there is a unique $\alpha \in H^1(G,\mathrm{Hom}(\Delta G, C_K)) = \mathrm{Ext}^1_G(\Delta G, C_K)$ with $\delta'\alpha = u_{K/k}$, the global fundamental class. We choose an exact sequence

$$0\to C_K\to\mathfrak{V}\to\Delta G\to 0$$

of G-modules with extension class α.

For $\mathfrak{p} \in S_*$, there is, locally, an analogous exact sequence

$$0\to K_\mathfrak{p}^\times\to V_\mathfrak{p}\to\Delta G_\mathfrak{p}\to 0$$

of $G_\mathfrak{p}$-modules with extension class $\alpha_\mathfrak{p} \in H^1(G_\mathfrak{p},\mathrm{Hom}(\Delta G_\mathfrak{p}, K_\mathfrak{p}^\times)) = \mathrm{Ext}^1_{G_\mathfrak{p}}(\Delta G_\mathfrak{p}, K_\mathfrak{p}^\times)$ satisfying $\delta'\alpha_\mathfrak{p} = u_{K_\mathfrak{p}/k_\mathfrak{p}}$, the local fundamental class in $H^2(G_\mathfrak{p}, K_\mathfrak{p}^\times)$.

For $\mathfrak{p} \in S'_*, \mathfrak{p} \notin S_*$ we use a different local sequence with $K_\mathfrak{p}^\times$ replaced by $U_\mathfrak{p}$, the units of $K_\mathfrak{p}$. By [GW], which is reviewed in Section 3, there is a special *inertial* $\mathbb{Z}G_\mathfrak{p}$-lattice $W_\mathfrak{p}$ and a canonical class $\beta_\mathfrak{p} \in H^1(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, U_\mathfrak{p})) = \mathrm{Ext}^1_{G_\mathfrak{p}}(W_\mathfrak{p}, U_\mathfrak{p})$. As before, we take an exact sequence

$$0 \to U_\mathfrak{p} \to V_\mathfrak{p} \to W_\mathfrak{p} \to 0$$

of $G_\mathfrak{p}$-modules with extension class $\beta_\mathfrak{p}$.

We apply $\mathrm{ind}^G_{G_\mathfrak{p}}$ to these local exact sequences and take the direct sum of them over $\mathfrak{p} \in S'_*$. Glueing on the unit ideles of the form $\prod_{\mathfrak{p} \notin S'} U_\mathfrak{p}$, we get an exact sequence

$$0 \to J \to V_{S'} \to W_{S'} \to 0$$

of G-modules,

with $J = \prod_{\mathfrak{p} \in S} K_\mathfrak{p}^\times \times \prod_{\mathfrak{p} \notin S} U_\mathfrak{p}$ the S-ideles,

with $V_{S'} = \left(\bigoplus_{\mathfrak{p} \in S'_*} \mathrm{ind}^G_{G_\mathfrak{p}} V_\mathfrak{p} \right) \times \prod_{\mathfrak{p} \notin S'} U_\mathfrak{p}$

and with $W_{S'} = \left(\bigoplus_{\mathfrak{p} \in S_*} \mathrm{ind}^G_{G_\mathfrak{p}} \Delta G_\mathfrak{p} \right) \oplus \left(\bigoplus_{\mathfrak{p} \in S'_* \setminus S_*} \mathrm{ind}^G_{G_\mathfrak{p}} W_\mathfrak{p} \right)$.

Here we are identifying the $\prod_{\mathfrak{p} \in \mathcal{B}} K_\mathfrak{p}^\times$ in J, for a G-orbit $\mathcal{B}$ of primes, with $\mathrm{ind}^G_{G_\mathfrak{p}} K_\mathfrak{p}^\times$ for $\mathfrak{p} \in \mathcal{B}_*$.

There is a canonical G-homomorphism $J \to C_K$. For $\mathfrak{p} \in S_*$, the inclusion $\Delta G_\mathfrak{p} \to \Delta G$ of $G_\mathfrak{p}$-modules induces a canonical G-homomorphism $\mathrm{ind}^G_{G_\mathfrak{p}} \Delta G_\mathfrak{p} \to \Delta G$. For $\mathfrak{p} \in S'_* \setminus S_*, W_\mathfrak{p}$ comes equipped with a canonical $G_\mathfrak{p}$-homomorphism $W_\mathfrak{p} \to \Delta G_\mathfrak{p}$, which, composed with the map of the last sentence, yields a canonical $W_{S'} \to \Delta G$.

THEOREM 1. *There exist surjective G-homomorphisms θ in*

diagram 1
$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & V_{S'} & \longrightarrow & W_{S'} & \longrightarrow & 0 \\ & & \downarrow & & {}^{\theta}\!\downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_K & \longrightarrow & \mathfrak{V} & \longrightarrow & \Delta G & \longrightarrow & 0 \end{array}$$

with exact rows and maps as described above.

The theorem is essentially due to the matching of the global and local fundamental classes. It is proved in Section 4.

Suppose now that θ is such an epimorphism. Since $J \to C_K$ has kernel E, the S-units of K, and cokernel cl, the S-classgroup of K, the snake lemma gives an exact sequence

$$0 \to E \to A_\theta \to R_{S'} \xrightarrow{s} cl \to 0$$

of finitely generated G-modules, with $A_\theta = \ker \theta$ cohomologically trivial because $V_{S'}$ and $\mathfrak{V}$ are (Section 4), and with $R_{S'} = \ker(W_{S'} \to \Delta G)$ a $\mathbb{Z}G$-lattice of known structure. We call this a “Tate” sequence for S_*. In Section 4 we describe a process by which it can be transformed into

$$0 \to E \to A_\theta \to B_{S'} \to \nabla_\theta \to 0,$$

where $B_{S'}$ is a finitely generated stably free $\mathbb{Z}G$-module and ∇_θ an extension of cl by a known $\mathbb{Z}G$-lattice $\overline{\nabla}_*$ which is independent of θ and S' (but not of $*$). The latter sequence will be referred to as a Tate sequence for S_*, since it gives back Tate's original sequence [TS, p. 54] provided that S is a larger set.

THEOREM 2. (a) *All surjective maps θ determine the same snake class $[s] \in H^0(G, \mathrm{Hom}(R_{S'}, cl))$. (Here, and always, H^0 is a Tate cohomology group.)*

(b) $[\nabla_\theta] \in \mathrm{Ext}^1_G(\overline{\nabla}_*, cl)$ *is independent of θ.*

(c) *The class $[A_\theta] \in K_0(\mathbb{Z}G)$ does not depend on θ.*

The proof of (a) and (b) is given in Section 5; (c) is shown in Section 6. Its main ingredients are the vanishing of $H^1(G, C_K)$ and the relation of $W_\mathfrak{p}$ to the decomposition group of $\tilde{K}/k$ for the ramified primes $\mathfrak{p} \in S'_* \setminus S_*$, where $\tilde{K}$ is again the Hilbert S-class field of K.

REMARK. A finitely generated cohomologically trivial $\mathbb{Z}G$-module A is, up to stable isomorphism, determined by its $\mathbb{Z}$-torsion submodule and its class $[A]$ in $K_0(\mathbb{Z}G)$ [GW, (2.3)]. Here $[A] = [P_1] - [P_2]$, if $0 \to P_2 \to P_1 \to A \to 0$ is a projective resolution for the cohomologically trivial A. It follows from Theorem 2(c) that the stable isomorphism class of A_θ is independent of θ, since A_θ has torsion μ_K, the roots of unity in K.

So far, the larger set S' has been kept fixed. Because of Theorem 2 we have for each larger set S' a uniquely determined extension class $[\nabla_{S'}] \in \mathrm{Ext}^1_G(\overline{\nabla}_*, cl)$. Equally well, there is a unique stable isomorphism class $[A_{S'}]$.

THEOREM 3. (a) *If $S' \subseteq S''$ are larger sets containing S, then the natural inclusion $R_{S'} \to R_{S''}$ induces an isomorphism*

$$H^0(G, \mathrm{Hom}(R_{S''}, cl)) \to H^0(G, \mathrm{Hom}(R_{S'}, cl))$$

which takes the snake class for S''_* to the snake class for S'_*.

(b) $[\nabla] = [\nabla_{S'}] \in \mathrm{Ext}^1_G(\overline{\nabla}_*, cl)$ *is independent of S'.*

(c) $[A_{S'}] - [B_{S'}] \in K_0(\mathbb{Z}G)$ *is independent of S'.*

The proof is given in Section 7.

Let r_S be the number of G-orbits of ramified primes of K/k outside S. Define the *Chinburg class* $\Omega_m^* \in K_0(\mathbb{Z}G)$ by setting

$$\Omega_m^* = [A] - [B] + r_S[\mathbb{Z}G]$$

in the notation of Theorem 3(c). The method of proof of Theorem 3 implies the

COROLLARY. $\Omega_m^* \in \mathrm{Cl}(\mathbb{Z}G)$ is independent of S (but not yet of $*$).

Each Tate sequence

$$0 \to E \to A \to B \to \nabla \to 0$$

defines an extension class $\tau \in \mathrm{Ext}_G^2(\nabla, E)$, whose uniqueness we discuss next. By Theorem 2(b), 3(b) we know that the ∇ which appears in a Tate sequence is uniquely determined up to *admissible* G-module isomorphisms, i.e. those which make

$$\begin{array}{ccccccccc} 0 & \longrightarrow & cl & \longrightarrow & \nabla & \longrightarrow & \overline{\nabla}_* & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & cl & \longrightarrow & \nabla' & \longrightarrow & \overline{\nabla}_* & \longrightarrow & 0 \end{array}$$

commute. So the strongest uniqueness statement possible appears to be

THEOREM 4. *Let $\tau \in \mathrm{Ext}_G^2(\nabla, E), \tau' \in \mathrm{Ext}_G^2(\nabla', E)$ be the extension classes of two Tate sequences for S_*. Then there exists an admissible isomorphism $h\colon \nabla \to \nabla'$ which takes τ' to τ.*

Most of the proof of this, i.e. variation of θ, is in Section 6, with the conclusion in Section 7. So the analogue of Tate's *canonical class* [TN] in $\mathrm{Ext}_G^2(\nabla, E)$ is now an orbit of the admissible automorphisms of ∇.

In addition to problems of uniqueness there is the question of identification of our objects. We next turn to a description of the snake map s. It is again based on the Galois theoretic behaviour of primes in the Hilbert S-class extension $\tilde{K}/k$. We outline the construction.

Galois theory provides, for each $\mathfrak{p} \in S'_*$, a natural commutative diagram

$$\begin{array}{ccccc} W(K_\mathfrak{p}^{nr}/K_\mathfrak{p}) & \hookrightarrow & W(K_\mathfrak{p}^{nr}/k_\mathfrak{p}) & \twoheadrightarrow & G_\mathfrak{p} \\ \downarrow & & \downarrow & & \downarrow \\ G(\tilde{K}/K) & \hookrightarrow & G(\tilde{K}/k) & \twoheadrightarrow & G, \end{array}$$

where $W(K_\mathfrak{p}^{nr}/K_\mathfrak{p}), W(K_\mathfrak{p}^{nr}/k_\mathfrak{p})$ denote the Weil groups of the maximal unramified extension of $K_\mathfrak{p}$ over $K_\mathfrak{p}$ and $k_\mathfrak{p}$, respectively (Section 3). By means of the *translationfunctor* $\mathbf{t}$ of Section 2, which takes short exact sequences $A \hookrightarrow X \twoheadrightarrow G$ of groups, with A abelian, to short exact G-module sequences $0\to A\to M\to \Delta G\to 0$, we get a corresponding diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & W_{\mathfrak{p}} & \longrightarrow & \Delta G_{\mathfrak{p}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G(\tilde{K}/K) & \longrightarrow & H & \longrightarrow & \Delta G & \longrightarrow & 0, \end{array}$$

in which the top row is the canonical one (and $\mathbb{Z}$ has been identified with $W(K_{\mathfrak{p}}^{nr}/K_{\mathfrak{p}})$) and in which the left vertical map sends $1 \in \mathbb{Z}$ to the Artin symbol $\left(\frac{\tilde{K}/K}{\mathfrak{p}}\right)$. In particular, if $\mathfrak{p} \in S$, then 1 is sent to 1 and so the map $W_{\mathfrak{p}}\to H$ factors through a map $\Delta G_{\mathfrak{p}}\to H$. Consequently, on putting all these diagrams, for $\mathfrak{p} \in S'_*$, together in the usual way we arrive at a well-defined map $\tilde{\sigma}\colon W_{S'}\to H$.

THEOREM 5. *The restriction of $\tilde{\sigma} : W_{S'}\to H$ to $R_{S'}$ takes values in $G(\tilde{K}/K)$ and, on identifying $G(\tilde{K}/K)$ with cl by means of the Artin symbol, is a snake map $\sigma\colon R_{S'}\to cl$.*

This is proved in Section 5. By the construction of the Tate sequence in Section 4, it follows that the extension class of ∇ is also explicitly described by Theorem 5.

Finally we discuss the dependence on the choice $*$ of G-orbit representatives of primes of K. Let $\Diamond$ be another such choice. For each $\mathfrak{p}$ distinguished by $*$ let $x_{\mathfrak{p}} \in G$ have the property that $x_{\mathfrak{p}}\mathfrak{p} = \mathfrak{p}'$ is distinguished by $\Diamond$. Such a system X of elements of G gives a *transport* $X : * \to \Diamond$. This means that it induces natural G-module transport maps $X : W_* \to W_\Diamond$, $\overline{\nabla}_* \to \overline{\nabla}_\Diamond$ (where $S \subseteq S'$ are now suppressed in the notation); these maps X are described at the end of Section 4, with some preparation at the end of Section 3.

Define X*-admissible* isomorphisms h between two ∇ to be those which make

$$\begin{array}{ccccccccc} 0 & \longrightarrow & cl & \longrightarrow & \nabla_* & \longrightarrow & \overline{\nabla}_* & \longrightarrow & 0 \\ & & \| & & {\scriptstyle h}\downarrow & & {\scriptstyle X}\downarrow & & \\ 0 & \longrightarrow & cl & \longrightarrow & \nabla_\Diamond & \longrightarrow & \overline{\nabla}_\Diamond & \longrightarrow & 0 \end{array}$$

commute. These allow the formulation of

THEOREM 6. *Assume a transport $X : * \to \Diamond$ has been chosen.*

(a) *The transport map $X : \overline{\nabla}_* \to \overline{\nabla}_\Diamond$ carries the extension class $[\nabla_\Diamond] \in \mathrm{Ext}^1_G(\overline{\nabla}_\Diamond, cl)$ to $[\nabla_*] \in \mathrm{Ext}^1_G(\overline{\nabla}_*, cl)$.*
(b) *Let $\tau_* \in \mathrm{Ext}^2_G(\nabla_*, E), \tau_\Diamond \in \mathrm{Ext}^2_G(\nabla_\Diamond, E)$ be the extension classes of Tate sequences for $S_*, S_\Diamond$ respectively. Then there exists an X-admissible isomorphism which takes $\tau_\Diamond$ to τ_*.*
(c) $\Omega^*_m = \Omega^\Diamond_m$.

The transport part of the proof is in Section 8, which is really a reduction to a suitable generalization of Theorem 4 in Section 6.

It now follows that the *Chinburg class* $\Omega_m = \Omega_m(K/k)$ depends only on K/k; this generalizes Theorem 3.1 of [C] to arbitrary sets S. Ω_m coincides with Chinburg's original class by [CB], because the construction there is essentially the same as ours for larger S.

Since $\overline{\nabla}_* = \Delta S$ for large S, which is independent of $*$, transport then does not need to be explicitly mentioned in the conclusions. For this reason it would be helpful to have, for arbitrary S, a "canonical model" for (the homotopy equivalence class of) $\overline{\nabla}_*$, which is independent of $*$. Theorem 6 is presently our only evidence for its existence.

2. Diagrammatic methods

The following terminology is convenient and useful. By $H^r(G, M)$ we always mean Tate cohomology for a finite group G : thus $H^0(G, \mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z}$. We call G-homomorphisms $s', s : M \to N$ *homotopic* (notation: $s' \sim s$) if $s' - s : M \to N$ factors through a projective $\mathbb{Z}G$-module. We write $[M, N]$ for the group of homotopy classes of G-homomorphisms.

LEMMA 1. (a) $[M, N] = H^0(G, \mathrm{Hom}(M, N))$ *if* M *is a* $\mathbb{Z}G$*-lattice.*

(b) *Let* M *be a* G*-module,* M' *a* $\mathbb{Z}G'$*-lattice, with* G' *a subgroup of* G*. Let* $h' : M' \to M$ *be a* G'*-homomorphism and* $h : \mathrm{ind}_{G'}^{G} M' \to M$ *the induced* G*-homomorphism. Then* $h \sim 0$ *if, and only if,* $h' \sim 0$.

Proof. (a) see, e.g., (5.1) of [GW]. (b) follows from (a) and Shapiro's lemma.

To describe the diagram manipulations we will need to do, we name the maps in certain numbered G-module diagrams by the convention

$$\text{diagram } n \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{t'} & M & \xrightarrow{t} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c''} & & \\ 0 & \longrightarrow & N' & \xrightarrow{b'} & N & \xrightarrow{b} & N'' & \longrightarrow & 0 \end{array}$$

of putting the diagram numbers as subscripts in the above pattern, if need be. Typically c' and c'' will be known maps, the rows will be exact with fixed extension classes, and the map c will be variable. Diagram 1 is the best example of this. *

If h', h are two instances of c, then $h' - h = b'dt$ for a unique G-homomorphism $d : M'' \to N'$, which we call the *diagonal deviation* from h to h'. And we define h', h to be *diatopic* if $d \sim 0$. Clearly the group $[M'', N']$ acts fixed point freely and transitively on the set of diatopy classes of the c.

LEMMA 2. (a) *Let* s', s *be the snake maps induced by* h', h*, respectively, which are two instances of* c*. If* d *is the diagonal deviation from* h *to* h' *then*

* The diagrams are numbered by type rather than strictly in the order in which they appear.

$$s' - s = j'dj''$$

where $j''\colon \ker c'' \to M''$, $j'\colon N' \to \operatorname{coker} c'$ *are the natural maps. In particular, if* h', h *are diatopic then* $s' \sim s$.

(b) *Given a diagram* n, *as above, and a map* $\tilde{c}''\colon M'' \to N''$ *which is homotopic to* c'', *then there exists* $\tilde{c}\colon M \to N$, *homotopic to* c, *so that replacing* c'', c *by* $\tilde{c}'', \tilde{c}$ *gives a diagram* $\tilde{n}$.

(c) *Given a diagram* n, *with all modules being* $\mathbb{Z}G$*-lattices, and* $\tilde{c}'$ *homotopic to* c', *then there exists* $\tilde{c} \sim c$, *so that replacing* c', c *by* $\tilde{c}', \tilde{c}$ *gives a diagram* $\tilde{n}$.

Proof. (a) is clear. For (b) write $\tilde{c}'' - c'' = qr$ with maps $r\colon M'' \to P$, $q\colon P \to N''$ and P projective. Since b is surjective, we can find $p\colon P \to N$ with $bp = q$, and set $\tilde{c} = c + prt$. For (c), take the $\mathbb{Z}$-dual of the diagram, apply (b), and then take $\mathbb{Z}$-duals again.

The *translation functor* $\mathbf{t}$, which is introduced in [G,10.5], is actually a pair of mutually inverse functors between categories **G** and **GM**, which we describe next.

The objects of **G** are group extensions $A \hookrightarrow X \twoheadrightarrow G$ of a (finite) group G by an abelian group A. The morphisms of **G** are triples of group homomorphisms which form the vertical arrows in a commutative diagram

$$\begin{array}{ccccc} A' & \hookrightarrow & X' & \twoheadrightarrow & G' \\ \downarrow & & \downarrow & & \downarrow \\ A & \hookrightarrow & X & \twoheadrightarrow & G. \end{array}$$

The objects of **GM** are pairs $(G; 0 \to A \to M \to \Delta G \to 0)$ consisting of a (finite) group G and an exact sequence of G-modules $0 \to A \to M \to \Delta G \to 0$ in which ΔG is the augmentation ideal of $\mathbb{Z}G$ with natural G-action. The morphisms of **GM** are again triples of vertical arrows making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & M' & \longrightarrow & \Delta G' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\kappa} & & \\ 0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & \Delta G & \longrightarrow & 0 \end{array}$$

commute, but now we insist that κ is induced by a group homomorphism $\kappa\colon G' \to G$ and that our vertical arrows are G'-homomorphisms when we view G-modules as G'-modules via κ.

PROPOSITION 1 [G]. **G** and **GM** *are naturally equivalent. The equivalence is given by the functors* $\mathbf{t}\colon \mathbf{G} \rightsquigarrow \mathbf{GM}$ *and* $\mathbf{t}\colon \mathbf{GM} \rightsquigarrow \mathbf{G}$ *described below.*

Given $A \hookrightarrow X \twoheadrightarrow G$ in **G**, we derive in due sequence $0 \to \Delta(X,A) \to \mathbb{Z}X \to \mathbb{Z}G \to 0, 0 \to \Delta(X,A) \to \Delta X \to \Delta G \to 0$ and $0 \to \frac{\Delta(X,A)}{\Delta(X,A)\Delta X} \to \frac{\Delta(X)}{\Delta(X,A)\Delta X} \to \Delta G \to 0$. The respective modules are viewed as left G-modules by letting $g \in G$ act by left multiplication by any preimage $x \in X$. The map $a \mapsto a - 1$

induces a G-isomorphism from A (with $g \in G$ acting by conjugation by x) to $\frac{\Delta(X,A)}{\Delta(X,A)\Delta X}$; an inverse is induced by the mapping $(a-1)r \mapsto a$ on the $\mathbb{Z}$-basis $\{(a-1)r\colon 1 \neq a \in A, X = \bigcup_r^{\cdot} Ar \text{ (disjoint union)}\}$ of $\Delta(X,A)$. Using this as an identification we define $\mathbf{t}$ of $A \hookrightarrow X \twoheadrightarrow G$ to be $(G; 0 \to A \to \frac{\Delta X}{\Delta(X,A)\Delta X} \to \Delta G \to 0)$.

Conversely, given $(G; 0 \to A \to M \to \Delta G \to 0)$ in **GM** we form semidirect products to get the exact sequence of (multiplicative) groups

$$A \rtimes 1 \hookrightarrow M \rtimes G \twoheadrightarrow \Delta G \rtimes G$$

Now $g \mapsto (g-1, g)$ is a group monomorphism $G \to \Delta G \rtimes G$. The pullback of the above sequence with respect to it gives $A \hookrightarrow X \twoheadrightarrow G$ with $X = \{(m,g) \in M \rtimes G : m \text{ has image } g-1 \text{ in } \Delta G\}$, which we define to be the $\mathbf{t}$-translate of $(G; 0 \to A \to M \to \Delta G \to 0)$.

It is clear how to define $\mathbf{t}$ on morphisms.

We need to check the compatibility of $\mathbf{t}$ with extension classes in the usual sense. If A is a G-module, then applying $\mathrm{Hom}(\cdot, A)$ to $0 \to \Delta G \to \mathbb{Z}G \to \mathbb{Z} \to 0$ and taking cohomology induces isomorphisms

$$\delta\colon\ H^q(G, \mathrm{Hom}(\Delta G, A)) \to H^{q+1}(G, A)\,,\ q \in \mathbb{Z},$$

because $\mathrm{Hom}(\mathbb{Z}G, A)$ is cohomologically trivial.

LEMMA 3 *Let* $\xi \in H^1(G, \mathrm{Hom}(\Delta G, A))$ *be the extension class of the* G*-module sequence* $0 \to A \to M \to \Delta G \to 0$. *Then* $\delta(\xi) \in H^2(G, A)$ *is the extension class of the* $\mathbf{t}$*-translate* $A \hookrightarrow X \twoheadrightarrow G$ *of* $(G; 0 \to A \to M \to \Delta G \to 0)$.

Proof. We include this instead of a list of our conventions. Letting $g \mapsto \xi_g \in \mathrm{Hom}(\Delta G, A)$ be a 1-cocycle representing ξ, we compute a 2-cocycle representing $\delta(\xi)$. Taking $\tilde{\xi}_g \in \mathrm{Hom}(\mathbb{Z}G, A)$, defined by $\tilde{\xi}_g(g') = \xi_g(g'-1)$ for $g' \in G$, as a preimage of ξ_g, then $\tilde{\xi}_{g,g'} = g\tilde{\xi}_{g'} - \tilde{\xi}_{gg'} + \tilde{\xi}_g$ belongs to $\mathrm{Hom}(\mathbb{Z}, A)$. Identifying this with A we find that

$$\xi_{g,g'} = \tilde{\xi}_{g,g'}(1) = g \cdot \xi_{g'}(g^{-1} - 1)$$

gives a 2-cocycle $(g, g') \mapsto \xi_{g,g'}$ representing $\delta(\xi)$.

Taking $\mathrm{Hom}(\Delta G, \cdot)$ and cohomology of our G-module sequence gives

$$\to \mathrm{Hom}_G(\Delta G, M) \to \mathrm{Hom}_G(\Delta G, \Delta G) \xrightarrow{\delta} H^1(G, \mathrm{Hom}(\Delta G, A)) \to$$

with $\delta(\mathrm{id}_{\Delta G}) = \xi$ by definition. This means that $\xi_g = gr - r$ for an appropriate pre-image $r \in \mathrm{Hom}(\Delta G, M)$ of $\mathrm{id}_{\Delta G}$, i.e. a section r to $M \to \Delta G$. From r we get a section r_1 of the group extension $A \rtimes 1 \hookrightarrow M \rtimes G \twoheadrightarrow \Delta G \rtimes G$ by setting $r_1(y, g) = (r(y), g), y \in \Delta G$. Since this induces a section r_2 for $A \hookrightarrow X \twoheadrightarrow G$, it follows that a 2-cocycle $(g, g') \mapsto \xi'_{g,g'}$ representing this extension is determined by

$$\begin{aligned} r_2(g)r_2(g') &= \xi'_{g,g'}r_2(gg'), \quad \text{i.e. } (r(g-1), g)\cdot(r(g'-1), g') \\ &= (\xi'_{g,g'}, 1)(r(gg'-1), gg'). \end{aligned}$$

Thus $\xi'_{g,g'} = \xi_g(g(g'-1))$ gives a 2-cocycle representing the $\mathbf{t}$-translate of $(G; 0 \to A \to M \to \Delta G \to 0)$.

Now define $a_{g'} = \xi_{g'}(g'-1) \in A$ for $g' \in G$. Then $ga_{g'} - a_{gg'} + a_g = \xi_{g,g'} - \xi'_{g,g'}$, so our two 2-cocycles represent the same element of $H^2(G, A)$, completing the proof.

Finally we need an analogous result for maps. Start from a diagram 0

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & M' & \longrightarrow & \Delta G' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow\kappa & & \\ 0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & \Delta G & \longrightarrow & 0 \end{array}$$

and let h', h be two instances of c_0. Then these two instances of diagram 0 may be viewed as two morphisms in **GM** provided h', h are G'-homomorphisms via $\kappa\colon G' \to G$. Taking d to be the diagonal deviation from h to h' defines an element $[d] \in H^0(G', \mathrm{Hom}(\Delta G', A))$.

Applying the translation functor to these two morphisms in **GM** gives two morphisms in **G** differing only in the group homomorphisms $u', u\colon X' \to X$

$$\begin{array}{ccccc} A' & \hookrightarrow & X' & \twoheadrightarrow & G' \\ \downarrow & & \downarrow & & \downarrow \\ A & \hookrightarrow & X & \twoheadrightarrow & G. \end{array}$$

Then $\lambda\colon G' \to A, g' \longmapsto u'(x')u(x')^{-1}$ for any preimage $x' \in X'$ of g', defines a 1-cocycle (cf. [AT, p. 178–179]), which clearly splits precisely when u', u are conjugate by an element of A.

LEMMA 4. *$\delta\colon H^0(G', \mathrm{Hom}(\Delta G', A)) \to H^1(G', A)$ takes $[d]$ to $-[\lambda]$.*

Proof. Taking $\tilde{d} \in \mathrm{Hom}(\mathbb{Z}G', A)$ so that $\tilde{d}(g') = d(g'-1)$ then $\delta[d]$ is represented by $g \mapsto g\tilde{d} - \tilde{d} \in \mathrm{Hom}(\mathbb{Z}, A) = A$, i.e., by $g \mapsto d(1-g)$.

Now $u(m', g') = (h(m'), \kappa(g'))$ for $(m', g') \in X'$, i.e. $t_0(m') = g' - 1$, hence $(d(g'-1), 1)u(m', g') = u'(m', g')$ and $\lambda(g') = (d(1-g'), 1)^{-1}$.

REMARK. We often use the translation functor as a convenience. However, it appears to be essential for Theorem 5.

3. Local considerations

The main part of this section is a review of [GW, Sections 11, 12]. However, using the translation functor allows us to derive diagram 2$\mathfrak{p}$ in a way which fits in well later.

We suppress the subscript $\mathfrak{p}$ in this section, in order to ease the notation. So K/k is a finite Galois extension of p-adic fields with group G, which has inertia subgroup G^0. We write $^-$ for the map $G \to \overline{G} = G/G^0$ to the corresponding residue field extension, and let φ denote the Frobenius generator of $\overline{G}$.

Define the *inertial* lattice of K/k to be the $\mathbb{Z}G$-lattice

$$W = \{(x, y) \in \Delta G \oplus \mathbb{Z}\overline{G} \colon \overline{x} = (\varphi - 1)y\}.$$

We need the following module-theoretic properties of W.

LEMMA 5.

(a) *$W \simeq \mathbb{Z}G$ if K/k is unramified.*

(b) *There are exact sequences*

$$0 \to \mathbb{Z} \to W \to \Delta G \to 0$$

$$0 \to \Delta(G, G^0) \to W \to \mathbb{Z}\overline{G} \to 0 \text{ of } G\text{-modules.}$$

We choose the map $\mathbb{Z} \to W$ to send 1 to $(0, 1 + \varphi + \cdots + \varphi^{f-1})$, $f = |\overline{G}|$.

(c) *Let $W^0 = \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z})$. There is a commutative diagram of G-modules*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & \mathbb{Z}G \oplus \mathbb{Z}G & \longrightarrow & W^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Delta G & \longrightarrow & \mathbb{Z}G & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

with exact rows. The map $W \to \Delta G$ is the canonical one of the first exact sequence in (b) and $W^0 \to \mathbb{Z}^0 = \mathbb{Z}$ is the $\mathbb{Z}$-dual of the map $\mathbb{Z} \to W$ there.

Proof. (a) is clear, (b) follows from the definition of W by projecting on the first and second components, respectively. And (c) is Lemma 4.1 of [GWL].

The relationship between W and Galois theory will follow from

LEMMA 6. *The $\mathfrak{t}$-translate of the first sequence $(G; 0 \to \mathbb{Z} \to W \to \Delta G \to 0)$ of Lemma 5(b) is the group extension*

$$\mathbb{Z} \hookrightarrow \tilde{G} \twoheadrightarrow G$$

up to canonical equivalence of extensions, with $\tilde{G} = \{(n, g) \in \mathbb{Z} \times G \colon \varphi^n = \overline{g}\}$. The map $\mathbb{Z} \to \tilde{G}$ takes 1 to $(f, 1)$ with $f = |\overline{G}|$, and the map $\tilde{G} \to G$ is the projection on the second component.

Proof. Writing $\mathbb{Z} \hookrightarrow X \twoheadrightarrow G$ for the $\mathfrak{t}$-translate, we construct an extension equivalence $X \to \tilde{G}$.

$$\begin{array}{ccccc} \mathbb{Z} & \hookrightarrow & X & \twoheadrightarrow & G \\ \| & & \downarrow & & \| \\ \mathbb{Z} & \hookrightarrow & \tilde{G} & \twoheadrightarrow & G. \end{array}$$

Since $X = \{(w,g) \in W \rtimes G : j(w) = g - 1\}$ with $j : W \to \Delta G$ given by Lemma 5(b), and since $w \in W$ is a pair $(x,y) \in \Delta G \oplus \mathbb{Z}\overline{G}$ we can define $X \to \tilde{G}$ by $(w,g) \mapsto (n,g)$ if $y \in \mathbb{Z}\overline{G}$ has augmentation n. Observe that if $\overline{g} = \varphi^m$ then $(\varphi - 1)y = \overline{x} = \overline{g} - 1 = (\varphi - 1)\sum_{i=0}^{m-1} \varphi^i$ implies $y - \sum_{i=0}^{m-1} \varphi^i = r\sum_{i=0}^{f-1} \varphi^i$ with $r \in \mathbb{Z}$, hence $n \equiv m \bmod f$, that is, $(n,g) \in \tilde{G}$.

If L/k is a Galois extension containing the maximal unramified extension k^{nr}/k, we denote the Weil group of L/k by $W(L/k) = \{g \in G(L/k) : \overline{g}$ is a $\mathbb{Z}$-power of the Frobenius of $\overline{L/k}\}$. Recall [W, appendix 2], that, if K^{ab}/K is the maximal abelian extension, then $W(K^{ab}/K)$ is the image of the reciprocity map $K^\times \hookrightarrow G(K^{ab}/K)$.

With $G = G(K/k)$, as usual, the field tower $k \subseteq K \subseteq K^{nr}$ defines an exact sequence $G(K^{nr}/K) \hookrightarrow G(K^{nr}/k) \twoheadrightarrow G$ of Galois groups.

LEMMA 7. *There is a unique commutative diagram*

$$\begin{array}{ccccc} \mathbb{Z} & \hookrightarrow & \tilde{G} & \twoheadrightarrow & G \\ \downarrow\simeq & & \downarrow\simeq & & \| \\ W(K^{nr}/K) & \hookrightarrow & W(K^{nr}/k) & \twoheadrightarrow & G \end{array}$$

with the map $\mathbb{Z} \to W(K^{nr}/K)$ *taking* 1 *to the unique Frobenius lift of* K^{nr}/K.

Proof. Choose a Frobenius lift $\tilde{\varphi} \in W(K^{nr}/k)$ and let $\tilde{\varphi}_K$ be its image in G. Define $\tilde{G} \to W(K^{nr}/k)$ by $(n,g) \mapsto \tilde{\varphi}^n g'$ where g' is the unique element of the inertia subgroup of K^{nr}/k which maps to $\tilde{\varphi}_K^{-n} g$ in G. This clearly gives such a commutative diagram. And there is a unique map $\tilde{G} \to W(K^{nr}/k)$ in it, because the 1-cocycle λ of Lemma 4 must be identically 1, by $H^1(G,\mathbb{Z}) = 0$ and $\mathbb{Z}$ central in $\tilde{G}$.

PROPOSITION 2. *The units* U *of* K *fit into a* G*-module diagram*

diagram $2\mathfrak{p}$
$$\begin{array}{ccccccccc} 0 & \to & U & \to & V & \to & W & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K^\times & \to & V' & \to & \Delta G & \to & 0 \end{array}$$

in which the bottom row has extension class α *and in which* $V \to V'$ *is an isomorphism. Moreover,* V *is then cohomologically trivial.*

Proof. There is a natural Weil group diagram

$$\begin{array}{ccccc} W(K^{ab}/K) & \hookrightarrow & W(K^{ab}/k) & \twoheadrightarrow & G \\ \downarrow & & \downarrow & & \| \\ W(K^{nr}/K) & \hookrightarrow & W(K^{nr}/k) & \twoheadrightarrow & G. \end{array}$$

The reciprocity isomorphism $K^\times \to W(K^{ab}/K)$ is a G-map which, by the local Šafarevič-Weil theorem [W], carries the local fundamental class $u_{K/k} \in H^2(G, K^\times)$ to the extension class of the top row of our diagram. Identifying the bottom row with $\mathbb{Z} \hookrightarrow \tilde{G} \twoheadrightarrow G$ by Lemma 7, and applying the translation functor gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^\times & \longrightarrow & V & \longrightarrow & \Delta G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & W & \longrightarrow & \Delta G & \longrightarrow & 0 \end{array}$$

by Lemma 6. The map $K^\times \to \mathbb{Z}$ here is the normalized valuation [S, p. 205], so has kernel U. The map $V \to W$ then also has kernel U and our diagram $2\mathfrak{p}$ follows, with $V \to V'$ the identity map, by suitably collapsing and introducing equality signs. That $0 \to K^\times \to V \to \Delta G \to 0$ has extension class α, in the sense of Section 1, follows from Lemma 3, and the cohomological triviality of V from [GW; 11.3].

REMARK. The special diagram $2\mathfrak{p}$ in the proof actually has the identity map $V \to V'$. This has the consequence, because of $\mathrm{Hom}_G(\Delta G, \mathbb{Z}) = 0$, that the extension class of the top row in diagram $2\mathfrak{p}$ is actually uniquely determined (cf. [GWL], 1.8), and this is the canonical class $\beta \in H^1(G, \mathrm{Hom}(W, U))$ of Section 1. For our purposes it is not essential that β always comes equipped with a reference to α so we need only use any diagram $2\mathfrak{p}$.

LEMMA 8. *The map* $H^0(G, \mathrm{Hom}(W, U)) \to H^0(G, \mathrm{Hom}(W, K^\times))$, *induced by the inclusion* $U \hookrightarrow K^\times$, *is surjective.*

Proof. We must show that the map δ in the long cohomology sequence induced by $0 \to U \to K^\times \to \mathbb{Z} \to 0$ is injective:

$$H^0(G, \mathrm{Hom}(W, \mathbb{Z})) \xrightarrow{\delta} H^1(G, \mathrm{Hom}(W, U)) \to H^1(G, \mathrm{Hom}(W, K^\times)).$$

Since, [GWL, Section 4], $H^0(G, \mathrm{Hom}(W, \mathbb{Z})) \simeq \mathbb{Z}/e\mathbb{Z}$ and $H^1(G, \mathrm{Hom}(W, U)) \simeq \mathbb{Z}/e\mathbb{Z} \oplus \mathbb{Z}/e\mathbb{Z}$, with e the ramification index of K/k, this will follow if we know that $H^1(G, \mathrm{Hom}(W, K^\times))$ is cyclic. However from $0 \to \mathbb{Z} \to W \to \Delta G \to 0$ we get

$$H^1(G, \mathrm{Hom}(\Delta G, K^\times)) \to H^1(G, \mathrm{Hom}(W, K^\times)) \to H^1(G, \mathrm{Hom}(\mathbb{Z}, K^\times))$$

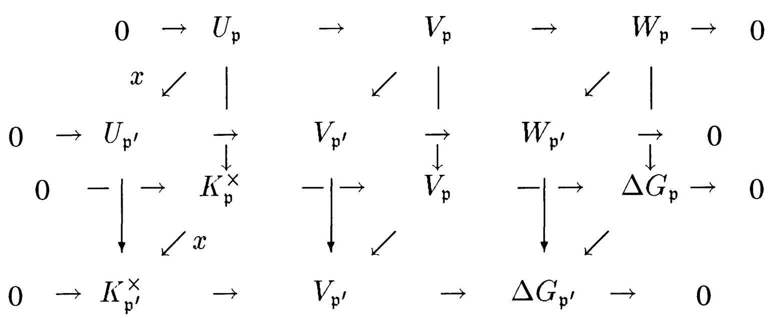
with $H^1(G, \mathrm{Hom}(\Delta G, K^\times)) \simeq H^2(G, K^\times) \simeq \mathbb{Z}/|G|\mathbb{Z}$ cyclic and $H^1(G, \mathrm{Hom}(\mathbb{Z}, K^\times)) = H^1(G, K^\times) = 0$.

We end this section with our first discussion of transport of structure. We revert to our global extension K/k with Galois group G, fix a prime $\mathfrak{p}$ of K and an element $x \in G$ and set $\mathfrak{p}' = x\mathfrak{p}$. Then $g \mapsto xgx^{-1}$ gives an isomorphism $G_\mathfrak{p} \to G_{\mathfrak{p}'}$. Thus $a \mapsto xax^{-1}$ gives transport maps $\Delta G_\mathfrak{p} \to \Delta G_{\mathfrak{p}'}$ and $W_\mathfrak{p} \to W_{\mathfrak{p}'}$; the latter works, in terms of the description of W as pairs, because $xG_\mathfrak{p}^0 x^{-1} = G_{\mathfrak{p}'}^0$, and $x\varphi_\mathfrak{p} x^{-1} = \varphi_{\mathfrak{p}'}$. The purpose for these maps in Section 4 will be that they induce transport maps $\mathrm{ind}_{G_\mathfrak{p}}^G W_\mathfrak{p} \to \mathrm{ind}_{G_{\mathfrak{p}'}}^G W_{\mathfrak{p}'}$ by $g \otimes w \mapsto gx^{-1} \otimes xwx^{-1}$, which are G-isomorphisms. For now, though, we can just view $W_\mathfrak{p} \to W_{\mathfrak{p}'}$ as a $G_\mathfrak{p}$-map when $G_\mathfrak{p}$ acts on $W_{\mathfrak{p}'}$ via $g \mapsto xgx^{-1}$.

The same applies to the $\mathbb{Z}$-duals of $\Delta G_\mathfrak{p}$ and $W_\mathfrak{p}$. But now the transport map $W_\mathfrak{p}^0 \to W_{\mathfrak{p}'}^0$ is obtained by taking the dual of the *inverse* map $W_{\mathfrak{p}'} \to W_\mathfrak{p}$, i.e. $w' \mapsto x^{-1}w'x$.

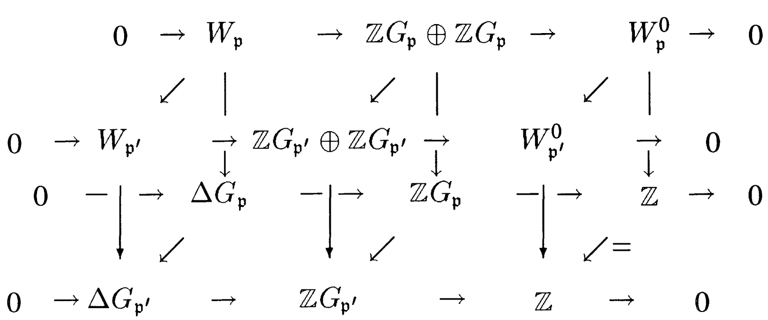
PROPOSITION 3. *The following commutative diagrams exist; the top and bottom faces are $G_\mathfrak{p}$-diagrams in the sense of transport of structure, while the back and front faces are $G_\mathfrak{p}$-diagrams and $G_{\mathfrak{p}'}$-diagrams in the ordinary sense.*

(a)



The front and back faces are diagram $2\mathfrak{p}'$ and $2\mathfrak{p}$, respectively. The action of x on K induces $K_\mathfrak{p} \to K_{\mathfrak{p}'}$ by continuity; this takes $U_\mathfrak{p}$ to $U_{\mathfrak{p}'}$.

(b)



The front and back faces are the diagrams in Lemma 5(c) for $\mathfrak{p}'$ and $\mathfrak{p}$.

Proof. (a) We begin by forming a commutative diagram

$$
\begin{array}{ccccccccccc}
 & & 0 & \to & K_{\mathfrak{p}}^{\times} & \to & V_{\mathfrak{p}} & \to & \Delta G_{\mathfrak{p}} & \to & 0 \\
 & & & {\scriptstyle x}\swarrow & | & \swarrow & | & \swarrow & |{\scriptstyle =} & & \\
0 & \to & K_{\mathfrak{p}'}^{\times} & \to & \downarrow & V_{\mathfrak{p}'} & \downarrow & \Delta G_{\mathfrak{p}'} & \downarrow & \to & 0 \\
 & & 0 & \to & \mathbb{Z} & \to & W_{\mathfrak{p}} & \to & \Delta G_{\mathfrak{p}} & \to & 0 \\
 & & \downarrow & \swarrow{\scriptstyle =} & & \downarrow & \swarrow & {\scriptstyle =}\downarrow & \swarrow & & \\
0 & \to & \mathbb{Z} & \to & & W_{\mathfrak{p}'} & \to & \Delta G_{\mathfrak{p}'} & \to & 0 &
\end{array}
$$

with top and bottom faces again being transport, e.g. the map $\Delta G_{\mathfrak{p}} \to \Delta G_{\mathfrak{p}'}$ is $d \mapsto xdx^{-1}$. The main point here is the existence of a map $V_{\mathfrak{p}} \to V_{\mathfrak{p}'}$ making the top face commute: this can be seen by applying the translation functor to the corresponding diagram of Weil groups (which clearly exists), or, equivalently, by recalling the behaviour of the fundamental class under transport of structure [S, XI, Section 3]. The front and back faces are those of the proof of Proposition 2 for $\mathfrak{p}'$ and $\mathfrak{p}$, and we get $W_{\mathfrak{p}} \to W_{\mathfrak{p}'}$ because $W_{\mathfrak{p}}$ is a pushout. This induces $\Delta G_{\mathfrak{p}} \to \Delta G_{\mathfrak{p}'}$, which must be $d \mapsto xdx^{-1}$, and then the diagram of (a) results by taking kernels as in the proof of Proposition 2. Note that it follows that $W_{\mathfrak{p}} \to W_{\mathfrak{p}'}$ is also $w \mapsto xwx^{-1}$: for this is the unique map which makes the bottom face commute, since the diagonal deviation is in $\mathrm{Hom}_{G_{\mathfrak{p}}}(\Delta G_{\mathfrak{p}}, \mathbb{Z}) = 0$.

(b) This follows by a straight-forward computation from the proof of Lemma 4.1 of [GWL], on using the maps $\phi_{\mathfrak{p}}, \theta_{\mathfrak{p}}$ there and $a \mapsto xax^{-1}$ on $\mathbb{Z}G_{\mathfrak{p}} \oplus \mathbb{Z}G_{\mathfrak{p}}$. These maps are described in terms of a special $\mathbb{Z}$-basis $\{w_g(\mathfrak{p}) : g \in G_{\mathfrak{p}}\}$ of $W_{\mathfrak{p}}$ and one observes that $xw_g(\mathfrak{p})x^{-1} = w_{xgx^{-1}}(\mathfrak{p}')$.

4. Existence of θ; the Tate sequence

The existence of a surjective θ in diagram 1 comes from local constructions. By Frobenius reciprocity, i.e. that induction and restriction are adjoint, it follows that a diagram 1 is equivalent to local diagrams

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_{\mathfrak{p}}^{\times} & \longrightarrow & V_{\mathfrak{p}} & \longrightarrow & \Delta G_{\mathfrak{p}} & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle \theta_{\mathfrak{p}}}\downarrow & & \downarrow & & \\
0 & \longrightarrow & C_K & \longrightarrow & \mathfrak{V} & \longrightarrow & \Delta G & \longrightarrow & 0
\end{array}
$$

diagram $1\mathfrak{p}$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & U_{\mathfrak{p}} & \longrightarrow & V_{\mathfrak{p}} & \longrightarrow & W_{\mathfrak{p}} & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle \theta_{\mathfrak{p}}}\downarrow & & \downarrow & & \\
0 & \longrightarrow & C_K & \longrightarrow & \mathfrak{V} & \longrightarrow & \Delta G & \longrightarrow & 0
\end{array}
$$

for $\mathfrak{p} \in S_*$ and for $\mathfrak{p} \in S'_* \setminus S_*$, respectively.

We first get diagrams $1\mathfrak{p}$ of the first kind for every $\mathfrak{p} \in S'_*$, in the following way. We consider the diagram of group extensions

$$\begin{array}{ccccc} K_\mathfrak{p}^\times & \hookrightarrow & F_\mathfrak{p} & \twoheadrightarrow & G_\mathfrak{p} \\ \downarrow & & \downarrow \cdot & & \downarrow \\ C_K & \hookrightarrow & \mathfrak{F} & \twoheadrightarrow & G \end{array}$$

in which the top row represents the local fundamental class $u_{K\mathfrak{p}/k\mathfrak{p}} \in H^2(G_\mathfrak{p}, K_\mathfrak{p}^\times)$ and the bottom row the global fundamental class $u_{K/k} \in H^2(G, C_K)$. The condition [AT, p. 180] for the existence of the dotted arrow is that res $u_{K/k}$ equals the image of $u_{K\mathfrak{p}/k\mathfrak{p}}$ in $H^2(G_\mathfrak{p}, C_K)$, and this condition is satisfied [TCF, p. 195–196]. Applying the translation functor of Section 2 now gives the first kind of diagram $1\mathfrak{p}$ for every $\mathfrak{p} \in S'_*$, because of Lemma 3.

Inducing from $G_\mathfrak{p}$ to G for $\mathfrak{p} \in S'_*$, taking direct sums and glueing on $\prod_{\mathfrak{p}\notin S'} U_\mathfrak{p}$, as in Section 1, gives

$$\text{diagram 3} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & J_{S'} & \longrightarrow & V_{S'} & \longrightarrow & \Delta_{S'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_K & \longrightarrow & \mathfrak{V} & \longrightarrow & \Delta G & \longrightarrow & 0 \end{array}$$

with $J_{S'}$ the S'-ideles and $\Delta_{S'} = \bigoplus_{\mathfrak{p}\in S'_*} \operatorname{ind}_{G_\mathfrak{p}}^G \Delta G_\mathfrak{p}$. Since S' is larger, the left vertical arrow in diagram 3 is surjective, by (i) of Section 1. We next show that (iii) of Section 1 forces the right one, c''_3, to be so as well. Set $H = \{h \in G : h - 1 \in \operatorname{im} c''_3\}$; then $H \supseteq \bigcup_{\mathfrak{p}\in S'_*} G_\mathfrak{p}$ and H is a subgroup of G by $hh' - 1 = h(h'-1) + (h-1)$. Thus $\bigcup_{g\in G} gHg^{-1} \supseteq \bigcup_{\mathfrak{p}\in S'} G_\mathfrak{p} = G$ implies $H = G$.

The $G_\mathfrak{p}$-modules $V_\mathfrak{p}$ are cohomologically trivial (Proposition 2), and the proof of this, which uses only class formations, also gives the cohomological triviality of $\mathfrak{V}$. The cohomological triviality of $V_{S'}$ follows from that of $\bigoplus_{\mathfrak{p}\in S'_*} \operatorname{ind}_{G_\mathfrak{p}}^G V_\mathfrak{p}$ and from (ii) of Section 1, because local units in unramified extensions are cohomologically trivial. The map $V_{S'} \to \mathfrak{V}$ in diagram 3 is then surjective by the snake lemma. This amounts to getting the Tate sequence for larger S', by, essentially, the method of [T,TN], especially in the form of [CB].

To get the diagram 1 we take a diagram $2\mathfrak{p}$ for $\mathfrak{p} \in S'_* \setminus S_*$, with isomorphisms $V_\mathfrak{p} \to V_\mathfrak{p}$: this is possible by Section 3. Since it is a $G_\mathfrak{p}$-module diagram we can apply $\operatorname{ind}_{G_\mathfrak{p}}^G$ to the whole diagram – not just the top row. Direct summing these over $\mathfrak{p} \in S'_* \setminus S_*$, we obtain

$$\text{diagram 2} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & V_{S'} & \longrightarrow & W_{S'} & \longrightarrow & 0 \\ & & \downarrow & & \simeq\downarrow & & \downarrow & & \\ 0 & \longrightarrow & J_{S'} & \longrightarrow & V_{S'} & \longrightarrow & \Delta_{S'} & \longrightarrow & 0, \end{array}$$

by using equalities on all $\mathfrak{p}$-components with $\mathfrak{p} \notin S'_* \setminus S_*$. Getting a diagram 1 with θ surjective is now just a matter of stacking diagram 2 on diagram 3. Theorem 1 is proved.

As has been said in Section 1, each surjective θ provides a snake map s together with a "Tate" sequence $0 \to E \to A_\theta \to R_{S'} \xrightarrow{s} cl \to 0$. We now turn to the derivation of the Tate sequence $0 \to E \to A_\theta \to B_{S'} \to \nabla_\theta \to 0$ from it.

By Lemma 5(c), composed with the inclusions $G_\mathfrak{p} \to G$, there is, for $\mathfrak{p} \in S_*^{\mathrm{ram}}$, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_\mathfrak{p} & \longrightarrow & \mathbb{Z}G_\mathfrak{p}^2 & \longrightarrow & W_\mathfrak{p}^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Delta G & \longrightarrow & \mathbb{Z}G & \longrightarrow & \mathbb{Z} & \longrightarrow & 0, \end{array}$$

of $G_\mathfrak{p}$-modules. For $\mathfrak{p} \in S_*$ there is a similar diagram with top row $0 \to \Delta G_\mathfrak{p} \to \mathbb{Z}G_\mathfrak{p} \to \mathbb{Z} \to 0$, and also one for unramified $\mathfrak{p} \in S'_* \setminus S_*$ with top row $0 \to W_\mathfrak{p} \to \mathbb{Z}G_\mathfrak{p} \to 0 \to 0$, by Lemma 5(a). Inducing these top rows to G and taking direct sums over $\mathfrak{p} \in S'_*$ gives the diagram

$$\text{diagram 4} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & W_{S'} & \longrightarrow & N_{S'} & \longrightarrow & M_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Delta G & \longrightarrow & \mathbb{Z}G & \longrightarrow & \mathbb{Z} & \longrightarrow & 0, \end{array}$$

of G-modules. Here $W_{S'}$ is as before, $N_{S'}$ is the free G-module

$$N_{S'} = \bigoplus \mathrm{ind}_{G_\mathfrak{p}}^G \mathbb{Z}G_\mathfrak{p} \oplus \bigoplus \mathrm{ind}_{G_\mathfrak{p}}^G (\mathbb{Z}G_\mathfrak{p})^2$$

where the first direct sum ranges over the $\mathfrak{p} \in S_*$ *and* the nonramified primes $\mathfrak{p} \in S'_* \setminus S_*$, the second over the ramified primes $\mathfrak{p} \in S'_* \setminus S_*$, and

$$M_* = \bigoplus_{\mathfrak{p} \in S_*} \mathrm{ind}_{G_\mathfrak{p}}^G \mathbb{Z} \oplus \bigoplus_{\mathfrak{p} \in S_*^{\mathrm{ram}}} \mathrm{ind}_{G_\mathfrak{p}}^G W_\mathfrak{p}^0$$

is independent of S' (but not of $*$). The G-homomorphisms $W_{S'} \to \Delta G$ and $M_* \to \mathbb{Z}$ in diagram 4 are fixed, but we allow any rows with the extension classes just constructed and any G-map $N_{S'} \to \mathbb{Z}G$ making the diagram commute to be considered as a diagram 4.

Taking kernels in diagram 4 gives an exact sequence $0 \to R_{S'} \to B_{S'} \to \overline{\nabla}_* \to 0$, which defines $B_{S'}$ and $\overline{\nabla}_*$. Observe that the extension class of this sequence does not depend on the choice of rows in diagram 4, and that $\overline{\nabla}_*$ is independent of S' (but not of $*$). Also $B_{S'} \oplus \mathbb{Z}G \simeq N_{S'}$ implies that $B_{S'}$ is stably free.

Now take the pushout along our given snake map s:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_{S'} & \longrightarrow & B_{S'} & \longrightarrow & \overline{\nabla}_* & \longrightarrow & 0 \\ & & s\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & cl & \longrightarrow & \nabla_\theta & \longrightarrow & \overline{\nabla}_* & \longrightarrow & 0. \end{array}$$

Since $B_{S'}$ is projective, the connecting homomorphism $H^0(G, \mathrm{Hom}(R_{S'}, cl)) \to H^1(G, \mathrm{Hom}(\overline{\nabla}_*, cl))$ is an isomorphism, under which $[s] \in [R_{S'}, cl]$ and $[\nabla_\theta] \in$

$\mathrm{Ext}^1_G(\overline{\nabla}_*, cl)$ uniquely correspond. We finally combine the "Tate" sequence $0 \to E \to A_\theta \to R_{S'} \xrightarrow{s} cl \to 0$ and the extension class $0 \to cl \to \nabla_\theta \to \overline{\nabla}_* \to 0$ by replacing ker $(R_{S'} \to cl)$ by ker $(B_{S'} \to \nabla_\theta)$. Thus we get the Tate sequence $0 \to E \to A_\theta \to B_{S'} \to \nabla_\theta \to 0$.

Finally we turn to the description of the transport maps X induced on G-modules by a transport $X : * \to \Diamond$. Writing $\mathfrak{p}' = x_\mathfrak{p}\mathfrak{p}$ as in Section 1, and $D_\mathfrak{p}$ for one of our local modules, the basic process, already explained at the end of Section 3, is to associate to a local transport map $f_\mathfrak{p} : D_\mathfrak{p} \to D_{\mathfrak{p}'}$, in the sense of Section 3, the G-module map $X : \mathrm{ind}^G_{G_\mathfrak{p}} D_\mathfrak{p} \to \mathrm{ind}^G_{G_{\mathfrak{p}'}} D_{\mathfrak{p}'}$ defined by $g \otimes d \mapsto gx_\mathfrak{p}^{-1} \otimes f_\mathfrak{p}(d)$; recall that $f_\mathfrak{p}(gd) = x_\mathfrak{p} g x_\mathfrak{p}^{-1}.f_\mathfrak{p}(d)$ for $g \in G_\mathfrak{p}, d \in D_\mathfrak{p}$. Rather than discussing this (simple) process in general we prefer to point out its peculiarities in the cases which actually arise.

$$\begin{array}{ccc} \mathrm{ind}^G_{G_\mathfrak{p}} K_\mathfrak{p}^\times & & \\ & \searrow & \\ X \downarrow & & \prod\limits_{\mathfrak{q}\in\mathcal{B}} K_\mathfrak{q}^\times \\ & \nearrow & \\ \mathrm{ind}^G_{G_{\mathfrak{p}'}} K_{\mathfrak{p}'}^\times & & \end{array}$$

If $\mathcal{B}$ is a G-orbit and $\mathfrak{p} \in \mathcal{B}_*, \mathfrak{p}' \in \mathcal{B}_\Diamond$, then the identification process before Theorem 1 gives the commutative triangle shown, and the same for U. This means that after this identification X becomes the identity map on J (which is, therefore, a "canonical model").

When we look at the transport map $X : W_* \to W_\Diamond$ we find it is not compatible with the given maps $W_* \to \Delta G, W_\Diamond \to \Delta G$. This comes from the presence of the right multiplication by $x_\mathfrak{p}^{-1}$ in the commutative square shown,

$$\begin{array}{ccc} \mathrm{ind}^G_{G_\mathfrak{p}} \Delta G_\mathfrak{p} & \longrightarrow & \Delta G \\ X \downarrow & & \downarrow x_\mathfrak{p}^{-1} \\ \mathrm{ind}^G_{G_{\mathfrak{p}'}} \Delta G_{\mathfrak{p}'} & \longrightarrow & \Delta G \end{array}$$

which has the effect of *perturbing* the canonical map c_1'', i.e. we only get a commutative

$$\begin{array}{ccc} W_* & & \\ & \searrow {}^{x}c''_{1*} & \\ X \downarrow & & \Delta G \\ & \nearrow c''_{1\Diamond} & \\ W_\Diamond & & \end{array}$$

triangle as shown. Here c''_{1*} is replaced by ${}^{x}c''_{1*}$, which is the composite

$$\bigoplus_{\mathfrak{p}\in S_*} \mathrm{ind}^G_{G_\mathfrak{p}} W_\mathfrak{p} \longrightarrow \bigoplus_{\mathfrak{p}\in S_*} \Delta G \longrightarrow \bigoplus_{\mathfrak{p}\in S_*} \Delta G \to \Delta G,$$

with canonical first map, with second map $(d_\mathfrak{p})_\mathfrak{p} \mapsto (d_\mathfrak{p} x_\mathfrak{p}^{-1})_\mathfrak{p}$ and with third map $(d_\mathfrak{p})_\mathfrak{p} \mapsto \sum_\mathfrak{p} d_\mathfrak{p}$.

So we do not get a transport map for R; it is this which necessitates the *perturbed* version of Theorem 4 in Section 6.

The observation necessary here is that ${}^{x}c''_{1*}$ is homotopic to c''_{1*}, which follows from right multiplication by $x_{\mathfrak{p}}^{-1}$, on ΔG, being homotopic to the identity map: for the difference is the composite of the inclusion $\Delta G \to \mathbb{Z}G$ with $\mathbb{Z}G \to \Delta G$, $z \mapsto z(x_{\mathfrak{p}}^{-1} - 1)$.

Turning to M, note that the triangle shown commutes (hence $\mathbb{Z}S$ is a "canonical model"). In particular, $X : M_* \to M_\Diamond$ is compatible with the augmentation maps $M \to \mathbb{Z}$, so we do get a transport map $X : \overline{\nabla}_* \to \overline{\nabla}_\Diamond$.

$$\begin{array}{ccc} \mathrm{ind}_{G_{\mathfrak{p}}}^{G}\mathbb{Z} & & \\ & \searrow & \\ X\downarrow & & \mathbb{Z}\mathcal{B} \\ & \nearrow & \\ \mathrm{ind}_{G_{\mathfrak{p}'}}^{G}\mathbb{Z} & & \end{array}$$

Of course, we then have a transport map $X : \overline{\nabla}_* \to \overline{\nabla}_\Diamond$ for each choice of transport $X : * \to \Diamond$. The next result shows that they all have the same effect in cohomology.

LEMMA 9. *The transport maps* $X : \overline{\nabla}_* \to \overline{\nabla}_\Diamond$ *are all homotopic.*

Proof. Composing transports $* \to *$ with a fixed transport $* \to \Diamond$ gives rise to all transports $* \to \Diamond$. So it suffices to show that if X is a transport $* \to *$ then the induced map $\overline{\nabla}_* \to \overline{\nabla}_*$ is homotopic to the identity.

Set $\tilde{S} = S \cup S^{\mathrm{ram}}$. Assembling the $\mathbb{Z}$-duals of the (first) local sequence of Lemma 5(b), the definition of $\overline{\nabla}_*$ implies a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \bigoplus_{\mathfrak{p}\in S_*^{\mathrm{ram}}} \mathrm{ind}_{G_{\mathfrak{p}}}^{G}(\Delta G_{\mathfrak{p}})^0 & \longrightarrow & \overline{\nabla}_* & \longrightarrow & \Delta\tilde{S} & \longrightarrow & 0 \\ & & X\downarrow & & X\downarrow & & \downarrow= & & \\ 0 & \longrightarrow & \bigoplus_{\mathfrak{p}\in S_*^{\mathrm{ram}}} \mathrm{ind}_{G_{\mathfrak{p}}}^{G}(\Delta G_{\mathfrak{p}})^0 & \longrightarrow & \overline{\nabla}_* & \longrightarrow & \Delta\tilde{S} & \to & 0 \end{array}$$

where we have used the last commutative triangle as an identification to get the identity on $\Delta\tilde{S}$.

The left X is homotopic to the identity. To see this note that X takes $g \otimes \eta$, with $\eta \in (\Delta G_{\mathfrak{p}})^0$ and $\mathfrak{p} \in S_*^{\mathrm{ram}}$, to $gx_{\mathfrak{p}}^{-1} \otimes \tilde{\eta}$ with $\tilde{\eta}(d) = \eta(x_{\mathfrak{p}}^{-1}dx_{\mathfrak{p}})$. Since $x_{\mathfrak{p}} \in G_{\mathfrak{p}}$ we have $gx_{\mathfrak{p}}^{-1} \otimes \tilde{\eta} = g \otimes x_{\mathfrak{p}}^{-1}\tilde{\eta}$ with $(x_{\mathfrak{p}}^{-1}\tilde{\eta})(d) = \tilde{\eta}(x_{\mathfrak{p}}d) = \eta(dx_{\mathfrak{p}})$. Thus this X is induced by right multiplication by $x_{\mathfrak{p}}$ on $\Delta G_{\mathfrak{p}}$, which was seen to be homotopic to the identity above.

By Lemma 2(c) we can replace the above diagram by one with left map the identity and a middle map $X' \sim X$. It then suffices to show X' is diatopic to the identity, i.e. that $[\Delta\tilde{S}, \bigoplus_{\mathfrak{p}\in S_*^{\mathrm{ram}}} \mathrm{ind}_{G_{\mathfrak{p}}}^{G}(\Delta G_{\mathfrak{p}})^0] = 0$.

This group is dual to $[\bigoplus_{\mathfrak{p}\in S_*^{\mathrm{ram}}} \mathrm{ind}_{G_{\mathfrak{p}}}^{G}(\Delta G_{\mathfrak{p}})^0, \Delta\tilde{S}]$. By Shapiro's lemma, we are reduced to showing $H^{-1}(G_{\mathfrak{p}}, \Delta\tilde{S}) = 0$ for $\mathfrak{p} \in S_*^{\mathrm{ram}}$, after dimension shifting by the dual of the augmentation sequence. Since $\mathfrak{p} \in \tilde{S}$, the sequence $0 \to \Delta\tilde{S} \to$

$\mathbb{Z}\tilde{S} \to \mathbb{Z} \to 0$ is $G_{\mathfrak{p}}$-split, by $1 \mapsto \mathfrak{p}$, so $H^{-1}(G_{\mathfrak{p}}, \Delta\tilde{S})$ embeds in $H^{-1}(G_{\mathfrak{p}}, \mathbb{Z}\tilde{S})$, which is 0 by Shapiro's lemma again.

5. Pinning down the snake map

We first turn to the proof of Theorem 2(a). By Lemma 2(a) it suffices to show that the diagonal deviation between θ and θ' maps to zero under $[W_{S'}, C_K] \to [R_{S'}, cl]$. So the result is a consequence of the stronger

PROPOSITION 4. $[W_{S'}, C_K] \to [W_{S'}, cl]$ *is the zero map.*

Proof. By the "semi-local" structure of $W_{S'}$ and Shapiro's lemma we must show that $H^0(G_{\mathfrak{p}}, \mathrm{Hom}(W_{\mathfrak{p}}, C_K)) \to H^0(G_{\mathfrak{p}}, \mathrm{Hom}(W_{\mathfrak{p}}, cl))$ is 0 for $\mathfrak{p} \in S'_* \setminus S_*$, and the analogous statement for $\mathfrak{p} \in S_*$ with $W_{\mathfrak{p}}$ replaced by $\Delta G_{\mathfrak{p}}$.

The case $\mathfrak{p} \in S_*$ is immediate because

$$H^0(G_{\mathfrak{p}}, \mathrm{Hom}(\Delta G_{\mathfrak{p}}, C_K)) \simeq H^1(G_{\mathfrak{p}}, C_K) = 0.$$

Turning to the case $\mathfrak{p} \in S'_* \setminus S_*$, we use the second sequence of Lemma 5(b),

$$0 \to \mathrm{ind}_{G_{\mathfrak{p}}^0}^{G_{\mathfrak{p}}} \Delta G_{\mathfrak{p}}^0 \to W_{\mathfrak{p}} \to \mathrm{ind}_{G_{\mathfrak{p}}^0}^{G_{\mathfrak{p}}} \mathbb{Z} \to 0,$$

to get the commutative square

$$\begin{array}{ccc} H^0(G_{\mathfrak{p}},\ \mathrm{Hom}(\mathrm{ind}_{G_{\mathfrak{p}}^0}^{G_{\mathfrak{p}}} \mathbb{Z}, C_K)) & \longrightarrow & H^0(G_{\mathfrak{p}},\ \mathrm{Hom}(W_{\mathfrak{p}}, C_K)) \\ \downarrow & & \downarrow \\ H^0(G_{\mathfrak{p}},\ \mathrm{Hom}(\mathrm{ind}_{G_{\mathfrak{p}}^0}^{G_{\mathfrak{p}}} \mathbb{Z}, cl)) & \longrightarrow & H^0(G_{\mathfrak{p}},\ \mathrm{Hom}(W_{\mathfrak{p}}, cl)) \end{array}$$

with the top arrow surjective by $H^0(G_{\mathfrak{p}}, \mathrm{Hom}(\mathrm{ind}_{G_{\mathfrak{p}}^0}^{G_{\mathfrak{p}}} \Delta G_{\mathfrak{p}}^0, C_K)) = 0$. This reduces us to proving that $H^0(G_{\mathfrak{p}}^0, C_K) \to H^0(G_{\mathfrak{p}}^0, cl)$ is zero.

In order to do this it will be convenient to have two more diagrams at our disposal. Let $\tilde{K}$ be the Hilbert S-class field of K. By the global Šafarevič-Weil theorem [AT] there is a commutative diagram of group extensions

$$\begin{array}{ccccc} C_K & \hookrightarrow & \mathfrak{F} & \twoheadrightarrow & G \\ \downarrow & & \downarrow & & \| \\ G(\tilde{K}/K) & \hookrightarrow & G(\tilde{K}/k) & \twoheadrightarrow & G, \end{array}$$

in which the top row represents the fundamental class $u_{K/k}$, and $C_K \to G(\tilde{K}/K)$ is the reciprocity map. Applying the translation functor gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_K & \longrightarrow & \mathfrak{V} & \longrightarrow & \Delta G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & cl & \longrightarrow & H & \longrightarrow & \Delta G & \longrightarrow & 0, \end{array}$$

where we have identified $G(\tilde{K}/K)$ with cl in the bottom row, via the Artin symbol. We regard this as the definition of H.

We come back to the assertion that $H^0(G_{\mathfrak{p}}^0, C_K) \to H^0(G_{\mathfrak{p}}^0, cl)$ is the zero map. To simplify notation we may assume that $G_{\mathfrak{p}}^0 = G$, i.e. that $\mathfrak{p}$ is totally ramified in K/k. It follows that the bottom row in the group extension diagram above splits, since the inertia group of a prime of $\tilde{K}$ above $\mathfrak{p}$ is a complement to $G(\tilde{K}/K)$ in $G(\tilde{K}/k)$. By Lemma 3, then the bottom row in the corresponding module extension diagram also splits. So, taking cohomology gives the square shown, with $\delta = 0$.

$$\begin{array}{ccc} H^{-1}(G, \Delta G) & \xrightarrow{\simeq} & H^0(G, C_K) \\ \| & & \downarrow \\ H^{-1}(G, \Delta G) & \xrightarrow{\delta} & H^0(G, cl). \end{array}$$

The isomorphism $\simeq$, which follows from the cohomological triviality of $\mathfrak{V}$, then completes the proof of the proposition and thus of Theorem 2(a).

In order to verify Theorem 2(b), we recall the isomorphism $[R_{S'}, cl] \simeq \mathrm{Ext}_G^1(\overline{\nabla}_*, cl)$ from the last section, under which $[s]$ and $[\nabla_\theta]$ correspond. Since $[s]$ is independent of θ, so is $[\nabla_\theta]$.

We now turn to the proof of Theorem 5, that is, to the construction of an explicit snake map $\sigma : R_{S'} \to cl$. Its basic idea has already been introduced in Section 1. Namely, we stack the diagram in Lemma 7 on the first diagram shown below and apply the translation functor in order to arrive at the second diagram below:

$$\begin{array}{ccccc} W(K_{\mathfrak{p}}^{nr}/K_{\mathfrak{p}}) & \hookrightarrow & W(K_{\mathfrak{p}}^{nr}/k_{\mathfrak{p}}) & \twoheadrightarrow & G_{\mathfrak{p}} \\ \downarrow & & \downarrow & & \downarrow \\ G(\tilde{K}/K) & \hookrightarrow & G(\tilde{K}/k) & \twoheadrightarrow & G \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & W_{\mathfrak{p}} & \longrightarrow & \Delta G_{\mathfrak{p}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & cl & \longrightarrow & H & \longrightarrow & \Delta G & \longrightarrow & 0, \end{array}$$

by Lemma 6. If $\mathfrak{p} \in S_*$, then the map $\mathbb{Z} \to cl$ is zero, so we modify the above diagram by replacing $\mathbb{Z} \to W_{\mathfrak{p}}$ by $0 \to \Delta G_{\mathfrak{p}}$.

Now we glue these together, for $\mathfrak{p} \in S'_*$, and get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}[S' \setminus S] & \longrightarrow & W_{S'} & \longrightarrow & \Delta_{S'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & cl & \longrightarrow & H & \longrightarrow & \Delta G & \longrightarrow & 0. \end{array}$$

As $R_{S'}$ is the kernel of the composite map $W_{S'} \to \Delta_{S'} \to \Delta G$, the map $\tilde{\sigma} : W_{S'} \to H$ when restricted to $R_{S'}$ takes values in cl. We denote this by σ and show that it is a snake map.

REMARK. Explicitly the map $\tilde{G}_\mathfrak{p} \to G(\tilde{K}/k)$ comes about as follows. Choose a prime $\tilde{\mathfrak{p}}/\mathfrak{p}$ of $\tilde{K}$ and a corresponding Frobenius lift $\tilde{\varphi}_{\tilde{\mathfrak{p}}}$ for $\tilde{K}/k$; let $\varphi_{\tilde{\mathfrak{p}}}$ denote its restriction to K. Then $(n, g) \in \tilde{G}_\mathfrak{p}$ is mapped to $\tilde{\varphi}_{\tilde{\mathfrak{p}}}^n \cdot g'$, where g' is the unique element in the inertia group of $\tilde{K}/k$ belonging to $\tilde{\mathfrak{p}}$ with image $\varphi_{\tilde{\mathfrak{p}}}^{-n} \cdot g$ in $G_\mathfrak{p}^0$.

The relation between σ and the snakes comes from Galois theory. It takes, though, the notion of *global Weil groups* to make this relation transparent. The relative Weil group of K^{ab}/k is, by definition,

$$W(K^{ab}/k) = W(k)/W(K)^c,$$

where $W(k)$ is an absolute Weil group of k, which is fixed once and for all. The notation is that of [TW, p. 3, 4]; observe, however, that we write $W(K^{ab}/k)$ rather than $W_{K/k}$, as Tate does.

From the argument at the end of page 4 and beginning of page 5 in [TW], the extension classes of

$$\text{(i)} \quad C_K \hookrightarrow \mathfrak{F} \twoheadrightarrow G \quad \text{and} \quad W(K^{ab}/K) \hookrightarrow W(K^{ab}/k) \twoheadrightarrow G$$

coincide in $H^2(G, C_K)$ after identifying C_K and $W(K^{ab}/K)$ by the reciprocity isomorphism. For $\mathfrak{p} \in S'_*$, choose an embedding $K^{ab} \to K_\mathfrak{p}^{ab}$ extending $K \to K_\mathfrak{p}$. Then there is a commutative diagram

$$\text{(ii)} \quad \begin{array}{ccc} W(K_\mathfrak{p}^{ab}/k_\mathfrak{p}) & \longrightarrow & G(K_\mathfrak{p}^{ab}/k_\mathfrak{p}) \\ \downarrow & & \downarrow \\ W(K^{ab}/k) & \longrightarrow & G(K^{ab}/k) \end{array}$$

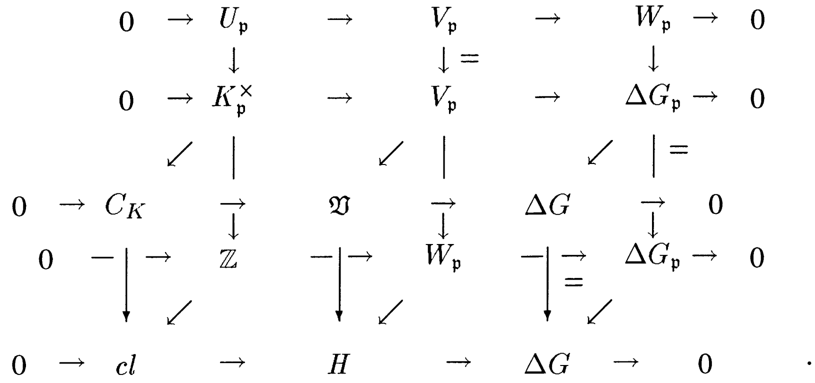
in which the horizontal maps are the canonical maps from the Weil to the appropriate Galois groups. The right vertical map comes from Galois theory and the left vertical map is induced by Tate's θ-map [TW, Prop. 1.6.1].

The following diagram reflects our Galois-theoretic set up, if it is read with every W replaced by G.

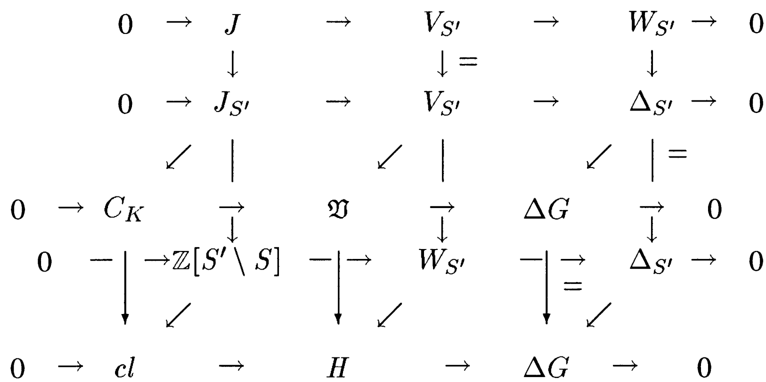
$$\begin{array}{ccccccc}
 & W(K_\mathfrak{p}^{ab}/K_\mathfrak{p}) & \hookrightarrow & W(K_\mathfrak{p}^{ab}/k_\mathfrak{p}) & \twoheadrightarrow & G_\mathfrak{p} & \\
 \swarrow & | & \swarrow & | & \swarrow & & \downarrow = \\
W(K^{ab}/K) & \hookrightarrow & W(K^{ab}/k) & \twoheadrightarrow & G & & \\
 & \downarrow W(K_\mathfrak{p}^{nr}/K_\mathfrak{p}) & \hookrightarrow & W(K_\mathfrak{p}^{nr}/k_\mathfrak{p}) & \twoheadrightarrow & G_\mathfrak{p} & \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow = & \swarrow & \\
G(\tilde{K}/K) & \hookrightarrow & G(\tilde{K}/k) & \twoheadrightarrow & G & &
\end{array}$$

Since the local Weil groups are subgroups of the corresponding Galois group, the only new information in the diagram arises from the maps in its top face. That we may indeed fill them in comes from diagram (ii) above.

We apply the translation functor and (i). For $\mathfrak{p} \in S'_* \setminus S_*$ we stack our old diagram $2\mathfrak{p}$ on the resulting diagram. So we obtain:



As the proof of Proposition 2 shows, we may choose this diagram $2\mathfrak{p}$ with the identity map $V_\mathfrak{p} \to V_\mathfrak{p}$ and with the map $V_\mathfrak{p} \to W_\mathfrak{p}$ coinciding with the one that was already there. For $\mathfrak{p} \in S_*$, we replace the top row above by the second row (with equality as vertical maps everywhere) and the third row (in the back face) by $0 \to 0 \to \Delta G_\mathfrak{p} \to \Delta G_\mathfrak{p} \to 0$. Taking $\mathrm{ind}_{G_\mathfrak{p}}^G$ and direct sums, and glueing on the unit ideles $\prod_{\mathfrak{p} \notin S'} U_\mathfrak{p}$ now gives



in which the map $W_{S'} \to H$ is, by construction, our $\tilde{\sigma}$. Since J is sent to 0 in cl, we have two more maps $W_{S'} \to H$. Namely, start with $w \in W_{S'}$ in the right upper corner, take a preimage $v \in V_{S'}$ and go down to H, either via $W_{S'}$ or via $\mathfrak{V}$. The first map is $\tilde{\sigma}$, since the image of $v \in V_{S'}$ via $V_{S'} = V_{S'} \to W_{S'}$ is w. The second map restricted to $R_{S'}$ takes values in cl and is a snake map s. As the diagram commutes we have $\sigma = s$. This finishes the proof of the theorem.

6. Tate canonical classes

For the proof of Theorem 4 we need to compare two instances of diagram 1 with θ replaced by θ'. Actually we will prove a stronger perturbed version of Theorem

4, which we need in Section 8, where we compare a diagram 1 with a diagram $\tilde{1}$ in which c_1'' is replaced by any $\tilde{c}_1''$ which is homotopic to c_1'' and $\tilde{\theta}$ is any surjective $\tilde{c}_1$ making the diagram commute; thus we write $\tilde{\theta}$ instead of θ'. Applying the process of Section 4 will give a perturbed Tate sequence, and we will show that this is essentially a Tate sequence.

Since $\tilde{c}_1'' \sim c_1''$ we can write $\tilde{c}_1'' - c_1'' = qr$ with $r\colon W_{S'} \to P$, $q\colon P \to \Delta G$ and P projective. Since b_1 is surjective, there exists $p\colon P \to \mathfrak{V}$ so that $q = b_1 p$. We set $V_{S'}^+ = V_{S'} \oplus P$, $W_{S'}^+ = W_{S'} \oplus P$ and build the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & J & \to & V_{S'}^+ & \to & W_{S'}^+ & \to & 0 \\
 & & {=}\swarrow \;\; | & & \cdot\swarrow \;\; |\, \tilde{\theta}_+ & & {}^{w}\swarrow \;\; | & & \\
0 & \to & J & \to & V_{S'}^+ & \to & W_{S'}^+ & \to & 0 \\
 & & \searrow \;\; \downarrow & & \theta_+ \searrow \;\; \downarrow & & \searrow \;\; \downarrow & & \\
0 & \to & C_K & \to & \mathfrak{V} & \to & \Delta G & \to & 0
\end{array} \quad .
$$

The front face is an instance of diagram $1+$, say, which is formed from diagram 1 by setting $c_{1+} = (c_1, p), c_{1+}'' = (c_1'', q), t_{1+} = t_1 \oplus \mathrm{id}_P$. The back face is a diagram $\tilde{1+}$ which is obtained from diagram $\tilde{1}$ in the same way. The w in the top face is the isomorphism $(x, y) \mapsto (x, y + rx)$. The obvious attempt at a dotted arrow is to take $v\colon V_{S'}^+ \to V_{S'}^+$, $(x, y) \mapsto (x, y + rt_1 x)$. Now $\theta_+ v$ need not agree with $\tilde{\theta}_+$, but otherwise the diagram commutes. We next show how to modify this v to give a dotted arrow so the whole diagram commutes. Let $d\colon W_{S'}^+ \to C_K$ be the diagonal deviation from $\theta_+ v$ to $\tilde{\theta}_+$ in diagram $\tilde{1+}$, i.e. $\tilde{\theta}_+ - \theta_+ v = b_{1+}' d t_{1+}$. By Proposition 5(a) below (on the Hom_G level), we can write $d = c_{1+}' d'$ for a G-homomorphism $d'\colon W_{S'}^+ \to J$. Then $v' = v + t_{1+}' d' t_{1+}$ still makes the top face commute and has $\theta_+ v' = \theta_+ v + b_{1+}' c_{1+}' d' t_{1+} = \tilde{\theta}_+$, as claimed.

Applying the snake lemma to front and back faces gives

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & E & \longrightarrow & A_{\tilde{\theta}}^+ & \longrightarrow & \tilde{R}_{S'}^+ & \xrightarrow{\tilde{s}_+} & cl & \longrightarrow & 0 \\
 & & \| & & v_+ \downarrow & & w_+ \downarrow & & \| & & \\
0 & \longrightarrow & E & \longrightarrow & A_{\theta}^+ & \longrightarrow & R_{S'}^+ & \xrightarrow{s_+} & cl & \longrightarrow & 0
\end{array}
$$

with the obvious notation, the presence of a $\tilde{}$ indicating a perturbed object.

Next we bring in the construction of Section 4 to get ∇ involved. Setting $N_{S'}^+ = N_{S'} \oplus P$, we build the diagram

$$\begin{array}{ccccccccc}
 & 0 & \to & W^+_{S'} & \to & N^+_{S'} & \to & M_* & \to & 0 \\
 & & {}^{w}\swarrow & | & {}^{\cdot}\swarrow & | & {}^{=}\swarrow & | & & \\
0 & \to & W^+_{S'} & \to & N^+_{S'} & \to & M_* & \to & 0 & \\
 & & \searrow & \downarrow & \searrow & \downarrow & \searrow & \downarrow & & \\
 & 0 & \to & \Delta G & \to & \mathbb{Z}G & \to & \mathbb{Z} & \to & 0
\end{array}$$

in which the front face is the diagram 4+, which is formed from diagram 4 by setting $c'_{4+} = (c'_4, q), c_{4+} = (c_4, b'_4 q), t'_{4+} = t'_4 \oplus \mathrm{id}_P$. The back face is a diagram $\tilde{4+}$ which is obtained from a diagram $\tilde{4}$ in the same way; here diagram $\tilde{4}$ means diagram 4 perturbed by Lemma 2(c), so $c'_4 = c''_1$ and c_4 are replaced by $\tilde{c}'_4 = \tilde{c}''_1$ and $\tilde{c}_4$, and the rest of diagram 4 remains. This diagram commutes for the w above and the identity on M_*, and we again fill in a dotted arrow. Since $H^1(G, \mathrm{Hom}(M_*, P)) = 0$ there exists $r_+ : N_{S'} \to P$ so that $r_+ t'_4 = r$, and defining $n\colon N^+_{S'} \to N^+_{S'}$, $(x, y) \mapsto (x, y + r_+ x)$, gives a dotted arrow making the top face commute, but with $c_{4+} n$ possibly not equal to c_{4+}. The procedure used to replace v by v', with Proposition 5(b) instead of 5(a), now modifies n to an n' so the whole diagram commutes.

Again applying the snake lemma to the diagram just constructed gives the top face of the diagram

$$\begin{array}{cccccccccc}
 & 0 & \to & \tilde{R}^+_{S'} & \to & \tilde{B}^+_{S'} & \to & \overline{\nabla}_* & \to & 0 \\
 & & {}^{w_+}\swarrow & |\ {}^{\tilde{s}_+} & {}^{n_+}\swarrow & | & {}^{=}\swarrow & |\ {}^{=} & & \\
0 & \to & R^+_{S'} & \to & B^+_{S'} & \to & \overline{\nabla}_* & \to & 0 & \\
 & 0 & \to & cl & \to & \nabla^+_{\tilde{\theta}} & \to & \overline{\nabla}_* & \to & 0 \\
 & & \downarrow\ {}^{s_+} & \swarrow{}^{=} & \downarrow & \swarrow{}^{\cdot} & \downarrow\ {}^{=} & \swarrow{}^{\cdot} & & \\
0 & \to & cl & \to & \nabla^+_{\theta} & \to & \overline{\nabla}_* & \to & 0 &
\end{array}$$

The front and back faces are the pushouts defining ∇ by the construction of Section 4 (and its perturbed analogue). The pushout property of $\nabla^+_{\tilde{\theta}}$ gives a dotted arrow $h\colon \nabla^+_{\tilde{\theta}} \to \nabla^+_{\theta}$ making the left cube commute, and this h induces the dotted arrow $\overline{h} : \overline{\nabla}_* \to \overline{\nabla}_*$ making the right cube commute. It follows that $\overline{h} = \mathrm{id}_{\overline{\nabla}_*}$, hence h is an admissible isomorphism. Applying the construction at the end of Section 4 now yields

$$\sharp \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & A^+_{\tilde{\theta}} & \longrightarrow & \tilde{B}^+_{S'} & \longrightarrow & \nabla^+_{\tilde{\theta}} & \longrightarrow & 0 \\
 & & \| & & v_+ \downarrow & & n_+ \downarrow & & h \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & A^+_{\theta} & \longrightarrow & B^+_{S'} & \longrightarrow & \nabla^+_{\theta} & \longrightarrow & 0
\end{array}$$

with h admissible.

The relation between θ and θ_+ is expressed by a similar diagram

$$\sharp \qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & E & \longrightarrow & A_\theta^+ & \longrightarrow & B_{S'}^+ & \longrightarrow & \nabla_\theta^+ & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E & \longrightarrow & A_\theta & \longrightarrow & B_{S'} & \longrightarrow & \nabla_\theta & \longrightarrow & 0 \end{array}$$

with admissible isomorphism $\nabla_\theta \to \nabla_\theta^+$, which is obtained by the same process as the previous one, except that it is even easier. One uses first

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & V_{S'} & \longrightarrow & W_{S'} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & V_{S'}^+ & \longrightarrow & W_{S'}^+ & \longrightarrow & 0, \end{array}$$

with the natural inclusions as vertical arrows, as the top face of a diagram relating diagram 1 to diagram 1+ for θ, and applies the snake lemma as before. Next we use

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_{S'} & \longrightarrow & N_{S'} & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & W_{S'}^+ & \longrightarrow & N_{S'}^+ & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

again with vertical inclusions, as the top face to relate diagram 4 to diagram 4+. Again the snake lemma gives the top face of a large diagram which yields an admissible isomorphism $\nabla_\theta \to \nabla_\theta^+$ by the pushout property.

In the same way one gets

$$\sharp \qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & E & \longrightarrow & A_{\tilde\theta} & \longrightarrow & \tilde B_{S'}^+ & \longrightarrow & \nabla_{\tilde\theta} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & A_{\tilde\theta}^+ & \longrightarrow & \tilde B_{S'}^+ & \longrightarrow & \nabla_{\tilde\theta}^+ & \longrightarrow & 0 \end{array}$$

relating $\tilde\theta$ and $\tilde\theta^+$, with admissible $\nabla_{\tilde\theta} \to \nabla_{\tilde\theta}^+$.

From the bottom face of the last big diagram shown we get $[\nabla_{\tilde\theta}^+] = [\nabla_\theta^+]$ in $\mathrm{Ext}_G^1(\overline{\nabla}_*, cl)$, and the analogous big diagrams for the comparisons to $+$ give $[\nabla_\theta^+] = [\nabla_\theta]$ and $[\nabla_{\tilde\theta}] = [\nabla_{\tilde\theta}^+]$: thus $[\nabla_{\tilde\theta}] = [\nabla_\theta]$ in $\mathrm{Ext}_G^1(\overline{\nabla}_*, cl)$.

Combining the three diagrams $\sharp$ then yields an admissible isomorphism $\nabla_\theta \to \nabla_{\tilde\theta}$ taking the (possibly perturbed) Tate class for $\tilde\theta$ to that for θ. In particular, this proves the independence from θ in Theorem 4; the independence from S' is delayed to Section 7.

Tracing through the above process we have $P = \mathrm{coker}(W_{S'} \to W_{S'}^+) = \mathrm{coker}(R_{S'} \to R_{S'}^+) = \mathrm{coker}(A_\theta \to A_\theta^+)$, hence $A_\theta^+ \simeq A_\theta \oplus P$ and, similarly, $A_{\tilde\theta}^+ \simeq A_{\tilde\theta} \oplus P$. Since $\tilde R_{S'}^+ \simeq R_{S'}^+$, we have $A_{\tilde\theta}^+ \simeq A_\theta^+$ and, combining, conclude $[A_{\tilde\theta}] = [A_\theta]$ in $K_0(\mathbb{Z}G)$. In particular, this proves Theorem 2(c).

LEMMA 10. *The usual 'inclusion' map $K_\mathfrak{p}^\times \to C_K$ induces an isomorphism*

$$H^0(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, K_\mathfrak{p}^\times)) \to H^0(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, C_K)).$$

Proof. By the cohomological triviality of $V_\mathfrak{p}$ and $\mathfrak{V}$ we have $H^0(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, K_\mathfrak{p}^\times)) \simeq H^{-1}(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, \Delta G_\mathfrak{p})) \simeq H^0(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, C_K))$, so it suffices to show that our map is injective.

By $H^0(G_\mathfrak{p}, \mathrm{Hom}(\Delta G_\mathfrak{p}, K_\mathfrak{p}^\times)) = 0$ the exact sequence $0 \to \mathbb{Z} \to W_\mathfrak{p} \to \Delta G_\mathfrak{p} \to 0$ induces

$$\begin{array}{ccccc} 0 \longrightarrow & H^0(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, K_\mathfrak{p}^\times)) & \longrightarrow & H^0(G_\mathfrak{p}, \mathrm{Hom}(\mathbb{Z}, K_\mathfrak{p}^\times)) \\ & \downarrow & & \downarrow \\ & H^0(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, C_K)) & \longrightarrow & H^0(G_\mathfrak{p}, \mathrm{Hom}(\mathbb{Z}, C_K)) \end{array}$$

so reducing to showing $H^0(G_\mathfrak{p}, K_\mathfrak{p}^\times) \to H^0(G_\mathfrak{p}, C_K)$ injective.

Finally, letting L be the fixed field of $G_\mathfrak{p}$ in K/k,

$$\begin{array}{ccc} H^0(G_\mathfrak{p}, K_\mathfrak{p}^\times) & \xrightarrow{(-, K_\mathfrak{p}/k_\mathfrak{p})} & G_\mathfrak{p}^{ab} \\ \downarrow & & \| \\ H^0(G_\mathfrak{p}, C_K) & \xrightarrow{(-, K/L)} & G_\mathfrak{p}^{ab} \end{array}$$

the standard relation between the local and global reciprocity isomorphisms completes the proof.

PROPOSITION 5.

(a) *$J \to C_K$ induces a surjective map $[W_{S'}, J] \to [W_{S'}, C_K]$.*
(b) *$c_1'' \colon W_{S'} \to \Delta G$ induces a surjective map $[M_*, W_{S'}] \to [M_*, \Delta G]$.*

Proof. (a) By Shapiro's lemma and $H^0(G_\mathfrak{p}, \mathrm{Hom}(\Delta G_\mathfrak{p}, C_K)) = 0$ this reduces to showing $H^0(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, J)) \to H^0(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, C_K))$ surjective for $\mathfrak{p} \in S_*' \backslash S_*$, for which it suffices that $H^0(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, U_\mathfrak{p})) \to H^0(G_\mathfrak{p}, \mathrm{Hom}(W_\mathfrak{p}, C_K))$ be surjective. However this map is the composite of two maps which are surjective by Lemma 8 and Lemma 10. (These ingredients will give a different proof of Proposition 4).

(b) From diagram 1 and the cohomological triviality of $V_{S'}$ and $\mathfrak{V}$ we get the left square

$$\begin{array}{ccccc}
[M_*, W_{S'}] & \xrightarrow{\simeq} & H^1(G, \mathrm{Hom}(M_*, J)) & \xleftarrow{\simeq} & [W_{S'}, J] \\
\downarrow & & \downarrow & & \downarrow \\
[M_*, \Delta G] & \xrightarrow{\simeq} & H^1(G, \mathrm{Hom}(M_*, C_K)) & \xleftarrow{\simeq} & [W_{S'}, C_K]
\end{array}$$

and the right square comes from the exact sequence $0 \to W_{S'} \to N_{S'} \to M_* \to 0$ of Section 4 with $N_{S'}$ projective.

REMARK. The group of admissible automorphisms of ∇ is, by Lemma 2(a), isomorphic to $\mathrm{Hom}_G(\overline{\nabla}_*, cl)$, and acts on $\mathrm{Ext}^2_G(\nabla, E)$ via the canonical map $\mathrm{Hom}_G(\overline{\nabla}_*, cl) \to [\overline{\nabla}_*, cl]$. In particular, it follows that there is a single Tate canonical class whenever cl is cohomologically trivial.

7. Independence from S'; the Chinburg class

For the proof of Theorem 3, let S'' be another larger set, which we may assume contains S'. There is then a natural embedding $W_{S'} \to W_{S''}$, which is the identity on the common components, and whose cokernel $P = \bigoplus_{\mathfrak{p}\in S''_*\setminus S'_*} \mathrm{ind}^G_{G_\mathfrak{p}} W_\mathfrak{p}$ is free, by Lemma 5(a). In the same component-wise manner we construct the top face to a diagram

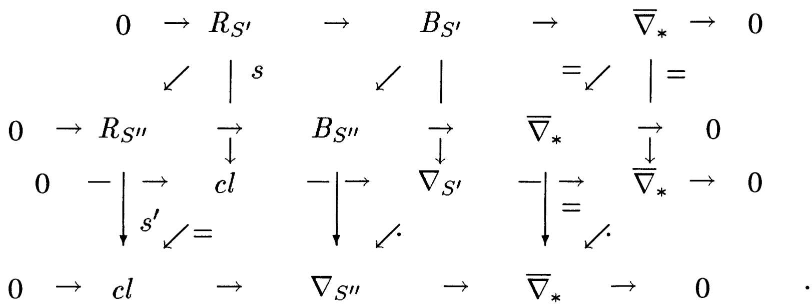
$$\begin{array}{ccccccccc}
 & 0 \to & J & \to & V_{S'} & \to & W_{S'} & \to 0 & \\
 & & {=}\swarrow \ \ | & & \swarrow \ \ |\,\theta & & \swarrow \ \ | & & \\
0 \to & J & \to & V_{S''} & \to & W_{S''} & \to 0 & & \\
 & & \searrow \ \downarrow & & \theta' \searrow \ \downarrow & & \searrow \ \downarrow & & \\
 & 0 \to & C_K & \to & \mathfrak{V} & \to & \Delta G & \to 0 &
\end{array}$$

in which the back face is any diagram 1 for S' and then θ is extended to a θ' (e.g. by the local method of Section 4) which gives a front face which is a diagram 1 for S''. Applying the snake lemma to the diagram 1 faces we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & A_\theta & \longrightarrow & R_{S'} & \xrightarrow{s} & cl & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & E & \longrightarrow & A_{\theta'} & \longrightarrow & R_{S''} & \xrightarrow{s'} & cl & \longrightarrow & 0
\end{array}$$

in which $R_{S'} \to R_{S''}$ clearly has the same cokernel P as $W_{S'} \to W_{S''}$. Since P is projective, the map $[R_{S''}, cl] \to [R_{S'}, cl]$ is an isomorphism and Theorem 3(a) follows from the commutativity of the snake map square above. It also follows that $A_\theta \to A_{\theta'}$ has cokernel P, hence, by Theorem 2(c), that $[A_{S''}] = [A_{S'}] + [P]$ in $K_0(\mathbb{Z}G)$.

We repeat the above procedure with the exact sequence $0 \to W_{S'} \to N_{S'} \to M_* \to 0$. Thus we note that it has a component-wise enlargement to S'', and, from diagram 4, we get the top face of the large diagram



Taking cokernels in this top face, we get $[B_{S''}] = [B_{S'}] + [P]$ in $K_0(\mathbb{Z}G)$, which, together with the last paragraph, completes the proof of Theorem 3(c). The front and back faces of the diagram above are the pushouts along s, s' which define ∇ in Section 4. The pushout property implies a map $h\colon \nabla_{S'} \to \nabla_{S''}$ filling in that dotted arrow and inducing the one $\overline{\nabla}_* \to \overline{\nabla}_*$, which must be $\mathrm{id}_{\overline{\nabla}_*}$. This bottom face now implies that $[\nabla_{S'}] = [\nabla_{S''}]$ in $\mathrm{Ext}^1_G(\overline{\nabla}_*, cl)$, proving Theorem 3(b).

We now turn to removing the dependence on S' in Theorem 4. We can use all of the above (which is parallel to Section 6). From the construction of the Tate Sequence in Section 4, the last two diagrams above imply

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E & \longrightarrow & A_\theta & \longrightarrow & B_{S'} & \longrightarrow & \nabla_{S'} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & h\downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & A_{\theta'} & \longrightarrow & B_{S''} & \longrightarrow & \nabla_{S''} & \longrightarrow & 0
\end{array}$$

with h admissible. This completes the proof of Theorem 4.

Finally we give a proof of the Corollary to Theorem 3(c). Let $rk\colon K_0(\mathbb{Z}G)\to\mathbb{Z}$ be the $\mathbb{Z}G$-rank, i.e. $\mathrm{rk}[\mathbb{Z}G] = 1$. We first use the Tate sequence to compute

$$\begin{aligned}
rk(\Omega^*_m) &= \frac{1}{|G|}\dim_{\mathbb{Q}}\mathbb{Q}\otimes E - \frac{1}{|G|}\dim_{\mathbb{Q}}\mathbb{Q}\otimes\nabla + r_S \\
&= \frac{1}{|G|}\dim_{\mathbb{Q}}\mathbb{Q}\otimes\Delta S - \frac{1}{|G|}\dim_{\mathbb{Q}}\mathbb{Q}\otimes\overline{\nabla}_* + r_S \\
&= -\sum_{\mathfrak{p}\in S^{\mathrm{ram}}_*}\frac{1}{|G|}\dim_{\mathbb{Q}}\mathbb{Q}\otimes\mathrm{ind}^G_{G_\mathfrak{p}} W^0_\mathfrak{p} + r_S = 0,
\end{aligned}$$

by Dirichlet's unit theorem and the exact sequence

$$0 \to \Delta S \to \overline{\nabla}_* \to \bigoplus_{\mathfrak{p}\in S^{\mathrm{ram}}_*} \mathrm{ind}^G_{G_\mathfrak{p}} W^0_\mathfrak{p} \to 0$$

which is immediate from the definition of $\overline{\nabla}^{\cdot}_*$ in Section 4. Thus Ω^*_m is in $Cl(\mathbb{Z}G)$.

Since $B_{S'}$ above is stably free, as we saw in Section 4, the above computation implies $\Omega_m^* = [A] - \mathrm{rk}(A)[A]$, still with S fixed. (Incidentally, this argument will slightly shorten the proof of Theorem 3(c)).

Now to see Ω_m^* is independent of S, let S_1 be another set of primes, and choose S' larger and containing $S \cup S_1$. We build particular Tate sequences for S and S_1, by using the method of Section 4 with the same S'. Thus we use the same diagram 3 and stack a diagram 2 for S, respectively S_1, on it to get a diagram 1 for S and S_1. The point now is that the maps $V_{S'} \to V_{S'}$ in the two copies of diagram 2 are isomorphisms, so the kernels $A_{S'}$ in our two copies of diagram 1 are isomorphic to the kernel in our single diagram 3. As we are free to use any S', by Theorem 3(c), this completes the proof of the Corollary.

8. Transport of structure

Finally we must analyse a transport $X: * \to \Diamond$ (with $S \subset S'$ now fixed). We start by building a particular diagram

$$\begin{array}{ccccccccc}
0 & \to & J & \to & V_* & \to & W_* & \to & 0 \\
 & & {=}\swarrow \;\; \downarrow & & \swarrow \;\; \downarrow & & X\swarrow \;\; \downarrow {}^{X}c''_{1*} & & \\
0 & \to & J & \to & V_\Diamond & \to & W_\Diamond & \to & 0 \\
 & & \searrow \;\; \downarrow & & \searrow \;\; \downarrow & & c''_{1\Diamond}\searrow \;\; \downarrow & & \\
0 & \to & C_K & \to & \mathfrak{B} & \to & \Delta G & \to & 0
\end{array}$$

with front face a diagram 1 for $\Diamond$ obtained by the process, from Section 4, of stacking a particular diagram 2 on any diagram 3. To get the rest of the diagram, we induce the diagram of Proposition 3(a) up to G, for each $\mathfrak{p} \in S_*$, and take direct sums of it in the usual way. This gives a big diagram with front face a diagram 2 for $\Diamond$ (which is the particular diagram 2 for $\Diamond$ mentioned above), with back face a diagram 2 for $*$, and with top and bottom faces consisting of transport maps X.

Composing the bottom face of the big diagram with the chosen diagram 3 for $\Diamond$ gives a diagram $3'$ for $*$. Composing the diagram 2 for $*$ with the diagram $3'$ gives the back face of the diagram we are constructing. The top face consists of transport maps X, which amount to the identity on J via our identification process, by the discussion at the end of Section 4.

This same discussion shows that the maps $J \to C_K$ are the canonical ones and that the map $W_* \to \Delta G$ obtained is ${}^{X}c''_{1*}$ which, moreover, is homotopic to c''_{1*}. Thus the back face of our diagram is a perturbed diagram 1 for $*$, in the sense of Section 6. So we write $\tilde{c}''_{1*}$ instead of ${}^{X}c''_{1*}$.

A similar, but simpler, use of Proposition 3(b) results in the commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & \to & W_* & \to & N_* & \to & M_* & \to & 0\\
 & & & \swarrow & \Big|\,{}^{x}c'_4 & \swarrow & \Big| & \swarrow & \Big| & & \\
0 & \to & W_\Diamond & \to & N_\Diamond & \to & M_\Diamond & \to & 0 & & \\
 & & c'_4\searrow & & \downarrow & \searrow & \downarrow & \searrow & \downarrow & & \\
 & & 0 & \to & \Delta G & \to & \mathbb{Z}G & \to & \mathbb{Z} & \to & 0
\end{array}$$

with $c'_4 = c''_1$ hence ${}^{x}c'_4 = \tilde{c}''_1$.

Proceeding in the familiar way the first diagram leads to

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E & \longrightarrow & \tilde{A}_* & \longrightarrow & \tilde{R}_* & \xrightarrow{\tilde{s}_*} & cl & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & E & \longrightarrow & A_\Diamond & \longrightarrow & R_\Diamond & \xrightarrow{s_\Diamond} & cl & \longrightarrow & 0
\end{array}$$

and the second to the top face of

$$\begin{array}{ccccccccccc}
 & & 0 & \to & \tilde{R}_* & \to & \tilde{B}_* & \to & \overline{\nabla}_* & \to & 0\\
 & & & \swarrow & \Big|\,\tilde{s}_* & \swarrow & \Big| & X\swarrow & \Big| = & & \\
0 & \to & R_\Diamond & \to & & B_\Diamond & \to & \overline{\nabla}_\Diamond & \to & 0 & \\
 & & 0 & \to & cl & \to & \tilde{\nabla}_* & \to & \overline{\nabla}_* & \to & 0\\
 & & \Big\downarrow s_\Diamond & \swarrow = & & \Big\downarrow & \swarrow h & \Big\downarrow = & \swarrow X & & \\
0 & \to & cl & \to & & \nabla_\Diamond & \to & \overline{\nabla}_\Diamond & \to & 0 &
\end{array}$$

with the rest of the diagram following because $\tilde{\nabla}_*$ is a pushout. It results that $X : \overline{\nabla}_* \to \overline{\nabla}_\Diamond$ carries $[\nabla_\Diamond] \in \mathrm{Ext}^1_G(\overline{\nabla}_\Diamond, cl)$ to $[\tilde{\nabla}_*] \in \mathrm{Ext}^1_G(\overline{\nabla}_*, cl)$ and that $h\colon \tilde{\nabla}_* \to \tilde{\nabla}_\Diamond$ is X-admissible.

Taking the last step in the construction we get

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E & \longrightarrow & \tilde{A}_* & \longrightarrow & \tilde{B}_* & \longrightarrow & \tilde{\nabla}_* & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \downarrow h & & \\
0 & \longrightarrow & E & \longrightarrow & A_\Diamond & \longrightarrow & B_\Diamond & \longrightarrow & \nabla_\Diamond & \longrightarrow & 0
\end{array}$$

so h takes the Tate classes in $\mathrm{Ext}^2_G(\nabla_\Diamond, E)$ to the perturbed Tate classes in $\mathrm{Ext}^2_G(\tilde{\nabla}_*, E)$. Moreover, from $W_\Diamond \simeq W_*$ we get $R_\Diamond \simeq \tilde{R}_*$ and thus $A_\Diamond \simeq \tilde{A}_*$ (and $B_\Diamond \simeq \tilde{B}_*$).

Comparing these last assertions with the corresponding ones in Section 6, which are listed just before the statement of Lemma 10, we recognize the assertions (a), (b), (c) of Theorem 6.

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