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An improved bound for the degree of smooth surfaces in \mathbf{P}^4 not of general type

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This is an addendum to the paper of Braun and Fløystad ([BF]) on the bound for the degree of smooth surfaces in \mathbf{P}^4 not of general type. (In fact, any statements made here without comment will be found there.) Using their construction and the regularity of curves in \mathbf{P}^3 , one may lower the bound a little more. We will prove the following:

THEOREM. *Let S be a smooth surface of degree d in \mathbf{P}^4 not of general type. Then either $d \leq 70$ or S lies on a hypersurface in \mathbf{P}^4 of degree 5 and $d \leq 80$. Thus $d \leq 80$.*

(We will also indicate how one may actually get the bound down to 76.)

The main result of [BF] is to bound $\chi_{\mathcal{O}_S}$ from below using generic initial ideal theory and information coming from a generic hyperplane section of S .

Let C be a generic hyperplane section of S , then C has associated to it two pieces of information relevant to this situation; *connected invariants* $\lambda_0 > \lambda_1 > \dots > \lambda_{s-1} > 0$ and *sporadic zeros* which are defined as follows:

DEFINITION. Let $\mathbf{C}[x_0, x_1, x_2, x_3]$ be the ring of polynomials of \mathbf{P}^3 , with reverse lexicographical ordering. Let C be a curve in \mathbf{P}^3 , then the generic initial ideal of C , $\text{gin}(I_C)$, is generated by elements of the form $x_0^i x_1^j x_2^k$. We say that a monomial $x_0^a x_1^b x_2^c$ is a *sporadic zero* if $x_0^a x_1^b x_2^c \notin \text{gin}(I_C)$, but there exists $c' > c$ such that $x_0^a x_1^b x_2^{c'} \in \text{gin}(I_C)$.

Let α_t is the number of sporadic zeros of degree t and m be the maximal degree of the sporadic zeros.

If π is the genus of C , then

$$\pi = 1 + \sum_{i=0}^{s-1} \left(\binom{\lambda_i}{2} + (i-1)\lambda_i \right) - \sum_{t=0}^m \alpha_t \quad (1)$$

and

$$1 + \sum_{i=0}^{s-1} \left(\binom{\lambda_i}{2} + (i-1)\lambda_i \right) \leq \frac{d^2}{2s} + (s-4)\frac{d}{2} + 1 = G(d, s) \tag{2}$$

for $d > (s-1)^2 + 1$.

(This is due to the work of Gruson and Peskine on the numerical invariants of points (see [GP]).)

In [BF] they show that

$$\chi \mathcal{O}_S \geq \sum_{t=0}^{s-1} \left(\binom{\lambda_t + t - 1}{3} - \binom{t-1}{3} \right) - \sum_{t=0}^m \alpha_t(t-1),$$

and if $s \geq 2$ and $d > (s-1)^2 + 1$,

$$\sum_{t=0}^{s-1} \left(\binom{\lambda_t + t - 1}{3} - \binom{t-1}{3} \right) \geq s \left(\frac{d}{s} + \frac{s-3}{3^2} \right) + 1 - \binom{s-1}{4}.$$

They Combine this with the double point formula (see [H, p. 434])

$$d^2 - 5d - 10(\pi - 1) + 2(6\chi - K^2) = 0,$$

and the fact that if the degree of S is $d > 5$ then $K^2 \leq 9$, to get

$$\begin{aligned} 18 \geq 2K^2 &= d^2 - 5d - 10(\pi - 1) + 12\chi \\ &\geq d^2 - 5d - 10 \left(\frac{d^2}{2s} + (s-4)\frac{d}{2} - \sum_{t=0}^m \alpha_t \right) \\ &\quad + 12 \left(s \left(\frac{d}{s} + \frac{s-3}{3^2} \right) + 1 - \binom{s-1}{4} - \sum_{t=0}^m \alpha_t(t-1) \right) \\ &= d^2 - 5d - 10 \left(\frac{d^2}{2s} + (s-4)\frac{d}{2} \right) \\ &\quad + 12 \left(s \left(\frac{d}{s} + \frac{s-3}{3^2} \right) + 1 - \binom{s-1}{4} \right) - \sum_{t=0}^m \alpha_t(12t - 22). \tag{3} \end{aligned}$$

We will use regularity conditions to find an upper bound for

$$A = \sum_{t=0}^m \alpha_t(12t - 22).$$

First we should note that if $s = \min\{k|h^0(\mathcal{I}_S(k)) \neq 0\}$. Then by the work of Ellingsrud and Peskine ([EP]) if S is a surface, of degree d , not of general type then either $d \leq 90$ or $s \leq 5$. Furthermore, if one imitates their argument with $s = 6$, one gets either $d \leq 70$ or $s \leq 6$. We will assume $s \leq 6$. We also know that if $s \leq 3$ then $d \leq 8$. Hence we only need to bound the degree of surfaces not of general type with $s = 4, 5$, or 6 .

By [BS] the regularity of an ideal I is equal to the regularity of $\text{gin}(I)$, which is the highest degree of a minimal generator of $\text{gin}(I)$. Furthermore, by [GLP], the regularity of a smooth curve of degree d in \mathbf{P}^3 is $\leq d - 1$. Hence the largest degree of the minimal generators of $\text{gin}(I_C)$ is $\leq d - 1$ and so all the sporadic zeros of C are in degree $\leq d - 2$.

We will now bound the number of sporadic zeros. Let $\gamma = G(d, s) - \pi$. Any bound on γ will also bound the number of sporadic zeros (see equations (1) and (2)). By [EP], $\gamma \leq \frac{d(s-1)^2}{2s}$. Furthermore, by the double point formula and the fact that if S is a surface not of general type $K^2 < 6\chi$, we get $\pi \geq \frac{d^2-5d+10}{10}$ and thus $\gamma \leq \frac{d^2}{2s} + (s-4)\frac{d}{2} + 1 - \frac{d^2-5d+10}{10}$. Taking the minimum of these bounds for γ , we get, for $s = 4, \gamma \leq \frac{9d}{8}$, for $s = 5, \gamma \leq d$ and for $s = 6, \gamma \leq \frac{d(90-d)}{60}$.

Furthermore, by connectedness of the invariants $\lambda_0 \leq \frac{d}{s} + s - 1$ and $\lambda_1 \leq \frac{d}{s} + s - 2$.

Putting all this together, we have

$$A \leq \sum_{t=\frac{d}{s}+s-1}^{d-2} (12t - 22) + \sum_{t=\frac{d}{s}+s-1}^{\gamma-d+\frac{2d}{s}+2s-2} (12t - 22)$$

and using the bounds on γ , we get

$$\text{for } s = 4, A \leq \frac{243}{32}d^2 - 9d + 192,$$

$$\text{for } s = 5, A \leq \frac{162}{25}d^2 - 16d + 300.$$

Substituting back into the original equation (3) above, we get

$$\text{for } s = 4, 0 \geq \frac{1}{8}d^3 - \frac{275}{32}d^2 + \frac{7}{2}d - 195 \text{ and hence } d \leq 68,$$

$$\text{for } s = 5, 0 \geq \frac{2}{25}d^3 - \frac{162}{25}d^2 + 4d - 318 \text{ and hence } d \leq 80.$$

Now suppose S does not lie on a hypersurface of degree 5. Then the degree of $S = d \leq 90$. By [EP], we know that either $d \leq 70$ or S is contained in a hypersurface of degree 6. Therefore it remains to show that there are no smooth surfaces not of general type with $s = 6$ and $71 \leq d \leq 90$.

Suppose such a surface existed. Then $\gamma \leq \frac{71(90-71)}{60}$, i.e. $\gamma \leq 22$ and so

$$A \leq \sum_{t=\frac{d}{6}+5}^{\frac{d}{6}+5+21} (12t - 22) = 44d + 3608.$$

Substituting as above we get

$$0 \geq \frac{d^3}{18} + \frac{2}{3}d^2 - \frac{119}{2}d - \frac{7321}{2}.$$

But this is a contradiction for $71 \leq d \leq 90$. □

By more careful consideration of the equations involved, it is possible to lower the bound a little more:

Making use of the equation

$$0 \leq \sum_{t=0}^m \alpha_t \leq \sum_{i=0}^{s-1} \left(\binom{\lambda_i}{2} + i(\lambda_i) \right) - \frac{d^2 + 5d}{10}$$

arising from the double point formula, and

$$\begin{aligned} 0 \geq & d^2 - 5d - 18 - 10 \left(\sum_{i=0}^{s-1} \left(\binom{\lambda_i}{2} + (i-1)(\lambda_i) \right) \right) \\ & + 12 \left(\sum_{i=0}^{s-1} \binom{\lambda_i + i - 1}{3} - \binom{s-1}{4} \right) - \sum_{t=0}^m \alpha_t (12t - 22) \end{aligned}$$

it is possible to show that $d \leq 76$ and $s \leq 8$. This is done by writing out all the possible connected invariants of length s and degree d and checking the inequalities on “Mathematica”. (I would like to thank Rich Liebling for sending me the program which calculates the invariants.)

The temptation at this point is to try to lower the bound by considering each degree individually and using ad hoc methods. For example, we know for certain configurations of the invariants the curve is Arithmetically Cohen-Macaulay and hence has no sporadic zeros. But, to make any significant improvements using these methods, one needs to understand which configurations of the sporadic zeros can occur.

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