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On diophantine equations involving sums of powers with quadratic characters as coefficients, II

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Abstract. Let d be the discriminant of a quadratic field. Denote by $\left(\frac{d}{\cdot}\right)$, $h(d)$ and $k_2(d)$ the Kronecker symbol, the class number and the order of the K_2-group of the ring of integers of a quadratic field with the discriminant d, respectively.

In this paper we shall be concerned with the equation

$$\left(\frac{d}{1}\right)1^k + \left(\frac{d}{2}\right)2^k + \cdots + \left(\frac{d}{xd}\right)(xd)^k = by^z$$

in the case of positive d. Using methods of [8] (based on the concept of [9]) we shall prove the above equation has only finitely many solutions in integers $x \geqslant 1$, $y, z > 1$ (with effective upper bounds for them), if $b \neq 0$, $k \geqslant 6$ are integers and $2 \nmid d$, $32 \nmid k_2(d)$. Moreover it is proved for all d satisfying $32 \nmid k_2(d)$ provided k and d are of different parities.

Key words: Sums of powers, generalized Bernoulli numbers and polynomials, class number, diophantine equations

1. Introduction

We follow the notation of [8]. Let d be the discriminant of a quadratic field. Denote by $\left(\frac{d}{\cdot}\right)$ its character. Write $\delta = |d|$. Let $\eta(5) = \frac{1}{5}$, $\eta(8) = \frac{1}{2}$, and $\eta(d) = 1$, otherwise. Put $\xi(-3) = \frac{1}{3}$, $\xi(-4) = \frac{1}{2}$, and $\xi(d) = 1$, otherwise. We have $\eta(d)k_2(d) = B_{2,\left(\frac{d}{\cdot}\right)}$ and $\xi(d)h(d) = -B_{1,\left(\frac{d}{\cdot}\right)}$, where $B_{k,\chi}$ denotes as usual the kth Bernoulli number belonging to the Dirichlet character χ. We let A_2 stand for the ring of polynomials over $\mathbb{Q}$ with 2-integral coefficients.

In this paper we discuss the equation (1.1) of [8] for positive d. Until now, no attempt has been made to consider this case. K. Dilcher [2] proved that:

'*the equation* (1.1) *of* [8] *has only finitely many integral solutions* $x \geqslant 1, y, z > 1$, *(with effective upper bounds for them)*' (D)

for some sequences of negative discriminants (listed in [8]). In [8] we extended Dilcher's list, but we were also concerned only with the case of negative d.

We shall prove the following:

THEOREM 1. *Let d be the odd discriminant of a real quadratic field, and let $k \geqslant 6$ be an integer. Then* (D) *is true if* $32 \nmid k_2(d)$.

THEOREM 2. *Let d be the discriminant of a real quadratic field, and let $k \geqslant 6$ be an integer. Then* (D) *is true if* $32 \nmid k_2(d)$, *and d and k are of different parities.*

REMARK. The conditions of Thms. 1 and 2 are satisfied in the following cases (for details and references, see [5] and [6]). We continue the numbering of the list after Thms. 1, 2 [8] (cases (xvii)–(xx) are with any $k \geqslant 6$, the others only with odd k). Here p and q are odd primes, and $\left(\frac{p}{q}\right)$ denotes the Legendre symbol.

(xvii)	$\delta = p \equiv -3 \pmod 8$,
(xviii)	$\delta = pq \equiv -3 \pmod 8$, $\left(\frac{p}{q}\right) = 1$ and $p \not\equiv -1 \pmod 8$,
(xix)	$\delta = p = u^2 - 2w^2, u > 0, u \equiv 3 \pmod 4, w \equiv 0 \pmod 4$,
(xx)	$\delta = pq, p \equiv q \equiv 3 \pmod 8$,
(xxi)	$\delta = 4p, p \equiv 7 \pmod{16}$,
(xxii)	$\delta = 4pq, p \equiv -q \equiv 3 \pmod 8$,
(xxiii)–(xxxi)	the same δ as in cases (viii)–(xiv) of [8].

2. Formulas used

We can rewrite the equation (1.1) of [8] in the form

$$P_{k+1}(x) = by^z, \tag{2.1}$$

where $P_{k+1}(x)$ is defined by the formula (3.1) [8]. Applying (3.2) [8] in the case of positive d we obtain

$$P_{k+1}(x) = \begin{cases} k(xd)^2 \sum_{i=1}^{\frac{k-1}{2}} \binom{k-1}{2(i-1)} \frac{b_{2i}(d)}{2i-1}(xd)^{k-2i-1}, & \text{if } k \text{ is odd,} \\ kxd \sum_{i=1}^{\frac{k}{2}} \binom{k-1}{2(i-1)} \frac{b_{2i}(d)}{2i-1}(xd)^{k-2i}, & \text{if } k \text{ is even,} \end{cases}$$

where for any $s \geqslant 1$

$$b_s(d) := \frac{B_{s,\left(\frac{d}{\cdot}\right)}}{s}.$$

Hence we get

$$P_{k+1}(x) = \begin{cases} k(xd)^2 \displaystyle\sum_{i=0}^{\frac{k-3}{2}} \binom{k-1}{2i} \frac{b_{2(i+1)}(d)}{2i+1}(xd)^{k-2i-3}, & \text{if } k \text{ is odd,} \\ kxd \displaystyle\sum_{i=0}^{\frac{k}{2}-1} \binom{k-1}{2i} \frac{b_{2(i+1)}(d)}{2i+1}(xd)^{k-2i-2}, & \text{if } k \text{ is even.} \end{cases} \tag{2.2}$$

3. Lemmas

The proofs of Thms. 1 and 2 will be divided into a sequence of lemmas. In the five lemmas below, let d and d^* be the discriminants of quadratic fields and let $k \geqslant 4$ be an even natural number. Assume that $d > 0$.

LEMMA 3. *(see Cor. 1 to Thm. 1 [7]) We have:*

(i) $\operatorname{ord}_2 b_k(d) \geqslant 1$, *if* $d \neq 8$, *and* $\operatorname{ord}_2 b_k(d) = 0$, *if* $d = 8$,
(ii) $\operatorname{ord}_2 b_k(d) = \nu$, $1 \leqslant \nu \leqslant 3 \Leftrightarrow 2^{\nu+1} \| k_2(d)$. □

LEMMA 4. *(see Cor. 2 to Thm. 1 [7]) If* $2 \nmid d$ *then we have:*

(i) $\operatorname{ord}_2 b_k(d) = 4 \Leftrightarrow 32 \| k_2(d)$, *and* $[k \equiv 2 \pmod 4$ or $(k \equiv 0 \pmod 4)$ *and* $16 | h(-4d))]$,
(ii) $\operatorname{ord}_2 b_k(d) \geqslant 5$, *otherwise.* □

LEMMA 5. *(see Cor. 3 to Thm. 2 [7]) If* $4 \mid d$, $d = -4d^*$ *then we have:*

(i) $\operatorname{ord}_2 b_k(d) = 4 \Leftrightarrow 32 \| k_2(d)$ *and* $\{(\frac{d^*}{2}) = 1$ *or* $[(\frac{d^*}{2}) = -1$ *and* $(8 \mid h(d^*)$ *or* $k \equiv 2 \pmod 4)]\}$,
(ii) $\operatorname{ord}_2 b_k(d) \geqslant 5$ *otherwise.* □

LEMMA 6. *(see Cor. 4 to Thm. 3 [7]) Let* $8 \mid d$ *and write* $d = \pm 8d^*$. *If* $d^* < 0$ *and* $(\frac{d^*}{2}) = 1$ *then we have:*

(i) $\operatorname{ord}_2 b_k(d) = 4 \Leftrightarrow \{32 \| k_2(d)$ *and* $[2 | h(d^*)$ *or* $(2 \nmid h(d^*)$ *and* $k \equiv 2 \pmod 4)]\}$ *or* $\{64 | k_2(d)$ *and* $2 \nmid h(d^*)$ *and* $k \equiv 0 \pmod 4\}$,

and if $d^* > 0$, *or* $d^* < 0$ *and* $(\frac{d^*}{2}) = -1$ *then we have:*

(ii) $\operatorname{ord}_2 b_k(d) = 4 \Leftrightarrow \{32 \| k_2(d)$ *and* $[16 | h(-d)$ *or* $(8 \| h(-d)$ *and* $k \equiv 2 \pmod 4)]\}$ *or* $\{64 | k_2(d)$ *and* $8 \| h(-d)$ *and* $k \equiv 0 \pmod 4\}$,

(iii) $\operatorname{ord}_2 b_k(d) \geqslant 5$, *otherwise.*

□

LEMMA 7. *The numbers $b_k(d) \in \mathbb{Q}$ are 2-integral and the congruence*

$$b_k(d) \equiv \tfrac{1}{2}\varepsilon_k(d) k_2(d) \eta(d) \pmod{2^{\operatorname{ord}_2 k_2(d)+1}}$$

holds with $\varepsilon_k(d) \in \{\pm 1\}$ defined as follows:

(a) *if d is odd*

$$\varepsilon_k(d) = \begin{cases} -1, & \text{if } 16 \nmid k_2(d), \text{ or } 16 \| k_2(d) \text{ and } 8 \mid h(-8d), \\ (-1)^{\frac{k}{2}}, & \text{if } 16 \| k_2(d) \text{ and } 4 \| h(-8d), \end{cases}$$

(b) *if $4 \| d$, $d = -4d^*$*

$$\varepsilon_k(d) = \begin{cases} 1, & \text{if } 16 \nmid k_2(d), \text{ or } 16 \| k_2(d), (\frac{d^*}{2}) = -1 \text{ and} \\ & 4 \| h(d^*), \text{ or } 32 \| k_2(d), (\frac{d^*}{2}) = 1 \text{ and } 2 \nmid h(d^*), \\ (-1)^{\frac{k}{2}+1}, & \text{if } 16 \| k_2(d), (\frac{d^*}{2}) = -1 \text{ and } 8 \mid h(d^*), \text{ or} \\ & (\frac{d^*}{2}) = 1 \text{ with } 16 \| k_2(d), \text{ or with } 32 \| k_2(d) \text{ and } 2 \mid h(d^*), \end{cases}$$

(c) *if $8 \mid d$, $d = \pm 8d^*$*

$$\varepsilon_k(d) = \begin{cases} 1, & \text{if } 16 \nmid k_2(d), \text{ or } 16 \| k_2(d), 8 \| h(-d) \text{ with} \\ & 2 \mid h(d^*), \text{ if } d^* < 0, (\frac{d^*}{2}) = 1, \\ (-1)^{\frac{k}{2}+1}, & \text{if } 16 \| k_2(d) \text{ with } 2 \nmid h(d^*), \text{ if } d^* < 0, (\frac{d^*}{2}) = 1, \\ & \text{and with } 16 \mid h(-d), \text{ otherwise.} \end{cases}$$

Proof. Let $2 \nmid d$. In this case, by Thm. 1 [7], we have the congruence

$$\begin{aligned} b_k(d) \equiv & \left(2k\left(\frac{d}{2}\right) + k + 2\right)\mu_k h(-4d) \\ & + \frac{3}{2}\left(-k - 2\left(\frac{d}{2}\right) + 1\right)\vartheta_k k_2(d)\eta(d) \pmod{64}, \end{aligned} \tag{3.1}$$

where μ_k, $\vartheta_k = 1$, if $k \geqslant 8$, and $\mu_4 \equiv \vartheta_4 \equiv \vartheta_6 \equiv 1 \pmod 4$ and $\mu_6 \equiv 1 \pmod 2$ are defined in [7]. On the other hand, by Cor. 1 to Thm. 1 [6], we get

$$k_2(d)\eta(d) \equiv 2h(-4d) \pmod{16}. \tag{3.2}$$

From this and from (3.1) we obtain

$$\begin{aligned} b_k(d) &\equiv \left(k\left(\frac{d}{2}\right)+\frac{k}{2}+1\right)k_2(d)\eta(d) \\ &\quad +\frac{3}{2}\left(-k-2\left(\frac{d}{2}\right)+1\right)k_2(d)\eta(d) \\ &\equiv \frac{1}{2}\left(5-6\left(\frac{d}{2}\right)\right)k_2(d)\eta(d) \pmod{16}, \end{aligned} \tag{3.3}$$

because $4 \mid k_2(d)$ (see [5] or [6]) and k is even.

The above gives the lemma for $2 \nmid d$, $16 \nmid k_2(d)$ because

$$5-6\left(\frac{d}{2}\right) \equiv -1 \pmod 4. \tag{3.4}$$

The same conclusion can be drawn for $16 \| k_2(d)$ and $8 \| h(-4d)$. Indeed, if $16 \| k_2(d)$ then, by Cor. 2 (iii) to Thm. 1 [6], we have $8 \| h(-4d)$ and $8 \mid h(-8d)$, or $16 | h(-4d)$ and $4 \| h(-8d)$. If $8 \| h(-4d)$ then the congruence (3.2) holds modulo 32, and by (3.1) so does the congruence (3.3). This together with (3.4) gives the congruence of the lemma in this case immediately. In order to get the lemma in the case of odd d, it remains to prove it for $16 \| k_2(d)$ and $16 | h(-4d)$. Then (3.1) implies the congruence

$$b_k(d) \equiv \tfrac{1}{2}(k+1)k_2(d)\eta(d) \pmod{32},$$

which yields the lemma at once.

Let $4 \| d$. Put $d = -4d^*$. Then, by Thm. 4 [7], the congruence

$$b_k(d) \equiv \tfrac{1}{2}k_2(d)\eta(d) \pmod{2^\nu} \tag{3.5}$$

holds with $\nu = 4$. This gives the lemma in the case $4 \| d$, $16 \nmid k_2(d)$.

If $16 \mid k_2(d)$ then we consider two cases according to $(\frac{d^*}{2}) = 1$, or -1. By Cor. 2 (i), (iii) to Thm. 2 [6], in both the cases we have $4 \mid h(-2d)$. Furthermore, Thm. 2 [7] states that

$$b_k(d) \equiv \tfrac{1}{2}(k-1)k_2(d) + \vartheta h(d^*)\xi(d^*) - 4(k-2)h(-2d) \pmod{64}, \tag{3.6}$$

where

$$\vartheta = -3\left(1-\left(\frac{d^*}{2}\right)\right)(k-2) + 8\left(1+\left(\frac{d^*}{2}\right)\right)\left(1-\left(\frac{-1}{k-1}\right)\right).$$

If $(\frac{d^*}{2}) = 1$ then, by definition, we deduce that $32 \mid \vartheta$. Consequently, in this case (3.6) implies the congruence

$$b_k(d) \equiv \tfrac{1}{2}(k-1)k_2(d)\eta(d) \pmod{2^\nu} \tag{3.7}$$

with $\nu = 5$. Hence the lemma for $16||k_2(d)$ and $(\frac{d^*}{2}) = 1$ follows easily.

Similar considerations apply to the case $32||k_2(d)$, $(\frac{d^*}{2}) = 1$ and $2|h(d^*)$. Then, by Cor. 2(i) to Thm. 2 [6], we have $8|h(-2d)$ and (3.7) with $\nu = 6$ holds.

If $32||k_2(d)$ and $2 \nmid h(d^*)$ then we have

$$k_2(d) \equiv 2^\tau h(d^*)\xi(d^*) \pmod{2^\nu} \tag{3.8}$$

with $\tau = 5$ and $\nu = 6$. Consequently, the congruence (3.6) in the case $(\frac{d^*}{2}) = 1$ yields

$$b_k(d) \equiv \frac{k-1}{2} k_2(d) + \left(\frac{k}{2} - 1\right) k_2(d) \pmod{2^\nu} \tag{3.9}$$

with $\nu = 6$. This clearly forces (3.5) with the same ν, and completes the proof of the lemma in the case $d = -4d^*$, $(\frac{d^*}{2}) = 1$.

We now turn to the case $(\frac{d^*}{2}) = -1$ and $16||k_2(d)$. If $(\frac{d^*}{2}) = -1$ then we have $\vartheta = -6(k-2)$ and, by Cor. 2(iii) to Thm. 2 [6], $16||k_2(d)$ implies $4 \mid h(d^*)$. If $8 \mid h(d^*)$ then the congruence (3.6) gives (3.7) with $\nu = 5$, and if $4||h(d^*)$ then it implies (3.9) and next (3.5) with the same ν because the congruence (3.8) with $\tau = 2$ and $\nu = 5$ holds then. This establishes the lemma in the case $4||d$.

The task is now to consider the case $8 \mid d$. Then, by Thm. 4 [7], the lemma for $8 \nmid k_2(d)$ follows immediately. We first deal with the case $d = 8d^*$. Then Thm. 3 [7] states

$$b_k(d) \equiv \frac{k-1}{2} k_2(d) + \vartheta_1 h(-d) + \vartheta_2 h(-4d^*) \pmod{64}, \tag{3.10}$$

where

$$\vartheta_1 = 13(k-2) + 16\lambda_k, \quad \vartheta_2 = -4\left(\frac{d^*}{2}\right)(k-2),$$

and $\lambda_4 = 1$, $\lambda_k = 0$, if $k > 4$. Let us assume $8||k_2(d)$. Then, by Cor. 2(ii) to Thm. 1 [6], we have $4||h(-d)$ and the congruence

$$k_2(d) \equiv 2h(-d) \pmod{2^\nu} \tag{3.11}$$

holds with $\nu = 4$. Consequently, the congruence (3.10) leads to (3.9) and next to (3.5) with the same ν.

If $16||k_2(d)$ then, by Cor. 2(iii) to Thm. 1 [6], we have either $8||h(-d)$ and $8 \mid h(-4d^*)$, or $16 \mid h(-d)$ and $4||h(-4d^*)$. If $8||h(-d)$ then the congruence (3.10) implies (3.9) with $\nu = 5$ as the congruence (3.11) with $\nu = 5$ holds. This forces (3.5) with the same ν. If $16 \mid h(-d)$ then (3.10) gives (3.7) with $\nu = 5$ at once, which proves the lemma in the case $d = 8d^*$ completely.

Let us consider the case $d = -8d^*$. Then Thm. 3 [7] states that

$$b_k(d) \equiv \frac{k-1}{2}k_2(d) + \mu h(-d) + 8(k-2)h(d^*)\xi(d^*) \pmod{64}, \tag{3.12}$$

where

$$\mu = \left(4\left(\frac{d^*}{2}\right) + 1\right)(k-2) + 16\lambda_k$$

and λ_k has the same meaning as in (3.10).

If $8 \| k_2(d)$ then, by Cor. 2(iii) to Thm. 2 [6] if $\left(\frac{d^*}{2}\right) = -1$ and by Cor. 1(iii) to Thm. 2 [6] if $\left(\frac{d^*}{2}\right) = 1$, we have $4 \| h(-d)$. Consequently, the congruence (3.12) gives (3.9) with $\nu = 4$, and next (3.5) with the same ν because of (3.11).

If $16 \| k_2(d)$ then the situation is a bit more complicated. If $\left(\frac{d^*}{2}\right) = -1$ then, by Cor. 2(iii) to Thm. 2 [6], we have either $16 \mid h(-d)$ and $2 \| h(d^*)$, or $8 \| h(-d)$ and $4 \mid h(d^*)$. If $16 \mid h(-d)$ then the congruence (3.12) yields (3.7) with $\nu = 5$ at once. If $8 \| h(-d)$ then it implies (3.9) with $\nu = 5$, and next (3.5) with the same ν because of (3.11). If $\left(\frac{d^*}{2}\right) = 1$ then, by Cor. 2(i) to Thm. 2 [6], we obtain $8 \| h(-d)$. Consequently, if $2 \mid h(d^*)$ then the congruences (3.11), (3.9), and next (3.5) with $\nu = 5$ hold. If $2 \nmid h(d^*)$ then, by (3.12), we deduce that

$$k_2(d) \equiv \frac{k-1}{2}k_2(d) + \left(\frac{k}{2} - 1\right) k_2(d) + \left(\frac{k}{2} - 1\right) k_2(d) \pmod{32}$$

because the congruences (3.11) and (3.8) with $\tau = 4, \nu = 5$ hold then. The above congruence clearly forces (3.7) with $\nu = 5$, which is our claim.
The lemma is proved completely. □

LEMMA 8. *(see [9], [3], [4] and [1]) Let $0 \neq b \in \mathbb{Z}$ and let $P(x) \in \mathbb{Q}[x]$ be a polynomial with at least three zeros of odd multiplicity and for any odd prime p, with at least two zeros of multiplicities relatively prime to p. Then the equation*

$$P(x) = by^z$$

has only finitely many integral solutions $x \geqslant 1$, $y, z > 1$ and these solutions can be effectively determined. □

LEMMA 9. *Let $l \geqslant 5$ be a natural number. Write $l \equiv \lambda \pmod 2, \lambda \in \{0,1\}$. Set*

$$\varphi_l(x) = \sum_{i=0}^{\frac{l-2+\lambda}{2}} \binom{l}{2i} c_{2i} x^{l-2i-2+\lambda}.$$

If $c_{2i} \in \mathbb{Q}$ $(i \geqslant 0)$ are 2-integral, $2 \nmid c_0$ and for $i \geqslant 1$ one of the following congruences holds

(i) $c_{2i} \equiv (-1)^{i+1}c_0 \pmod 4$,

(ii) $c_{2i} \equiv (-1)^{i}c_0 \pmod 4$,

(iii) $c_{2i} \equiv c_0 \pmod 4$,

then the polynomial $x^{2-\lambda}\varphi_l(x)$ *satisfies the hypothesis of Lemma 8 for any* l *in the case* (i), *and for even* l *in the cases* (ii) *and* (iii).

Proof. (i) (a) The case of even l.

The proof of the lemma in this case is similar to the proof of Lemma 9(ii) [8] with k, $f_k(x)$, and a_{2i} replaced by l, $\varphi_l(x)$, and c_{2i}, respectively. Since in each of the cases (i), (ii) and (iii) of the lemma we have $a_0 \equiv a_{2i} \equiv c_{2i} \equiv c_0 \pmod 2$, all the congruences modulo 2 of the proof of Lemma 9(ii) [8] work with k and $f_k(x)$ replaced by l and $\varphi_l(x)$ respectively. Therefore in the same manner we can exclude the equality

$$\varphi_l(x) = qt^p(x) \tag{3.13}$$

for any prime $p \geqslant 3$, 2-integral $q \in \mathbb{Q}$ and $t \in A_2$ provided $l \neq 2^\theta$. Also in all these cases, if $l = 2^\theta$, $\theta \geqslant 3$ then analysis similar to that in the proof of Lemma 9(ii) [8] excludes (3.13) for any prime p and the equality

$$\varphi_l(x) = w(x)u^2(x) \tag{3.14}$$

for $w, u \in A_2$, $\deg w = 2$.

What is left is to exclude the equalities (3.13) with $p = 2$ and (3.14) if $l \neq 2^\theta$. Let us again note that to exclude them for the polynomial $\varphi_l(x)$ it is sufficient to do it for the polynomial $\psi_l(x)$ defined by the formula

$$\psi_l(x) = (x+1)^2\varphi_l(x+1) = \sum_{i=0}^{\frac{l-2}{2}} \binom{l}{2i} c_{2i}(x+1)^{l-2i}. \tag{3.15}$$

We first exclude (3.13) with $p = 2$ and with $\varphi_l(x)$ replaced by $\psi_l(x)$. As in the proof of Lemma 9(ii) [8], (3.13) with $p = 2$ implies the congruence

$$\begin{aligned}\psi_l(x) \;&\equiv\; qx^l + 2x^{\frac{l}{2}} + q + 2(x^{l_1} \cdots + x^{l_m}) \\ &\quad +2(x^{l_1+\frac{l}{2}} + \cdots + x^{l_m+\frac{l}{2}}) \pmod 4\end{aligned} \tag{3.16}$$

(see (4.11) of [8]), where $l_1 > \cdots > l_m \geqslant 0$.

On the other hand denoting by γ_j, $0 \leqslant j \leqslant l$ the coefficient of x^j in the polynomial $\psi_l(x)$, by (3.15), we deduce that

$$\gamma_l = c_0, \tag{3.17}$$

and

$$\begin{aligned}
\gamma_0 &= \sum_{i=0}^{\frac{l-2}{2}} \binom{l}{2i} c_{2i} \equiv c_0\left[1+\sum_{i=1}^{\frac{l-2}{2}} \binom{l}{2i}(-1)^{i+1}\right] \\
&\equiv c_0\left[2-\sum_{i=0}^{\frac{l-2}{2}} \binom{l}{2i}(-1)^{i}\right] \\
&\equiv c_0\left[2+\sum_{\substack{0\leqslant i\leqslant l-2\\ 2|i}} \binom{l}{i} - 2\sum_{\substack{0\leqslant i\leqslant l-2\\ 4|i}} \binom{l}{i}\right] \\
&\equiv c_0\left[2+\left(2^{l-1}-1\right)-\left(2^{l-1}+2^{\frac{l}{2}}\cos\frac{l\pi}{4}-1-(-1)^{\frac{l}{2}}\right)\right] \\
&\equiv c_0\left[2-2^{\frac{l}{2}}\cos\frac{l\pi}{4}+(-1)^{\frac{l}{2}}\right] \\
&\equiv c_0[2\pm 2^{\frac{l}{2}-1}((-1)^{\frac{l}{2}}+1)+(-1)^{\frac{l}{2}}] \pmod 4,
\end{aligned}$$

i.e.,

$$\gamma_0 \equiv (-1)^{\frac{l}{2}+1} c_0 \pmod 4 \tag{3.18}$$

because $l \geqslant 4$.

Moreover using the same arguments as in the proof of Lemma 9(ii) [8], by (3.15), for $0 < j < l$ we obtain

$$\begin{aligned}
\gamma_j &= \sum_{i=0}^{[\frac{l-j}{2}]} \binom{l}{2i}\binom{l-2i}{j} c_{2i} = \binom{l}{j}\sum_{i=0}^{[\frac{l-j}{2}]} \binom{l-j}{2i} c_{2i} \\
&\equiv c_0\binom{l}{j}\left[2-\sum_{i=0}^{[\frac{l-j}{2}]} \binom{l-j}{2i}(-1)^{i}\right] \\
&\equiv c_0\binom{l}{j}\left[2+\sum_{\substack{0\leqslant i\leqslant l-j\\ 2|i}} \binom{l-j}{i} - 2\sum_{\substack{0\leqslant i\leqslant l-j\\ 4|i}} \binom{l-j}{i}\right] \\
&\equiv c_0\binom{l}{j}\left[2+2^{l-j-1}-\left(2^{l-j-1}+2^{\frac{l-j}{2}}\cos\frac{(l-j)\pi}{4}\right)\right] \\
&\equiv c_0\binom{l}{j}\times\begin{cases} [2\pm 2^{\frac{l-j}{2}-1}((-1)^{\frac{l-j}{2}}+1)] \pmod 4, & \text{if } j \text{ is even,} \\ (2-(-1)^{\frac{(l-j)^2-1}{8}} 2^{\frac{l-j-1}{2}}) \pmod 4, & \text{if } j \text{ is odd.} \end{cases}
\end{aligned}$$

Thus we get

$$\gamma_j \equiv \begin{cases} 2\binom{l}{j} \pmod 4, & \text{if } j \leqslant l-3, \\ l \pmod 4, & \text{if } j \geqslant l-2. \end{cases}$$

Consequently the above together with (3.17) and (3.18) give

$$\psi_l(x) \equiv c_0 x^l + l x^{l-1} + l x^{l-2} + 2\sum_{j=1}^{l-3} \binom{l}{j} x^j + (-1)^{\frac{l}{2}+1} c_0 \pmod 4. \quad (3.19)$$

Combining this with (3.16) we can assert that $l_m = 0$ because, by Lemma 8(i) [8], we have $2 \mid \binom{l}{l/2}$. Moreover we get $c_0 \equiv q \pmod 4$ and $(-1)^{\frac{l}{2}+1} c_0 \equiv q+2 \pmod 4$, and in consequence $l \equiv 0 \pmod 4$.

Furthermore, since $l \neq 2^\theta$ there exists $1 \leqslant j \leqslant l-3$ such that $2 \nmid \binom{l}{j}$. Combining this with (3.16) yields $2 \nmid \binom{l}{j+\frac{l}{2}}$, and this contradicts Lemma 8(ii) [8]. Indeed, let $2^\theta \| l$. Then $2 \nmid \binom{l}{j}$ implies $2^\theta \mid j$. This yields $2^{\theta-1} \| j + \frac{l}{2}$ and, by Lemma 8(ii) [8] again, we get $2 \mid \binom{l}{j+\frac{l}{2}}$. Contradiction.

Now, we exclude (3.14) with $\varphi_l(x)$ replaced by $\psi_l(x)$. Put $w(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{Q}$ are 2-integral and $a \neq 0$. Then as in the proof of Lemma 9(ii) [8] we get the congruence

$$\begin{aligned} \psi_l(x) \equiv\ & c_0 x^l + (-1)^{\frac{l}{2}+1} c_0 + (1 + (-1)^{\frac{l}{2}+1})(x^{l-2} + \cdots + x^4 + x^2) \\ & + 2\sum_{t=3}^{l-3} x^t + (l-2)x + (l-2)x^2 + 2x^{l-1} \\ & + b(x^{l-1} + \cdots + x^3 + x) \pmod 4 \end{aligned} \quad (3.20)$$

(see (4.18) and (4.19) [8]), where b is even.

Thus if $l \equiv 0 \pmod 4$ the congruence (3.20) takes the form

$$\begin{aligned} \psi_l(x) \equiv\ & c_0 x^l - c_0 + 2\sum_{t=3}^{l-3} x^t + (l-2)x + (l-2)x^2 + 2x^{l-1} \\ & + b(x^{l-1} + \cdots + x^3 + x) \pmod 4. \end{aligned} \quad (3.21)$$

This contradicts (3.19) because of the coefficients of x^2 (the above congruence gives $\gamma_2 \equiv l - 2 \pmod 4$, and (3.19) implies $\gamma_2 \equiv l \pmod 4$).

If $l \equiv 2 \pmod 4$ then (3.20) takes the form

$$\psi_l(x) \equiv c_0x^l + c_0 + 2 \sum_{\substack{3\leqslant t\leqslant l-3 \\ 2\nmid t}} x^t + 2x^{l-2} + 2x^2 + 2x^{l-1}$$
$$+b(x^{l-1} + \cdots + x^3 + x) \pmod 4. \tag{3.22}$$

Thus, by (3.19) (because of the coefficients of x^{l-1}), we find that $b+2 \equiv l \pmod 4$, i.e., $b \equiv 0 \pmod 4$. Therefore (3.22) takes the form

$$\psi_l(x) \equiv c_0x^l + c_0 + 2 \sum_{\substack{3\leqslant t\leqslant l-1 \\ 2\nmid t}} x^t + 2x^{l-2} + 2x^{l-1} + 2x^2 \pmod 4,$$

which together with (3.19) imply $2 \mid \binom{l}{j}$ for all even $2 < j < l-2$. Since l is not a power of 2, these divisibilities contradict Lemma 8(ii) [8].

Part (i) of the lemma in the case of even l is proved. □

(b) The case of odd l.

As in the proof of Lemma 9(ii) [8] we shall prove that the polynomial $\psi_l(x)$ defined by the formula

$$\psi_l(x) := \varphi_l(x+1) = \sum_{i=0}^{\frac{l-1}{2}} \binom{l}{2i} c_{2i}(x+1)^{l-2i-1}$$

is an Eisenstein polynomial with respect to $p = 2$ (and so irreducible over $\mathbb{Q}$).

Let us denote by γ_j $(0 \leqslant j \leqslant l-1)$ the coefficient of x^j in the polynomial $\psi_l(x)$. We have:

$$\gamma_{l-1} = c_0, \tag{3.23}$$

i.e., γ_{l-1} is odd.

Moreover, by definition, we get

$$\begin{aligned}\gamma_0 &= \sum_{i=0}^{\frac{l-1}{2}} \binom{l}{2i} c_{2i} \equiv c_0\left[2 - \sum_{i=0}^{\frac{l-1}{2}} \binom{l}{2i}(-1)^i\right] \\ &\equiv c_0\left[2 + \sum_{\substack{0\leqslant i\leqslant l-1 \\ 2\mid i}} \binom{l}{i} - 2\sum_{\substack{0\leqslant i\leqslant l-1 \\ 4\mid i}} \binom{l}{i}\right] \\ &\equiv c_0\left[2 + 2^{l-1} - \left(2^{l-1} + 2^{\frac{l}{2}}\cos\frac{l\pi}{4}\right)\right] \equiv c_0(2 \pm 2^{\frac{l-1}{2}}) \pmod 4\end{aligned}$$

i.e.,

$$\gamma_0 \equiv 2c_0 \equiv 2 \pmod 4, \tag{3.24}$$

if $l \geqslant 5$.

Furthermore, analysis similar to that in the proof of Lemma 9(ii) [8] shows that

$$\gamma_j \equiv 0 \pmod 2,$$

if $1 \leqslant j \leqslant l-2$ (we only use in this place congruences modulo 2), as required. This together with (3.23) and (3.24) give the desired conclusion, which completes the proof of (i) of the lemma. □

(ii) and (iii). Let l be even. As was mentioned at the beginning of the proof of (i) of the lemma, it suffices to exclude the equalities (3.13) with $p = 2$ and (3.14) (with $\varphi_l(x)$ replaced by $\psi_l(x)$). By (3.15), in case (ii) we get

$$\gamma_0 \equiv c_0 \sum_{i=0}^{\frac{l-2}{2}} \binom{l}{2i} (-1)^i \equiv (-1)^{\frac{l}{2}+1} c_0 \pmod 4,$$

and in case (iii) it follows that

$$\gamma_0 \equiv c_0 \sum_{i=0}^{\frac{l-2}{2}} \binom{l}{2i} \equiv c_0(2^{l-1} - 1) \equiv -c_0 \pmod 4,$$

(cf. the congruences before (3.18)). Moreover for $0 < j < l$, in case (ii) we obtain

$$\gamma_i \equiv c_0 \binom{l}{j} \sum_{i=0}^{[\frac{l-j}{2}]} \binom{l-j}{2i} (-1)^i \equiv \begin{cases} 0 \pmod 4, & \text{if } j \leqslant l-2, \\ l \pmod 4, & \text{if } j = l-1 \end{cases}$$

(cf. the congruences before (3.19)), and in case (iii) we find that

$$\gamma_i \equiv c_0 \binom{l}{j} \sum_{i=0}^{[\frac{l-j}{2}]} \binom{l-j}{2i} \equiv c_0 \binom{l}{j} 2^{l-j-1} \equiv \begin{cases} 0 \pmod 4, & \text{if } j \leqslant l-3, \\ l \pmod 4, & \text{if } j \geqslant l-2. \end{cases}$$

Therefore, by (3.17), we deduce that

$$\psi_l(x) \equiv c_0 x^l + l x^{l-1} + (-1)^{\frac{l}{2}+1} c_0 \pmod 4 \tag{3.25}$$

in case (ii), and

$$\psi_l(x) \equiv c_0 x^l + l x^{l-1} + l x^{l-2} - c_0 \pmod 4 \tag{3.26}$$

in case (iii).

We first exclude (3.13) with $p = 2$ and $\varphi_l(x)$ replaced by $\psi_l(x)$. Write $t(x) = t_{l/2}x^{l/2} + \cdots + t_0$, where $t_i \in \mathbb{Q}$ are 2-integral. We see at once that the congruences (3.25) and (3.26) imply $2 \nmid t_{l/2}, t_0$. Consequently we have $\gamma_l = qt_{l/2}^2 \equiv q \pmod 4$, and $\gamma_0 = qt_0^2 \equiv q \pmod 4$ in both the cases. This excludes (3.13) in case (iii) for all l and in case (ii) for $4 \mid l$. If $l \equiv 2 \pmod 4$ then (3.25) takes the form $\psi_l(x) \equiv c_0 x^l + c_0 \pmod 4$. This contradicts (3.16), which implies $l_m = 0$ (because of the coefficient of $x^{l/2}$), and next $\gamma_l \not\equiv \gamma_0 \pmod 4$.

In order to exclude (3.14), let us note that in case (ii) (3.14) implies (3.20), and in case (iii) it gives (3.21). Combining (3.25) with (3.20) yields

$$1 + (-1)^{\frac{l}{2}+1} + (l-2) \equiv 0 \pmod 4$$

because of the coefficients of x^2. This is impossible, of course. Next combining (3.26) with (3.21) gives $l \equiv 2 \pmod 4$ (because of the coefficient of x^2). Contradiction because of the coefficients of x^{l-2} in these formulas.

The lemma is proved completely. □

REMARKS. For odd $l \geqslant 5$ the polynomial $\psi_l(x) := \varphi_l(x+1)$ is not an Eisenstein polynomial with respect to $p = 2$ because

$$\gamma_0 \equiv c_0 \sum_{i=0}^{\frac{l-1}{2}} \binom{l}{2i} (-1)^i \equiv 0 \pmod 4$$

in case (ii) (cf. the congruences before (3.23)), and

$$\gamma_0 \equiv c_0 \sum_{i=0}^{\frac{l-1}{2}} \binom{l}{2i} \equiv c_0 2^{l-1} \equiv 0 \pmod 4$$

in case (iii).

In the case of odd l we have

$$\binom{l}{2i} \equiv (-1)^i \binom{l-1}{2i} \pmod 4. \tag{3.27}$$

Hence we obtain

$$\varphi_l(x) \equiv c_0 \sum_{i=0}^{\frac{l-1}{2}} \binom{l}{2i} (-1)^i x^{l-2i-1} \equiv c_0 \sum_{i=0}^{\frac{l-1}{2}} \binom{l-1}{2i} x^{2i} \pmod 4,$$

i.e.,

$$\varphi_l(x) \equiv c_0 \frac{(x+1)^{l-1} + (x-1)^{l-1}}{2} \pmod 4$$

in case (ii). Thus in case (ii) we get

$$\begin{aligned} \varphi_l(x+1) &\equiv c_0\left[x^{l-1} + (l-1)x^{l-2} + (l-1)x^{l-3}\right] \\ &\equiv c_0 x^{l-3}\left[x^2 + (l-1)x + (l-1)\right] \pmod 4. \end{aligned} \tag{3.28}$$

Similarly, by (3.27) in case (iii) we have

$$\begin{aligned} \varphi_l(x) &\equiv c_0 \sum_{i=0}^{\frac{l-1}{2}} \binom{l}{2i} x^{l-2i-1} \\ &\equiv c_0 \left[\sum_{\substack{0 \leqslant i \leqslant \frac{l-1}{2} \\ 2 \mid i}} \binom{l-1}{2i} x^{l-1-2i} - \sum_{\substack{0 \leqslant i \leqslant \frac{l-1}{2} \\ 2 \nmid i}} \binom{l-1}{2i} x^{l-1-2i} \right] \\ &\equiv c_0 \left[\sum_{i=0}^{\frac{l-1}{2}} \binom{l-1}{2i} x^{l-1-2i} - 2 \sum_{\substack{0 \leqslant i \leqslant \frac{l-1}{2} \\ 2 \nmid i}} \binom{l-1}{2i} x^{l-2i-1} \right] \\ &\equiv c_0 \left[\sum_{i=0}^{\frac{l-1}{2}} \binom{l-1}{2i} x^{2i} - (l-1) \sum_{i=0}^{\frac{l-1}{2}} \binom{l-2}{2i-1} x^{2i-1} \right] \\ &\equiv c_0 \frac{(x+1)^{l-1} + (x-1)^{l-1}}{2} \\ &\quad + (l-1) \frac{(x+1)^{l-2} - (x-1)^{l-2}}{2} \pmod 4 \end{aligned}$$

because

$$(l-1)\binom{l-2}{2i-1} \equiv 0 \pmod 4,$$

if $2 \mid i$, $i > 0$. Thus in case (iii) we obtain

$$\varphi_l(x+1) \equiv c_0[x^{l-1} + (l-1)x^{l-2} + (l-1)x^{l-3}] - (l-1)x^{l-3}$$
$$\equiv c_0 x^{l-2}(x+l-1) \pmod 4.$$

With this and (3.28) we get in both cases

$$\varphi_l(x) \equiv c_0(x-1)^{l-1} \pmod 4,$$

if $l \equiv 1 \pmod 4$, and

$$\varphi_l(x) \equiv c_0(x-1)^{l-3}(x^2 \pm 1) \pmod 4,$$

if $l \equiv 3 \pmod 4$.

Thus it is not possible to exclude (3.13) and (3.14) for odd l using only congruences modulo 4.

4. Proofs of the theorems

Put $l = k-1$. We shall apply Lemma 9, or Lemma 9(ii) of [8] (with $k, \kappa, f_k(x), a_{2i}$ replaced by $l, \lambda, \varphi_l(x)$ and c_{2i} respectively) for the polynomial

$$\varphi_l(x) := 2^{1-\operatorname{ord}_2 k_2(d)}(l+1)^{-1}x^{\lambda-2}P_{l+2}(xd^{-1}) \in \mathbb{Q}[x],$$

where $l \equiv \lambda \pmod 2$, $\lambda \in \{0,1\}$. Then, by (2.2), we get

$$\varphi_l(x) = \sum_{i=0}^{\frac{l-2+\lambda}{2}} \binom{l}{2i} c_{2i} x^{l-2i-2+\lambda},$$

where for $i \geqslant 0$ we have

$$c_{2i} := c_{2i}(d) = 2^{1-\operatorname{ord}_2 k_2(d)} \frac{b_{2(i+1)}(d)}{2i+1} \in \mathbb{Q}.$$

Moreover, by Lemmas 3–6 and by the assumptions on d of both the theorems, we get

$$\operatorname{ord}_2 b_{2i}(d) = \operatorname{ord}_2 k_2(d) - 1,$$

if $i \geqslant 1$ and in consequence the coefficients c_{2i} are 2-integral and $c_{2i} \equiv 1 \pmod 2$. Furthermore, by Lemma 7, the assumptions on d lead to the congruences (i) of Lemma 9 or to the congruence

$$c_{2i} \equiv -c_0 \pmod 2$$

in case of Theorem 1, and to the congruences (ii) or (iii) of Lemma 9 in case of Theorem 2. Therefore in order to prove Theorem 1 it is sufficient to combine

Lemma 9(i) or Lemma 9(ii) [8] with Lemma 8 applied to the equation (2.1). In case of Theorem 2 it suffices to apply Theorem 1 if l is even, or to combine Lemma 9(ii),(iii) with Lemma 8 applied to the same equation if l is odd.
The theorem is proved. □

REMARKS. According to Remark after the proof of Lemma 9, it is not possible to prove this lemma under the assumptions (ii) or (iii) for odd l in the same manner. We can extend Lemma 7 for some even d by proving similar congruences modulo $2^{\mathrm{ord}_2 k_2(d)+2}$ which imply congruences for c_{2i} modulo 8. Unfortunately these new congruences are still too weak to give Lemma 9(ii),(iii) for odd l by the same methods.

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