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On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields

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The main goal of this paper is to prove the existence of cuspidal automorphic representations for some series of examples of S-arithmetic subgroups of reductive groups over number fields which give rise to non-vanishing cuspidal cohomology classes. In order to detect these cuspidal automorphic representations we combine two techniques, both of which can be seen as special cases of Langlands functoriality. Prior to that, we have to extend to the S-arithmetic case the definition of cuspidal cohomology and to show it is a direct summand in the cohomology. We emphasize that in our framework the class of S-arithmetic groups contains the class of arithmetic groups.

Before describing the contents of the paper we recall some facts about the cohomology of S-arithmetic groups. Let G be a semisimple group over a number field k. As usual S is a finite set of places of k containing all the archimedean ones. Let (ϕ, E) be a finite dimensional complex representation of G_S trivial on G_{S_f} and let Γ be an S-arithmetic subgroup of $G(k)$. The cohomology groups $H^{\cdot}(\Gamma; E)$ of Γ with values in E can be computed using the differential cohomology of G_S in the space of smooth functions on $\Gamma\backslash G_S$:

$$H^{\cdot}(\Gamma;\, E) = H^{\cdot}_d(G_S;\, C^\infty(\Gamma\backslash G_S) \otimes E).$$

The space of cusp forms plays a central role; accordingly it is natural to investigate the corresponding space of cohomology called the cuspidal cohomology:

$$H^{\cdot}_{\text{cusp}}(\Gamma;\, E) := H^{\cdot}_d(G_S;\, \mathbf{L}^2_{\text{cusp}}(\Gamma\backslash G_S)^\infty \otimes E)$$

where

$$\mathbf{L}^2_{\text{cusp}}(\Gamma\backslash G_S)^\infty$$

is the space of smooth vectors, in the space of square integrable functions, generated by cusp forms. It is a subspace of the cohomology with respect to the discrete spectrum

$$H^{\cdot}_{\text{disc}}(\Gamma; E) := H^{\cdot}_d(G_S; \mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S)^\infty \otimes E)$$

which is in turn a subspace of the $\mathbf{L}^2$-cohomology

$$H^{\cdot}_{(2)}(\Gamma; E) := H^{\cdot}_d(G_S; \mathbf{L}^2(\Gamma\backslash G_S)^\infty \otimes E).$$

The natural inclusions

$$\begin{aligned}\mathbf{L}^2_{\mathrm{cusp}}(\Gamma\backslash G_S)^\infty &\to \mathbf{L}^2_{\mathrm{disc}}(\Gamma\backslash G_S)^\infty \otimes E \\ &\to \mathbf{L}^2(\Gamma\backslash G_S)^\infty \otimes E \to C^\infty(\Gamma\backslash G_S) \otimes E \end{aligned} \tag{1}$$

yield therefore natural homomorphisms

$$H^{\cdot}_{\mathrm{cusp}}(\Gamma; E) \xrightarrow{\mu} H^{\cdot}_{\mathrm{disc}}(\Gamma; E) \xrightarrow{\nu} H^{\cdot}_{(2)}(\Gamma; E) \xrightarrow{\sigma} H^{\cdot}(\Gamma; E). \tag{2}$$

When the discrete group Γ is cocompact, all the maps in (1) are isomorphisms, hence so are those in (2). A first goal of this paper is to provide some information on (2) in the isotropic case. In particular we show that $\sigma \circ \nu \circ \mu$ is injective. A second goal is to construct, in some cases, non-trivial cohomology classes in $H^{\cdot}_{\mathrm{cusp}}(\Gamma; E)$.

From now on, we assume for convenience in this introduction that G is almost absolutely simple over k. We are only concerned with the isotropic case and assume that $rk_k(G) > 0$, so that, in particular, $G(k_v)$ is not compact for any $v \in S$.

In Part I we prove a decomposition theorem for functions of uniform moderate growth on $\Gamma\backslash G_S$ (see Section 2 for the statement), originally established by R. Langlands in the arithmetic case, i.e. when $S = S_\infty$. His argument had so far only been sketched in a letter [Lan2] and elaborated upon in an unpublished preprint of the first named author. We take this opportunity to present a complete proof for any S. In the case $S = S_\infty$, another one is contained in [Ca2] (see 4.6).

In Part II we define and study the cuspidal cohomology of Γ. We generalize to the S-arithmetic situation the regularization theorem of [B5]. From this and the decomposition theorem it follows that the cohomology space $H^{\cdot}(\Gamma; E)$ is canonically a direct sum

$$H^{\cdot}(\Gamma; E) = \bigoplus_{\mathcal{P} \in \mathcal{A}} H^{\cdot}_{\mathcal{P}}(\Gamma; E) \tag{3}$$

where $\mathcal{P}$ runs through the set $\mathcal{A}$ of classes of associate parabolic k-subgroups of G (5.4). Of main interest to us is the summand indexed by G, which we shall also denote by $H^{\cdot}_{\mathrm{cusp}}(\Gamma; E)$ and call the cuspidal cohomology, since it can be identified to the space so denoted above. This shows in particular that the homomorphism

$$\sigma \circ \nu \circ \mu : H^{\cdot}_{\mathrm{cusp}}(\Gamma; E) \to H^{\cdot}(\Gamma; E)$$

(see (2)) is injective, a fact already established in the arithmetic case in [B4], by a different argument.

Let r_f be the sum of the k_v-ranks of G/k_v for $v \in S_f$. For $v \in S_f$ the Steinberg representation is the only irreducible unitary representation of $G(k_v)$ with non trivial cohomology, besides the trivial representation. Then arguments similar to those of [BW:XIII] show that for all $i \in \mathbb{Z}$

$$H^{i+r_f}_{\text{cusp}}(\Gamma;\, E) = \bigoplus_{\pi} H^i_d(G_\infty; I^\infty_\pi \otimes E), \tag{4}$$

where $I_\pi \otimes H_{\pi_{S_f}}$ is the isotypic subspace of $(\pi, H_\pi) = (\pi_\infty \otimes \pi_{S_f}, H_{\pi_\infty} \otimes H_{\pi_{S_f}})$ in $\mathbf{L}^2_{\text{cusp}}(\Gamma\backslash G_S)$ and the sum runs over the set of equivalence classes of irreducible unitary representations $\pi = \pi_\infty \otimes \pi_{S_f}$ for which π_{S_f} is the Steinberg representation of G_{S_f} (6.5).

Assume that S_f is non-empty. A straightforward adaptation of an argument of N. Wallach [W] shows that

$$H^{\cdot}_{\text{disc}}(\Gamma;\, E) = H^{\cdot}_{\text{cusp}}(\Gamma;\, E) \oplus H^{\cdot}_d(G_\infty;\, E). \tag{5}$$

In Section 7 it is shown that (5) gives the whole $\mathbf{L}^2$-cohomology i.e. that ν in (2) is an isomorphism. For congruence subgroups of simply connected groups similar results are contained in [BFG]; moreover it is shown there, by use of the main result of [F], that (5) is equal in that case to the full cohomology of Γ with coefficients in E, i.e. that σ in (2) is an isomorphism under those assumptions. We shall not need this fact.

In Part III, we return to the general case where S_f may be empty, and we produce in Sections 10 and 11 examples of S-arithmetic groups Γ containing a subgroup Γ_1 of finite index for which

$$H^{\cdot}_{\text{cusp}}(\Gamma_1;\, E) \neq 0. \tag{6}$$

Our basic tool in Section 10 will be the twisted trace formula, the twist being given by a rational automorphism α of G. Some technical preliminaries are carried out in Sections 8 and 9: in Section 8 we construct and study, in the twisted case, analogues of Euler-Poincaré functions due to Clozel–Delorme for real Lie groups and to Kottwitz for $\mathfrak{p}$-adic groups; the twisted analogues will be called Lefschetz functions. In Section 9 we establish in the twisted case a simple form of the trace formula; this is a variant of a theorem due to J. Arthur.

It suffices to find a Γ' commensurable with Γ with property (6). Such a Γ' may be taken to be an arbitrary small congruence subgroup; then we may draw on our understanding of automorphic representations in the adelic setting. The relation (4) above shows that proving (6) amounts to finding cuspidal automorphic adelic representations (π, H_π) such that

$$H^{\cdot}_d(G_S;\, H^\infty_{\pi_S} \otimes E) \neq 0. \tag{7}$$

Observe that in this new setting, as far as existence (or non vanishing) assertions are concerned, we are free to enlarge S whenever it is convenient, even if one is primarily interested in arithmetic groups. Beyond the classical case of groups G whose archimedean component G_∞ has discrete series, very little is known about such automorphic representations (see [Sch] for a discussion of the state of art in 1989). In order to detect some in some cases, we combine two techniques, both of which can be seen as special cases of Langlands functoriality.

The first one, used in Section 10, is based on a very crude and preliminary form of what should be the stabilization of the twisted trace formula. The non-vanishing of the cuspidal cohomology for a small enough Γ' (or equivalently the existence of cuspidal automorphic representations such that (7) holds) is clear whenever one can prove the non-vanishing of the cuspidal Lefschetz number:

$$\mathrm{Lef}_{\mathrm{cusp}}(\alpha, h, \Gamma'; E) := \sum(-1)^i \operatorname{trace}(\alpha \times h \mid H^i_{\mathrm{cusp}}(\Gamma'; E)), \tag{9}$$

for some suitably chosen Hecke correspondence h for some small enough Γ'. This turns out to be the case if (and only if) the automorphism α of G is such that the S-local Lefschetz number:

$$\mathrm{Lef}(\alpha, G_S; H_{\pi_S} \otimes E) = \sum(-1)^i \operatorname{trace}(\alpha \mid H^i_d(G_S; H^\infty_{\pi_S} \otimes E)) \tag{8}$$

does not vanish identically for representations of G_S (10.4). This is the case if, at archimedean places, α is a 'Cartan-type automorphism' (10.5). Our technique is a variant of the one used by Rohlfs and Speh to exhibit cohomology in some cases [RS1] [RS2]; but we have to refine their argument to get cuspidal cohomology. The twisted trace formula allows us to compute the Lefschetz number in the discrete spectrum. As one may guess from (5) the non-vanishing of the cuspidal Lefschetz number is easier to prove if S_f is non empty; as already observed this particular case is enough for our needs. Particular cases of our result appear in [RS3]. The existence of Cartan-type rational automorphism of G is easily seen when G is split over a totally real field k; also, if G' is defined over a CM-field k' i.e. a quadratic totally imaginary extension k' of a totally real field k, the complex conjugation induces a Cartan-type automorphism of the group $G = \mathrm{Res}_{k'/k}G'$ (this was observed by Clozel) (10.6).

Conjecturally, cuspidal representations with non-vanishing S-local Lefschetz numbers, which contribute non trivially to the twisted trace formula, should be liftings from cuspidal representations, of some twisted endoscopic group, with non-vanishing S-local Euler-Poincaré numbers. Cartan-type automorphisms do not exist in general and one may try to use other Langlands functorialities.

In Section 11 we shall use cyclic base change for $\mathrm{GL}(n)$. Representations that are 'Steinberg' at some finite place are well behaved with respect to this lifting: they remain cuspidal after base change. Base change preserves the non-vanishing of cohomology; combined with (10.6) this allows us to show that $\Gamma = \mathrm{SL}(n, \mathcal{O}_k)$, where $\mathcal{O}_k$ is the ring of integers of k, and k is obtained by a tower of cyclic extensions from a totally real number field k_0, has subgroups of finite index with non-vanishing cuspidal cohomology. This generalizes to all n's a result proved in [LS] for $n = 2$ or 3.

I. A DECOMPOSITION THEOREM FOR FUNCTIONS OF UNIFORM MODERATE GROWTH ON $\Gamma\backslash G_S$

0. Assumptions and notation

0.1. The following notation and assumptions are to be used throughout the paper:

- k is a number field, $\mathcal{V}$ the set of places of k and $\mathcal{V}_\infty$ (resp. $\mathcal{V}_f$) the set of infinite (resp. finite) places of k. We denote by $|\cdot|_v$ the normalized absolute value on the completion k_v of k at v. For $v \in \mathcal{V}_f$, $\mathfrak{O}_v$ is the ring of integers of k_v. For any finite set of places Σ, we let k_Σ be the direct sum of the k_v for $v \in \Sigma$; for $x \in k_\Sigma$ let
$$|x|_\Sigma = \prod_{v\in\Sigma} |x_v|_v.$$
For any connected component L of a k-group, we let
$$L_\Sigma = L(k_\Sigma) = \prod_{v\in\Sigma} L(k_v), \quad L_\infty = \prod_{v\in\mathcal{V}_\infty} L(k_v).$$
- S will denote a finite set of places of k, containing $\mathcal{V}_\infty$. Thus $S = S_\infty \cup S_f$, where $S_\infty = \mathcal{V}_\infty$ and $S_f = S \cap \mathcal{V}_f$.
- G is a connected reductive k-group.
- $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra of left-invariant vector fields on G_∞.

0.2. We fix a maximal compact subgroup K_v of G_v, assumed to be 'good' for $v \notin \mathcal{V}_\infty$. Let $K_\infty = \prod_{v\in\mathcal{V}_\infty} K_v$, $K_f = \prod_{v\in S_f} K_v$ and $K = K_\infty K_f$.

0.3. We shall use a height $\|\ \|$ on G_S defined by means of a faithful finite dimensional representation of G over k. The height is a product of local heights. For each archimedean place it is a Hilbert-Schmidt norm on endomorphisms; for finite places it is a sup norm on endomorphisms.

0.4. In the sequel Γ denotes an S-arithmetic subgroup of $G(k)$. It is viewed as a discrete subgroup of G_S via the diagonal embedding. The group G is the almost direct product of a central torus Z^0 and of its derived group $\mathcal{D}G$, which is semisimple. Γ is commensurable with the product of the intersections $\Gamma \cap Z^0(k)$ and $\Gamma \cap \mathcal{D}G(k)$, which are S-arithmetic in $Z^0(k)$ and $\mathcal{D}G(k)$ respectively.

0.5. If f, g are strictly positive functions on a set X, we write $f \prec g$ if there exists a constant $c > 0$ such that $f(x) \leqslant cg(x)$ for all $x \in X$, and then say that f is essentially bounded by g.

1. The vector space $\mathfrak{a}_P$ and the function $\hat{\tau}_P$

1.1. Let P be a connected linear k-group, N its unipotent radical, M a Levi k-subgroup. Denote by $X(P)_k$ the group of k-rational characters of P, and let

$$\mathfrak{a}_P = \mathrm{Hom}_{\mathbb{Z}}(X(P)_k, \mathbb{R});$$

we shall also use its dual

$$\mathfrak{a}_P^* = X(P)_k \otimes_{\mathbb{Z}} \mathbb{R}.$$

We shall denote by

$$H_P : P_S \to \mathfrak{a}_P$$

the map defined as follows: for any $\lambda = \chi \otimes r \in X(P)_k \otimes_{\mathbb{Z}} \mathbb{R}$

$$\mathrm{e}^{\langle \lambda, H_P(x) \rangle} = |\chi(x)|_S^r.$$

The kernel of H_P will be denoted P_S^1; it contains the unipotent radical and all compact or S-arithmetic subgroups of P. To M is associated canonically a section of the homomorphism H_P, the image of which is the connected component $A_M(\mathbb{R})^0$ of the group of real points of the maximal $\mathbb{Q}$-split torus A_M of the center of $\mathrm{Res}_{k/\mathbb{Q}} M$. Not to overburden notation we shall sometimes denote by A_M^0 or even simply A^0 the vector group $A_M(\mathbb{R})^0$. We denote a_P the compositum of the map H_P and of this section; hence any $x \in P_S$ can be written uniquely as

$$x = y.a_P(x)$$

with $y \in P_S^1$. Given $\lambda \in \mathfrak{a}_P^*$ we have

$$a_P(x)^\lambda = \mathrm{e}^{\langle \lambda, H_P(x) \rangle}.$$

1.2. Now assume that P is a parabolic k-subgroup in G. The subgroup M is the centralizer of A and is therefore determined by A, and A runs through the set of maximal $\mathbb{Q}$-split tori in $\mathrm{Res}_{k/\mathbb{Q}} P$. The pair (P, A) is called a p-pair. The set of k_S-points of a parabolic k-subgroup P of G with Levi subgroup M, has a Langlands decomposition

$$P_S = N_S A^0 M_S^1.$$

Moreover one has $G_S = P_S K$. This allows us to extend the function a_P to a function on G_S by the formula $a_P(pk) = a_P(p)$.

There is a natural map $\mathfrak{a}_P \to \mathfrak{a}_G$, the kernel of which will be denoted by $\mathfrak{a}_P^G$. The set $\Delta(P, A)$ of simple roots of P with respect to A may be identified with a subset of $\mathfrak{a}_P^*$. Given a real number $t > 0$ let

$$A_t = \{a \in A^0 | a^\alpha > t, \quad \forall \alpha \in \Delta(P, A)\}.$$

We denote by $\hat{\Delta}(P, A)$ the set of fundamental weights which is the dual basis to the basis in $\mathfrak{a}_P^G$ given by the simple coroots. The cone generated by the dominant regular weights is the positive Weyl chamber; its dual cone in $\mathfrak{a}_P$ is sometimes denoted ${}^+\mathfrak{a}_P$. As in [A1] we let $\hat{\tau}_P$ be the characteristic function of the latter:

$$\hat{\tau}_P(H) = 1 \Longleftrightarrow \varpi(H) > 0, \quad \forall \varpi \in \hat{\Delta}(P, A), \quad H \in \mathfrak{a}_P.$$

1.3. Let ω be a relatively compact open subset in P_S^1. The Siegel set $\mathfrak{S}(P, \omega, t)$ in G_S, relatively to the p-pair (P, A), is by definition

$$\mathfrak{S}(P, \omega, t) = \omega A_t K. \tag{1}$$

If the set ω is a product $\omega_\infty \omega_f$ with $\omega_\infty \subset P_\infty^1$ and $\omega_f \subset P_{S_f}^1$, the Siegel set $\mathfrak{S}(P, \omega, t)$ is the product

$$\mathfrak{S}(P, \omega, t) = (\omega_\infty A_t K_\infty)(\omega_f K_f) \tag{2}$$

of a Siegel set $\omega_\infty A_t K_\infty$ of G_∞ by a relatively compact open subset in G_{S_f}.

1.4. Given a minimal parabolic k-subgroup P_0 there exist a Siegel set $\mathfrak{S}_0$ relative to P_0 and a finite subset $C \subset G(k)$ such that

$$G_S = \Gamma C \mathfrak{S}_0. \tag{1}$$

For G_∞ this is classical ([B2], Section 15). The general case follows from 8.4 and the proof of 8.5 in [B1].

We recall that $C\mathfrak{S}_0$ has the 'Siegel property', namely, it meets only finitely many of its translates under Γ (cf. [B2] Section 15 and [B1] 8.4).

1.5. A function $f \in C^\infty(G_S)$ is of *moderate growth* if there exists $m \in \mathbb{N}$, such that for every $x \in G_S$

$$|f(x)| \prec ||x||^m. \tag{1}$$

Its restriction to any Siegel set $\mathfrak{S}(P, \omega, t)$ satisfies an inequality

$$|f(x)| \prec a_P(x)^\lambda ||a_G(x)||^{m'}$$

for some $\lambda \in \mathfrak{a}_P^*$, which is zero on $\mathfrak{a}_G$, and some $m' \in \mathbb{N}$. Conversely if now f is Γ-invariant and if this condition is satisfied for the functions $x \mapsto f(cx)$ with $c \in C$, in the notation of 1.4 (1), then f is of moderate growth. (If $S = S_\infty$ see [HC] lemma 6 p. 9. The general case again follows from 8.4 and 8.5 in [B1].)

Let f be Γ-invariant; if for any Siegel set $\mathfrak{S}(P, \omega, t)$, any $\lambda \in \mathfrak{a}_P^*$, which is zero on $\mathfrak{a}_G$, and some $m' \in \mathbb{N}$, the function f satisfies an inequality

$$|f(x)| \prec a_P(x)^\lambda ||a_G(x)||^{m'}$$

we say that the function is rapidly decreasing.

2. The decomposition theorem

2.1. A function $f \in C^\infty(G_S)$ is of *uniform moderate growth* (u.m.g.) if it is uniformly locally constant under right translations on G_{S_f} and if there exists $m \in \mathbb{N}$, such that for every $D \in \mathcal{U}(\mathfrak{g})$ and $x \in G_S$

$$|Df(x)| \prec ||x||^m. \tag{1}$$

Let $V_\Gamma = C^\infty_{\text{umg}}(\Gamma \backslash G_S)$ be the space of smooth functions on $\Gamma \backslash G_S$ of uniform moderate growth. By 1.1, applied to the case $P = G$, we have $G_S = G_S^1 \times A_G^0$ and $\Gamma \subset G_S^1$. Similarly we define V_Γ^1 to be the set of functions on $\Gamma \backslash G_S^1$ of uniform moderate growth.

2.2. As usual $\Gamma_P = \Gamma \cap P(k)$, $\Gamma_N = \Gamma \cap N(k)$ but $\Gamma_M = \Gamma_N \backslash \Gamma_P$. If f is a Γ_P-left-invariant function, then its constant term f^P along P is:

$$f^P(x) = \int_{\Gamma_N \backslash N_S} f(nx)\, dn,$$

the Haar measure being normalized so that the quotient has volume one. For $x \in G_S$, the function $f_x^P : m \mapsto f^P(mx)$ ($m \in M_S^1$) is Γ_M-left-invariant. A function $f \in V_\Gamma$ is said to be *negligible along* P if for all $x \in G_S$ the function f_x^P is orthogonal to the cusp forms on $\Gamma_M \backslash M_S^1$.

Let $\mathcal{A}$ be the set of classes of associate parabolic k-subgroups. For $\mathcal{P} \in \mathcal{A}$, denote by $V_\Gamma(\mathcal{P})$ the space of elements of $f \in V_\Gamma$ which are negligible along Q for every parabolic k-subgroup $Q \notin \mathcal{P}$. We shall also need the space $V_\Gamma^1(\mathcal{P}) = V_\Gamma(\mathcal{P}) \cap V_\Gamma^1$. We recall the

2.3 PROPOSITION. *If a function f is negligible along all parabolic k-subgroups it is zero, and if it is negligible along all proper parabolic k-subgroups, it is cuspidal.*

Proof. We refer to [Lan1] Lemma 3.7 and its corollary p. 58. The statement and the proof extend to the adelic case (see for example [MW], Proposition 1.3.4) and also to the S-arithmetic case. Note that in the case of congruence subgroups it follows directly from the adelic case by taking functions invariant under a suitable open compact subgroup outside of S.

As a consequence, the sum

$$V_\Gamma(\mathcal{A}) := \sum_{\mathcal{P}\in\mathcal{A}} V_\Gamma(\mathcal{P})$$

is direct. We shall prove in Section 4 that $V_\Gamma = V_\Gamma(\mathcal{A})$. This will complete the proof of the

2.4 THEOREM. *(Langlands)*

$$V_\Gamma = \bigoplus_{\mathcal{P}\in\mathcal{A}} V_\Gamma(\mathcal{P}).$$

In the case $S = S_\infty$, this theorem and a sketch of proof were communicated by R. P. Langlands in a letter to the first named author [Lan 2]. The following proof is a variant of the original one in our slightly more general setting; there are two new ingredients:

(a) The truncation operator in the form given by J. Arthur. This replaces an inductive construction of a (smooth) truncation, and a delicate analysis of the geometry of Siegel sets to prove that the truncated function is rapidly decreasing.
(b) The Dixmier–Malliavin theorem [DM], which allows one to work up to a convolution; this makes the passage from moderate growth to uniform moderate growth easy.

3. Preliminaries on E-series and constant terms

3.1. Let P be a parabolic k-subgroup of G and $P = MN$ a Levi decomposition over k. Let α be a smooth compactly supported function on G_S and $T_P \in \mathfrak{a}_P$.

LEMMA. *Let $f \in C^\infty(\Gamma_P N\backslash G_S)$ and E_{P,f,T_P} the series defined by*

$$E_{P,f,T_P}(x) = \sum_{\gamma\in\Gamma_P\backslash\Gamma} \hat{\tau}_P(H_P(\gamma x) - T_P) f(\gamma x) \quad (x \in G_S). \tag{1}$$

(i) *The series E_{P,f,T_P} converges absolutely and locally uniformly.*
(ii) *Assume that f is of u.m.g and let α be a smooth compactly supported function on G_S. Then $E_{P,f,T_P} * \alpha \in V_\Gamma$.*

Proof. The convergence of the series and the moderate growth of its sum (which already follows if f has moderate growth) is Corollary 5.2 in [A1]. Its convolution with a smooth compactly supported function is then of u.m.g.

3.2. Let $\mathcal{Q}$ be an associate class of parabolic k-subgroups in M. We denote by $\mathcal{Q}^G$ the associate class in $\mathcal{A}$ which consists of the parabolic k-subgroups of G having a Levi k-subgroup conjugate to M_Q, where M_Q is a Levi k-subgroup of an element $Q \in \mathcal{Q}$.

LEMMA. *We keep the assumptions of 3.1. Assume that f is of u.m.g. and that the functions f_x^P belong to $V_{\Gamma_M}(\mathcal{Q})$ for all $x \in G_S$. Then*

$$E_{P,f^P,T_P} * \alpha \in V_\Gamma(\mathcal{Q}^G).$$

Proof. We have to prove that if R is a parabolic k-subgroup of G not belonging to $\mathcal{Q}^G$, the constant term of E_{P,f^P,T_P} along R is negligible:

$$\int_{\Gamma_{M_R}\backslash M^1_{R,S}} \psi(m).(E_{P,f^P,T_P})^R(my)\,\mathrm{d}m = 0, \tag{1}$$

for all cusp forms ψ on $\Gamma_{M_R}\backslash M^1_{R,S}$ and $y \in G$. This vanishing result is a variant for our E-series of the classical vanishing result for constant terms of Eisenstein series constructed from cusp forms. In the case of arithmetic subgroups see [Lan1] Lemma 4.4 or also [HC], pp. 34–39, in particular the proof of Corollary 3 p. 39. In the adelic case see [MW] II.1.8 and II.2.1. The proof in our case uses the same formal manipulations, which is allowed since our E-series are absolutely convergent.

4. Proof of the theorem

4.1. We note first that V_Γ may be viewed as a union of differentiable G_S-modules $V_{G,m} (m \in \mathbb{N})$, where $V_{G,m}$ is the set of $f \in V_\Gamma$ whose derivatives Df are essentially bounded by $||x||^m$, endowed with the semi-norms $\sup_x|Df(x)|.||x||^{-m}$. Therefore, since at finite places our functions are uniformly locally constant, it follows from the Dixmier–Malliavin theorem [DM] that any $f \in V_\Gamma$ is a finite linear combination of terms $h * \alpha$, with $h \in V_\Gamma$ and $\alpha \in C_c^\infty(G_S)$ (with support in a prescribed neighborhood of 1). In other words,

$$V_\Gamma = \bigcup_{\alpha \in C_c^\infty(G_S)} V_\Gamma * \alpha. \tag{1}$$

4.2. We have the decomposition

$$\mathbf{L}^2(\Gamma\backslash G_S^1) = \bigoplus_{\mathcal{P}\in\mathcal{A}} \mathbf{L}^2_{\mathcal{P}}(\Gamma\backslash G_S^1), \tag{1}$$

where $\mathbf{L}^2_{\mathcal{P}}(\Gamma\backslash G_S^1)$ is the space of $\mathbf{L}^2$-functions on $\Gamma\backslash G_S^1$ which are negligible along the parabolic k-subgroups $Q \notin \mathcal{P}$. This follows from 2.3 (see also [MW] II.2.4). We have then

$$\mathbf{L}^2(\Gamma\backslash G_S^1)^\infty = \bigoplus_{\mathcal{P}\in\mathcal{A}} \mathbf{L}^2_{\mathcal{P}}(\Gamma\backslash G_S^1)^\infty. \tag{2}$$

But, again by [DM], the elements of $\mathbf{L}^2_{\mathcal{P}}(\Gamma\backslash G^1_S)^\infty$ are finite sums of terms $f * \alpha$ and hence have u.m.g. (cf. [BJ] 1.6, which refers to [HC] I.3, Lemma 9 and its corollary p. 10), therefore belong to $V^1_\Gamma(\mathcal{P})$:

$$\mathbf{L}^2_{\mathcal{P}}(\Gamma\backslash G^1_S)^\infty \subset V^1_\Gamma(\mathcal{P}). \tag{3}$$

As a consequence,

$$\mathbf{L}^2(\Gamma\backslash G^1_S)^\infty \subset \bigoplus_{\mathcal{P}\in\mathcal{A}} V^1_\Gamma(\mathcal{P}) = V^1_\Gamma(\mathcal{A}). \tag{4}$$

4.3. By 2.1, $G_S = G^1_S \times A^0_G$ and $\Gamma \subset G^1_S$. Given a function f on $\Gamma\backslash G_S$ and $a \in A^0_G$ we let f_a be the function on $\Gamma\backslash G^1_S$ defined by $x \mapsto f(xa)$.

LEMMA. *Let $f \in V_\Gamma$. Assume that for $a \in A^0_G$, $f_a \in \mathbf{L}^2(\Gamma\backslash G^1_S)$ and that its $\mathbf{L}^2$-norm is slowly increasing, i.e.*

$$||f_a||_2 \prec ||a||^m$$

for some $m \in \mathbb{N}$. Let $\alpha \in C^\infty_c(G_S)$. Then $f * \alpha \in V_\Gamma(\mathcal{A})$.

Proof. We apply 4.2(2) to f_a, with $a \in A^0_G$. There exists then a finite set of functions $(f_{\mathcal{P}})_{\mathcal{P}\in\mathcal{A}}$ on $\Gamma\backslash G_S$, such that

$$f = \sum_{\mathcal{P}\in\mathcal{A}} f_{\mathcal{P}} \tag{1}$$

with $f_{\mathcal{P},a} \in V^1_\Gamma(\mathcal{P})$. We show first that each $f_{\mathcal{P}}$ is locally integrable on G_S. (In an earlier version, we had overlooked this point. N. Wallach drew our attention to it and provided the argument.) By hypothesis f is square integrable on $\Gamma\backslash G_S = \Gamma\backslash G^1_S \times A^0_G$ for the measure

$$\mathrm{d}\mu(xa) = \mathrm{d}x \otimes ||a||^{-n}\,\mathrm{d}a$$

if n is large enough, where $\mathrm{d}x$ and $\mathrm{d}a$ are Haar measures on G^1_S and A^0_G respectively. But $f_{\mathcal{P}}$ is the image of f by the orthogonal projection on

$$\mathbf{L}^2(\Gamma\backslash G^1_S)_{\mathcal{P}} \hat{\otimes} \mathbf{L}^2(A^0_G, ||a||^{-n}\,\mathrm{d}a);$$

it is in particular square integrable for the above measure and hence locally integrable for any Haar measure on G_S. It remains to show that

$$f_{\mathcal{P}} * \alpha \in V_\Gamma(\mathcal{P}). \tag{2}$$

The function $f_{\mathcal{P}} * \alpha$ is smooth. Since the formation of constant terms commutes with convolution, $f_{\mathcal{P}} * \alpha$ is negligible outside $\mathcal{P}$. There remains to see that it is of u.m.g.

For each a, $||f_a||_2^2$ is a sum of the $||f_{\mathcal{P},a}||_2^2$. Therefore there exists $m \in \mathbb{N}$ such that

$$||f_{\mathcal{P},a}||_2 \prec ||a||^m. \tag{3}$$

Let $D \in \mathcal{U}(\mathfrak{g})$. We have

$$D(f_{\mathcal{P}} * \alpha)(y) = (f_{\mathcal{P}} * \beta)(y) = \int_{\Gamma\backslash G_S} f_{\mathcal{P}}(x) K_\beta(x,y)\, \mathrm{d}x \tag{4}$$

where $D\alpha = \beta$ and

$$K_\beta(x,y) = \sum_{\gamma\in\Gamma} \beta(x^{-1}\gamma y) \quad (x, y \in G_S). \tag{5}$$

We claim that this series converges absolutely, uniformly in x, locally uniformly in y and that there exist $r \in \mathbb{N}$ and a compact set C in A_G^0, which depend only on the support of α, such that

$$|K_\beta(xa, yb)| \prec ||y||^r \tag{6}$$

for $D \in \mathcal{U}(\mathfrak{g})$, $x, y \in G_S^1$ and $a, b \in A_G^0$. Moreover $K_\beta(xa, yb) = 0$ if $ab^{-1} \notin C$. This follows from standard arguments (see for example [HC] Lemma 9, [A1] Lemma 4.3 or [MW] Lemma I.2.4). We can write

$$(f_{\mathcal{P}} * \beta)(yb) = \int_{bC} \mathrm{d}a \int_{\Gamma\backslash G_S^1} f_{\mathcal{P},a}(x) K_\beta(xa, yb)\, \mathrm{d}x$$

for $y \in G_S^1$ and $b \in A_G^0$. In view of (3) and (6)

$$|D(f_{\mathcal{P}} * \alpha)(yb)| = |(f_{\mathcal{P}} * \beta)(yb)| \prec ||y||^r \mathrm{vol}(\Gamma\backslash G_S^1) ||b||^m$$

with r and m independent of D. This implies that $f_{\mathcal{P}} * \alpha$ is of u.m.g.

4.4. We now use Arthur's truncation operator Λ^T. The parameter T belongs to $\mathfrak{a}_{P_0}$ where P_0 is a fixed minimal parabolic k-subgroup of G; it will be assumed to be far enough in the positive Weyl chamber. Given a function f on $\Gamma\backslash G_S$, $\Lambda^T f$ is a sum, over a set $\mathcal{X}$ of representatives of Γ-conjugacy classes of parabolic k-subgroups of G, of E-series of the type studied in 3.1:

$$\Lambda^T f = \sum_{P\in\mathcal{X}} (-1)^{\mathrm{prk}\,(P) - \mathrm{prk}\,(G)} E_{P, f^P, I_P(T)}$$

where $I_P : \mathfrak{a}_{P_0} \to \mathfrak{a}_P$ is the linear map defined in [Mü] p. 488 and $\mathrm{prk}(P)$ the parabolic k-rank of P. Given a function f on $\Gamma\backslash G_S$ of u.m.g., the fundamental property of the truncation operator is that $\Lambda^T f$ is rapidly decreasing on $\Gamma\backslash G_S^1$.

More precisely, if $\mathfrak{S}_0$ is a Siegel set in G_S for a minimal parabolic k-subgroup P_0, let $\mathfrak{S}_0^1 = \mathfrak{S}_0 \cap G_S^1$. Then

$$\sup_{(x,a)\in\mathfrak{S}_0^1\times A_G^0} |\Lambda^T f(xa)|\, ||x||^n ||a||^{-m} \prec \sup_{D\in\mathcal{U}(\mathfrak{g}),y\in G_S} |Df(y)|\cdot||y||^{-r}$$

for all n. In the adelic setting, this a variant of Lemma 1.4 page 95 of [A2], where we have replaced the $\mathbf{L}^1$-norm over some parameter space S for a measure $d\sigma$ by the sup-norm over A_G^0. In the arithmetic case, this is theorem 5.2 of [OW]; a proof may be found in Section 7 of [OW]. Again in view of 8.4 and 8.5 of [B1], it extends to the S-arithmetic case. This shows that $(\Lambda^T f)_a$ is square integrable on $\Gamma\backslash G_S^1$ and that the $\mathbf{L}^2$-norm of $(\Lambda^T f)_a$ is a slowly increasing function on A_G^0.

4.5. Combined with 4.2 and 4.3, the last assertion of 4.4 shows that $(\Lambda^T f) * \alpha$ belongs to $V_\Gamma(\mathcal{A})$. To finish the proof we proceed by induction on the semisimple k-rank. By construction, $f - \Lambda^T f$ is an alternated sum of E-series E_{P,f^P,T_P}, where $T_P = I_P(T)$, for P in a set of representatives of Γ-conjugacy classes of proper parabolic k-subgroups of G. By Lemma 3.2 we know that $E_{P,f^P,T_P} * \alpha$ belongs to $V_\Gamma(\mathcal{A})$ since the constant term f^P is of u.m.g. and the induction assumption implies that the functions $m \mapsto f^P(mh)$ belong to $V_{\Gamma_M}(\mathcal{A}_M)$ for all $h \in K$.

4.6. REMARK. W. Casselman has shown the existence of a similar decomposition theorem in the arithmetic case ($S = S_\infty$) for the Schwartz space of $\Gamma\backslash G_S$, i.e. the space of functions which are uniformly rapidly decreasing (u.r.d.) (i.e., all $\mathcal{U}(\mathfrak{g})$-derivatives are rapidly decreasing in the sense of 1.5) [Ca2]. The above proof also yields this, in the S-arithmetic case, once it is noted that the argument of 3.1 also shows that if f is u.r.d., then $E_{P,f^P,T_P} * \alpha$ is u.r.d. Conversely 2.4 for $S = S_\infty$ follows from 1.16 and 4.7 in [Ca2].

II. COHOMOLOGY OF S-ARITHMETIC GROUPS

5. Decomposition of cohomology

Let (ϕ, E) be a finite dimensional irreducible complex rational representation of G_∞. We view E as a representation of G_S which is trivial on G_{S_f}. For convenience, we assume it to be irreducible. It is therefore a tensor product of representations E_v, where E_v is an irreducible representation of $G(k_v)$ ($v \in \mathcal{V}_\infty$).

5.1. Assume that $S = \mathcal{V}_\infty$. It is shown in [B5] that there is a canonical isomorphism

$$H^\cdot(\Gamma;\, E) = H^\cdot(\mathfrak{g},\, K; V_\Gamma \otimes E). \tag{1}$$

In 2.4, the direct summands are obviously $(\mathfrak{g}, K)$ submodules, therefore we get

5.2 THEOREM. *Assume that $S = \mathcal{V}_\infty$. Then there is a canonical direct sum decomposition*

$$H^\cdot(\Gamma; E) = \bigoplus_{\mathcal{P}\in\mathcal{A}} H^\cdot_{\mathcal{P}}(\Gamma; E), \tag{1}$$

where

$$H^\cdot_{\mathcal{P}}(\Gamma; E) = H^\cdot(\mathfrak{g}, K; V_\Gamma(\mathcal{P}) \otimes E). \tag{2}$$

REMARK. For $\mathcal{P} = \{G\}$

$$V_\Gamma(\{G\}) = \mathbf{L}^2_{\{G\}}(\Gamma\backslash G)^\infty := \mathbf{L}^2_{\text{cusp}}(\Gamma\backslash G)^\infty \tag{3}$$

is the space of cuspidal functions. The corresponding summand is therefore the cuspidal cohomology $H^\cdot_{\text{cusp}}(\Gamma; E)$. Consequently (1) proves again that the inclusion

$$\mathbf{L}^2_{\text{cusp}}(\Gamma\backslash G)^\infty \subset C^\infty(\Gamma\backslash G)$$

yields an injective map

$$H^\cdot_{\text{cusp}}(\Gamma; E) \to H^\cdot(\Gamma; E)$$

[B4]. Moreover, it exhibits a natural complement to the cuspidal cohomology.

5.3. We now return to the case of a general S-arithmetic group and want to extend the foregoing in the framework of continuous (or differentiable) cohomology, for which we refer to [BW:IX]. By [BW:XIII, 1.1]

$$H^\cdot(\Gamma; E) = H^\cdot_d(G_S; C^\infty(\Gamma\backslash G_S) \otimes E). \tag{1}$$

5.4 PROPOSITION. *The inclusion*

$$\iota : V_\Gamma \to C^\infty(\Gamma\backslash G_S)$$

induces an isomorphism

$$\iota^\cdot : H^\cdot_d(G_S; V_\Gamma \otimes E) \xrightarrow{\sim} H^\cdot_d(G_S; C^\infty(\Gamma\backslash G_S) \otimes E). \tag{1}$$

Proof. We can operate either in differentiable cohomology, denoted $H^\cdot_d$ in [BW] or, by passing to K-finite vectors, in the variant $H^\cdot_e$ of [BW:X, 5]. We use the former. The spaces $H_d^{\cdot}(G_S; V_\Gamma \otimes E)$ and $H_d^{\cdot}(G_S; C^\infty(\Gamma\backslash G_S) \otimes E)$ are the abutments of spectral sequences (E_r) and (E'_r) [BW:IX, 4.3] in which

$$E_2^{p,q} = H_d^p(G_{S_f}; H_d^q(G_\infty; V_\Gamma \otimes E)) \tag{2}$$

and

$$E'^{p,q}_2 = H_d^p(G_{S_f};\, H_d^q(G_\infty;\, C^\infty(\Gamma\backslash G_S) \otimes E)). \tag{3}$$

The inclusion ι induces a homomorphism $(E_r) \to (E'_r)$ of spectral sequences. It suffices therefore to show that $E_2 \tilde{\to} E'_2$ is an isomorphism and for this, it is enough to prove that the homomorphism

$$\iota^{\cdot}\colon H_d^{\cdot}(G_\infty; V_\Gamma \otimes E) \to H_d^{\cdot}(G_\infty;\, C^\infty(\Gamma\backslash G_S) \otimes E) \tag{4}$$

is an isomorphism. The group G_{S_f} operates by right translations on these cohomology groups and $\iota^{\cdot}$ is G_{S_f}-equivariant. By going over to the K_∞-finite elements in V_Γ and $C^\infty(\Gamma\backslash G_S)$, we may replace the differentiable cohomology by the relative Lie algebra cohomology $H^{\cdot}(\mathfrak{g}_\infty, K_\infty; \cdot)$. The latter commutes with inductive limits. The two spaces of functions under consideration are inductive limits of the subspaces of elements fixed under a compact open subgroup K'_f of G_{S_f}. We are then reduced to showing that the natural homomorphism

$$H^{\cdot}\left(\mathfrak{g}_\infty, K_\infty; V_\Gamma^{K'_f}\right) \to H^{\cdot}(\mathfrak{g}_\infty, K_\infty;\, C^\infty(\Gamma\backslash G_S/K'_f)), \tag{5}$$

is an isomorphism for every compact open subgroup K'_f of G_{S_f}. But the space $\Gamma\backslash G_S/K'_f$ is isomorphic to a finite disjoint union of arithmetic quotients:

$$\Gamma\backslash G_S/K'_f = \coprod_c \Gamma_c\backslash G_\infty$$

[BJ:4.3]. We are thus reduced to the case dealt with in [B5].

5.5. Using 5.3, 5.4 and the decomposition Theorem 2.4, we can therefore write $H^{\cdot}(\Gamma; E)$ as a direct sum

$$H^{\cdot}(\Gamma; E) = \bigoplus_{\mathcal{P}\in\mathcal{A}} H_{\mathcal{P}}^{\cdot}(\Gamma;\, E), \tag{1}$$

where, by definition,

$$H_{\mathcal{P}}^{\cdot}(\Gamma;\, E) = H_d^{\cdot}(G_S; V_\Gamma(\mathcal{P}) \otimes E). \tag{2}$$

In particular, $V_\Gamma(\{G\})$ is the space of cuspidal functions and the corresponding summand is the cuspidal cohomology of Γ, also to be denoted $H_{\text{cusp}}^{\cdot}(\Gamma;\, E)$.

6. Cohomology with coefficients in the discrete spectrum

6.1. Let F be a non-archimedean local field, L a connected reductive F-group and P_0 a minimal parabolic F-subgroup of L. We shall denote by $\mathrm{St}(L)$ the *Steinberg representation* of $L(F)$. We recall that it can be defined as the natural representation of $L(F)$ in the quotient of $C^\infty(P_0(F)\backslash L(F))$ by the invariant subspace generated by the functions which are left-invariant under a parabolic subgroup $Q \supsetneq P_0$. It is the space of C^∞-vectors of an irreducible representation of $L(F)$ which is square integrable modulo the split component of the center of $L(F)$ (see for example [B3]). It is the trivial representation if $L(F)$ is compact or if L is a torus. The center of $L(F)$ belongs to its kernel. If L is an almost direct product of two F-subgroups L_1, L_2 then $\mathrm{St}(L) = \mathrm{St}(L_1) \otimes \mathrm{St}(L_2)$. From 4.7 in [BW:X] we see then that

$$H_d^i(L(F); \mathrm{St}(L)) = \begin{cases} 0, & i \neq \mathrm{rk}_F(\mathcal{D}L) \\ \mathbb{C}, & i = \mathrm{rk}_F(\mathcal{D}L) \end{cases} \quad (i \in \mathbb{Z}), \tag{1}$$

where $\mathcal{D}L$ is the derived group of L. Moreover, if L is almost absolutely simple over F, we deduce from results of W. Casselman (see [Ca1] or [BW:XI, 3.9]) that the only irreducible admissible unitarizable representations (π, H) of $L(F)$ for which $H_d^{\cdot}(L(F); H) \neq 0$ are the Steinberg representation and the trivial representation. In the latter case, we have

$$H_d^i(L(F); \mathbb{C}) = \begin{cases} \mathbb{C}, & i = 0 \\ 0 & i \neq 0 \end{cases} \quad (i \in \mathbb{Z}). \tag{2}$$

6.2. We now come back to the S-arithmetic groups. In the remainder of Section 6 we shall assume that G is semisimple, almost absolutely simple over k of strictly positive k-rank. Let $\tilde{G}$ be the universal covering of G and τ the canonical isogeny $\tilde{G} \to G$. It induces a morphism $\tau_S : \tilde{G}_S \to G_S$ with finite kernel and cokernel.

6.3 LEMMA. *We keep the previous assumptions. Let (π, H) be an irreducible unitary representation of G_S which occurs discretely in $\mathbf{L}^2(\Gamma\backslash G_S)$. Assume that π has a non-compact kernel. Then π is the trivial representation if either $G = \tilde{G}$ or*

$$H_d^{\cdot}(G_S; H^\infty \otimes E) \neq 0.$$

Proof. Assume first that G is simply connected. By our assumptions, G is absolutely almost simple over k and G_v is not compact for every $v \in S$. Therefore G_v is simple modulo its center as an abstract group [T]. There exists then $v \in S$ such that G_v is in the kernel of π. By hypothesis, H^∞ is realized as a space of functions on G_S right-invariant under G_v and left-invariant under Γ. Since G_v is normal, they are also left-invariant under G_v. The group G being assumed simply connected, strong approximation is valid and implies that $G_v.\Gamma$ is dense in G_S. Therefore the elements of H^∞ are left-invariant under G_S, hence are constant functions.

We now drop the assumption $G = \tilde{G}$. The group $G' = \tau_S(\tilde{G}_S)$ is normal of finite index in G_S. As a $\tilde{G}_S$-module, H is the direct sum of finitely many irreducible G'-modules $H_i (i \in I)$ which are permuted transitively by G_S (cf. [Sil] or Lemma A.2 (ii) in the Appendix). Because of this last fact, their kernels are isomorphic, and in particular either all compact or all non-compact. Since G' is open of finite index in G_S, its intersection with $\ker \pi$ is not compact, hence $\ker \pi_i$ is not compact and the previous argument shows that H_i is a trivial G'-module $(i \in I)$, i.e. that $G' \subset \ker \pi$. Consequently, H is an irreducible representation of the finite group G_S/G', in particular is finite dimensional. Write it as a tensor product $(\pi, H) = \otimes_{s \in S}(\pi_s, H_s)$, where (π_s, H_s) is an irreducible finite dimensional representation of $G(k_s)$. Since $H_d^{\cdot}(G_S; H^\infty \otimes E) \neq 0$ and the Künneth rule holds [BW:XIII, 2.2], we have

$$H_d^{\cdot}(G(k_s); H_s \otimes E) \neq 0, \ (s \in \mathcal{V}_\infty), \ H_d^{\cdot}(G(k_s); H_s) \neq 0 \quad (s \in S_f).$$

If $s \in S_f$ this forces H_s to be trivial (6.1). Now let s be archimedean. We are dealing with relative Lie algebra cohomology with coefficients in a finite dimensional representation, hence $H_s \otimes E$ must contain the trivial representation, i.e. E must be the contragredient representation to H_s. In particular it must have a kernel of finite index. Since E is a rational representation, it must be trivial, and then so is H_s $(s \in \mathcal{V}_\infty)$.

REMARK. The first part of the proof is just a variant of 3.4 in [BW:XIII] and 2.2 in [BW:VII]. The second part of the previous argument allows one to suppress the assumption $G = \tilde{G}$ in [BW:XIII, 3.4], hence also in 3.5 there.

6.4. An irreducible unitary representation (π, H) of G_S can be written uniquely as $(\pi_\infty, H_\infty) \otimes (\pi_{S_f}, H_f)$, where (π_∞, H_∞) (resp. (π_{S_f}, H_f)) is an irreducible unitary representation of G_∞ (resp. G_{S_f}). If T is a finite subset of $\mathcal{V}_f$ we let

$$\mathrm{St}(G_T) := \bigotimes_{v \in T} \mathrm{St}(G_v)$$

be the Steinberg representation of G_T. We let $\hat{G}_{\mathrm{cusp}}(S, \Gamma, \mathrm{St})$ (resp. $\hat{G}_{\mathrm{disc}}(S, \Gamma, \mathrm{St})$) be the set of equivalence classes of irreducible unitary representations of G_S which occur in the cuspidal (resp. $\mathbf{L}^2$-discrete) spectrum of $\Gamma \backslash G_S$ and in which π_{S_f} is the Steinberg representation. Assume that S_f is non-empty, then

$$\hat{G}_{\mathrm{cusp}}(S, \Gamma, \mathrm{St}) = \hat{G}_{\mathrm{disc}}(S, \Gamma, \mathrm{St}). \tag{1}$$

Since the Steinberg representation is tempered, this follows from an argument of Wallach's [W], which asserts that an irreducible unitary representation $\pi = \otimes_{s \in S} \pi_s$

belonging to the discrete spectrum and tempered at one place belongs to the cuspidal spectrum. Wallach's argument is carried out only at infinity (i.e. for $S = \mathcal{V}_\infty$), but it extends very easily to the S-arithmetic case, in the same way as it was done by Clozel [Cl3] in the adelic case.

Recall that if $A^\cdot$ is a module graded by $\mathbb{Z}$, given $m \in \mathbb{Z}$, then $A^\cdot[m]$ denotes the graded module defined by $(A^\cdot[m])^i = A^{m+i}$ $(i \in \mathbb{Z})$.

6.5 THEOREM. *Let r_f be the sum over $s \in S_f$ of the k_s-ranks of the groups G/k_s. Let us denote by $I_\pi \otimes H_{\pi_{S_f}}$ the isotypic subspace of $(\pi, H_\pi) = (\pi_\infty \otimes \pi_{S_f}, H_{\pi_\infty} \otimes H_{\pi_{S_f}})$ in $\mathbf{L}^2_{\text{cusp}}(\Gamma\backslash G_S)$. Under the assumptions of 6.2,*

$$H^\cdot_{\text{cusp}}(\Gamma; E) = \bigoplus_\pi H^\cdot_d(G_\infty; I^\infty_\pi \otimes E)[-r_f], \tag{1}$$

where π runs through $\hat{G}_{\text{cusp}}(S; \Gamma, \text{St})$. If moreover S_f is non-empty, the discrete part

$$H^\cdot_{\text{disc}}(\Gamma; E) := H^\cdot_d(G_S; \mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S)^\infty \otimes E)$$

of $H^\cdot_{(2)}(\Gamma; E)$ is given by

$$H^\cdot_{\text{disc}}(\Gamma; E) = H^\cdot_{\text{cusp}}(\Gamma; E) \oplus H^\cdot_d(G_\infty; E). \tag{2}$$

Proof. First we show that $H^\cdot_d(G_S; \mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S)^\infty \otimes E)$ is finite dimensional.
(a) Assume that G is simply connected. We shall use the variant $H^\cdot_e$ of the cohomology introduced in [BW:X, 5.1]. Let $V = \mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S)^\infty$; denote by V_f the space of K-finite vectors in V. By [BW:XII, 2.5]

$$H^\cdot_e(G_S; V_f \otimes E) = H^\cdot_d(G_S; V \otimes E).$$

By [BW:X, 5.3], there is a spectral sequence abutting to $H^\cdot_e(G_S; V_f \otimes E)$, in which

$$E_2^{p,q} = H^p_e(G_{S_f}; H^q_e(G_\infty; V_f \otimes E)). \tag{3}$$

The space V_f is the inductive limit of the spaces $V^{K'_f}$, where K'_f runs through the compact open subgroups of G_{S_f}. Let Y be the Tits building of G_{S_f}. The cohomology of G_{S_f} can be computed as that of a simplicial sheaf on a chamber $C \simeq G_{S_f}\backslash Y$ which associates to a face σ of C the vector space.

$$H^q_d(G_\infty; \mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S)^\infty \otimes E)^{K_\sigma}, \tag{4}$$

where K_σ is the isotropy subgroup of σ in G_{S_f} [BW:X, 2.5]. Since the representations of K_σ are fully reducible, taking fixed points commutes with the formation of cohomology, hence (4) can also be written as

$$H_d^q(G_\infty; \mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S/K_\sigma)^\infty \otimes E). \tag{5}$$

As already pointed out in 5.4, $\Gamma\backslash G_S/K_\sigma$ is a disjoint union of finitely many quotients $\Gamma_c\backslash G_\infty$, where Γ_c is arithmetic; the $\mathbf{L}^2$-discrete spectrum is the direct sum of the $\mathbf{L}^2$-discrete spectra of the quotients $\Gamma_c\backslash G_\infty$. But it is proved in [BG] that the relative Lie algebra cohomology with coefficients in the $\mathbf{L}^2$-discrete spectrum of an arithmetic group is finite dimensional. Therefore $E_2^{\cdot,q}$ is the cohomology of G_{S_f} with coefficients in a finite dimensional space, which is in particular an admissible G_{S_f}-module. It is then finite dimensional [BW:X, 6.3]. As a consequence, E_2 is finite dimensional, therefore so is the abutment of the spectral sequence and our assertion is proved in case (a).

(b) Let $\tilde{G}$, τ, G' be as in 6.2, 6.3 and $N = \ker \tau_S$. The group N is finite and acts trivially on $\mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S)^\infty \otimes E$, therefore the spectral sequence of $\tilde{G}_S$ modulo N yields an isomorphism

$$H_d^\cdot(G'; \mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S)^\infty \otimes E) \simeq H_d^\cdot(\tilde{G}_S; \mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S)^\infty \otimes E). \tag{6}$$

On the other hand, the spectral sequence of G_S modulo G' degenerates to an isomorphism

$$H_d^\cdot(G_S; \mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S)^\infty \otimes E) \simeq H_d^\cdot(G'; \mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S)^\infty \otimes E)^{G_S/G'}, \tag{7}$$

which, together with (6), provides the reduction to case (a).

This implies of course also that $H_e^\cdot(G_S; \mathbf{L}^2_{\text{cusp}}(\Gamma\backslash G_S)^\infty \otimes E)$ is finite dimensional, but this already follows from the fact that it injects into $H^\cdot(\Gamma; E)$ (5.5), since the latter is known to be finite dimensional [BS].

Let (π, H_π) be an irreducible representation of G_S contained in the $\mathbf{L}^2$-discrete spectrum. By the Künneth rule

$$H_d^\cdot(G_S; H_\pi^\infty \otimes E) = H_d^\cdot(G_\infty; H_{\pi_\infty}^\infty \otimes E) \otimes H_d^\cdot(G_{S_f}; H_{\pi_{S_f}}^\infty). \tag{8}$$

If π is trivial, then by 6.1(2)

$$H_d^\cdot(G_S; H_\pi^\infty \otimes E) = H_d^\cdot(G_\infty; E). \tag{9}$$

Let now π be non-trivial. By 6.1, 6.3, the left-hand side of (8) can be non-zero only if π_{S_f} is the Steinberg representation, hence if $\pi \in \hat{G}_{\text{disc}}(S, \Gamma, \text{St})$. Moreover, its contribution to cohomology is equal to

$$H_d^\cdot(G_\infty; H_{\pi_\infty}^\infty \otimes E)[-r_f], \tag{10}$$

in view of the Künneth rule (8) and 6.1 (1). If we write $\mathbf{L}^2_{\text{disc}}(\Gamma\backslash G_S)$ as a Hilbert direct sum of G_S-invariant irreducible subspaces, then only finitely many can contribute to the cohomology. Let then V be the Hilbert direct sum of those with respect to which the cohomology of G_S is zero. The space $H^{\cdot}(G_S;\, V^{\infty})$ is finite dimensional in view of our initial argument. Then it is equal to zero by [BW:XIII, 1.6]. In view of (10) and 6.4(1), this concludes the proof of (1) and also shows that

$$H^{\cdot}_{\text{disc}}(\Gamma;\, E) = H^{\cdot}_d(G_\infty;\, E) \oplus \bigoplus_{\pi} H^{\cdot}_d(G_\infty;\, I^{\infty}_{\pi} \otimes E)[-r_f], \tag{11}$$

where π runs through $\hat{G}_{\text{disc}}(S,\, \Gamma,\, \mathrm{St})$, so that (2) now follows from (1) and 6.4(1).

6.6. REMARK. In the anisotropic case, $H^{\cdot}_{\text{disc}}(\Gamma;\, E) = H^{\cdot}(\Gamma; E)$. The equality (11) above so modified is proved in [BW], XIII, 3.5, for G simply connected, but, as already remarked in 6.3, the argument there allows one to suppress that restriction.

7. $\mathbf{L}^2$-cohomology

We prove the vanishing of the cohomology with coefficients in direct integrals (1) below when S_f is not empty and $\mathfrak{a}_P \neq 0$. This implies that the $\mathbf{L}^2$-cohomology of Γ is reduced to $H^{\cdot}_{\text{disc}}(\Gamma;\, E)$, provided that the complement of the discrete spectrum in the $\mathbf{L}^2$-spectrum is the sum, over Γ-conjugacy classes of classes of Levi k-subgroups of proper parabolic k-subgroups of G, of direct integrals of discrete spectra for Levi subgroups. It is likely that Langlands' proof [Lan1] extends to the S-arithmetic case but we do not know of a reference in this more general case. For congruence subgroups this follows easily from the corresponding adelic result (cf. [MW] VI.2.1).

7.1. Let P be a proper parabolic k-subgroup of G, N its unipotent radical and M a Levi k-subgroup. Recall (2.1) that $M_S = M^1_S \times A^0$. Let V be the $\mathbf{L}^2$-discrete spectrum of $\Gamma_M\backslash M^1_S$. It is viewed as a representation of P^1_S trivial on N_S. Let $I(P, V, \mu)$ be the induced representation from $V \otimes \mathbb{C}_{\rho+i\mu}$, where $\rho = \rho_P$ is as usual and $\mathbb{C}_\nu$ denotes $\mathbb{C}$ on which A^0 acts by ν ($\nu \in \mathfrak{a}^*_P \otimes \mathbb{C}$). Let $I_{P,V}$ be the direct integral

$$I_{P,V} = \int^{\oplus} I(P, V, \mu)\, \mathrm{d}\mu \qquad \text{(over the positive Weyl chamber in } \mathfrak{a}^*_P). \tag{1}$$

We want to prove that

$$H^{\cdot}_d(G_S; I_{P,V} \otimes E) = 0 \quad \text{if } S_f \neq \emptyset. \tag{2}$$

7.2. As in [BC:3.4] we use Shapiro's lemma and are reduced to consider

$$H_d^{\cdot}\left(P_S; E \otimes V \otimes \int^{\oplus} \mathbb{C}_{\rho+i\mu}\, \mathrm{d}\mu\right). \tag{3}$$

It is the abutment of a spectral sequence in which

$$E_2^{p,q} = H_d^p\left(M_S;\, H_d^q(N_S;\, E) \otimes V \otimes \int^{\oplus} \mathbb{C}_{\rho+i\mu}\, \mathrm{d}\mu\right). \tag{4}$$

We have

$$H_d^{\cdot}(N_S; E) = H_d^{\cdot}(N_\infty;\, E) \otimes H_d^{\cdot}(N_{S_f}; \mathbb{C}). \tag{5}$$

The archimedean factors are given by Kostant's theorem, as in *loc. cit.* For $v \in S_f$, $H_d^{\cdot}(N_v;\, \mathbb{C})$ is reduced to $\mathbb{C}$ in dimension 0 by [BW:X, 4.1].

The center of M contains a non-trivial k-split torus T. Consider a finite place $v \in S_f$. To compute the cohomology of M_S we use the spectral sequence of M_S modulo T_v. Its term E_2 is

$$H_d^{\cdot}\left(M_S/T_v;\, H_d^{\cdot}\left(T_v;\, \mathbb{C}_\rho \otimes V \otimes \int^{\oplus} \mathbb{C}_{i\mu}\, \mathrm{d}\mu\right) \otimes H^{\cdot}(N_S;\, E)\right),$$

since T_v acts trivially on $H^{\cdot}(N_S;\, E)$; the fact that we can factor out $H^{\cdot}(N_S;\, E)$ is the essential difference with the archimedean case and the main point here. Let C_v be the maximal compact subgroup of T_v. Using the spectral sequence of T_v modulo C_v and remembering that the cohomology with respect to a compact subgroup reduces to the invariants in dimension 0, we get

$$H_d^{\cdot}\left(T_v;\, V \otimes \mathbb{C}_\rho \otimes \int^{\oplus} \mathbb{C}_{i\mu}\, \mathrm{d}\mu\right) = H_d^{\cdot}\left(T_v/C_v;\, V^{C_v} \otimes \mathbb{C}_\rho \otimes \int^{\oplus} \mathbb{C}_{i\mu}\, \mathrm{d}\mu\right)$$

which is equal to

$$V^{C_v} \otimes H_d^{\cdot}\left(T_v/C_v; \int^{\oplus} \mathbb{C}_{\rho+i\mu}\, \mathrm{d}\mu\right)$$

Now T_v/C_v is a finitely generated free abelian group. We may identify its cohomology with the Lie algebra cohomology of a commutative Lie algebra with coefficients in a direct integral. It is zero by [BC:3.2] since $\rho \neq 0$.

III. CONSTRUCTION OF CUSPIDAL COHOMOLOGY CLASSES

In this part α is an automorphism of G defined over k, of finite order ℓ. Denote by $\widetilde{L}$ the semi-direct product $G \rtimes \langle\alpha\rangle$; this is a non-connected algebraic group whose identity component equals G. Here, $\langle\alpha\rangle$ is viewed as a k-group all elements of which are rational over k and $\widetilde{L}$ as a semi-direct product over k. In particular $\widetilde{L}(k)$ is Zariski-dense in $\widetilde{L}$. Let L be the coset defined by α. If A is a k-algebra, we denote by $L(A)^+$ the group generated by $L(A)$. Note that in general $L(A)^+ \subsetneq \widetilde{L}(A)$.

8. Lefschetz functions for automorphisms of finite order

8.1. Let F be the completion of k at some place. Let E be a finite dimensional representation of $L(F)^+$, assumed to be trivial if F is non-archimedean. Let (π, H_π) be an admissible irreducible representation of $L(F)^+$. A description of admissible irreducible representations of $L(F)^+$ in terms of admissible irreducible representations of $G(F)$ is given in the Appendix. Consider an element β in $L(F)$. By abuse of notation we denote again by β the automorphism of the differentiable cohomology groups $H_d^{\cdot}(G(F); H_\pi \otimes E)$ induced by β. The Lefschetz number of β with respect to $H_\pi \otimes E$ is by definition

$$\mathrm{Lef}(\beta, G(F); H_\pi \otimes E) = \sum (-1)^i \operatorname{trace}(\beta \mid H_d^i(G(F); H_\pi^\infty \otimes E)).$$

Since $G(F)$ acts trivially on the differentiable cohomology groups, this number is independent of the choice of β in the coset $L(F)$, and we shall sometimes denote it by $\mathrm{Lef}(\alpha, G(F); H_\pi \otimes E)$ instead of $\mathrm{Lef}(\beta, G(F); H_\pi \otimes E)$.

Fix a minimal parabolic F-subgroup P_0 of G over F with Levi decomposition $P_0 = M_0 N_0$ over F. Since all minimal parabolic subgroups and Levi decompositions over F are conjugate under $G(F)$ we may choose β_0 in the coset defined by α such that P_0 and M_0 are β_0-stable: $\beta_0 P_0 \beta_0{}^{-1} = P_0$ and $\beta_0 M_0 \beta_0{}^{-1} = M_0$. Let P be a β_0-stable parabolic F-subgroup with a β_0-stable Levi decomposition $P = MN$. We denote by $P(F)^+$ (resp. $M(F)^+$) the subgroup generated by $P(F)$ and β_0 (resp $M(F)$ and β_0).

When dealing with representations induced from representations of a parabolic subgroup we shall use normalized induction (as in 7.1): it differs from ordinary induction by a shift by the square root of the modulus function of the parabolic subgroup, so that it preserves unitarity. We shall first exhibit some properties of Lefschetz numbers for F non-archimedean.

8.2 PROPOSITION. *Let F be non-archimedean.*

(1) $\mathrm{Lef}(\alpha, G(F); H_\pi) = 0$ *whenever the restriction of π to $G(F)$ is a constituent of a representation induced from a unitary representation of a proper parabolic subgroup, trivial on the unipotent radical.*

(2) *The map* $\pi \mapsto \mathrm{Lef}(\alpha, G(F); H_\pi)$ *does not vanish identically.*

Proof. We recall that, according to a theorem due to W. Casselman, admissible irreducible representations of $G(F)$ with non-trivial cohomology are the irreducible subquotients of the right regular representation in the space of smooth functions on $P_0(F)\backslash G(F)$ (cf. [BW:X, 4.12]). Hence, if π has non-trivial cohomology, π is a subrepresentation of $i_{P_0}^G(\delta_{P_0}^{1/2})$, the semi-simplification of the representation of $G(F)$ induced from $\delta_{P_0}^{1/2}$. On the other hand let $P = MN$ be a proper parabolic F-subgroup of G and assume that π is a constituent of the representation of $G(F)$ induced from an irreducible unitary representation σ of $M(F)$ extended trivially to the unipotent radical $N(F)$. By Frobenius reciprocity σ is a subrepresentation of the semi-simplification of the Jacquet module $r_P(i_{P_0}^G(\delta_{P_0}^{1/2}))$. The semi-simplification of this Jacquet module is a sum of representations of the form $i_{M\cap P_0}^M(\delta_{P_0}^{1/2} \circ w)$ of $M(F)$, where w runs over a subset of the Weyl group and where $\delta_{P_0}^{1/2} \circ w$ is considered as a representation of $M_0(F)$, extended trivially to $N_0(F) \cap M(F)$ (cf. [BDK:5.4]). Hence the unitary representation σ itself should be a subrepresentation of $i_{M\cap P_0}^M(\delta_{P_0}^{1/2} \circ w)$ for some w. But if $M \neq G$ the representation $i_{M\cap P_0}^M(\delta_{P_0}^{1/2} \circ w)$ has a non-unitary central character; this is a contradiction. This proves assertion (1).

To prove assertion (2) we first observe that since P_0 is β_0-stable

$$P_0(F)\backslash G(F) = P_0(F)^+\backslash L(F)^+$$

and hence I_{P_0} has a canonical extension to $L(F)^+$. Now the Steinberg representation $\mathrm{St}(G(F))$ of $G(F)$ is the unique irreducible quotient of I_{P_0}; hence the Steinberg representation is β_0-stable and has a canonical extension to a representation $\mathrm{St}(L(F)^+)$ of $L(F)^+$. By [BW:X, 4.7], we know that $H_d^q(G(F); \mathrm{St}(G(F))) = 0$ unless $q = \mathrm{rk}\ G(F)$, in which case the cohomology space is canonically isomorphic to $H^q(G(F); I_{P_0})$ and is one-dimensional. The automorphism induced by β_0 acts by 1 on the non-trivial cohomology space, and one has

$$\mathrm{Lef}(\beta_0, G(F); \mathrm{St}(L(F)^+)) = \mathrm{Lef}(\alpha, G(F); \mathrm{St}(L(F)^+)) = (-1)^{\mathrm{rk}\, G(F)}.$$

8.3. Let $K \subset G(F)$ be a maximal compact subgroup. If F is an archimedean field, K may and will be chosen α-stable; it is well known that it is possible and can be seen as follows: the symmetric space attached to $G(F)$ is the set of maximal compact subgroups of $G(F)$; it is a complete simply connected Riemannian manifold of negative curvature; α acts on this space and generates a finite group of isometries, which has a fixed point, say K, by a well known theorem of É. Cartan ([Hel] Chap. I Th.13.5 p.75). If F is non-archimedean, K is assumed to be special.

Let f be a smooth K-finite function with compact support on $G(F)$; we shall denote by f^* the function on $L(F)$ defined by $f^*(x \rtimes \alpha) = f(x)$. We have to recall two definitions. According to J. Arthur (cf. [A5], Section 7, p. 538) a function f^* on $L(F)$ is said to be *cuspidal* if the trace of $\pi(f^*)$ vanishes whenever π is a representation of $L(F)^+$ induced from an irreducible representation (τ, H_τ) of $P(F)^+$ trivial on $N(F)$, where $P = MN$ is a β_0-stable proper parabolic F-subgroup. A smooth compactly supported function f^* is said to be *very cuspidal* if for any β_0-stable proper parabolic F-subgroup P with Levi decomposition $P = MN$, the constant term with respect to P (or along P)

$$f_P^*(m) = \delta_P(m)^{1/2} \int_K \int_{N(F)} f^*(k^{-1}mnk)\, \mathrm{d}n\, \mathrm{d}k$$

vanishes for all $m \in M(F)\beta_0$. As suggested by the terminology very cuspidal functions are cuspidal, since $\operatorname{trace}\pi(f^*) = \operatorname{trace}\tau(f_P^*)$ if π is induced from a representation τ of $P(F)^+$ trivial on $N(F)$.

8.4 PROPOSITION. *(1) There exist K-finite, cuspidal, compactly supported smooth functions $f_{\alpha,E}$ on $G(F)$ such that for any admissible representation π of $L(F)^+$ of finite length*

$$\operatorname{trace}\pi(f_{\alpha,E}^*) = \operatorname{Lef}(\alpha, G(F); H_\pi \otimes E).$$

(2) If F is archimedean, the functions $f_{\alpha,E}^*$ may be chosen to be very cuspidal.

Such functions are called *Lefschetz functions*.

Proof. Assume first that F is non-archimedean; then E is assumed to be trivial. We have to generalize a construction due to Kottwitz [Ko]. We use the notation of [BW:X, 2]. The automorphism α acts on the Tits building Y associated to $G(F)$. Let s be a face of Y; denote by $L(F)_s^+$ (resp. $G(F)_s$) the stabilizer of s in $L(F)^+$ (resp. $G(F)$). Denote by sign_s the function on $L(F)^+$ equal to the signature of the permutation of the vertices of s induced by x if $x \in L(F)_s^+$ and equal to zero otherwise. Choose a chamber C in Y; since $G(F)$ acts transitively on the chambers there is a $\beta_1 \in L(F)$ which fixes C. If H_π is the space of an admissible representation π of $L(F)^+$, the differentiable cohomology $H_d^{\cdot}(G(F); H_\pi)$ is isomorphic to the cohomology of a complex whose terms are:

$$C^q(Y; H_\pi)^{G(F)} = \bigoplus_{\dim(s)=q} H_\pi^s$$

where H_π^s is the largest subspace of H_π on which $G(F)_s$ acts by the character $sign_s$, and the sum is over the faces s of C ([BW:X, 2.5]); β_1 acts on this complex. Consider the function on $G(F)$

$$x \mapsto f_\alpha(x) = \sum (-1)^{\dim(s)} \operatorname{meas}(G(F)_s)^{-1} \operatorname{sign}_s(x \rtimes \alpha),$$

where the sum is over the faces s of C such that $\beta_1(s) = s$. We have

$$\mathrm{trace}(\pi(f_\alpha^*)) = \sum(-1)^{\dim(s)}\mathrm{sign}_s(\beta_1)\mathrm{trace}(\beta_1|H_\pi^s)$$

and hence

$$\mathrm{trace}(\pi(f_\alpha^*)) = \mathrm{Lef}(\beta_1, G(F); H_\pi) = \mathrm{Lef}(\alpha, G(F); H_\pi).$$

To prove that such functions are cuspidal we have to show that

$$\pi \mapsto \mathrm{Lef}(\alpha, G(F); H_\pi) = \mathrm{Lef}(\beta_0, G(F); H_\pi)$$

vanishes on representations (π, H_π) of $L(F)^+$ induced from a representation (τ, H_τ) of $P(F)^+$, where $P = MN$ is a proper β_0-stable parabolic F-subgroup, and τ is irreducible and trivial on the unipotent radical $N(F)$. By [BW:X, 4.2] we have

$$\mathrm{Lef}(\beta_0, G(F); H_\pi) = \mathrm{Lef}(\beta_0, M(F); H_{\tau_1})$$

where $\tau_1 = \tau \otimes \delta_P^{1/2}$. Since all cohomology groups $H_d^q(M(F); H_{\tau_1})$ vanish unless the center of $M(F)$ acts trivially, we are reduced to prove the vanishing of the Lefschetz numbers if this center acts trivially. If $M \neq G$, the center of M^+ contains a non-trivial split torus A on which β_0 acts trivially. Let $A(F)^1$ be the maximal compact subgroup of $A(F)$. Using the Hochschild-Serre spectral sequence associated to the exact sequence

$$1 \to A(F)/A(F)^1 \to M(F)^+/A(F)^1 \to M(F)^+/A(F) \to 1$$

and since β_0 acts trivially on A, we see that

$$\mathrm{Lef}(\beta_0, M(F); H_{\tau_1}) = \mathrm{Lef}(\beta_0, M(F)/A(F); H_{\tau_1})\mathrm{Lef}(1, A(F)/A(F)^1; \mathbb{C}),$$

where $A(F)$ acts trivially on $\mathbb{C}$. Now $A(F)$ is the group of F-points of a non-trivial F-split torus, thus the group $A(F)/A(F)^1$ is isomorphic to $\mathbb{Z}^n$ for some $n > 0$ and its Euler-Poincaré characteristic vanishes: $\mathrm{Lef}(1, A(F)/A(F)^1; \mathbb{C}) = 0$. This implies the vanishing of all Lefschetz numbers for $M(F)^+$ and concludes the proof of assertion (1) for non-archimedean fields.

Assume now that F is archimedean. For $\alpha = 1$ the result is known: Euler-Poincaré functions were first constructed by Clozel and Delorme using their trace Paley-Wiener theorem [CD]; then Laumon, in a letter to J. Arthur, has shown that a more direct construction, due to N. Wallach and which uses 'multipliers', yields very cuspidal Euler-Poincaré functions (see [Lab]). For arbitrary α, some functions $f_{\alpha,E}$ have been constructed by this latter method in [Lab] Proposition 12. This takes care of assertion (1). To prove assertion (2) we have to check that one can extend to the twisted case the arguments in [Lab] Section 4. Let us denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G(F)$ and K respectively. Since K is α-stable, the same is true for the Cartan decomposition $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$. Let $\mathfrak{a}_0$ be a maximal abelian subspace in $\mathfrak{p}$. Choose a set of simple coroots $\check{\Delta}_0$ for $(\mathfrak{g}, \mathfrak{a}_0)$. The pairs $(\mathfrak{a}_0, \check{\Delta}_0)$ are all conjugate under K and hence we may choose $\beta_1 = k_1 \rtimes \alpha$ in $K \rtimes \alpha$ which preserves $\mathfrak{a}_0$ and the set of simple coroots. This defines a β_1-stable minimal parabolic subgroup Q_0. Since P_0 and Q_0 are conjugate under K there exists $k_0 \in K$ such that $\beta_1 = k_0^{-1}\beta_0 k_0$. A β_1-stable subset $\check{\Delta} \subset \check{\Delta}_0$ defines a β_1-stable subalgebra $\mathfrak{a} \subset \mathfrak{a}_0$. The dimension of the subalgebra of β_1-fixed vectors $\mathfrak{a}^{\beta_1} \subset \mathfrak{a}$ is the number of β_1-orbits in $\check{\Delta}$. Any β_0-stable proper parabolic subgroup P is K-conjugate to a parabolic subgroup Q with Levi decomposition $Q = MN$, where M is the centralizer of an abelian β_1-stable subalgebra $\mathfrak{a} \subset \mathfrak{a}_0$ as above. In fact, M is also the centralizer of the subalgebra $\mathfrak{a}^{\beta_1}$. Now, if t belongs to $(K \cap M).\beta_1$ then $\mathrm{Ad}(t)$ acts trivially on the non-trivial subalgebra $\mathfrak{a}^{\beta_1} \subset \mathfrak{p}$, and hence

$$\sum(-1)^i \operatorname{trace}(\mathrm{Ad}(t) \mid \Lambda^i\mathfrak{p}) = \det(1 - \mathrm{Ad}(t) \mid \mathfrak{p}) = 0.$$

Let us denote by $\chi_{\alpha,E}$ the function on K defined by

$$x \mapsto \operatorname{trace}(x \rtimes \alpha|\check{E})\sum(-1)^i \operatorname{trace}(\mathrm{Ad}(x \rtimes \alpha) \mid \Lambda^i\mathfrak{p})$$

where $\check{E}$ is the contragredient of E. Let $\mu_{\alpha,E}$ be the measure supported on K which is the product of $\chi_{\alpha,E}$ by the normalized Haar measure of K (cf. [Lab] p. 616). According to [Lab] Section 4 the constant term along Q of the measure $\mu_{\alpha,E}$ is the product of the function on $(K \cap M)$: $m \mapsto \chi_{\alpha,E}(m.k_1)$ by the Haar measure on $(K \cap M)$ and hence it vanishes. In other words $\mu_{\alpha,E}$ is very cuspidal. The function $f_{\alpha,E}$ is defined by applying a multiplier to $\mu_{\alpha,E}$. Since taking constant terms commutes with multipliers [A3] we see that $f^*_{\alpha,E}$ is very cuspidal. This proves assertion (2).

Remark: The second statement is likely to hold also for non-archimedean fields. For $G = \mathrm{GL}(n)$ and $\alpha = 1$ such functions have been constructed by Laumon using results of Waldspurger ([Lau] Chapter 5).

8.5. The previous construction immediately extends to groups over S. Let E be the tensor product over S of finite dimensional representations E_v of L_v^+, trivial for finite places. Consider $f_{\alpha,E} = \otimes_{v\in S} f_{\alpha,E_v}$ where f_{α,E_v} is a Lefschetz function for each $v \in S$. Given an admissible representation π_S of $L_S^+ = G_S \rtimes \langle\alpha\rangle$ which is the restriction of a tensor product $\otimes_{v\in S}\pi_v$ of representations of G_v^+ one has

$$\begin{aligned}\operatorname{trace} \pi_S(f^*_{\alpha,E}) &= \mathrm{Lef}(\alpha, G_S; H_{\pi_S} \otimes E) = \prod_{v\in S} \mathrm{Lef}(\alpha, G_v; H_{\pi_v} \otimes E_v) \\ &= \sum(-1)^i \operatorname{trace}(\alpha \mid H^i_d(G_S, H_{\pi_S} \otimes E))\end{aligned}$$

Such functions are called *Lefschetz functions for α and E (over S).*

9. A simple form of the trace formula

9.1. Let F be a global field; we denote by $\mathbb{A}_F$ the ring of adèles of F. Let $\mathbb{A}_k^S$ be the subring of adèles of k with null component in S. There is a natural action ρ of the semi-direct product

$$L(\mathbb{A}_k)^+ = G(\mathbb{A}_k) \rtimes \langle\alpha\rangle$$

on $\mathbf{L}^2(G(k)\backslash G(\mathbb{A}_k))$:

$$(\rho(y \rtimes \alpha^r)f)(x) = f(\alpha^{-r}(xy)).$$

The discrete spectrum $\mathbf{L}^2_{\text{disc}}(G(k)\backslash G(\mathbb{A}_k))$ is invariant under this action. Let ρ_{disc} denote the representation of $L(\mathbb{A}_k)^+$ on the discrete spectrum. Denote by $m_{\text{disc}}(\pi)$ the multiplicity with which the irreducible representation π of $L(\mathbb{A}_k)^+$ occurs in the discrete spectrum. We have

$$\operatorname{trace} \rho_{\text{disc}}(f) = \sum m_{\text{disc}}(\pi) \operatorname{trace} \pi(f).$$

Here we use the fact that $\rho_{\text{disc}}(f)$ is of trace class [Mü]; in particular the above expansion is absolutely convergent, and a partial summation indexed by absolute values of norms of the imaginary part of infinitesimal characters, as in [A5] Theorem 4.4 and in 9.2 below, is not necessary here.

We want to compute the trace of $\rho_{\text{disc}}(f)$ for functions supported on $L(\mathbb{A}_k)$ of the form $f = f^*_{\alpha,E} \otimes h^*$ where $f_{\alpha,E}$ is a Lefschetz function for α over S and $h \in C_c^\infty(G(\mathbb{A}_k^S))$. We assume here $\operatorname{Card}(S) \geqslant 2$ and we would like to use the 'simple form of the trace formula' ([A5] Theorem 7.1 p. 538) which should be valid for functions $f = \otimes f_v$ such that f_v is cuspidal for at least two places. This simple form is established by J. Arthur using the invariant form of the trace formula. The proof of this invariant trace formula ([A5], Theorems 3.3 and 4.4) relies on the twisted trace Paley-Wiener theorem. Unfortunately, while this theorem has been proved for non-archimedean fields [Ro], it is still not known to be true for archimedean fields in full generality (as far as we know it is proved in the non-twisted case [CD], and in the case of base change [D]).

However one can bypass this difficulty: mimicking the proof of Theorem 7.1 in [A5], we shall get in 9.5 an unconditional proof of the simple form of the trace formula we need, using the ordinary – non-invariant – trace formula and functions $f = \otimes f_v$ satisfying slightly more stringent assumptions: the functions f_v will be assumed to be very cuspidal at one place and cuspidal at another one.

The proof is quite technical and the reader may skip Section 9 if he is willing to take 9.5 for granted. We shall assume the reader to be familiar with the contents of Section 7 in [A5]. In particular, we shall use the notation adopted there without further explanation. Fix a minimal parabolic k-subgroup P_0^0 with Levi decomposition $P_0^0 = M_0^0 N_0$ over k. A parabolic k-subgroup P^0 containing M_0^0 has a unique Levi k-subgroup M^0 containing M_0^0. Denote by $\widetilde{P}$ the normalizer of P^0 in $\widetilde{L}$ and by $\widetilde{M}$ the normalizer of M^0 in $\widetilde{P}$. Let $P = \widetilde{P} \cap L$ and $M = \widetilde{M} \cap L$; they are called a parabolic subset and a Levi subset respectively. Since all minimal parabolic subgroups and Levi decompositions over k are conjugate under $G(k)$, the cosets P_0 and M_0 have a common rational point over k; but for an arbitrary parabolic subgroup P^0 the sets $P(k)$ and $M(k)$ may be empty. Let $\mathcal{L}(L)$ denote the set of Levi subsets containing M_0. As usual, upper indices are often omitted if $L' = L$ in the various distributions $J_M^{L'}$ that will show up.

9.2 PROPOSITION. *Let $f = \otimes f_v$ be a function in $C_c^\infty(L(\mathbb{A}_k))$. Assume that*

(a) *At a place v_0 the function f_{v_0} is very cuspidal.*
(b) *At a place $v_1 \neq v_0$, the function f_{v_1} is cuspidal.*

Then there exists a finite set $S(f)$ of places of k such that, if Σ is a finite set of places of k containing $S(f)$, the trace formula can be written:

$$\begin{aligned} J(f) &= \sum_{\gamma \in (L(k))_{G,\Sigma}} a^L(\Sigma, \gamma) J_L(\gamma, f) \\ &= \sum_{t \geqslant 0} \sum_{\pi \in \Pi_{\mathrm{disc}}(L,t)} a^L_{\mathrm{disc}}(\pi) J_L(\pi, f). \end{aligned}$$

Proof. Given f, for a large enough finite set of places Σ, the ordinary trace formula can be written (cf. [A5], p. 508 and 521):

$$\begin{aligned} J(f) &= \sum_{M \in \mathcal{L}(L)} \frac{|W^M|}{|W^L|} \sum_{\gamma \in (M(k))_{M,\Sigma}} a^M(\Sigma, \gamma) J_M^L(\gamma, f) \\ &= \sum_{t \geqslant 0} \sum_{M \in \mathcal{L}(\mathcal{L})} \frac{|W^M|}{|W^L|} \int_{\Pi(M,t)} a^M(\pi) J_M^L(\pi, f)\, \mathrm{d}\pi. \end{aligned}$$

The proposition is easily proved combining this formula and the following lemma.

9.3 LEMMA. *Let $f = \otimes f_v$ be a function in $C_c^\infty(L(\mathbb{A}_k))$ which satisfies assumptions* (a) *and* (b) *in 9.2 above. Then $J_M^L(*, f) = 0$ if $M \neq L$; moreover $J_L(\pi, f) = 0$ if π is induced from a representation of a proper parabolic subgroup, trivial on the unipotent radical.*

Proof. Since the distributions $J_M^L(*, f)$ arise from a (L, M)-family one may use the descent and splitting formulas ([A4], Proposition 7.1 p. 357 and Corollary 7.4 p. 358). The splitting formula ([A4] Corollary 7.4) shows that $J_M^L(*, f)$ can be expressed as a sum of terms indexed by pairs of Levi subsets (L_1, L_2); to each such pair of Levi subsets is attached a pair of parabolic subsets (Q_1, Q_2) via a 'section' (cf. [A4] p. 356–357) so that:

$$J_M^L(\, *\, , f) = \sum \mathrm{d}_M^L(L_1, L_2) J_M^{L_1}(\, *\, , f_{v_0, Q_1}) J_M^{L_2}(\, *\, , f_{Q_2}^{v_0}).$$

One should note that the constant terms f_{Q_i} along parabolic subsets Q_i with Levi subsets L_i that show up in the above formula cannot in general be replaced by their invariant avatars f_{L_i} unless the distributions are invariant; recall that unless $M = L$ the distributions $J_M^L(*, f)$ are non-invariant.

Since f_{v_0} is very cuspidal, $f_{v_0,Q} = 0$ unless $Q = L$ and since $d_M^L(L, L') = 0$ unless $M = L'$, all terms vanish except if $L_1 = L$ and $L_2 = M$:

$$J_M^L(*, f) = J_M^L(*, f_{v_0}) J_M^M(*, f_M^{v_0}).$$

We have used the equality $d_M^L(L, M) = 1$ and the fact that J_M^M is an invariant distribution.

By assumption there is a place $v_1 \neq v_0$ where f_{v_1} is cuspidal, so that $f_M^{v_0} = 0$ if $M \neq G$, and hence $J_M^L(*, f) = 0$ if $M \neq L$, as expected. It remains to see that $J_L(\pi, f) = 0$ if π is an induced representation, but this follows from the cuspidality of f_v for at least one place v.

9.4. If at one place, say w, the support of f_w is inside the set of regular elements (in particular they are semisimple), the summation over the set $(L(k))_{G,\Sigma}$ in the geometric side of the trace formula can be replaced by a sum over $(L(k))_{G,\mathrm{reg}}$, the set of regular $G(k)$-conjugacy classes in $L(k)$. We recall that

$$J_L(\gamma, f) = \int_{G_\gamma(\mathbb{A}_k)\backslash G(\mathbb{A}_k)} f(x^{-1}\gamma x)\, \mathrm{d}x,$$

where G_γ is the identity component of the centralizer of γ in G. If γ is regular then $a^L(\Sigma, \gamma)$ equals 0 unless γ is elliptic in which case it equals the volume of $G_\gamma(k)\backslash G_\gamma(\mathbb{A}_k)$ divided by the order of $G_\gamma(k)$ in the centralizer of γ in $G(k)$; this expression is independent of Σ and will be denoted $a^L(\gamma)$.

9.5 PROPOSITION. *Let $f = \otimes f_v$ be a function in $C_c^\infty(L(\mathbb{A}_k))$. Assume that f satisfies assumptions* (a) *and* (b) *in 9.2 and*

(c) *There is a place v such that* trace $\pi_v(f_v) = 0$, *whenever π_v is a constituent of a representation induced from a unitary representation of a proper parabolic subgroup, trivial on the unipotent radical.*
(d) *At one place w the support of f_w is inside the set of regular elements.*

Then

$$\operatorname{trace} \rho_{\mathrm{disc}}(f) = \sum_{\gamma \in (L(k))_{G,\mathrm{reg}}} a^L(\gamma) J_L(\gamma, f).$$

Proof. As in the proof of Corollary 7.3 in [A5] we see that if there is a place v such that $\operatorname{trace} \pi_v(f_v) = 0$, whenever π_v is a constituent of a representation induced from a unitary representation of a proper parabolic subgroup, trivial on the unipotent radical (by the way this implies that f_v is cuspidal), then

$$a^L_{\mathrm{disc}}(\pi) J_L(\pi, f) = m_{\mathrm{disc}}(\pi) \operatorname{trace} \pi(f).$$

We conclude by using 9.2.

10. Non-vanishing results

In this section we assume, as in 6.2, that G is an almost absolutely simple connected algebraic group over k of strictly positive k-rank.

10.1. We shall say that the cuspidal cohomology of G over S, with coefficients in E, does not vanish if every S-arithmetic subgroup has a subgroup of finite index with non-zero cuspidal cohomology with respect to E. If $E = \mathbb{C}$, the coefficients will not be mentioned.

To get information about the non-vanishing of the cuspidal cohomology of deep enough S-arithmetic subgroups of $G(k)$, we shall first study the group $H^{\cdot}_{\mathrm{cusp}}(G, S; E)$, which we shall call the *cuspidal cohomology group for G over S with values in E*, which is by definition the inductive limit over *congruence* S-arithmetic subgroups Γ of $G(k)$ of the cuspidal cohomology groups:

$$H^{\cdot}_{\mathrm{cusp}}(G, S; E) = \varinjlim H^{\cdot}_{\mathrm{cusp}}(\Gamma; E).$$

This inductive limit is isomorphic to the cohomology of the adelic cuspidal spectrum:

$$H^{\cdot}_{\mathrm{cusp}}(G, S; E) = H^{\cdot}_d(G_S; \mathbf{L}^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}_k))^{\infty} \otimes E).$$

The non-vanishing of the cuspidal cohomology for G over S, which will be proved below for some groups G, implies the non-vanishing of the cuspidal cohomology for deep enough S-arithmetic subgroups Γ of $G(k)$, not necessarily of congruence type:

10.2 PROPOSITION. *Assume that $H^p_{\mathrm{cusp}}(G, S; E)$ is non-trivial for some p, then any S-arithmetic subgroup Γ of $G(k)$, contains a subgroup Γ_1 of finite index such that the cuspidal cohomology $H^p_{\mathrm{cusp}}(\Gamma_1; E)$ is non-trivial.*

Proof. By hypothesis there exists a congruence S-arithmetic subgroup Γ_0 for which $H^p_{\text{cusp}}(\Gamma_0; E)$ is non-trivial. The subgroup $\Gamma_0 \cap \Gamma$ is of finite index in both Γ and Γ_0 and contains a subgroup Γ_1 of finite index, invariant in Γ_0. The usual argument to show that the cohomology of Γ_0 injects in the cohomology of Γ_1 also applies to the cuspidal cohomology: we have

$$\mathbf{L}^2_{\text{cusp}}(\Gamma_0 \backslash G_S)^\infty = \left(\mathbf{L}^2_{\text{cusp}}(\Gamma_1 \backslash G_S)^\infty\right)^{\Gamma_0/\Gamma_1},$$

and since the formation of cohomology commutes with taking fixed points with respect to finite groups this implies

$$H^p_{\text{cusp}}(\Gamma_0; E) = H^p_{\text{cusp}}(\Gamma_1; E)^{\Gamma_0/\Gamma_1}$$

so that the restriction map

$$H^p_{\text{cusp}}(\Gamma_0; E) \to H^p_{\text{cusp}}(\Gamma_1; E)$$

is injective (this also follows from the representation theoretic description of the cuspidal cohomology groups in theorem 6.5) and our claim follows.

We shall use the trace formula to prove an $\mathbf{L}^2$-Lefschetz formula and then apply the latter to show the non-vanishing of the cuspidal cohomology in some cases.

For α as in Section 9 and $h \in C_c^\infty(G(\mathbb{A}_k^S))$, the $\mathbf{L}^2$-Lefschetz number on the discrete part of the $\mathbf{L}^2$-spectrum is, by definition,

$$\begin{aligned}&\text{Lef}_{\text{disc}}(\alpha, h, G; E)\\ &:= \sum (-1)^i \operatorname{trace}(\alpha \times h \mid H^i_d(G_S; \mathbf{L}^2_{\text{disc}}(G(k) \setminus G(\mathbb{A}_k))^\infty \otimes E).\end{aligned}$$

Remark that only the irreducible representations π of $L(\mathbb{A}_k)^+$ whose restriction σ to $G(\mathbb{A}_k)$ remain irreducible contribute non-trivially to the α-twisted trace (see Corollary A.4 in the appendix). If π is one, we may decompose σ into a tensor product $\sigma = \sigma_S \otimes \sigma^S$ of representations of G_S and $G(\mathbb{A}_k^S)$ respectively; according to A.3 the representations σ_S and σ^S may be extended to representations π_S and π^S of L_S^+ and $L(\mathbb{A}_k^S)^+$ respectively, in such a way that π is the restriction of $\pi_S \otimes \pi^S$ to $L(\mathbb{A}_k)^+$; (observe that $L(\mathbb{A}_k)^+$ is of index ℓ in $L_S^+ \times L(\mathbb{A}_k^S)^+$). The spectral decomposition (Theorem 6.5) and 6.5(8) show that

$$\text{Lef}_{\text{disc}}(\alpha, h, G; E) = \sum m_{\text{disc}}(\pi) \operatorname{Lef}(\alpha, G_S; \pi_S \otimes E) \operatorname{trace} \pi^S(h^*).$$

We have shown in Section 8 that there exist Lefschetz functions $f_{\alpha,E}$ for α and E over S. Since

$$\operatorname{Lef}(\alpha, G_S; \pi_S \otimes E) = \operatorname{trace} \pi_S(f^*_{\alpha,E})$$

we have

$$\mathrm{Lef}_{\mathrm{disc}}(\alpha, h, G; E) = \mathrm{trace}\Big(\rho(\alpha)\rho(f_{\alpha,E} \otimes h) \mid \mathbf{L}^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}_k))\Big)$$

10.3 PROPOSITION. *Assume that S contains at least one finite place v and that at some place $w_1 \notin S$ the support of $h^*_{w_1}$ is inside the set of elliptic regular elements. Then the Lefschetz number is given by*

$$\mathrm{Lef}_{\mathrm{disc}}(\alpha, h, G;\, E) := \sum_{\gamma \in (L(k))_{G,\mathrm{reg}}} a^L(\gamma)\; J_L(\gamma, f^*_{\alpha,E} \otimes h^*)$$

where

$$\begin{aligned} &J_L(\gamma, f^*_{\alpha,E} \otimes h^*) \\ &= \int_{G_\gamma(k_S)\backslash G(k_S)} f^*_{\alpha,E}(x^{-1}\gamma x)\, \mathrm{d}x \int_{G_\gamma(\mathbb{A}_k^S)\backslash G(\mathbb{A}_k^S)} h^*(x^{-1}\gamma x)\, \mathrm{d}x. \end{aligned}$$

Proof. We shall apply Proposition 9.5; we have to check that the various assumptions are satisfied. By Proposition 8.4 the functions f^*_{α,E_v} may be chosen to be cuspidal for $v \in S_f$ and very cuspidal for $v \in \mathcal{V}_\infty$. Since $S \neq S_\infty$ assumptions 9.2 (a) and (b) are satisfied. By 8.2(1) the assumption 9.5(c) is fulfilled for $v \in S_f$; finally 9.5(d) is part of our hypothesis.

Let us define the cuspidal Lefschetz number by

$$\mathrm{Lef}_{\mathrm{cusp}}(\alpha, h, G; E) := \mathrm{trace}\Big(\rho(\alpha)\rho(f_{\alpha,E} \otimes h) \mid \mathbf{L}^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}_k))\Big)\,.$$

10.4 THEOREM. *Assume that the Lefschetz number at infinity*

$$\pi_\infty \mapsto \mathrm{Lef}(\alpha, G_\infty; H_{\pi_\infty} \otimes E)$$

where $(\pi_\infty, H_{\pi_\infty})$ varies through the set of equivalence classes of irreducible unitary representations of the Lie group G_∞, is not identically zero. Then the cuspidal cohomology $H^{\cdot}_{\mathrm{cusp}}(G, S;\, E)$ does not vanish.

Proof. Recall that we have assumed G to be almost absolutely simple over k and that the k-rank of G is at least 1. To prove the theorem we are free to enlarge S arbitrarily since, according to Theorem 6.5, if $S \subset \Sigma$

$$H^{\cdot}_d(G_\Sigma, \mathbf{L}^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}_k))^\infty \otimes E)[-r_f(G_\Sigma)]$$

injects in

$$H^{\cdot}_d(G_S, \mathbf{L}^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}_k))^\infty \otimes E)[-r_f(G_S)].$$

It suffices to show that for S big enough and some $h \in C_c^\infty(G(\mathbb{A}_k^S))$ the cuspidal Lefschetz number $\mathrm{Lef}_{\mathrm{cusp}}(\alpha, h, G; E)$ does not vanish. We shall assume that S contains at least one finite place v. Theorem 6.5 also shows that only the trivial representation may contribute to the non-cuspidal discrete spectrum. From this we get that

$$\mathrm{Lef}_{\mathrm{disc}}(\alpha, h, G; E) = \mathrm{Lef}_{\mathrm{cusp}}(\alpha, h, G; E) + \mathbf{1}(f_{\alpha,E} \otimes h).$$

By assumption for archimedean places and by 8.2.(2) for non-archimedean ones the Lefschetz number is not identically zero; hence there is a representation π_S of G_S such that the Lefschetz number $\mathrm{Lef}(\alpha, G_S; \pi_S \otimes E)$ is not zero. But

$$\mathrm{Lef}(\alpha, G_S; \pi_S \otimes E) = \mathrm{trace}\, \pi_S(f^*_{\alpha,E})$$

can be computed, using Weyl's integration formula, as an integral of orbital integrals $J_L(x, f^*_{\alpha,E})$ against the character of π_S for some measure on the set of G_S-conjugacy classes in L_S, recalling that the character of π_S is given by a locally integrable function on L_S. This fact – due to Harish-Chandra for connected reductive groups over local fields of characteristic zero – has been proved for non-connected reductive real Lie groups by Bouaziz [Bou], and for non-connected reductive groups over non-archimedean local fields of characteristic zero by Clozel [Cl1].

The non-vanishing of the Lefschetz number over S implies that for some semisimple element $x_0 \in L_S$, the orbital integral of the Lefschetz function $J_L(x_0, f^*_{\alpha,E})$ does not vanish. The same will be true for some, close enough, rational regular elliptic element $\gamma_0 \in L(k)$, which can always be found since these elements are dense in L_S. If h^* has a small enough support in a neighborhood of γ_0, the Lefschetz formula in Proposition 10.3 reduces to:

$$\mathrm{Lef}_{\mathrm{cusp}}(\alpha, h, G; E) + \mathbf{1}(f_{\alpha,E} \otimes h) = a^L(\gamma_0) J_L(\gamma_0, f^*_{\alpha,E}) J_L(\gamma_0, h^*).$$

Fix the function h^*_v at all places $v \notin S$ but one, say w, with a non-vanishing orbital integral for γ_0; we take h^*_w to be, up to scalars, the characteristic function of a decreasing sequence of (sufficiently small) neighborhoods of γ_0 so that

$$J_L(\gamma_0, h^*) = c_1 \int_{G_{\gamma_0}(k_w)\backslash G(k_w)} h^*_w(x^{-1}\gamma_0 x)\, \mathrm{d}x \equiv 1,$$

while

$$\mathbf{1}(f_{\alpha,S} \otimes h) = c_2 \int_{G(k_w)} h_w(x)\, \mathrm{d}x$$

goes to zero when the neighborhood decreases. Such sequences of neighborhoods and scalars exist since the variety $G_{\gamma_0}(k_w)\backslash G(k_w)$ has a strictly smaller dimension than $G(k_w)$. To finish the proof we simply have to recall that for γ_0 regular elliptic the number $a^L(\gamma_0)$ is the volume of $G_{\gamma_0}(k)\backslash G_{\gamma_0}(\mathbb{A}_k)$ divided by some integer and, in particular, it is non-zero.

10.5. To apply 10.4 to a given pair G, E we have to exhibit an automorphism α of G defined over k such that E extends to a representation of L_∞^+ and such that the Lefschetz number for α at infinity $\pi_\infty \mapsto \mathrm{Lef}(\alpha, G_\infty; \pi_\infty \otimes E)$ is not identically zero. We say that G over k admits a Cartan-type automorphism if there is a k-rational automorphism α of G such that on the Lie group G_∞, for some $x \in G_\infty$, the automorphism $\theta = \alpha \circ \mathrm{Ad}(x)$ is a Cartan involution.

10.6 THEOREM. *Let G be an absolutely almost simple group G over k that admits a Cartan-type automorphism. When $E = \mathbb{C}$, the cuspidal cohomology over S does not vanish.*

Proof. As observed by Rohlfs and Speh ([RS1], Proposition 4.3 p. 493), the Lefschetz number for a Cartan involution θ is never identically zero: in fact since θ acts by -1 on $\mathfrak{g}/\mathfrak{k}$, the Lefschetz number for θ in the $\mathfrak{g}$-K_∞-cohomology of the trivial representation of L_∞^+ is strictly positive. But on the differentiable cohomology at infinity θ and α have the same Lefschetz number. Hence we may apply Theorem 10.4.

This theorem bears some resemblance to some of the results of Rohlfs and Speh in [RS1], where they prove non-vanishing results for Lefschetz numbers on the total cohomology of arithmetic groups. Here we work with cuspidal cohomology and we allow S-arithmetic discrete subgroups for arbitrary large S. More general finite dimensional representations E could be used (see [RS1] and [LS:5.4]) but from now on we shall restrict ourselves to $E = \mathbb{C}$.

10.7 COROLLARY. *Assume that G is k-split and k totally real or $G = \mathrm{Res}_{k'/k}G'$ where k' is a CM-field. Then the cuspidal cohomology of G over S does not vanish.*

Proof. If G is split semisimple over a totally real field k, a Cartan involution of G_∞ induces, up to an inner automorphism, an automorphism of the Dynkin diagram which is itself induced by a rational automorphism α of finite order. Let G' be defined over a CM-field k' ; it is a quadratic totally imaginary extension of a totally real field k; the complex conjugation induces a Cartan-type automorphism of the group $G = \mathrm{Res}_{k'/k}G'$.

11. A base change construction

11.1. Let k/k_0 be a cyclic extension of number fields, of prime degree. Consider an irreducible cuspidal automorphic representation $\pi = \otimes\pi_v$ of $\mathrm{GL}_n(\mathbb{A}_{k_0})$. J. Arthur and L. Clozel have shown ([AC] Chap. 3 Theorem 4.2 (c) and 5.2) that there exists a unique representation $\Pi = b_{k/k_0}(\pi)$ of $\mathrm{GL}_n(\mathbb{A}_k)$, which is locally everywhere the base change lift of π. The representation Π is not necessarily cuspidal, but it is always induced from a cuspidal representation of a Levi subgroup of a parabolic subgroup, extended trivially to the unipotent radical. Now, suppose that a local component π_v of π at a finite place v of k_0 is the Steinberg representation. Then, for any place w above v the component Π_w is again the Steinberg representation (cf. [AC], Chap. 1, Lemma 6.12 or p. 56) and cannot be properly induced. Thus, Π is cuspidal in that case.

More generally, consider a finite extension k/k_0 such that there is a tower

$$k = k_m \supset k_{m-1} \supset \cdots \supset k_1 \supset k_0 \tag{1}$$

of intermediate fields so that k_{i+1}/k_i is a cyclic extension of prime degree. The above base change maps b_{k_{i+1}/k_i} can be composed to define a map b_{k/k_0}.

From this, a base change which associates to a given irreducible cuspidal automorphic representation σ of $\mathrm{SL}_n(\mathbb{A}_{k_0})$ an L-packet $b_{k/k_0}(\sigma)$ of automorphic representations of $\mathrm{SL}_n(\mathbb{A}_k)$ has been defined in [LS] 3.6 and 4.5. We recall that it is constructed by means of a bijection between representations of GL_n up to twists by characters and L-packets of representations of SL_n [LS:3.6], (see also [Cl2] p. 136–138).

This can be summed up in the following proposition.

11.2 PROPOSITION. *Let k/k_0 be as above. If σ is an irreducible cuspidal automorphic representation of $\mathrm{SL}_n(\mathbb{A}_{k_0})$ such that for some finite place v of k_0 the local component σ_v of σ is the Steinberg representation, then the L-packet $b_{k/k_0}(\sigma)$, contains a representation which is cuspidal and the local component of which at any place above v is the Steinberg representation.*

We may now state and prove the following generalization to SL_n of [LS:6.3] for extensions of the type 11.1(1) in which k_0 is totally real.

11.3 THEOREM. *Let k_0 be a totally real field and let k/k_0 be as in 11.1(1). Then the cuspidal cohomology for SL_n/k over S with trivial coefficients does not vanish.*

Proof. By 6.5 and 10.7 there is a cuspidal automorphic representation σ occurring in the cuspidal spectrum $\mathbf{L}^2_{\mathrm{cusp}}(\mathrm{SL}_n(k_0)\backslash\mathrm{SL}_n(\mathbb{A}_{k_0}))$ which is a Steinberg representation at $v \in S_f$ and the infinite component σ_∞ of which has non-trivial differentiable cohomology. On the other hand the base change lift for SL_n preserves the non-vanishing of differentiable cohomology: at finite places this is clear since Steinberg representations lift to Steinberg representations; at infinity it is checked in [LS] Section 5 (a more sophisticated reader might look in [J]). Therefore the base change lift b_{k/k_0} provides us with an L-packet which contains cuspidal automorphic representations that contribute non-trivially to the cuspidal cohomology.

11.4. For more general groups a proof of the existence of cyclic base change, at least for cuspidal representations that are Steinberg representations at some finite places, should be soon available. Using [J] one would be able to extend Theorem 11.3 to all simple split groups over a tower of cyclic extensions of a totally real number field. The use of some other Langlands functoriality (non-cyclic base change; comparison with inner forms...) should produce other non-vanishing results (as in [LS]). In particular it seems likely that if G is a split simple group over any global field k, and if $E = \mathbb{C}$, the cuspidal cohomology over S does not vanish.

Appendix

A.1. Let H^+ be a locally compact topological group with a closed invariant subgroup H, such that H^+/H is compact abelian. We consider a category of continuous representations in complex vector spaces of H and similarly one for H^+, called *good* representations, for which Schur's lemma is valid (e.g. unitary representations or admissible representations of $\mathfrak{p}$-adic groups) and compatible with the restriction from H^+ to H. Let π be a good irreducible representation of H^+ and let $X(\pi)$ be the group of characters χ of the abelian group H^+/H such that $\pi \otimes \chi \simeq \pi$. For any character χ of H^+/H we choose an intertwining operator I_χ between π and $\pi \otimes \chi$; if $\chi \in X(\pi)$ we may and will choose I_χ invertible, otherwise define I_χ to be zero. Of course I_χ is a self-intertwining operator for the restriction σ of π to H, and the various I_χ for $\chi \in X(\pi)$ are linearly independent operators. Now let U be a self-intertwining operator for σ; for any character χ of the abelian group H^+/H we define an intertwining operator between π and $\pi \otimes \chi$:

$$U_\chi = \int_{H^+/H} \chi(x)\pi(x)U\pi(x)^{-1}\,\mathrm{d}\dot{x}.$$

Here $\mathrm{d}\dot{x}$ is the normalized Haar measure on the compact abelian group H^+/H. Since U_χ intertwines π and $\pi \otimes \chi$ and π is irreducible, there are scalars c_χ such that $U_\chi = c_\chi I_\chi$.

A.2 LEMMA. *Let σ be the restriction to H of an irreducible representation π of H^+.*

(i) *The representation σ is reducible if and only if there exists a non-trivial character χ of H^+/H such that $\pi \otimes \chi$ is equivalent to π.*
(ii) *Assume that $X(\pi)$ is finite. The algebra $I(\sigma)$ of self-intertwining operators of σ is finite dimensional: its dimension equals the order of the group $X(\pi)$. It is semisimple and the representation σ is a finite direct sum of irreducible representations.*
(iii) *If $X(\pi)$ is cyclic of order ℓ, the algebra $I(\sigma)$ is commutative; the representation σ is a direct sum of ℓ inequivalent irreducible representations.*

Proof. If $X(\pi)$ is of finite order, say ℓ, for each $\chi \in X(\pi)$ the operator I_χ can be chosen such that $I_\chi^\ell = 1$; moreover we have by Fourier inversion,

$$U = \sum_{\chi \in X(\pi)} c_\chi I_\chi.$$

In other words the operators I_χ $(\chi \in X(\pi))$ form a basis of the algebra $I(\sigma)$ of self-intertwining operators of σ. Hence, its dimension equals the order of the group $X(\pi)$; this is the first part of assertion (ii). In particular if $X(\pi)$ is trivial, $I(\sigma)$ is reduced to scalars. This last remark and Schur's lemma imply (i). To prove (iii) observe that the algebra $I(\sigma)$ is generated over $\mathbb{C}$ by I_χ if χ generates the group $X(\pi)$. We still have to conclude the proof of (ii). Denote by $H(\pi)$ the intersection of the kernels of characters $\chi \in X(\pi)$, and note that $X(\pi)$ is a product of finite cyclic groups; by using induction on the number of cyclic factors in the quotient group $H^+/H(\pi)$ and the previous remarks, we see readily that the representation σ is a finite direct sum of irreducible representations and that the algebra $I(\sigma)$ is semisimple.

A.3. Assume now that H^+/H is cyclic of order ℓ. We shall denote by α an element of H^+ whose class modulo H generates the quotient group; in particular α^ℓ belongs to H. Let σ be a good representation of H and σ^α the representation $x \mapsto \sigma(\alpha x \alpha^{-1})$. The representation σ is called α-invariant if σ^α is equivalent to σ , i.e. if there exists an operator A in the space of σ such that

$$\sigma^\alpha(x) = A\,\sigma(x)\,A^{-1}.$$

For example, the restriction σ to H of a representation π of H^+ is α-invariant.

If σ is α-invariant and irreducible, by Schur's lemma, $\sigma(\alpha^{-\ell})A^\ell$ is a scalar. We may choose A such that $A^\ell = \sigma(\alpha^\ell)$ and define an extension π of σ to H^+, by letting for $x \in H$

$$\pi(x\alpha^i) = \sigma(x)A^i.$$

The representation π is uniquely defined by σ up to tensor products with characters χ of the cyclic group H^+/H. Since π restricted to H is irreducible, the above lemma implies that $\pi \otimes \chi$ can be equivalent to π only if $\chi = 1$.

More generally, let r be the smallest integer such that $\sigma^{\alpha^r} \simeq \sigma$; then σ can be extended to a representation π_r of the subgroup H_r generated by H and α^r; then π_r, induced from H_r to H^+, yields an irreducible representation π of H^+. We have $\pi \otimes \chi \simeq \pi$ if and only if $\chi(\alpha)$ is a root of unity of order r.

The following corollary is elementary but very useful. We say that the character of π exists as a distribution if $\phi \mapsto \operatorname{trace} \pi(\phi)$ is a continuous linear form on $C_c^\infty(H^+)$.

A.4 COROLLARY. *Assume that the character of π exists as a distribution. If π is irreducible but its restriction to H is reducible, the character vanishes on the subspace of smooth functions compactly supported on αH.*

Proof. By A.2(i), $X(\pi)$ is non-trivial. For $\chi \in X(\pi)$ we have $\pi \otimes \chi \simeq \pi$ and hence

$$\operatorname{trace} \pi(\phi) = \operatorname{trace}(\chi \otimes \pi)(\phi) = \chi(\alpha) \operatorname{trace} \pi(\phi),$$

hence trace $\pi(\phi)$ vanishes if $\chi(\alpha) \neq 1$.

This applies in particular to the pair $G(\mathbb{A}_k), G(\mathbb{A}_k)^+$ considered in 10.2.

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