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A weak Néron model with applications to p-adic dynamical systems

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1. Introduction and notation

1.1.

Let K be a local field which is complete with respect to a discrete valuation v. Considering a finite morphism ϕ: $V \to V$ on a smooth, projective variety V over K, Call and Silverman introduced the notion of weak Néron model associated to the pair $(V/K, \phi)$ in order to study the canonical local heights defined in [6]. They raised the question of whether or not a weak Néron model always exists. This paper is the attempt to answer their question. It turns out that the answer is closely related to the theory of dynamical systems associated with the given morphism ϕ over non-archimedean fields.

First, let us fix the following data which will be used throughout this paper:

K	a local field, complete with respect to a discrete valuation v;
$\mid\ \mid_v$	the absolute value induced by v;
$\mathcal{O}_v$	the ring of integers $= \{x \in K : v(x) \geqslant 0\}$;
$\mathcal{M}$	the maximal ideal of $\mathcal{O}_v = \{x \in K : v(x) > 0\}$;
π	a uniformizer of $\mathcal{M}$;
k	the residue field of $\mathcal{O}_v = \mathcal{O}_v/\mathcal{M}$. We will assume the residue field is algebraically closed;
$\bar{K}$	an algebraic closure of K;
$\mathbf{C}_v$	the completion of $\bar{K}$ with respect to an extension of v.

For the convenience of readers, we repeat the definition of weak Néron model given in [6]. There is an alternative notion of weak Néron model, see [2].

Let $S = \operatorname{Spec}(\mathcal{O}_v)$ and V/K be a smooth variety over K. Let $\phi : V/K \to V/K$ be a finite morphism over K. We will write $(V/K, \phi)$ to denote a given morphism on a variety described as above.

DEFINITION. An S-scheme $\mathcal{V}$ is called a *weak Néron model* of $(V/K, \phi)$ if it is smooth, separated and of finite type over S and if there exists a finite morphism $\Phi\colon \mathcal{V} \longrightarrow \mathcal{V}$ over S so that the following axioms hold :

(1) The generic fibre $\mathcal{V}_K$ of $\mathcal{V}$ is isomorphic to V over K.
(2) $V(K) \cong \mathcal{V}(\mathcal{O}_v)$.
(3) The restriction of the morphism Φ to the generic fibre of $\mathcal{V}$, denoted Φ_K, is ϕ.

EXAMPLE 1.1. If V is an abelian variety and ϕ is an endomorphism of V with finite kernel, then $(V/K, \phi)$ has a weak Néron model, namely, the Néron model of the abelian variety V. Of course, a Néron model satisfies the property called the universal mapping property which is stronger than axiom (3) in our definition.

EXAMPLE 1.2. Another example of weak Néron model which does not come from Néron model can be given as follows:

Let $\phi(z) = f(z)/g(z)$, where $f(z)$ and $g(z)$ are two polynomials in the polynomial ring $K[z]$. If $f(z)$ and $g(z)$ are coprime, then it is clear that $\phi(z)$ defines a morphism over K from $\mathbf{P}^1$ to itself. Moreover, by multiplying elements in $\mathcal{O}_v$, one can assume both $f(z)$ and $g(z)$ are in $\mathcal{O}_v[z]$. If the resultant of $f(z)$ and $g(z)$ is a unit of $\mathcal{O}_v$, then $\phi(z)$ extends to a finite morphism over S from $\mathbf{P}^1$ to itself. Therefore $\mathbf{P}^1/S$ gives a weak Néron model of the pair $(\mathbf{P}^1/K, \phi)$.

We prove our main results in Section 3. The first result (Theorem 3.1) is to give a necessary condition for the existence of a weak Néron model. We consider the rational n-periodic points $p \in V(K)$. By n-periodic point we mean a point $p \in V(\bar{K})$ such that $\phi^n(p) = p$ where ϕ^n denotes the nth iterate of ϕ. If $(V/K, \phi)$ has a weak Néron model, then it's necessary that the linear map $(\phi^n)^*$ on the cotangent space at the point p has all its eigenvalues integral over $\mathcal{O}_v$. Since there exist morphisms on $\mathbf{P}^1$ having repelling periodic points (see the remark in Section 3.1), one can not expect a weak Néron model always to exist. Therefore we consider the case that $(V/K, \phi)$ does not have a weak Néron model and proceed as follows:

Assume that $(V/K, \phi)$ does not have a weak Néron model and let X be an S-scheme with generic fibre V. We assume that every K-rational point of V extends to a unique section on the smooth locus $(X)_{\text{smooth}}$ of X. This is possible by the process of smoothening, see [2, Chapter 3]. The map ϕ extends at least to an S-rational map

$$\varphi\colon (X)_{\text{smooth}} \dashrightarrow (X)_{\text{smooth}}$$

on $(X)_{\text{smooth}}$. By blowing up some closed subscheme of the special fibre of X, φ extends to an S-morphism from another scheme X' to X. We test ϕ on the new scheme X'. Again, ϕ extends to an S-rational map from $(X')_{\text{smooth}}$ to itself. We continue this process inductively. We see that either ϕ extends to an S-morphism Φ_i on some scheme $(X_i)_{\text{smooth}}$ or one needs to repeat the process of blowing-up infinitely many times and gets a family of schemes $\{X_i\}_{i=0}^{\infty}$ and a family of S-rational maps $\{\varphi_i\}_{i=0}^{\infty}$, where φ_i is the S-rational maps on X_i represented by ϕ. In the first case, we let $X_j = X_{j+1} = \cdots$ for j large enough. We define the set $F_\phi(K)$ consisting of $p \in V(K)$ such that the section P extending p is contained in the domain of φ_m on schemes X_m for m large enough. We have the result that the family of morphisms $\{\phi^i\}_{i=0}^{\infty}$ is equicontinuous on the set $F_\phi(K)$ (Proposition 3.2). Here, we consider V as a rigid analytic space and ϕ as an analytic map. In the case that V is a smooth projective curve, $F_\phi(K)$ is exactly the set where $\{\phi^i\}_{i=0}^{\infty}$ is equicontinuous (Theorem 3.3).

It is Theorem 3.3 that relates the problem of weak Néron models to the theory of dynamical systems over a non-archimedean field. In Section 4, we restrict ourselves to the case that $V = \mathbf{P}^1$ and give some applications to p-adic dynamical systems. First, we consider the dual graphs of the special fibres of the family of schemes $\{X_i\}_{i=0}^{\infty}$. Since the generic fibre of X_i is $\mathbf{P}^1$, the dual graph of its special fibre is a finite tree, denoted by $T_{\phi,i}$. Corresponding to the family of schemes $\{X_i\}_{i=0}^{\infty}$ is a family of finite trees $\{T_{\phi,i}\}_{i=0}^{\infty}$. The family of trees $\{T_{\phi,i}\}_{i=0}^{\infty}$ becomes a direct system over non-negative integers via the injective maps $\mu_{ij} : T_{\phi,i} \hookrightarrow T_{\phi,j}$ corresponding to the strict transformation of the blowing-ups $P_{ij} : X_j \to X_i$ for $i \leqslant j$. Let $T_{\phi,K}$ denote the direct limit of $\{T_{\phi,i}\}_{i=0}^{\infty}$. As an equivalent statement of Theorem 3.3, we show that its boundary $\partial T_{\phi,K}$ corresponds to the complement of $F_\phi(K)$ which we denote by $J_\phi(K)$ (Theorem 4.3). Borrowing the terminology from the theory of dynamical systems, we call $F_\phi(K)$ the rational Fatou set and $J_\phi(K)$ the rational Julia set.

Theorem 4.3 can be used to study the property of the rational Julia set of the dynamical systems over a non-archimedean field. We are able to show that the rational Julia set is compact in $\mathbf{P}^1(K)$ in the case that ϕ is a polynomial map on $\mathbf{P}^1$ (Theorem 4.8). Note that in our case the result does not follow automatically, since $\mathbf{P}^1(K)$ is not compact in our case.

In the last paragraph, we give two examples to illustrate the general theory. The first one is a well known example in complex dynamical systems which gives that the Julia set is the whole space $\mathbf{P}^1(\mathbf{C})$. However, in our situation, we show that the same map has empty Julia set when one consider the dynamical systems being over a non-archimedean field. In the second example, we give a map with Julia set contained in the integers $\mathcal{O}_F$ of some finite extension F over $\mathbf{Q}_p$. Moreover, we show the dynamics of ϕ on $J_\phi(F)$ is symbolic dynamics. For the definition of symbolic dynamics, see [7] or Example 4.11.

1.2.

We will fix V to be a smooth, projective variety over K and the pair $(V/K, \phi)$ is as described above. For the convenience of discussion, we will say a smooth scheme has the *e.e.p.* (abbreviation for extending the étale points) property if it satisfies axioms (1) and (2) in the definition of weak Néron model. As pointed out in [4], a flat scheme over the spectrum of a discrete valuation ring is determined by its generic fibre and its formal completion along its special fibre. We will follow this principle to study the weak Néron model and describe the obstruction to its existence in terms of the rigid analytic structure on the analytification of the variety V.

In the following, we use letters $X, Y, Z, \ldots$, to denote S-schemes and $\hat{X}, \hat{Y}, \hat{Z}, \ldots$, to denote their formal completion along their special fibres. We will be studying the formal analytic varieties associated to these formal schemes. We denote the formal analytic varieties associated to these formal schemes by $\hat{X}_K, \hat{Y}_K, \hat{Z}_K, \ldots$. For the definition and detailed discussion of formal analytic varieties and their relationship with formal schemes, see [3], [4] and [5].

There is a natural chordal distance (v-adic distance) function defined on the projective n-space $\mathbf{P}^n$. We use $\| \; , \; \|_v$ to denote the chordal distance on $\mathbf{P}^n$. If V is embedded into $\mathbf{P}^n$, we use $\| \; , \; \|_V$ to denote the chordal distance on V induced by the embedding. We will use the following notations:

$$\begin{aligned}
\mathbf{B}(p, \pi^r) &= \{x \in \mathbf{P}^n_{\bar{K}} \mid \|x, p\|_v \leqslant |\pi^r|_v\}, \\
\mathbf{B}_+(p, \pi^r) &= \{x \in \mathbf{P}^n_{\bar{K}} \mid \|x, p\|_v < |\pi^r|_v\}, \\
V(p, \pi^r) &= \{x \in V(\bar{K}) \mid \|x, p\|_V \leqslant |\pi^r|_v\} \quad \text{and} \\
(V)_+(p, \pi^r) &= \{x \in V(\bar{K}) \mid \|x, p\|_V < |\pi^r|_v\}.
\end{aligned}$$

Let X be a scheme of finite type over the base S and let Y_k be a closed subscheme of the special fibre of X. Assume that Y_k is given the induced reduced structure defined by the sheaf of ideals I_{Y_k}. Let $\tilde{X}$ be the blowing-up with respect to I_{Y_k} on X. Following [2], we call the open subscheme $\widetilde{X_{\pi, Y_k}}$ of $\tilde{X}$ the *dilation* of Y_k in X, where

$$\widetilde{X_{\pi, Y_k}} = \{x \in \tilde{X} : \text{ the sheaf of ideals } I_{Y_k} \cdot O_{\tilde{X}, x} \text{ is generated by } \pi\}.$$

We use the following notation to express how an S-rational map

$$\varphi\colon X \dashrightarrow Y$$

extends to a morphism on a third scheme Z.

DEFINITION. Given schemes X, Y, Z and an S-rational map

$$\varphi\colon X \dashrightarrow Y,$$

assume that there are S-morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ so that when f restricts to $f^{-1}(\mathrm{dom}(\varphi))$, the following diagram commutes.

$$\begin{tikzcd}
f^{-1}(\mathrm{dom}(\varphi)) \arrow[d, "f"'] \arrow[dr, "g"] & \\
\mathrm{dom}(\varphi) \arrow[r, "\varphi"] & Y
\end{tikzcd}$$

Then we will say the following diagram commutes.

$$
\begin{tikzcd}
Z \arrow[d, "f"'] \arrow[dr, "g"] & \\
X \arrow[r, dashed, "\varphi"] & Y
\end{tikzcd}
$$

2. Preliminaries

Let $X \subset \mathbf{P}^n_{\mathcal{O}_v}$ be a quasi-projective S-scheme and let $p \in X_K(K)$ be a rational point. Assume that p extends to a section P on X. Let $\bar{p}$ denote the closed point where P meets X_k. We see that p extends to a section P_1 on the dilation $\widetilde{X}_{\pi,\bar{p}}$ by the universal property of dilation [2, Prop. 3.2.1(b)]. We still denote the closed point where P_1 meets with the special fibre of $\widetilde{X}_{\pi,\bar{p}}$ by $\bar{p}$. We will need to blow up $\bar{p}$ consecutively. We let $\widetilde{X}^{(l)}_{\pi,\bar{p}}$ denote the l-th dilation of $\bar{p}$.

LEMMA 2.1. *Let $X \subset \mathbf{P}^n_{\mathcal{O}_v}$ be a quasi-projective S-scheme having a reduced special fibre. Let $p \in X_K(K)$ be a rational point and let $P \subseteq X$ be an S-section that extends p. Then the formal analytic variety $(\hat{X}^{(l)}_{\pi,\bar{p}})_K$ is isomorphic to $X_K(p, \pi^l)$.*

Proof. Since X is quasi-projective, the dilation of $\bar{p}$ on X can be realized as a subscheme of the dilation of $\bar{p}$ on $\mathbf{P}^n_{\mathcal{O}_v}$. It is enough to show the proposition in the case that $X = \mathbf{P}^n_{\mathcal{O}_v}$. Let $p = [x_0, \ldots, x_n]$ be a point of $\mathbf{P}^n_K$ and let P be the section extending p on $\mathbf{P}^n_{\mathcal{O}_v}$. Without loss of generality, we may assume that $x_0 = 1$ and $x_i \in \mathcal{O}_v$ for $i = 0, \ldots, n$. That is, p is in the affine patch $\mathbf{A}^n_0$ of $\mathbf{P}^n_{\mathcal{O}_v}$. The formal analytic variety associated with the formal completion of $\mathbf{A}^n_0$, is isomorphic to $SpfK\langle z_1, \ldots, z_n\rangle$. The chordal metric is just the norm on the unit ball $SpfK\langle z_1, \ldots, z_n\rangle$ induced from $\mathbf{C}^n_v$. The formal analytic variety associated with the dilation $\mathbf{P}^n_{\mathcal{O}_v,\pi}$ of $\bar{p}$ on $\mathbf{P}^n_{\mathcal{O}_v}$ is isomorphic to $\mathbf{B}(p,\pi) \simeq SpfK\langle(z_1 - x_1)/\pi, \ldots, (z_n - x_n)/\pi\rangle$.

By induction, $(\widehat{\mathbf{P}}^{n(l)}_{\mathcal{O}_v,\bar{p}})_K$ is isomorphic to

$$\mathbf{B}(p, \pi^l) \simeq SpfK\langle(z_1 - x_1)/\pi^l, \ldots, (z_n - x_n)/\pi^l\rangle.$$

This proves the proposition. $\square$

LEMMA 2.2. *Let X, Y be S-schemes locally of finite type and flat over S having reduced special fibres X_k, Y_k. Let $\phi : X_K \to Y_K$ be a K-morphism between the generic fibres and let*

$$\varphi : X \dashrightarrow Y$$

be the rational map represented by ϕ.

Let $\alpha \in X_K(K)$ *be a point such that*

$$\|\phi(z), \phi(\alpha)\|_{Y_K} < 1 \quad \text{for all} \quad z \in X_K(\bar{K}) \quad \text{with} \quad \|z, \alpha\|_{X_K} \leqslant |\pi^l|_v.$$

Assume that α *extends to a section* $Sec(\alpha)$ *on* X, *and consider the* l*-th dilation* $X^{(l)}_{\pi,\bar{\alpha}}$ *of* $\bar{\alpha}$, *then we have the following commutative diagram:*

$$\begin{tikzcd}
X^{(l)}_{\pi,\bar{\alpha}} \arrow[d, "P_\alpha"'] \arrow[dr, "\tilde{\varphi}"] & \\
X \arrow[r, dashed, "\varphi"] & Y
\end{tikzcd}$$

where P_α *is the* l*-th dilation* $P_\alpha \colon X^{(l)}_{\pi,\bar{\alpha}} \to X$ *and* $\tilde{\varphi}$ *is a morphism representing the rational map* $\varphi \circ P_\alpha$.

Proof. Let $U = Spf B$ be a formal neighborhood of $(Y_K)_+(\phi(\alpha), 1)$, where B is an K-affinoid algebra.

By Lemma 2.1, the formal analytic variety associated with the l-th dilation $X^{(l)}_{\pi,\bar{\alpha}}$ is $X_K(\alpha, \pi^l)$. Since, on the generic fibre, we have the K-morphism $\phi : X_K \to Y_K$ and its analytification $\phi^{\text{an}} \colon \hat{X}_K \to \hat{Y}_K$. The assumption on α implies that

$$\phi^{\text{an}} \colon (\hat{X}^{(l)}_{\pi,\bar{\alpha}})_K \to SpB$$

is an analytic map between affinoid varieties. It therefore gives rise to an analytic map of formal analytic varieties.

By [4, Prop. 1.3], we conclude that ϕ extends to an S-morphism

$$\tilde{\varphi} \colon X^{(l)}_{\pi,\bar{\alpha}} \to Y. \qquad \square$$

LEMMA 2.3. *Let* X, Y *be smooth* S*-schemes of finite type over* S *and having reduced special fibres. Assume that both* X *and* Y *satisfy the e.e.p. property and* X_K, Y_K *are smooth and projective over* K *and let* $\|,\|_{X_K}, \|,\|_{Y_K}$ *denote the chordal metric on* X_K *and* Y_K *respectively. Let* ϕ *be an* S*-morphism from* X *to* Y. *Then, for any* $\alpha \in X_K(K)$, *we have:*

$$\|\phi(z), \phi(\alpha)\|_{Y_K} \leqslant \|z, \alpha\|_{X_K}$$

for any $z \in X_K(\bar{K})$.

Proof. If $\|z, \alpha\|_{X_K} = 1$, the assertion is trivial since the chordal metric is bounded by 1. Therefore, we may assume that $\|z, \alpha\|_{X_K} < 1$. Since X satisfies the *e.e.p.* property, α extends to a section over S, denoted by Sec(α). We see that z also extends to a section over S and meets with Sec(α) at the same point on the special fibre of X. Let Sec(z) denote the section extending z. By making a finite extension L of K and replacing π by τ^e, where τ is a uniformizer of $\mathcal{O}_L$, we may assume that z is rational over K and $\|z,\alpha\|_{X_K} = |\pi^l|_v$ for some positive integer l. Let $\bar{z},\bar{\alpha}$ denote the points where $\mathrm{Sec}(z),\mathrm{Sec}(\alpha)$ meet with the special fibre of X.

We will prove the inequality between chordal metrics by induction on l.

(i) $l=0$: as explained above, the assertion is true,

(ii) assume that the proposition is true for all integers less than or equal to l, with $l \geqslant 0$.

(iii) suppose that $\|z,\alpha\|_{X_K} = |\pi^{l+1}|_v$:
Since $\|z,\alpha\|_{X_K} \leqslant |\pi^l|_v$, we have $z \in X_K(\alpha,\pi^l)$. It follows from the hypothesis of the induction that

$$\|\phi(x),\phi(\alpha)\|_{Y_K} \leqslant \|x,\alpha\|_{X_K} \leqslant |\pi^l|_v \quad \text{for all} \quad x \in X_K(\alpha,\pi^l).$$

By Lemma 2.1, we have that $X_K(\alpha,\pi^l) \simeq (\hat{X}^{(l)}_{\pi,\bar{\alpha}})_K$ and the image under ϕ is contained in $Y_K(\phi(\alpha),\pi^l) \simeq (\hat{Y}^{(l)}_{\pi,\bar{\phi}(\alpha)})_K$. The K-analytic map $\phi^{\mathrm{an}} : X_K^{\mathrm{an}} \to Y_K^{\mathrm{an}}$ induces an analytic map on the formal analytic varieties:

$$(\phi^{(l)})^{\mathrm{an}} : (\hat{X}^{(l)}_{\pi,\bar{\alpha}})_K \to (\hat{Y}^{(l)}_{\pi,\phi(\bar{\alpha})})_K.$$

It follows from [4, Prop. 1.3], the K-morphism $\phi : X_K \to Y_K$ extends uniquely to an S-morphism $\Phi^{(l)} \colon X^{(l)}_{\pi,\bar{\alpha}} \to Y^{(l)}_{\pi,\phi(\bar{\alpha})}$. Let $X^{(l+1)}_{\pi,\bar{\alpha}}$ be the dilation of $\bar{\alpha}$ on $X^{(l)}_{\pi,\bar{\alpha}}$ and consider the following commutative diagram.

$$\begin{tikzcd}
X^{(l+1)}_{\pi,\bar{\alpha}} \arrow[d, "P_{l+1}"'] \arrow[dr, "\widetilde{\Phi^{(l)}}"] & \\
X^{(l)}_{\pi,\bar{\alpha}} \arrow[r, "\Phi^{(l)}"] & Y^{(l)}_{\pi,\phi(\bar{\alpha})}
\end{tikzcd}$$

Since $\|z,\alpha\|_{X_K} = |\pi^{l+1}|_v$, $\bar{z}=\bar{\alpha}$ on $X^{(l)}_{\pi,\bar{\alpha}}$, $\mathrm{Sec}(z)$ factors through $X^{(l+1)}_{\pi,\bar{\alpha}}$ by the universal property of dilation. On the other hand, $\widetilde{\Phi^{(l)}}$ is an S-morphism and $\widetilde{\Phi^{(l)}} = \Phi^{(l)} \circ P_{l+1}$, we see that on special fibres, $(\widetilde{\Phi^{(l)}})_k$ factors through the point $\phi(\bar{\alpha})$. Since the scheme $X^{(l+1)}_{\pi,\bar{\alpha}}$ is flat over S, it follows that $\widetilde{\Phi^{(l)}}$ factors through $Y^{(l+1)}_{\pi,\phi(\bar{\alpha})}$ by the universal property of dilation, where $Y^{(l+1)}_{\pi,\phi(\bar{\alpha})}$ denotes the dilation of $\phi(\bar{\alpha})$ on $Y^{(l)}_{\pi,\phi(\bar{\alpha})}$.

By Lemma 2.1, we conclude that $\|\phi(z), \phi(\alpha)\|_{Y_K} \leqslant |\pi^{l+1}|_v = \|z, \alpha\|_{X_K}$. This establishes the induction steps and proves the lemma. □

3. Main results

In this section, we will prove our main results. One of the results gives a necessary condition for a pair $(V/K, \phi)$ to have a weak Néron model; the other one gives a partial converse to the necessary condition in the case that V is a smooth projective curve over K.

3.1. A NECESSARY CONDITION

We begin with the following definition:

DEFINITION. A point $p \in V(K)$ is called a rational periodic point associated with ϕ if $\phi^n(p) = p$ for some positive integer n, where $\phi^n = \phi \circ \phi \cdots \circ \phi$ denotes the n-th iterate of ϕ. The set consisting of all the rational periodic points is denoted by $\mathrm{Per}_\phi(K)$.

THEOREM 3.1. *Let V be a smooth variety over K and let $\phi\colon V/K \to V/K$ be a finite morphism. Assume that $\mathrm{Per}_\phi(K)$ is non-empty and let $p \in \mathrm{Per}_\phi(K)$ such that $\phi^n(p) = p$. If $(V/K, \phi)$ has a weak Néron model, then all the eigenvalues of the linear map*

$$(\phi^n)^*\colon \Omega_{V/K,p} \to \Omega_{V/K,p}$$

on the cotangent space at p are integral over $\mathcal{O}_v$.

Proof. Let $\mathcal{V}/S$ be a weak Néron model for $(V/K, \phi)$ and let

$$\Phi\colon \mathcal{V} \to \mathcal{V}$$

be the morphism such that $\Phi_K = \phi$ on $\mathcal{V}_K \simeq V$. Let $\Omega_{\mathcal{V}/S}$ be the sheaf of relative differentials of $\mathcal{V}$ over S.

Since $\mathcal{V}$ is a weak Néron model for $(V/K, \phi)$, it follows that the rational point p extends to a section

$$P\colon S \to \mathcal{V}$$

over S by the *e.e.p.* property of weak Néron model. Moreover, we have $\Phi^n \circ P = P$ since $\phi^n(p) = p$. It follows

$$\begin{aligned} P^* &= (\Phi^n \circ P)^* && (1)\\ &= P^* \circ (\Phi^n)^*. && (2) \end{aligned}$$

Note that P^* gives a functor from the category of coherent sheaves on $\mathcal{V}$ to the category of coherent sheaves on S, see [10, Prop. II.5.8(b)]. In particular, $P^*\Omega_{\mathcal{V}/S}$ is a coherent sheaf on S. Let $M = \Gamma(S, P^*\Omega_{\mathcal{V}/S})$, then M is a finitely generated free $\mathcal{O}_v$-module, since $\mathcal{V}$ is smooth over S.

On the other hand, $(\Phi^n)^*$ induces a morphism

$$\Omega_{\mathcal{V}/S} \to (\Phi^n)^*\Omega_{\mathcal{V}/S}$$

of sheaves. Applying the functor P^*, we get a map

$$P^*\Omega_{\mathcal{V}/S} \to P^*(\Phi^n)^*\Omega_{\mathcal{V}/S}.$$

By equation (1) and (2), P^* and $P^* \circ (\Phi^n)^*$ are the same functor, hence

$$P^*\Omega_{\mathcal{V}/S} = P^* \circ (\Phi^n)^*\Omega_{\mathcal{V}/S}.$$

By taking the functor $\Gamma(S, \cdot)$, we get a $\mathcal{O}_v$-homomorphism of $\mathcal{O}_v$-modules. Namely,

$$\Gamma(S, P^*\Omega_{\mathcal{V}/S}) \to \Gamma(S, P^*(\Phi^n)^*\Omega_{\mathcal{V}/S}).$$

By definition, $\Omega_{V/K,p}$ is just the vector space obtained by tensoring M with K over $\mathcal{O}_v$ and $(\phi^n)^*$ is just the map gotten from the above $\mathcal{O}_v$-homomorphism by tensoring with K.

Since M is a free $\mathcal{O}_v$-module and $(\phi^n)^*$ restricted to M gives an endomorphism on M, a standard determinant argument shows that the eigenvalues of $(\phi^n)^*$ are integral over $\mathcal{O}_v$. $\square$

REMARK. This theorem shows that there are many counterexamples to the existence of a weak Néron model. For example, let $\phi(z) = z^2 + z - 1/\pi^2$ be a morphism on $\mathbf{P}^1$ over K. Then, $\phi(1/\pi) = 1/\pi$ and $\phi'(1/\pi) = 2/\pi + 1$ which is not integral over $\mathcal{O}_v$, provided $v(2) = 0$. By the above theorem, $(\mathbf{P}^1/K, \phi)$ does not have a weak Néron model.

3.2. SEQUENCE OF BLOWING-UPS

From Theorem 3.1, we know that one cannot expect a weak Néron model always to exist. A natural question to ask then is to find the obstruction to the existence of a weak Néron model. It turns out the obstruction is closely related to a set called the *Julia set* defined in the theory of dynamical systems associated with the given morphism ϕ.

Since V/K is projective, there is a (smooth) closed embedding $V \hookrightarrow \mathbf{P}_K^N$ into some projective N-space over K. Let $X \subseteq \mathbf{P}_{\mathcal{O}_v}^N$ be the schematic closure of V in $\mathbf{P}_{\mathcal{O}_v}^N$. In general, one can not expect X to be smooth. However, due to the smoothening process, we may assume that the smooth locus X_{smooth} has the *e.e.p.* property.

The morphism $\phi\colon V/K \to V/K$ extends at least to an S-rational map

$$\varphi\colon X \dashrightarrow X$$

over S. Restricting φ to the open subscheme X_{smooth} of X, if φ induces a finite morphism $\Phi : X_{\text{smooth}} \to X_{\text{smooth}}$ over S then X_{smooth} can be taken to be a weak Néron model for $(V/K, \phi)$.

We assume that $(V/K, \phi)$ does not have a weak Néron model. In the following we give an algorithm consisting of sequences of blowing-ups. Starting with $X_0 = X$, let $i \geqslant 0$ and assume that we have X_i and an S-rational map

$$\varphi_i\colon X_i \dashrightarrow X_i$$

such that:

(a) $(X_i)_{\text{smooth}}$ satisfies the *e.e.p.* property.
(b) φ_i is the rational map represented by $\phi \simeq (\varphi_i)_K$.

It is well known that one can eliminate the points of indeterminacy of a rational map by blowing up a coherent sheaf of ideals determined by the points of indeterminacy (see [10, II.7]). Applying the smoothening process, we may assume that there exists a morphism

$$P_{i+1} : X_{i+1} \to X_i$$

such that:

(1) P_{i+1} is a composition of blowing-ups and $(X_{i+1})_{\text{smooth}}$ satisfies the *e.e.p.* property.
(2) There is a unique morphism

$$\widetilde{\varphi_i} : (X_{i+1})_{\text{smooth}} \to (X_i)_{\text{smooth}}$$

which represents the rational map $\varphi_i \circ P_{i+1}$ from $(X_{i+1})_{\text{smooth}}$ to $(X_i)_{\text{smooth}}$. That is, in our notation, the following diagram commutes.

$$\begin{tikzcd}
(X_{i+1})_{\text{smooth}} \arrow[d, "P_{i+1}"'] \arrow[dr, "\widetilde{\varphi_i}"] & \\
(X_i)_{\text{smooth}} \arrow[r, dashed, "\varphi_i"] & (X_i)_{\text{smooth}}
\end{tikzcd}$$

Let

$$\varphi_{i+1}\colon X_{i+1} - - \to X_{i+1}$$

be the rational map represented by the morphism

$$\phi \simeq (\varphi_{i+1})_K\colon (X_{i+1})_K \to (X_{i+1})_K$$

on the generic fibre of X_{i+1}.

If $\varphi_{i+1}|_{(X_{i+1})_{\text{smooth}}}$ is a morphism from $(X_{i+1})_{\text{smooth}}$ to $(X_{i+1})_{\text{smooth}}$, then we stop the above process and let

$$X_{i+1} = X_{i+2} = \cdots = X_m \quad \text{for all} \quad m \geqslant i+1.$$

Note that, in this case we do not require $\varphi_{i+1}|_{(X_{i+1})_{\text{smooth}}}$ to be a finite morphism. If $\varphi_{i+1}|_{(X_{i+1})_{\text{smooth}}}$ is not a morphism, then repeat the process.

Inductively, we get a sequence of schemes $\{X_i\}_{i=0}^{\infty}$ satisfying (1) and (2). For any $\alpha \in V(K)$, we let $\mathrm{Sec}(\alpha)_i$ denote the section extending α on X_i. This is possible since X_i satisfies the *e.e.p.* property. We denote the point where $\mathrm{Sec}(\alpha)_i$ meets with $(X_i)_k$ by $(\bar{\alpha})_i$. If no confusion will arise, we'll drop the subscript i.

We define the following set:

DEFINITION. Let $F_\phi(K) \subseteq V(K)$ be a subset of $V(K)$ so that:

For any $\alpha \in F_\phi(K)$, there exists an integer N_α such that $\mathrm{Sec}(\alpha)_i \in \mathrm{dom}(\varphi_i)$ for all $i \geqslant N_\alpha$.

3.3. Properties of $F_\phi(K)$

Consider the metric $\|\,,\,\|_V$ induced by the embedding $V \hookrightarrow \mathbf{P}_K^N$, as described in 1.2. Our goal is to describe the set $F_\phi(K)$ in terms of the behavior of the iterates of ϕ with respect to the chordal metric on the variety V.

PROPOSITION 3.2. *The family of morphisms $\{\phi^i\}_{i=1}^{\infty}$ is equicontinuous on the set $F_\phi(K)$.*

Proof. Let $\alpha \in F_\phi(K)$, by definition, there exists an integer N such that $\bar{\alpha} \in \mathrm{dom}(\varphi_i)$ for all $i \geqslant N$. Let $\|\,,\,\|_{X_N}$ denote the chordal metric on X_N induced by an embedding of X_N into $\mathbf{P}_{\mathcal{O}_v}^M$ for some integer M, then we have

$$|\pi^r|_v \|z, \alpha\|_{X_N} \leqslant \|z, \alpha\|_{X_0} \leqslant \|z, \alpha\|_{X_N} \quad \text{for some positive integer } r.$$

The right half of the inequality is Lemma 2.3. As for the other half of the inequality, one can deduce it from the fact that there is a birational morphism $X_N \to X_0$ coming from a finite sequence of blowing-up and then apply Lemma 2.1.

We see that $\| \ , \ \|_{X_N}$ is equivalent to $\| \ , \ \|_{X_0}$ and it is enough to prove the proposition in the case that $N = 0$, therefore we assume $N = 0$. Let us consider the following diagram:

$$\begin{tikzcd}
X_i \arrow[d, "P_i"'] \arrow[dr, "\widetilde{\varphi_{i-1}}"] & & & \\
X_{i-1} \arrow[r, dashed, "\varphi_{i-1}"] \arrow[d, "P_{i-1}"'] \arrow[dr] & X_{i-1} \arrow[d] \arrow[dr] & & \\
\vdots & \vdots & \vdots \arrow[dr] & \\
X_0 \arrow[r, dashed, "\varphi_0"] & X_0 \arrow[r, dashed, "\varphi_0"] & \cdots \arrow[r] & X_0
\end{tikzcd}$$

Since $\bar{\alpha} \in \mathrm{dom}(\varphi_i)$ for $i \geqslant 0$, there exists an open neighborhood $U_i \subseteq X_i$ of $\bar{\alpha}$ such that $P_{i+1}^{-1}(U_i) \simeq U_i$. By induction, we conclude that there is an open neighborhood $U_0 \subseteq X_0$ of $\bar{\alpha}$ in X_0 such that $(P_1 \circ P_2 \circ \cdots \circ P_i)^{-1}(U_0) \simeq U_0$. It follows that $\bar{\alpha} \in \mathrm{dom}(\varphi_0^i)$ on X_0. Let V_i denote the open subscheme $\mathrm{dom}(\varphi_0^i)$ of $(X_0)_{\mathrm{smooth}}$ for $i \geqslant 0$. We see that φ_0^i is a morphism from V_i into $(X_0)_{\mathrm{smooth}}$.

Because the domain of an S-rational map is stable under flat base change ([2, Prop. 2.6]), we are free to make any finite extension of K. We may apply Lemma 2.3. By Lemma 2.3, we conclude that $\|\phi^i(z), \phi^i(\alpha)\|_V \leqslant \|z, \alpha\|_V$ for $z \in V_{i,K}(\bar{K})$. This shows that the family of morphisms $\{\phi^i\}_{i=0}^{\infty}$ is equicontinuous at α. Since α is any point of $F_\phi(K)$, $\{\phi^i\}_{i=0}^{\infty}$ is equicontinuous on $F_\phi(K)$. □

In the case that V is a curve over K, we have the following stronger result.

THEOREM 3.3. *Let V be a smooth projective curve over K and let $\phi\colon V \to V$ be a morphism. Then, $\alpha \in F_\phi(K)$ if and only if the family of morphisms $\{\phi^i\}_{i=0}^{\infty}$ is equicontinuous at α.*

We need to prove the following lemma first:

LEMMA 3.4. *Let X, Y, Z be S-schemes such that their generic fibres X_K, Y_K, Z_K are smooth, projective curves over K. Let $\phi\colon X_K \to Y_K$ be a finite morphism, and let*

$$f\colon Z \to X, \qquad g\colon Z \to Y$$

be morphisms over S.

Assume that Z and X are flat over S and the following diagram commutes.

$$\begin{tikzcd} Z \arrow[d, "f"'] \arrow[dr, "g"] & \\ X \arrow[r, dashed, "\phi"] & Y \end{tikzcd}$$

Then, there exists an S-scheme $X'$and a sequence of dilations $P\colon X' \to X$ and $f'\colon Z \to X'$ such that the following diagram is commutative.

$$\begin{tikzcd} & & X \arrow[dd, dashed, "\phi"] \\ Z \arrow[urr, "f"] \arrow[r, "f'"] \arrow[drr, "g"'] & X' \arrow[ur, "P"] \arrow[dr, "\tilde{\phi}"] & \\ & & Y \end{tikzcd}$$

Proof. Since we only need to blow up closed points on X, we may assume that Z_k and X_k are irreducible. The map $f_k\colon Z_k \to X_k$ is either constant or surjective since Z_k and X_k are curves over Spec(k).

Suppose that f_k is constant, let $x = f_k(Z_k)$ be the image of f_k on the special fibre of Z. If $x \in \mathrm{dom}(\varphi)$, then we can take X' such that x is not in the center of the blowing-up and we are done. Therefore, assume $x \notin \mathrm{dom}(\varphi)$, let ${X'}_{\pi,x}$ be the dilation of x in X. Since Z is flat over S and $f_k(Z_k) = x$, f factors through $X'_{\pi,x}$ by the universal property of dilation. Let $f'\colon Z \to {X'}_{\pi,x}$ denote this map.

By considering the rational map

$$\varphi'\colon X'_{\pi,x} \xrightarrow{P_x} X \overset{\varphi}{\dashrightarrow} Y,$$

we have the following commutative diagram

$$\begin{tikzcd} Z \arrow[d, "f'"'] \arrow[dr, "g"] & \\ X'_{\pi,x} \arrow[r, dashed, "\varphi'"] & Y \end{tikzcd}$$

Replacing X by $X'_{\pi,x}$, we can repeat the above argument. Since only finitely many steps of blowing-up are needed, we may therefore assume that

$$f_k \colon Z_k \to X_k$$

is surjective.

Since X_k is a curve over k and f_k is surjective, it follows that f_k is flat. X and Z are flat over S, it follows from [17, IV 5.9] that f is flat.

Since f is surjective, it is faithfully flat. The rational map $g = f \circ \phi$ is an S-morphism by assumption. We conclude that ϕ is an S-morphism ([9, 20.3.11]). There is no need to blow up in this case ($X' = X$). The lemma is proved. □

Proof of Theorem 3.3. The necessary condition is Proposition 3.2.

Assume $\alpha \notin F_\phi(K)$ but $\{\phi^n\}$ is equicontinuous at α. By definition, there exist a sequence of integers $\{n_i\}_{i=1}^{\infty}$ such that $n_i < n_{i+1}$ and $\bar{\alpha}_{n_i}$ is in the center of the blowing-up $p_{n_i+1} \colon X_{n_i+1} \to X_{n_i}$.

Since $\{\phi^n\}$ is equicontinuous at α, given $\epsilon = 1$ there exists a $\delta > 0$ such that $\|\phi^n(z), \phi^n(\alpha)\|_V < 1$ for all $z \in V(\mathbf{C}_v)$ with $\|z, \alpha\|_V < \delta$ and all ϕ^n. By letting δ be smaller, we may assume that $\delta = \pi^l$ for some positive integer l and $\|z, \alpha\|_V \leqslant |\pi^l|_v$.

Let $X_{\bar\alpha}^{(l)}$ be the lth dilation of $\bar\alpha$ on X_0. Note that $\bar z = \bar\alpha$ in $X_{\bar\alpha}^{(l)}$ if and only if $\|z, \alpha\|_v \leqslant |\pi^l|$. Since $\bar\alpha$ is in the center of blowing-up on X_{n_i}, we have $X_{\bar\alpha}^{(l)} \subseteq X_n$ for some n. Let $m \in \{n_i\}$ and $m > n$ and let $P \colon X_m \to X_n$ be the sequence of blowing-ups $X_m \to X_{m+1} \cdots \to X_n$. Let $X_{\bar\alpha}^{(l)*}$ be the inverse image of $X_{\bar\alpha}^{(l)}$ in X_m. We have the following diagram:

$$\begin{tikzcd}
X_{\bar\alpha}^{(l)*} \subseteq X_m \arrow[d, "P_m"'] \arrow[dr, "\widetilde{\varphi_{m-1}}"] & & & \\
X_{m-1} \arrow[r, dashed, "\varphi_{m-1}"] \arrow[d] \arrow[dr] & X_{m-1} \arrow[d] \arrow[dr] & & \\
\vdots & \vdots & \vdots \arrow[dr] & \\
X_0 \arrow[r, dashed, "\varphi_0"] & X_0 \arrow[r, dashed, "\varphi"] & \cdots \arrow[r] & X_0
\end{tikzcd}$$

Let $\widetilde{\Phi_m}$ denote the composition of morphisms $\widetilde{\varphi_{m-1}}, \ldots, \widetilde{\varphi_0}$ and consider the diagram:

$$\begin{tikzcd} X_{\bar{\alpha}}^{(l)*} \subseteq X_m \arrow[d, "\widetilde{\Phi_m}"'] \arrow[dr, dashed, "\widetilde{\Phi_{m+1}}"] & \\ X_0 \arrow[r, dashed, "\varphi_0"] & X_0 \end{tikzcd}$$

where $\widetilde{\Phi_{m+1}}$ denotes the rational map represented by

$$\phi^{m+1}\colon (X_m)_K \to (X_0)_K.$$

Since $\|z, \alpha\| < |\pi^l|_v$ if and only if $\mathrm{Sec}(z) \subseteq X_{\bar{\alpha}}^{(l),*}$, we have $\|\phi^i(z), \phi^i(\alpha)\|_V < 1$ for all z with $\mathrm{Sec}(z) \subseteq X_{\bar{\alpha}}^{(l)*}$ and all i, by the assumption on the equicontinuity of morphisms $\{\phi^i\}$. By Lemma 2.2, $\widetilde{\Phi_{m+1}}$ is a morphism when it restricts to $X_{\bar{\alpha}}^{(l)*}$. Let us consider the following diagram:

$$\begin{tikzcd} X_{\bar{\alpha}}^{(l)*} \arrow[d, "\widetilde{\Phi_m}"'] \arrow[dr, "\widetilde{\Phi_{m+1}}"] & \\ X_0 \arrow[r, dashed, "\varphi_0"] & X_0 \end{tikzcd}$$

Let I_{ϕ_0} be the coherent sheaf of ideals of the blowing-up $P_1\colon X_1 \to X_0$. From the algorithm in 3.2, we see that $\widetilde{\Phi_m}^{-1}(I_{\phi_0})$ is the sheaf of ideals of the blowing-up $P_{m+1} : X_{m+1} \to X_m$. We only need to show that $\widetilde{\Phi_m}^{-1}(I_{\phi_0})$ is invertible on $\widetilde{X_{\bar{\alpha}}^{(l)}}$, but this is clear by Lemma 3.4. Since m is any integer greater than n, this shows that $\bar{\alpha}$ is not in the center of blowing-ups for $m > n$ which contradicts the assumption that $\alpha \notin F_\phi(K)$. This proves the Theorem. □

REMARK. In complex dynamical systems, the set of points where the family of morphisms $\{\phi^n\}_{i=1}^{\infty}$ is equicontinuous is called the *Fatou set*. Borrowing this terminology, we will call $F_\phi(K)$ the *rational Fatou set*. Its complement, $J_\phi(K) \stackrel{\text{def}}{=} V(K) \setminus F_\phi(K)$, is called the *rational Julia set*.

4. Applications

In this section we give some applications of the main theorems to the dynamical systems associated with morphisms on $\mathbf{P}^1$.

Throughout this section, V_K will be $\mathbf{P}^1$ and $\phi(z) = f(z)/g(z)$ will be a non-constant rational map on $\mathbf{P}^1$, where $f(z), g(z)$ are coprime polynomials with coefficients in K. The following terminology is standard in the classical theory of dynamical systems:

DEFINITION. Let $\phi : \mathbf{P}^1_K \to \mathbf{P}^1_K$ be a non-constant morphism and let $p \in \mathrm{Per}^n_\phi$ be a periodic point of period n for some positive integer n. We say p is a repelling periodic point if $|(\phi^n)'(p)|_v > 1$; otherwise it is a non-repelling periodic point.

4.1. THE JULIA SET

By Theorem 3.1, if $(V/K, \phi)$ has a weak Néron model, then the linear map $(\phi^n)^*$ on the cotangent space of a rational periodic point $p \in \mathrm{Per}^n_\phi$ must have integral eigenvalues. The following proposition is just a corollary to Theorem 3.1.

PROPOSITION 4.1. *Let ϕ: $\mathbf{P}^1_K \to \mathbf{P}^1_K$ be a non-constant morphism and let $p \in \mathrm{Per}^n_\phi(K)$ for some n. If $(\mathbf{P}^1_K, \phi)$ has a weak Néron model, then p is a non-repelling periodic point.*

Proof. Since $\mathbf{P}^1$ is smooth and of dimension one over K, the vector space $\Omega_{\mathbf{P}^1/K,p}$ is a one dimensional vector space over K, the linear map $(\phi^n)^*$ is just multiplication by an element c in K. By Theorem 3.1, the linear map

$$(\phi^n)^* \colon \Omega_{\mathbf{P}^1/K,p} \to \Omega_{\mathbf{P}^1/K,p}$$

has all the eigenvalues integral over $\mathcal{O}_v$, therefore, $c \in \mathcal{O}_v$.

It is an elementary fact that $(\phi^n)'(p) = c$, therefore $|(\phi^n)'(p)|_v \leqslant 1$. That is, p is non-repelling. □

4.2. TREES

Assuming that $(\mathbf{P}^1/K, \phi)$ does not have a weak Néron model, we start with $X_0 = \mathbf{P}^1_{\mathcal{O}_v}$ and perform sequences of blowing-ups as described in 3.2. We still denote by $\{X_i\}_{i=0}^\infty$ the family of schemes that satisfy the *e.e.p.* property obtained by blowing-ups. Note that the irreducible components of the special fibres of X_i's are isomorphic to $\mathbf{P}^1_k$ over Spec(k).

DEFINITION. The dual graph of $X_{i,k}$, denoted by $T_{\phi,i}$, is the graph with a vertex for each component of $X_{i,k}$ and an edge between vertices corresponding to intersecting components.

We have therefore a family of finite graphs $\{T_{\phi,i}\}_{i=0}^\infty$ corresponding to $\{X_i\}_{i=0}^\infty$. The family of graphs satisfy the following properties:

PROPOSITION 4.2. (1) *For each i, $T_{\phi,i}$ is a finite tree.*

(2) *For each pair of integers $i, j \geqslant 0$ such that $i \leqslant j$, there is a map of finite graphs $\mu_{ij} : T_{\phi,i} \to T_{\phi,j}$ corresponding to the blowing-up $p_{ij}\colon X_j \to X_i$ such that:*

(a) *μ_{ij} is injective,*
(b) *$\mu_{ii} =$ identity,*
(c) *$\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leqslant j \leqslant k$.*

(3) *The S-rational map*

$$\varphi_i\colon X_i \to X_i$$

represented by the K-morphism ϕ on the generic fibres induces a map

$$\varphi_i^\sharp\colon T_{\phi,i} \to T_{\phi,i}$$

such that $\varphi_i^\sharp$ is compatible with μ_{ij}. More precisely, we have the following commutative diagram:

$$\begin{tikzcd} T_{\phi,i} \arrow[r, "\varphi_i^\sharp"] \arrow[d, "\mu_{ij}"'] & T_{\phi,i} \arrow[d, "\mu_{ij}"] \\ T_{\phi,j} \arrow[r, "\varphi_j^\sharp"] & T_{\phi,j} \end{tikzcd}$$

for all i, j such that $i \leqslant j$.

Proof. (1) To show the graph $T_{\phi,i}$ is a tree, we need to show:

(a) $T_{\phi,i}$ is connected,
(b) $T_{\phi,i}$ does not have loops.

(a) follows from the fact that X_i's are projective over S and $(f_i)_* O_{X_i} = O_S$, where f_i is the structural morphism $f_i\colon X_i \to S$, then the special fibres $X_{i,k}$ are connected, see [10, Cor. III.11.3].

(b) is clear from the construction. Since each irreducible component of $X_{i,k}$ is non-singular, it follows that each component does not intersect with itself. Two components intersect at most at one point since the X_i's are obtained by sequence of blowing-ups.

The finiteness of the graph follows from the fact that the X_i's are of finite type over S.

(2) By construction, for each pair of integers i, j with $i \leqslant j$, we have $P_{ij} : X_j \to X_i$ consisting of sequence of blowing-ups. We define P_{ii} to be the identity map on X_i.

Let $\{E_{i,r}\}_{r=1}^{l}$ be the set of irreducible components of $X_{i,k}$. Consider the strict transformation of $E_{i,r}$ under the birational morphism P_{ij}, denoted by $E_{i,r}^*$. Then

$\{E_{i,r}^*\}_{r=1}^{l}$ is a subset of the irreducible components of $X_{j,k}$. Define:

$$\begin{aligned} \mu_{ij} : T_{\phi,i} &\rightarrow T_{\phi,j} \quad \text{by} \\ e_{i,r} &\mapsto e_{i,r}^*, \end{aligned}$$

where $e_{i,r}(e_{ir}^*)$ are the vertices corresponding to $E_{i,r}(E_{i,r}^*)$. Then, (a) and (b) are clear from the definition of μ_{ij}. (c) follows from the commutativity of the following diagram:

$$\begin{tikzcd} X_k \arrow[d, "P_{ik}"'] \arrow[dr, "P_{jk}"] & \\ X_i & X_j \arrow[l, "P_{ij}"] \end{tikzcd}$$

(3) The S-rational map

$$\varphi_i \colon X_i \dashrightarrow X_i$$

is defined at points of codimension $\leqslant 1$ since X_i is normal. Consider the generic points of the irreducible components of $X_{i,k}$ which are of codimension 1 in X_i, φ_i is defined at these points. It follows that φ_i sends each irreducible component of $X_{i,k}$ into one of the irreducible components. Therefore, φ_i induces a map among vertices of $T_{\phi,i}$. To show that φ_i indeed induces a map $\varphi_i^\sharp : T_{\phi,i} \rightarrow T_{\phi,i}$ of trees. We simply note that for two intersecting components of $X_{i,k}$, φ_i either sends them into the same component or two intersecting components.

The generic points $\zeta_{i,r}$ of $E_{i,r}$ are in the domain of φ_i, therefore $\varphi_j(P_{ij}^{-1}(\zeta_{i,r})) = P_{ij}^{-1}(\varphi_i(\zeta_{i,r}))$. This shows $\varphi_i^\sharp$ is compatible with μ_{ij}. □

By the above proposition, $\{T_{\phi,\mu_{ij}}\}$ is a direct system over integers $\mathbf{N}\bigcup\{0\}$. Let $T_{\phi,K} = \varinjlim_{i} T_{\phi,i}$ denote the direct limit of graphs. It is obvious that $T_{\phi,K}$ is a finite graph if the process of blowing-up stops after a finite stage; otherwise, it is a infinite graph. It is also clear that $T_{\phi,K}$ is a tree.

By (3) of the above proposition, we see that the K-morphism $\phi\colon \mathbf{P}_K^1 \rightarrow \mathbf{P}_K^1$ induces a map

$$\phi^\sharp : T_{\phi,K} \longrightarrow T_{\phi,K}.$$

4.3. Tree and the Julia set

A tree T can be viewed as a metric space by defining the distance $d(e, e')$ between two vertices e, e' to be the infimum of the number of edges connecting e and e'.

Fixing a vertex e, we can define the following sets:

$$\begin{aligned} T_n &= \{e' \in T \mid d(e,e') \leqslant n\} \quad \text{and} \\ \lambda_n : \ & T_n \to T_n \quad \text{by} \\ \lambda_n(e) &= \text{ the only vertex in } T_{n-1} \text{ adjacent to } e \text{ in } T. \end{aligned}$$

It is easy to see $\{T_n, \lambda_n\}$ form an inverse system of trees. Its inverse limit, denoted by

$$\partial T = \varprojlim_n T_n.$$

∂T consists of half lines of T (or *ends* of T). It is a topological space. If, furthermore, T is locally finite, then ∂T is compact and totally disconnected, see [16, 2.2], where T is locally finite if for every vertex there are only finitely many edges originated from the vertex.

One can define an injective map

$$\imath\colon \partial T_{\phi,K} \hookrightarrow \mathbf{P}^1_K(K)$$

sending half lines of $T_{\phi,i}$ into $\mathbf{P}^1_K(K)$. $\imath$ is defined as follows:

Let $l \in \partial T_{\phi,K}$ be a half line, then l is represented by an infinite sequence of vertices $\{e = e_0, e_1, \ldots, e_n, \ldots,\}$. We may assume e_0 corresponds to the special fibre of $\mathbf{P}^1_{\mathcal{O}_v}$, then e_1 corresponds to the special fibre of the dilation of a closed point on $\mathbf{P}^1_k$. For arbitrary j, e_{j+1} corresponds to the special fibre of the dilation of a closed point on E_j which corresponds to e_j. Then, $\imath(l) = \cap_i(\hat{E}_i)_K$. Since K is complete, $\imath(l)$ is a point in $\mathbf{P}^1_K(K)$. One can also define $\imath(l)$ in another equivalent way, cf. [13].

As an equivalent statement of Theorem 3.3, we have:

THEOREM 4.3. $\imath(\partial T_{\phi,K}) = J_\phi(K)$.

Proof. If $J_\phi(K) = \emptyset$, then, by definition, $F_\phi(K) = \mathbf{P}^1_K(K)$. We see that for any point $\alpha \in \mathbf{P}^1_K(K)$, $\bar{\alpha}$ is in the center for at most finitely many blowing-ups. This shows $T_{\phi,K}$ is bounded, that is $\partial T_{\phi,K}$ is empty.

Assume $J_\phi(K)$ is non-empty, let $\omega \in J_\phi(K)$. By Theorem 3.3, $\omega \in J_\phi(K)$ if and only if $\bar{\omega}$ is in the center of infinitely many blowing-ups. It is clear from the definition of $T_{\phi,K}$, $\bar{\omega}$ is in the center of infinite blowing-ups if and only if there is a half line $l_\omega \in \partial T_{\phi,K}$ such that $\imath(l_\omega) = \omega$. This completes the proof of the theorem. □

The following theorem which is proved in [15] is useful in our situation.

THEOREM 4.4. *Let $\mathcal{C}$ be a projective curve. Let λ, μ be non-singular embeddings of $\mathcal{C}(\mathbf{C}_v)$ in $\mathbf{P}^N(\mathbf{C}_v), \mathbf{P}^M(\mathbf{C}_v)$ respectively. There exists a positive constant $C = C(\lambda, \mu)$ such that for all $x, y \in \mathcal{C}(\mathbf{C}_v)$,*

$$\left(\frac{1}{C}\right) \|\lambda(x), \lambda(y)\|_v \leqslant \|\mu(x), \mu(y)\|_v \leqslant C\|\lambda(x), \lambda(y)\|_v.$$

Proof. [15, pp. 27–29]. □

The above theorem shows that smooth embeddings into projective spaces give equivalent chordal metrics on a curve. As a result, we see that the equicontinuity of families of morphisms is independent of smooth embeddings.

THEOREM 4.5. *The set $J_\phi(K)$ is independent of the scheme X_0- the first scheme that one starts with in the algorithm of blowing-ups.*

4.4. THE JULIA SETS OF POLYNOMIAL MAPS

We give several applications of Theorem 4.3 to the dynamical systems over non-archimedean fields. First, let's prove the following lemma:

LEMMA 4.6. *Let $\phi : \mathbf{P}^1_K \to \mathbf{P}^1_K$ be a non-constant morphism. Let $\alpha \in \mathbf{P}^1_K(K)$ be a rational point, then there exists a constant $C = C(\phi) \geqslant 1$ such that*

$$||\phi(p), \phi(\alpha)||_v \leqslant C||p, \alpha||_v$$

for all $p \in \mathbf{P}^1_K(\bar{K})$.

Proof. Since the chordal metric is bounded by 1, the assertion is certainly true if $||p, \alpha||_v = 1$. Therefore, we will assume $||p, \alpha||_v < 1$. That is, p and α are in the same affine patch of $\mathbf{P}^1_K$. Let z be an affine coordinate and let $z(p) = x, z(\alpha) = y$ be the coordinates of p and α, where $x, y \in \bar{\mathcal{O}}_v$ (the integral closure in $\bar{K}$). We write $\phi(z) = [f(z), g(z)]$, where $f(z), g(z) \in \mathcal{O}_v[z]$. Since $\phi(z)$ is a K-morphism, there exist polynomials $h_f(z), h_g(z) \in \mathcal{O}_v[z]$ such that

$$h_f(z)f(z) + h_g(z)g(z) \;=\; \pi^\mu \quad \text{for some integer } \mu \geqslant 0. \tag{3}$$

Let $||\phi(x)||_v = \max(|f(x)|_v, |g(x)|_v)$ and let $||\phi||_v = \max(||f||_v, ||g||_v)$, where $||f||_v$ ($||g||_v$) denote the maximal absolute value of the coefficients of f (g respectively). By the definition of the chordal metric, we have:

$$||p, \alpha||_v \;=\; |x - y|_v \quad \text{and} \tag{4}$$
$$||\phi(p), \phi(\alpha)||_v \;=\; |f(x)g(y) - f(y)g(x)|_v/(||\phi(x)||_v||\phi(y)||_v). \tag{5}$$

On the other hand, we have the following estimates:

$$|f(x)g(y) - f(y)g(x)|_v$$
$$= |f(x)[g(y) - g(x)] + g(x)[f(x) - f(y)]|_v \tag{6}$$
$$\leqslant ||\phi(x)||_v \max(|f(x) - f(y)|_v, |g(x) - g(y)|_v) \tag{7}$$
$$\leqslant ||\phi(x)||_v||\phi||_v|x - y|_v. \tag{8}$$

Substituting (6) into (4),

$$\|\phi(p), \phi(\alpha)\|_v \leqslant \|\phi\|_v |x - y|_v / \|\phi(y)\|_v.$$

By (3),

$$\|\phi(y)\|_v \geqslant |\pi^\mu|_v.$$

Therefore,

$$\begin{aligned}\|\phi(p), \phi(\alpha)\|_v &\leqslant (\|\phi\|_v / |\pi^\mu|_v)|x - y|_v \\ &\leqslant C\|p, \alpha\|_v,\end{aligned}$$

where $C = \max(1, \|\phi\|_v/|\pi^\mu|_v)$. □

As a result of the above lemma, we have

PROPOSITION 4.7. $J_{\phi,K} = J_{\phi^n,K}$ *for all integers* $n \geqslant 1$.

Proof. Instead of proving the assertion about the Julia set, we prove the equivalent assertion about the Fatou set.

By the definition of the Fatou set, it's clear that $F_\phi(K) \subseteq F_{\phi^n}(K)$. Let $\omega \in F_{\phi^n}(K)$ and let $\epsilon' = \epsilon/C^n$, where C is the constant in Lemma 4.6. By definition, the family of morphisms $\{\phi^{rn}\}_{r=0}^\infty$ is equicontinuous on $F_{\phi^n}(K)$. Given any $\epsilon' > 0$, there exists a $\delta > 0$ such that whenever $x, y \in F_{\phi^n}(K)$ *and* $\|x, y\|_v < \delta$ we have $\|\phi^{rn}(x), \phi^{rn}(y)\|_v < \epsilon'$ for all integer $r \geqslant 1$.

Fix r and consider the set $\mathbf{B}_\delta(\omega) = \{z \in F_{\phi^n}(K) | \ \|z, \omega\|_v < \delta\}$. Let $\mathbf{B}_{\epsilon'}(\omega)$ denote the image of $\mathbf{B}_\delta(\omega)$ under ϕ^{rn}. Let us apply the morphisms $\{\phi, \phi^2, \ldots, \phi^{n-1}\}$ to $\mathbf{B}_{\epsilon'}(\omega)$. Since the radius of $\mathbf{B}_{\epsilon'}(\omega)$ is no more than ϵ', by Lemma 4.6 we see that $\phi^i(\mathbf{B}_{\epsilon'}(\omega))$ has radius less than $C^i\epsilon'$ for $i = 0, 1, \ldots, n-1$. Now $\epsilon' = \epsilon/C^n$, $\phi^i(\mathbf{B}_{\epsilon'}(\omega))$ has radius less than ϵ for $i = 0, 1, \ldots, n-1$.

We have shown $\|\phi^{rn+i}(z), \phi^{rn+i}(\omega)\|_v < \epsilon$ for $i = 0, 1, \ldots, n-1$ whenever $\|z, \omega\|_v < \delta$. Since $\{\phi^m\}_{m=1}^\infty = \{\phi^{rn+i}\}_{r=0,i=0}^{\infty,i=n-1}$ and the argument is independent of the integer r, the family of morphisms $\{\phi^m\}_{m=1}^\infty$ is equicontinuous at ω. This shows $F_{\phi^n}(K) \subseteq F_\phi(K)$ and completes the proof. □

THEOREM 4.8. *Let* $\phi(z) \in K[z]$ *be a polynomial map on* $\mathbf{P}^1_K$. *Then, the rational Julia set,* $J_\phi(K)$ *is compact with respect to the* v*-adic topology on* $\mathbf{P}^1$.

Before proving Theorem 4.8, we need the following lemma. We use the notation $\{X_i\}_{i=0}^\infty$ to denote the family of schemes obtained by blowing-up as described in 3.2, where $X_0 = \mathbf{P}^1_{\mathcal{O}_v}$.

LEMMA 4.9. *Let* ϕ *be the polynomial map in Theorem 4.8 and let* $l = \{e_0, \ldots, e_i, \ldots\}$ *be a half line in* $\partial T_{\phi,K}$. *For any* $e_i \in l$, *let* E_i *be the corresponding irreducible component in* $X_{M,k}$ *for some* M. *Let*

$$\varphi_M \colon X_M \dashrightarrow X_M$$

be the rational map represented by ϕ. Then, there exists an integer $N = N(E_i)$ such that

$$\varphi_M^j(\zeta_{E_i}) = \bar{\infty} \in E_0 \quad \text{for all} \quad j \geqslant N,$$

where ζ_{E_i} denotes the generic point of E_i and E_0 denotes the strict transformation of the special fibre of X_0 by the blowing-up $X_M \to X_0$.

Proof. To ease the notation, we let $X = X_M, F = E_i$ and $\varphi = \varphi_M$. Suppose the lemma is false for F and consider the iteration of φ on X. Let

$$F \to F_1 \to \cdots \to F_i \cdots$$

be the orbit of the irreducible components of X_k under the iteration of φ. We see that none of the F_i's satisfies the condition of the lemma, otherwise F would satisfy the condition. Moreover, since there are only finitely many components of X_k, the orbit of F is stationary. There are integers μ, r such that $\varphi^\mu(\zeta_{F_r}) \in F_r$. On the other hand, ϕ induces a map

$$\partial\phi^\sharp \colon \partial T_{\phi,K} \to \partial T_{\phi,K}.$$

There corresponds an orbit of half lines under the iteration of $\partial\phi^\sharp$. We let

$$l \to l_1 \to \cdots \to l_r \cdots$$

denote the orbit and $\omega_r = \imath(l_r)$. By definition of the tree $T_{\phi,K}$, we see that F_r corresponds to a vertex of l_r, denoted by e', and $\omega_r \in (\hat{F}_r)_K$.

Since φ^μ sends F_r into F_r, e' is fixed by $(\partial\phi^\sharp)^\mu$. This implies that e' also belongs to $l_{2r}, \ldots, l_{nr}, \ldots$. We have $\phi^{n\mu}(\omega_r) \in (\hat{F}_r)_K$ for all integer n. Let $d(e_0, e') = u =$ number of edges connecting e_0 to e' in l_r and let t be a local coordinate of $(\hat{F}_r)_K$. Then, $(\hat{F}_r)_K \simeq SpfK\langle t\rangle$ and $z = \omega_r + \pi^u t$. We have $\phi^\mu(\omega_r) = \omega_r + \pi^u\alpha$ for some $\alpha \in \mathcal{O}_v$, since $\phi^\mu(\omega_r) \in (\hat{F}_r)_K$.

Since ϕ is a polynomial map, ϕ^μ is also a polynomial map. By substituting t into z, we have:

$$\phi^\mu(\omega_r + \pi^u t) = \omega_r + \pi^u\alpha + \pi^u h(t)$$

where $h(t)$ is a polynomial.

φ^μ takes F_r to F_r, we see that $h(t)$ must be a polynomial in $\mathcal{O}_v[t]$, otherwise one can factor out π^a in the denominator for some positive integer a, then φ^μ would take F_r to some component other than F_r. Therefore, ϕ^μ induces a map:

$$(\phi^\mu)^{\text{an}} \colon (\hat{F}_r)_K \to (\hat{F}_r)_K.$$

We see that $\{\phi^{n\mu}\}_{n=0}^\infty$ is equicontinuous on $(\hat{F}_r)_K$. However, $\omega_r \in J_\phi(K)$, $\{\phi^m\}_{m=0}^\infty$ is not equicontinuous at ω_r. By Propositon 4.7, $\{\phi^{n\mu}\}_{n=0}^\infty$ is not equicontinuous at $\omega_r \in (\hat{F}_r)_K$, a contradiction. The lemma is proved. □

Proof of Theorem 4.8. If the degree of $\phi(z)$ is 1, then the statement is trivial. Therefore, we only need to show the theorem in the case that $J_\phi(K)$ is non-empty and the degree of $\phi(z)$ is at least 2. The property that $J_\phi(K)$ is compact is equivalent to that the corresponding tree $T_{\phi,K}$ is locally compact. Furthermore, since $\deg\phi \geqslant 2$ and $J_\phi(K)$ is independent of the X_0 that one starts with, by conjugating the fractional linear map $h(z) = z/\pi^m$ for some m, we may choose $\phi(z)$ to be the form:

$$\phi(z) = f(z)/\pi^d \quad \text{for some integer } d > 0,$$

where $f(z) \in \mathcal{O}_v[z]$ is a polynomial with the leading coefficient being a unit in $\mathcal{O}_v$. Let E_0 be the special fibre of $\mathbf{P}^1_{\mathcal{O}_v}$ and let e_0 be the corresponding vertex in $T_{\phi,K}$. Let I_0 be the sheaf of ideals to be blown-up on X_0, it is obvious that $\mathrm{Supp} I_0 =$ roots of $\bar{f}(\bar{z})$, where $\bar{f}(\bar{z})$ denotes the polynomial of f reduced modulo π. We use the notation in 3.2, at each stage of the blowing-up, we have the following commutative diagram:

$$\begin{tikzcd}
X_{i+1} \arrow[d, "p_{i+1}"'] \arrow[dr, "\widetilde{\varphi_i}"] & \\
X_i \arrow[r, dashed, "\varphi_i"] & X_i
\end{tikzcd}$$

Let $l = \{e_0, \ldots, e_n, \ldots\} \subseteq \partial T_{\phi,K}$. We show the proposition by induction on the vertices e_n of l.

(i) $n = 0$:

Consider $E_{0,i} \subseteq X_i$, the strict transformation of E_0 under the blowing-up $X_i \to \cdots \to X_0$. Let $\widetilde{\Phi_{i-1}} = \widetilde{\varphi_0} \circ \cdots \circ \widetilde{\varphi_{i-1}}$ and let the restriction of $\widetilde{\Phi_{i-1}}$ to $E_{0,i}$ be denoted by $\widetilde{\Phi_{i-1,E_0}}$. Then we have the diagram:

$$\begin{tikzcd}
E_{0,i} \arrow[d] \arrow[dr, "\widetilde{\Phi_{i-1,E_0}}"] & \\
X_0 \arrow[r, dashed, "\varphi_0"] & X_0
\end{tikzcd}$$

Note that $(\widetilde{\Phi_{i-1}})_K = \phi^i$.

Since $E_{0,i}$ is the strict transformation of E_0, we may use $\bar{z}$ as a local coordinate on $E_{0,i}$. It is not hard to see that $\mathrm{supp}(\widetilde{\Phi_{i-1,E_0}})^{-1} I_0 \subseteq$ roots of $\bar{f}(\bar{z})$. This shows that $E_{0,i}$ intersects with other components of $X_{i,k}$ at roots of $\bar{f}(\bar{z})$ for all integer $i \geqslant 0$. Therefore, only finitely many edges originate from e_0.

(ii) Assume that there are only finitely many edges originating from e_{n-1} for $n \geqslant 1$.

(iii) Let E be the component in $X_{m,k}$ corresponding to e_n for some m and consider the rational map on X_m:

$$\varphi_m : X_m \dashrightarrow X_m$$

represented by ϕ on the generic fibre. By Lemma 4.9, there exists an integer N such that $\varphi_m^r(\zeta_E) = \bar{\infty} \in E_0$ for all $r \geqslant N$. Let $r \geqslant \max(m, N)$ and consider the following diagram:

$$\begin{tikzcd}
E_r^* \subseteq X_r \arrow[d] \arrow[dr, "\widetilde{\varphi_{r-1}}"] & & & \\
X_{r-1} \arrow[r, dashed, "\varphi_{r-1}"] \arrow[d] \arrow[dr] & X_{r-1} \arrow[d] \arrow[dr] & & \\
\vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[dr, dashed] & \\
X_0 \arrow[r, dashed, "\varphi_0"] & X_0 \arrow[r, dashed, "\varphi_0"] & \cdots & X_0
\end{tikzcd}$$

where E_r^* denotes the strict transformation of E under the blowing-up $X_r \to X_{r-1} \to \cdots \to X_m$.

Then, $\widetilde{\Phi_{r-1}}(\zeta_{E_{r-1}^*}) = \bar{\infty} \in \mathbf{P}_k^1$, where $\widetilde{\Phi_{r-1}} = \widetilde{\varphi_{r-1}} \circ \cdots \circ \widetilde{\varphi_0}$. Consider the formal analytic variety $\widehat{E_K}$ and let t be a local coordinate on $\widehat{E_K}$. Let $\omega = \imath(l)$. We see that $\omega \in \widehat{E_K}$ and $z = \omega + \pi^n t$. Since $(\widetilde{\Phi_{r-1}})_K \simeq \phi^r$ and $\widetilde{\Phi_{r-1}}(\zeta_{E_{r-1}}^*) = \bar{\infty} \in \mathbf{P}_k^1$, it is equivalent to saying:

$$\phi^r(\omega + \pi^n t) = \frac{g(t)}{\pi^d} \tag{9}$$

for some $d > 1$ and some polynomial $g(t) \in \mathcal{O}_v[t]$.

Then $\mathrm{Supp}(\widetilde{\Phi_{r-1}})^{-1} I_0 \subseteq$ roots of $\bar{g}(\bar{t})$. For any $r' > r$, we still have $\widetilde{\Phi_{r'-1}}(\zeta_{r'-1}^*) = \bar{\infty}$, but

$$\phi^{r'}(\omega + \pi^n t) = \phi^{r'-r}(\phi^r(\omega + \pi^n t)).$$

Because ϕ^r has the form (9), one can check $\mathrm{Supp}(\widetilde{\Phi_{r'-1}})^{-1} I_0 \subseteq$ roots of $\bar{g}(\bar{t})$. This shows that E only intersects with other components at some fixed finite set of closed points on E, that is, only finitely many edges originate from e_n. This completes the steps of induction and proves the theorem. □

REMARKS. (1) Theorem 4.8 is trivial in the theory of dynamical systems over the complex numbers because in that case $\mathbf{P}^1(\mathbf{C})$ is compact with respect to the complex topology. However, in the non-archimedean case, $\mathbf{P}^1_K(K)$ is no longer compact with respect to the v-adic topology, since K is not a locally compact field. (K is locally compact if and only if k is finite.)

(2) It is easy to give counterexamples to Theorem 4.8 for non-polynomial maps, for example: let

$$\phi(z) = (z-1)^2 z/(z+\pi) \quad ch(k) \neq 2.$$

An easy calculation shows the tree $T_{\phi,K}$ is not locally compact, therefore $J_\phi(K)$ is not compact.

4.5. Examples

In this paragraph, we compute the following two examples.

EXAMPLE 4.10. Let E be an elliptic curve over K and let $G = \{\pm I\} \subseteq \mathrm{Aut}(E)$ consist the inverse and the identity maps. By identifying $\mathbf{P}^1_K \simeq E/G$, we see that the multiplication by m-map induces a rational map:

$$\phi_m \colon \mathbf{P}^1_K \to \mathbf{P}^1_K \quad m \geqslant 2.$$

In the case that K is the field of complex numbers, ϕ_m provides an example that the Julia set is the whole space $\mathbf{P}^1(\mathbf{C})$ (see, for example: [1]). In the case that K is a non-achimedean field, we contend that $(\mathbf{P}^1_K, \phi_m)$ has a weak Néron model and the rational Julia set $J_{\phi_m}(K) = \emptyset$.

Our contention can be seen as follows:

Let $\mathcal{E}/S$ be the Néron model of E. By the universal mapping property of the Néron model and a theorem of Mumford's ([14, Theorem III.12.1]), we see that the action of G extends on $\mathcal{E}$ and $X = \mathcal{E}/G$ is a separated, smooth scheme of finite type over S. The same arguments and the fact that ϕ_m commutes with the action of G show that ϕ_m extends to a finite S-morphism Φ_m on X. Therefore, $(X/S, \Phi_m)$ is a weak Néron model for $(\mathbf{P}^1_K, \phi_m)$ and $J_{\phi_m}(K) = \emptyset$.

EXAMPLE 4.11. Let F be a p-adic number field and let $l \geqslant 2$ be an integer such that F contains the $(l-1)$-th roots of unity and p does not divide $(l-1)$. Let $\phi(z) = f(z)/\pi_F$ be a rational map such that $f(z)$ is a monic polynomial and $f(z) \equiv z^l - z (\mathrm{mod} \pi_F)$, where π_F is a uniformizer in $\mathcal{O}_F$.

We will show that $J_\phi(F) \subseteq \mathcal{O}_F$ and the dynamical systems on $J_\phi(F)$ associated with ϕ is symbolic dynamics in l-symbols. By symbolic dynamics we mean that:

(1) Each element $\omega \in J_\phi(F)$ is represented by an infinite sequence $(\alpha_0, \alpha_1, \alpha_2, \ldots,)$, where $\alpha_i \in \{1, \ldots, l\}$.
(2) $\phi((\alpha_0, \alpha_1, \alpha_2, \ldots,)) = (\alpha_1, \alpha_2, \alpha_3, \ldots,)$.

To prove our claim, we first perform the sequence of blowing-ups as described in 3.2:

Starting with $X_0 = \mathbf{P}^1_{\mathcal{O}_F}$, we see that we need to blow up the roots of the polynomial $\bar{f}(\bar{z})$ (the reduction of the polynomial $f(z)$ modulo π_F). By hypothesis, we have that $f(z)$ splits into linear factor over F and the roots of $\bar{f}(\bar{z})$ are distinct. Therefore, l distinct closed points on the special fibre of $\mathbf{P}^1_{\mathcal{O}_F}$ need to be blown up. Let us denote these l closed points by $\lambda_1, \lambda_2, \ldots, \lambda_l$ and their dilation by $E_1, \ldots, E_l$. Let X_1 denote the scheme obtained by the first blowing-up. Using the notations in 3.2, we have the following commutative diagram:

$$\begin{tikzcd} (X_1)_{\text{smooth}} \arrow[d, "P_1"'] \arrow[dr, "\widetilde{\varphi}_0"] & \\ X_0 \arrow[r, dashed, "\varphi_0"] & X_0 \end{tikzcd}$$

where φ_0 is the rational map represented by $\phi : \mathbf{P}^1_F \to \mathbf{P}^1_F$. Let E_0 denote the special fibre of $\mathbf{P}^1_{\mathcal{O}_F}$. It is not hard to see that $\widetilde{\varphi}_0$ takes $E_{i_1}, 1 \leqslant i_1 \leqslant l$, onto E_0 and the restriction $\widetilde{\varphi}_0 \colon E_{i_1} \to E_0$ is of degree one.

As a result, we need to blow up l distinct closed points on each E_{i_1}. Let $E_{i_1 i_2}, 1 \leqslant i_1 \leqslant l, 1 \leqslant i_2 \leqslant l$, denote the dilation of these l closed points on E_{i_1}. Let X_2 be the resulting scheme after we blow up l closed points on all E_{i_1}, we have the following diagram:

$$\begin{tikzcd} (X_2)_{\text{smooth}} \arrow[d, "P_2"'] \arrow[dr, "\widetilde{\varphi}_1"] & \\ (X_1)_{\text{smooth}} \arrow[r, dashed, "\varphi_1"] & (X_1)_{\text{smooth}} \end{tikzcd}$$

where φ_1 is the rational map represented by ϕ. We arrange the indices so that $\widetilde{\varphi}_1$ maps $E_{i_1 i_2}$ onto E_{i_2}.

Inductively, the special fibre of the scheme X_r consist of $E_{i_1 i_2 \cdots i_s}$ where $1 \leqslant i_j \leqslant l$ and $1 \leqslant s \leqslant r$ and $\widetilde{\varphi}_{r-1}$ maps $E_{i_1 \cdots i_s}$ onto $E_{i_2 \cdots i_s}$. The dual graph of the special fibres of X_r consists of vertices $\{e_0, \ldots, e_{i_1 \cdots i_s} | 1 \leqslant i_j \leqslant l, 1 \leqslant s \leqslant r\}$ corresponding to $E_{i_1 \cdots i_s}$. As $r \to \infty$, we get the infinite tree $T_{\phi,F}$ defined in 4.2. One can check that each half line of $\partial T_{\phi,F}$ consists vertices $\{e_0, e_{i_1}, \ldots, e_{i_1 \ldots i_r}, \ldots\}$. By Theorem 4.3, $J_\phi(F) = \imath(\partial T_{\phi,F})$. We can represent each point $\omega \in J_\phi(F)$ by this sequence of vertices, or equivalently, by sequence of indices $(i_1, i_2, \ldots, i_r, \ldots)$. By the construction of $T_{\phi,F}$ and Proposition 4.2, $\phi((i_1, i_2, \ldots)) = (i_2, i_3, \ldots)$. It is also clear that $J_\phi(F) \subseteq \mathcal{O}_F$. This completes the proof of the claim.

As a result of this claim, we have the following conclusions:

(i) All the periodic points of ϕ, except $\{\infty\}$, are contained in $J_\phi(F)$, therefore all the periodic points are rational over F.
(ii) Let $\mathrm{Per}'_\phi = \mathrm{Per}_\phi \setminus \{\infty\}$, then the rational Julia set $J_\phi(F)$ is the closure of Per'_ϕ with respect to the p-adic topology.
(iii) If $l = q = |\mathcal{O}_F/(\pi_F)|$, then $J_\phi(F) = \mathcal{O}_F$.

We would like to end this paper by posing the following questions concerning the structure of the rational Julia set:

(1) Over complex numbers, the Julia set can also be characterized as the closure of the repelling periodic points. Does the same characterization still hold in the case of p-adic number fields ? Namely, is the rational Julia set the closure of the repelling rational periodic points ?
(2) Does there exist a morphism ϕ: $\mathbf{P}^1 \to \mathbf{P}^1$ over a p-adic number field F such that $J_\phi(F) = \mathbf{P}^1(F)$?
(3) Given a morphism $\phi : \mathbf{P}^1 \to \mathbf{P}^1$ over K, can one determine a finite set of numerical invariants associated to ϕ so that whether or not $(\mathbf{P}^1/K, \phi)$ has a weak Néron model is determined by these numerical invariants ?
(4) If all the periodic points $\mathrm{Per}_\phi = \cup_n \mathrm{Per}^n_\phi$ are non-repelling, is it true that $(\mathbf{P}^1/K, \phi)$ has a weak Néron model?

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