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Singularity of the moduli space of stable bundles on surfaces

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1. Introduction

Let (S, H) be a polarized algebraic surface defined over $\mathbb{C}$. For given divisor D on S and an integer c, let $M_H(r, D, c)$ be the moduli space of rank r torsion-free sheaves E on S which are Gieseker semistable with respect to H, with $\det E \cong \mathcal{O}_S(D)$, $c_2(E) = c$. Recently the singularity of $M_H(r, D, c)$ has been studied by several authors. J.-M.Drezet considered the case $S = \mathbb{P}^2$ and proved that $M_H(r, D, c)$ is locally factorial ([D]). For arbitrary surface S, J. Li proved that $M_H(2, D, c)$ is normal if c is sufficiently large ([L]) and, under additional assumption on the canonical divisor K_S, D. Huybrechts showed that $M_H(2, D, c)$ is a $\mathbb{Q}$-Gorenstein variety ([H]). The purpose of this note is to generalize their results to the case $r > 2$. Our main result is the following

THEOREM. *For $r \geqslant 2$ and sufficiently large c, $M_H(r, D, c)$ is normal. If we assume further that there exist integers m, $l (m \neq 0)$ such that $mK_S = lH$, then $M_H(r, D, c)$ is a $\mathbb{Q}$-Gorenstein variety.*

Our proof of the above theorem rests on two results concerning $M_H(r, D, c)$. One is the generic smoothness of $M_H(r, D, c)$ for sufficiently large c and another is the construction of determinant line bundles on it([O1], [O2], [LP]).

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2. Determinant line bundles

In what follows, all varieties are defined over the complex number field $\mathbb{C}$. In this section we describe the determinant bundle formalism in [O1], [LP] for higher rank sheaves. Let C be a smooth curve. Let $\mathcal{F}$ be a family of sheaves of rank r on C, parametrized by a scheme T, which we assume to be irreducible. We let $\mathrm{Det}(\mathcal{F})$ be the line bundle on X defined by

$$\mathrm{Det}(\mathcal{F}) = (\det(p_T)_!\mathcal{F})^\vee.$$

LEMMA 1.1. *Let C, T, $\mathcal{F}$ be as above. Assume that there exist line bundles L_1(resp. L_2) on C (resp. on T) such that* $\det \mathcal{F} \cong p_C^* L_1 \otimes p_T^* L_2$ *and assume further that $\chi(\mathcal{F}_x) = 0$ for all $x \in T$. Then*

$$c_1(\mathrm{Det}(\mathcal{F})) = (p_T)_* \left(c_2(\mathcal{F}) - \frac{r-1}{2r} c_1(\mathcal{F})^2 \right).$$

Proof. By Grothendieck-Riemann-Roch theorem, we obtain

$$c_1(\mathrm{Det}(\mathcal{F})) = (p_T)_*(c_2(\mathcal{F}) - \tfrac{1}{2}c_1(\mathcal{F})^2 + \tfrac{1}{2}c_1(\mathcal{F}) \,.\, p_C^*[K_C]).$$

The claim follows from the assumption $\chi(\mathcal{F}_x) = 0$ for all $x \in T$. □

Let S be a smooth projective surface and $H = \mathcal{O}_S(1)$ an ample divisor on S. As in the introduction, we denote by $M_H(r, D, c)$ the moduli of semistable sheaves with the given invariants. There exists an integer n such that for all $F \in M_H(r, D, c)$, $F(n) = F \otimes \mathcal{O}_S(n)$ is globally generated and $h^i(F(n)) = 0$ for $i > 0$. We fix such n and let $N = h^0(F(n))$.

For integers n, N as above, let $\mathrm{Quot}(r, D, c)$ denote Grothendieck's Quot-scheme parametrizing all quotient sheaves $\mathcal{O}_S(-n) \otimes \mathbb{C}^N \to F$ such that $\det(F) \cong \mathcal{O}_S(D)$ and $c_2(F) = c$. Let $p_S : S \times \mathrm{Quot} \to S$, $p_Q : S \times \mathrm{Quot} \to \mathrm{Quot}$ be the natural projections. There exists a universal quotient morphism

$$\theta : p_S^*(\mathcal{O}_S(-n) \otimes \mathbb{C}^N) \to \mathcal{F}$$

on $\mathrm{Quot}(r, D, c) \times S$. We denote by $\mathrm{Quot}^{ss} = \mathrm{Quot}(r, D, c)^{ss}$ the open subset consisting of semistable points $x \in \mathrm{Quot}(r, D, c)$ such that

$$\theta_x : \mathcal{O}_S(-n) \otimes \mathbb{C}^N \to \mathcal{F}_x$$

induces an isomorphism

$$\mathbb{C}^N \cong H^0(\mathcal{F}_x(n)).$$

$M_H(r, D, c)$ is constructed as the good quotient by $PGL(N)$ of Quot^{ss} ([Ma]). Let Σ be an irreducible component of the Picard scheme $\mathrm{Pic}(S)$ containing $\mathcal{O}_S(D)$. Similarly we define $M_H(r, \Sigma, c)$ and $\mathrm{Quot}(r, \Sigma, c)$ as the scheme parametrizing sheaves F with $\det(F) \in \Sigma$. Let

$$\pi : \mathrm{Quot}^{ss} \to M_H(r, D, c)$$

denote the quotient morphism.

Using the above $\mathcal{F}$, we define the linear map

$$\nu_{\mathcal{F}} : \mathrm{Pic}(S) \otimes \mathbb{Q} \to \mathrm{Pic}(\mathrm{Quot}^{ss}) \otimes \mathbf{Q}$$

by

$$\nu_{\mathcal{F}}([C]) = (p_Q)_* \left(\left(c_2(\mathcal{F}) - \frac{r-1}{2r} c_1(\mathcal{F})^2 \right) . p_S^*[C] \right).$$

For a curve $C \subset S$, let $\mathcal{F}^C = \mathcal{F}_{|C \times \mathrm{Quot}^{ss}}$. We have

LEMMA 1.2. *Let C be a smooth complete curve of genus $g(C)$ on S. Assume that for every point $x \in$ Quot^{ss}, we have $\deg \mathcal{F}_x^C = D.C = rd$ for some integer d. Then for every line bundle L on C with $\deg L = -d + g(C) - 1$, we have the following equality in* Pic(Quot^{ss}).

$$c_1(\mathrm{Det}(\mathcal{F}^C \otimes p_C^* L)) = \nu_{\mathcal{F}}([C]).$$

Proof. Let $p_Q^C : C \times \mathrm{Quot}^{ss} \to \mathrm{Quot}^{ss}$ be the projection. $\mathcal{F}^C$ clearly satisfies the first assumption in Lemma 1.1. Furthermore, we have $\chi(\mathcal{F}_x^C \otimes p_C^* L) = 0$ for all $x \in \mathrm{Quot}^{ss}$. Hence we obtain

$$\begin{aligned} c_1(\mathrm{Det}(\mathcal{F}^C \otimes p_C^* L)) &= (p_Q^C)_* \left(c_2(\mathcal{F}^C \otimes p_C^* L) - \frac{r-1}{2r} c_1(\mathcal{F}^C \otimes p_C^* L)^2 \right) \\ &= (p_Q^C)_* \left(c_2(\mathcal{F}^C) - \frac{r-1}{2r} c_1(\mathcal{F}^C)^2 \right). \end{aligned}$$

If we denote by $\tau : C \times \mathrm{Quot}^{ss} \hookrightarrow S \times \mathrm{Quot}^{ss}$ the inclusion map, we have $c_i(\mathcal{F}^C) = \tau^* c_i(\mathcal{F})$ for $i = 1, 2$. Therefore

$$\begin{aligned} c_1(\mathrm{Det}(\mathcal{F}^C \otimes p_C^* L)) &= (p_Q)_* \left(\left(c_2(\mathcal{F}) - \frac{r-1}{2r} c_1(\mathcal{F})^2 \right) . p_S^*[C] \right) \\ &= \nu_{\mathcal{F}}([C]). \end{aligned}$$

□

PROPOSITION 1.3. *Let C be a smooth irreducible curve with $C \in |rnH|$ for a positive integer n. Let $d(n) = -nD \,.\, H + g(C) - 1$. Then for every $L \in$ Pic$^{d(n)}(C)$, there exists a line bundle $\mathrm{Det}_{\mathcal{F}}(C)$ on $M_H(r, D, c)$ such that*

$$\pi^* \mathrm{Det}_{\mathcal{F}}(C) \cong \mathrm{Det}(\mathcal{F}^C \otimes p_C^* L).$$

Proof. This is essentially a consequence of [LP, Théoréme (2.5)]. We note that Le Potier's theorem cannot be applied directly to our case since we don't know whether $\mathcal{F}_x^C$ is semistable for all $x \in \mathrm{Quot}^{ss}$. However, the argument used in its proof, which we sketch below, works without change.

We have a GL(N)-action on $\mathrm{Det}(\mathcal{F}^C \otimes p_C^* L)$, which is induced from the natural GL(N)-action on $\mathcal{F}$. Since $\chi(\mathcal{F}_x^C \otimes p_C^* L) = 0$ for every $x \in \mathrm{Quot}^{ss}$

and $L \in \mathrm{Pic}^{d(n)}(C)$, the center of $\mathrm{GL}(N)$ acts trivially on $\mathcal{F}$. Hence this action descends to a $\mathrm{PGL}(N)$-action. By a descent lemma [LP, Lemma (1.4)] which generalizes [D-N, Theorem 2.3], it suffices to show that for every point $x \in \mathrm{Quot}^{ss}$ with the closed $\mathrm{PGL}(N)$-orbit, the stabilizer $\mathrm{Stab}(x)$ of x acts trivially on the fiber of $\mathrm{Det}(\mathcal{F}^C \otimes p_C^* L)$ at x. If a point x has the closed orbit, then $\mathcal{F}_x \cong F_1^{m_1} \oplus \cdots F_s^{m_s}$ where F_i are pairwise non-isomorphic stable sheaves of $\mathrm{rk} F_i = r_i$ satisfying

$$\frac{\chi(F_i)}{r_i} = \frac{\chi(F)}{r}, \quad \frac{c_1(F_i) \,.\, H}{r_i} = \frac{D \,.\, H}{r}.$$

Then we have $\mathrm{Stab}(x) \cong \prod_{i=1}^s \mathrm{GL}(m_i)$ and it acts on the fiber $\mathrm{Det}(\mathcal{F}^C \otimes p_C^* L)_x$ via the character defined by

$$(g_1, \ldots, g_s) \mapsto \prod_{i=1}^{s} (\det g_i)^{\chi_i},$$

where $\chi_i = \chi(F_{i|C} \otimes L)$. For each i, we have

$$\begin{aligned} \chi(F_{i|C} \otimes L) &= c_1(F_i) \,.\, C + r_i(d(n) + 1 - g(C)) \\ &= r_i(nD \,.\, H + d(n) + 1 - g(C)) = 0. \end{aligned}$$

It follows that $\mathrm{Stab}(x)$ acts trivially on $\mathrm{Det}(\mathcal{F}^C \otimes p_C^* L)_x$. This proves the claim. □

3. Singularity of the moduli space

In this section we prove our main result on the singularity of the moduli space. Let $M_H(r, D, c)$, $M_H(r, \Sigma, c)$ be as in the previous section. We define the expected dimensions of $M_H(r, D, c)$ and $M_H(r, \Sigma, c)$ as follows

$$\begin{aligned} d(r, D, c) &= 2rc - (r-1)D^2 - (r^2 - 1)\chi(\mathcal{O}_S), \\ d(r, \Sigma, c) &= d(r, D, c) + h^1(\mathcal{O}_S). \end{aligned}$$

For a torsion-free sheaf F on S, let $\mathrm{Ext}^2(F, F)^0$ denote the kernel of the trace map

$$\mathrm{tr} : \mathrm{Ext}^2(F, F) \to H^2(\mathcal{O}_S).$$

Let M^0 be the open subset of $M_H(r, D, c)$ defined as follows:

$$M^0 = \{F \in M_H(r, D, c) | \mathrm{Ext}^2(F, F)^0 = 0\}.$$

By [Mu], we see that M^0 is smooth. The following result is due to K.G.O'Grady.

THEOREM 2.1 ([O2]). *There exists an integer c_0 such that for all $c \geq c_0$, the followings hold.*

(1) *$M_H(r, D, c)$ has pure dimension $d(r, D, c)$;*

(2) $\mathrm{codim}(M_H(r,D,c) \setminus M^0) \geq 2$.

We define Q^0 to be the inverse image of M^0 by the morphism $\pi : \mathrm{Quot}^{ss} \to M_H(r,D,c)$. For a universal quotient sheaf $\mathcal{F}$ on $S \times Q$, let $\mathcal{F}_0 = \mathcal{F}_{S\times Q^0}$. We denote by $p_S : S \times Q^0 \to S$ and $p_Q : S \times Q^0 \to Q^0$ the projections. The following is a generalization of [L, Theorem 1.2].

PROPOSITION 2.2. *For sufficiently large c, we have*

(1) $\mathrm{Quot}(r,\Sigma,c)^{ss}$ *has pure dimension* $e(r,\Sigma,c) = d(r,\Sigma,c) + N^2 - 1$*;*

(2) $\mathrm{Quot}(r,\Sigma,c)^{ss}$ *is normal and locally complete intersection.*

Proof. Let P be the identity component of $\mathrm{Pic}(S)$ and let $\hat{P}$ be the quotient of P by the subgroup of r-torsion points. Then $\hat{P}$ is a smooth group scheme which acts freely on $M_H(r,\Sigma,c)$ and $M_H(r,D,c)$ is isomorphic to $M_H(r,\Sigma,c)/\hat{P}$(cf. [L, p.11]). From Theorem 2.1 it follows that $M_H(r,\Sigma,c)$ has pure dimension $d(r,\Sigma,c)$ and is smooth in codimension two for $c >> 0$ since $M_H(r,\Sigma,c)$ is a principal bundle over $M_H(r,D,c)$. Hence by construction of $M_H(r,\Sigma,c)$, $\mathrm{Quot}(r,\Sigma,c)^{ss}$ has dimension at most $e(r,\Sigma,c)$. On the other hand, the argument in [L, Sect. 1] shows that locally $\mathrm{Quot}(r,\Sigma,c)^{ss}$ is defined by an ideal $J \subset \mathbb{C}[t_1,\ldots,t_k]$ which is generated by at most $k - e(r,\Sigma,c)$ elements. Therefore $\mathrm{Quot}(r,\Sigma,c)^{ss}$ is normal and locally complete intersection. □

PROPOSITION 2.3. *For sufficiently large c,* $M_H(r,D,c)$ *is normal and* $M_H(r,D,c)^s$ *is locally comlete intersection.*

Proof. From Proposition 2.2 we deduce that $M_H(r,\Sigma,c)$ is normal and $M_H(r,\Sigma,c)^s$ is locally complete intersection for $c \gg 0$. The claims for $M_H(r,D,c)$ and $M_H(r,D,c)^s$ follow from that fact that they are quotients by a smooth group scheme $\hat{P}$. □

LEMMA 2.4. *Let* T_{M^0} *denote the tangent bundle of* M^0*. In* $\mathrm{Pic}(Q^0) \otimes \mathbb{Q}$*, we have*

$$c_1(\pi^* T_{M^0}) = -r(p_Q)_* \left(\left(c_2(\mathcal{F}_0) - \frac{r-1}{2r} c_1(\mathcal{F}_0)^2 \right) \cdot p_S^*[K_S] \right).$$

Proof. Let $\mathcal{E}xt^i_{p_Q}(\mathcal{F},\mathcal{F})$ be the relative extension sheaf and let $\mathcal{E}xt^i_{p_Q}(\mathcal{F},\mathcal{F})^0$ be the kernel of the trace map

$$\mathrm{tr}\colon\ \mathcal{E}xt^i_{p_Q}(\mathcal{F},\mathcal{F}) \to R^i p_{Q*}\mathcal{O}_{S\times Q}.$$

We have $\mathcal{E}xt^i_{p_Q}(\mathcal{F},\mathcal{F})^0_{|Q^0} = 0$ for $i = 0, 2$ and

$$\mathcal{E}xt^1_{p_Q}(\mathcal{F},\mathcal{F})^0_{|Q^0} \cong \pi^* T_{M^0}.$$

We choose a locally free resolution of $\mathcal{F}$

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0.$$

Then Grothendieck-Riemann-Roch yields

$$\begin{aligned} c_1(\pi^* T_{M^0}) &= -c_1\left(\sum(-1)^i \mathcal{E}xt^i_{p_Q}(\mathcal{F},\mathcal{F})^0_{|Q^0}\right) \\ &= (p_Q)_*((\mathrm{ch}(\mathcal{F}_1^\vee \otimes \mathcal{F}_0) - \mathrm{ch}(\mathcal{F}_2^\vee \otimes \mathcal{F}_0)) . p_S^* \mathrm{td}(S)) \\ &= -r(p_Q)_* \left(\left(c_2(\mathcal{F}_0) - \frac{r-1}{2r} c_1(\mathcal{F}_0)^2 \right) . p_S^*[K_S] \right). \end{aligned}$$

□

THEOREM 2.5. *Let (S, H) be a polarized surface with $mK_S = lH$ for some integers l, $m (m \neq 0)$. Then $M_H(r, D, c)$ is a $\mathbb{Q}$-Gorenstein variety for sufficiently large c.*

Proof. We treat the case $l > 0$ only since the proof in the case $l \leq 0$ is similar. For sufficiently large n, we choose a smooth irreducible curve $C \in |rlnH|$ and $L \in \mathrm{Pic}^{d(ln)}(C)$. Let K_{M^0} denote the canonical bundle of M^0. By Lemma 1.2 and Lemma 2.4 we have the following equality in $\mathrm{Pic}(M^0) \otimes \mathbb{Q}$.

$$\begin{aligned} c_1(\mathrm{Det}_{\mathcal{F}}(C))_{|M^0} &= \pi_*(p_Q)_* \left(\left(c_2(\mathcal{F}_0) - \frac{r-1}{2r} c_1(\mathcal{F}_0)^2 \right) . p_S^*[rmnK_S] \right) \\ &= rmn\pi_*(p_Q)_* \left(\left(c_2(\mathcal{F}_0) - \frac{r-1}{2r} c_1(\mathcal{F}_0)^2 \right) . p_S^*[K_S] \right) \\ &= mnc_1(K_{M^0}). \end{aligned}$$

It follows that there exists an integer N_0 such that

$$\mathrm{Det}_{\mathcal{F}}(C)^{\otimes N_0}_{|M^0} \cong K_{M^0}^{\otimes N_0 mn}.$$

This completes the proof of the theorem since for sufficiently large c, $M_H(r, D, c)$ is normal and $\mathrm{codim}(M_H(r, D, c) \setminus M^0) \geq 2$. □

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