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On the representations of $\mathrm{U}(m, n)$ unitarily induced from derived functor modules

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Abstract. We prove a decomposition formula of a representation of $\mathrm{U}(m, n)$ unitarily induced from a derived functor module, which enables us to reduce the problem of irreducible decompositions to the study of derived functor modules. In particular, we show such an induced representation is decomposed into a direct sum of irreducible unitarily induced modules from derived functor modules.

0. Introduction

The easiest nontrivial decomposition of a unitarily induced representation of a real reductive Lie group from a parabolic subgroup is the decomposition of non-spherical principal series of $\mathrm{SL}(2, \mathbb{R})$ at the singular parameter into two limits of discrete series, which are realized as Hardy spaces on the upper and lower half-planes respectively. This decomposition is certainly 'easiest'. However, it is not very easy and it is extremely important. For example, thanks to the recent progress of the Beilinson–Bernstein theory, we can reduce Schmid's character identity at the wall ([27, 28]) to the easiest decomposition for $\mathrm{SL}(2, \mathbb{R})$, which is nothing but the easiest Schmid identity, using certain change of polarization (cf. [37], [23], [7], [29, 30], [22]). In view of the Knapp–Zuckerman theory on the irreducible tempered representations ([18]), this means that the reducibilities of unitarily induced modules from discrete series arise from the above easiest decomposition.

We can consider the degenerate setting. Namely, we consider unitary degenerate series instead of unitary principal series and derived functor modules (cf. [36, 38, 39, 41], [43], [17], [44, 45], etc.) instead of discrete series. Our object of study is the decomposition of unitarily induced modules of $\mathrm{U}(m, n)$ from derived functor modules (We call such induced modules generalized unitary degenerate series.).

Kashiwara and Vergne found a remarkable decomposition into $n + 1$ distinct irreducible factors of the degenerate series of $\mathrm{SU}(n, n)$ with respect to the Siegel parabolic subgroup at the most singular parameter ([15]). Two of the irreducible factors are representations realized as the Hardy spaces on the Siegel upper and lower half-planes. Their embeddings into degenerate series are nothing but the

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boundary value map to the Shilov boundary. Perhaps, the embeddings of other irreducible factors can be regarded as certain boundary value maps of 'cohomological Hardy spaces'. The case of $\mathrm{SU}(2,2)$ is considered in [21]. Although a complex analytical interpretation of the decomposition is not clear at this time, the corresponding algebraic interpretation is available, thanks to [2], Theorem 4.2. Each irreducible factor is a derived functor module.

In this article, we show the Kashiwara–Vergne decompositions play a role on our degenerate setting for $\mathrm{U}(m,n)$ similar to the role of the easiest decomposition for $\mathrm{SL}(2,\mathbb{R})$ on the Knapp–Zuckerman theory.

Since we shall consider the Vogan–Zuckerman cohomological inductions of Harish-Chandra modules (cf. [36]) with respect to various parabolic subalgebras of the complexified Lie algebra of $\mathrm{U}(m,n)$, we introduce the following terminology. We call cohomological induction with respect to a parabolic subalgebra stable under a Cartan involution (resp. the complex conjugation with respect to the real form) elliptic induction (resp. hyperbolic induction). Namely, elliptic induction is cohomological parabolic induction in the sense of [39] Chapter 6, and hyperbolic induction is usual parabolic induction with respect to a parabolic subgroup. A derived functor module is nothing but the elliptic induction from a one-dimensional unitary representation satisfying a certain positivity condition ('weakly fair' in the sense of [41]). So, generalized unitary degenerate series are hyperbolic induction of elliptic induction from certain one-dimensional representations.

In Section 2, we consider exchange of the order of applications of these two inductions. Actually, it is possible under a certain regularity condition ('good enough' see 2.1 below.), and the result is that generalized unitary degenerate series in the good enough range can be written as the elliptic induction of the external tensor product of unitary degenerate series with respect to Siegel parabolic subgroups and one-dimensional unitary representations (Theorem 2.2.3, Corollary 2.2.5). The main ingredient of our proof is the theory of change of polarization for standard module originated by Vogan ([37]) and completed by Hecht, Miličić, Schmid, and Wolf ([22], [29–31]). Actually, they worked on $\mathcal{D}$-modules, but as is mentioned in [29], we can translate their result to the cohomological induction setting using the Hecht–Miličić–Schmid–Wolf duality theorem ([11]).

In Section 3, combining the above result on the change of the polarization and the decomposition formula of unitary degenerate series with respect to the Siegel parabolic subgroups [15], [2], [14], we derive a decomposition formula of generalized unitary degenerate series in the good enough range into non-zero irreducible generalized degenerate series (Theorem 3.2.2). The decomposition formula (Theorem 3.3.1) in the general case is deduced from the formula for the good enough range using a precise version of the translation principle developed in [41]. The direct summand appeared in the formula in the general case is either irreducible or zero. In order to obtain the 'true' decomposition formula, we must determine the ghost direct summands. We easily see this problem is reduced to the problem of determination of the parameters in the fair range such that the corrresponding derived functor module is zero. Although some partial results are known (cf. [20] Theorem 1.1, [41] Theorem 6.5, [19] Sect. 5), that remains to be solved in future.

Our result suggests that, for any real linear reductive group, generalized unitary degenerate series are decomposed into a direct sum of irreducible generalized unitary degenerate series. Unfortunately, this is a mere wishful thinking. The easiest counterexample is known by K. Gross [10]. Namely, for $\mathrm{Spin}(5,\mathbb{C})$ we consider the unitary degenerate series for a parabolic subgroup whose Levi part corresponds to the long simple root. The degenerate series decomposed into two irreducible components at the most singular parameter. (Cf. [1]) This decomposition gives a counterexample. This reducibility of the degenerate series arises from the failure of the birationality of the moment map (to its image) of the cotangent bundle of the corresponding generalized flag variety. (For more precise theory, see [3].)

Probably, more mysterious decomposition is the Kashiwara–Vergne decomposition ([15]) for $\mathrm{Sp}(2n+1,\mathbb{R})$, which seems very likely to be another counterexample. At least, some irreducible factors are not the natural candidates, namely 'cohomological Hardy spaces', which are derived functor modules arising from certain Cayley transforms. (For example, it follows from [25, 26].) If all the irreducible factors should be derived functor modules, we would have a counterexample for the Barbasch–Vogan conjecture on the quantized Kostant–Sekiguchi correspondence. This decomposition is very nicely explained in terms of the dual reductive pair. However, I do not know any reasonable description of some irreducible factors in terms of $\mathrm{Sp}(2n+1,\mathbb{R})$ itself.

I was inspired through the effort to understand what happens on the unitary axis in a result in [12]. (There is a tiny overlap between their results and ours.)

List of Symbols

1.1

$A-B$,

$\mathfrak{S}_\ell$,

$[V]$,

1.2

$\mathbb{C}_{\rho(\mathfrak{u})}$,

$\mathcal{R}^i_{\mathfrak{q}}((\mathfrak{l}, L\cap K)\uparrow(\mathfrak{g},K);V), \mathcal{R}^i_{\mathfrak{q}}(\mathfrak{l}\uparrow\mathfrak{g};V)$,

$\mathsf{Ind}(P\uparrow G;V), \mathsf{Ind}^G_P(V)$,

$\mathcal{A}(\mathfrak{q}\uparrow\mathfrak{g};\lambda), \mathcal{A}^{\mathfrak{g}}_{\mathfrak{q}}(\lambda)$,

1.3

$M_\ell(\mathbb{C})$,

$B(X,Y)$,

$\mathbb{C}^\ell[p,q]$,

$e_1, \dots, e_\ell$,
J_ℓ,
$\mathrm{U}(m,n), \mathfrak{u}(m,n), \mathfrak{g}(m,n)$,
$\mathbb{P}_\ell(m,n), \mathbb{P}(m,n)$,
$(\underline{\mathbf{m}}, \underline{\mathbf{n}})$,
$\mathrm{U}(\underline{\mathbf{m}}, \underline{\mathbf{n}}),\ \mathfrak{g}(\underline{\mathbf{m}}, \underline{\mathbf{n}}), \mathrm{K}(\underline{\mathbf{m}}, \underline{\mathbf{n}}), \mathfrak{k}(\underline{\mathbf{m}}, \underline{\mathbf{n}})$,
$(\underline{\mathbf{m}}, \underline{\mathbf{n}})^\sigma$,
$\Phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}, \phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}$,
$\mathbb{P}_d(r), \mathbb{P}(r)$,
$\underline{\mathbf{r}} \star (\underline{\mathbf{p}}, \underline{\mathbf{q}}), \mathrm{U}(\underline{\mathbf{r}}, \underline{\mathbf{r}}) \star \mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$,

1.4
$C_i, c_r(X)$,
$\mathbb{GL}(m, \mathbb{C})$,
$\mathrm{S}(m), \mathfrak{s}(m)$,
$\mathfrak{h}(m,n;s), \Delta(m,n;s), \mathrm{W}(m,n;s)$,
$[m,n;s]_i$,
$\langle\ ,\ \rangle$,
$\mathcal{P}(m,n;s), \mathcal{P}_u(m,n;s)$,
$\mathfrak{h}(m,n;s)^*_c$,
$\Delta^+(m,n;s;\lambda)$,
$\rho_+(m,n;s), \rho_-(m.n;s), \rho(m,n;s)$,
$\mathfrak{z}(m,n), [[m,n]]$,
$\mathfrak{b}_x(m,n;s)$,
$\Delta_i(m,n;s), \Delta_r(m,n;s), \Delta_c(m,n;s)$,
$\mathbb{S}(\underline{\mathbf{m}}, \underline{\mathbf{n}})$,
$\mathfrak{h}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}}), \mathrm{H}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}}), \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}}), \underline{\Delta}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}}), \mathrm{W}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}}), \underline{W}(\underline{\mathbf{m}}, \underline{\mathbf{n}}, \underline{\mathbf{s}}), \Delta_i(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}})$,
$\Delta_r(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}}), \Delta_c(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}})$,
$\mathbb{C}_\lambda(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}})$,
$[\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}}; i]_j, [[\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}}; i]]_+, [[\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}}; i]]_-, [[\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{s}}; i]]$,
$\underline{W}(\underline{\mathbf{r}})$,
$\mathfrak{gl}_{LR}(\underline{\mathbf{r}}, \mathbb{C}), \mathfrak{b}_{LR}(\underline{\mathbf{r}}, \mathbb{C})$,
$\mathrm{S}(\underline{\mathbf{r}}), \mathfrak{s}(\underline{\mathbf{r}})$,

2.1
$\mathfrak{l}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}), L(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$,
$\mathfrak{c}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$,
$[[\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; i]]$,

$\bar{\Delta}^+(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}})$,
$\mathfrak{v}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}), \mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$,
$S_{\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}}$,
$\bar{\rho}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$,
$\overline{\mathcal{P}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}), \overline{\mathcal{P}}^{\star}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}), \overline{\mathcal{P}}^{\star}_u(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$,
$\mathbb{C}_\lambda(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$,
$\mathfrak{m}(\underline{\mathbf{r}}; p, q), \mathrm{M}(\underline{\mathbf{r}}; p, q)$,
$\underline{\Delta}(\underline{\mathbf{r}}; p, q; \underline{\mathbf{t}})$,
$\mathfrak{p}(\underline{\mathbf{r}}; p, q), \mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$,
$\overline{\mathcal{P}}^{+}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}), \overline{\mathcal{P}}^{+}_u(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$,
$\tau(\lambda)$,
$\overline{\overline{\lambda}}_{(k)}, \underline{\underline{\lambda}}_{(k)}$,
$\mathcal{G}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$,
$\mathfrak{q}(\underline{\mathbf{m}}, \underline{\mathbf{m}}), \mathfrak{v}(\underline{\mathbf{m}}, \underline{\mathbf{m}}), \bar{\rho}(\underline{\mathbf{m}}, \underline{\mathbf{m}}), \overline{\mathcal{P}}^{+}(\underline{\mathbf{m}}, \underline{\mathbf{m}}), \mathcal{G}(\underline{\mathbf{m}}, \underline{\mathbf{m}})$,

2.2
$\bar{\Delta}^+_2(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \lambda)$,
$\mathfrak{w}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda), \mathfrak{f}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$,
$\bar{\rho}_2(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$,
$\mathfrak{q}_1(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$,
$f_{\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda}$,
$\sigma(\mu)$,

2.3
$\underline{W}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}), \underline{\Delta}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}), \underline{\Delta}^+(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta), \underline{\mathfrak{n}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta), \underline{\mathfrak{b}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta)$,
$\underline{\mathfrak{b}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta), \underline{\Pi}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta)$,
$\underline{\rho}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; i), \underline{\rho}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}), \rho(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}})$,
$\mathrm{W}(; \underline{\mathbf{t}}), \rho(; \underline{\mathbf{t}}), \underline{\Delta}^+(; \underline{\mathbf{t}})_x, \underline{\Pi}(; \underline{\mathbf{t}})_x, \underline{\mathfrak{b}}(; \underline{\mathbf{t}})_x, \underline{\mathfrak{n}}(; \underline{\mathbf{t}})_x$,
$S_{\underline{\mathbf{t}}, x}$,
$\mathfrak{b}(; \underline{\mathbf{t}})_x, \Delta^+(; \underline{\mathbf{t}})_x, \Pi(; \underline{\mathbf{t}})_x$,
$\mathfrak{b}_1(; \underline{\mathbf{t}}; \lambda)_x, \Delta^+_1(; \underline{\mathbf{t}}; \lambda)_x, \Pi_1(; \underline{\mathbf{t}}; \lambda)_x$,

2.4
$\{i\}_j, \mathrm{ht}(\Delta^+)$,

3.1
$\varepsilon(r)$,
$\lambda(z, \mu)$,

3.2
$\lambda(\underline{\mathbf{z}}, \underline{\mu})$,
$\mathcal{X}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}), \mathcal{Y}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$,
$e(\sigma; i), \hat{\sigma}(i), \underline{\mathbf{r}} \mid \underline{\mathbf{k}}$,

3.3
$C(\lambda)$.

1. Notations

1.1. General notations

In this article, we use the following notations.

As usual we denote the complex number field, the real number field, the rational number field, the ring of integers, and the set of non-negative integers by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively.

For a complex vector space V, we denote by V^* the dual vector space. For a real vector space V_0, we denote by V_0' the real dual vector space of V_0. We denote by $\emptyset$ the empty set and denote by $A - B$ the set theoretical difference of A from B. For each set A, we denote by card A the cardinality of A. For a complex number a (resp. a matrix X over $\mathbb{C}$), we denote by $\bar{a}$ (resp. $\bar{X}$) the complex conjugation. If $p > q$, we put $\Sigma_{i=p}^{q} = 0$.

Let R be a ring and let M be a left R-module. We denote by $\mathrm{Ann}_R(M)$ the annihilator of M in R.

In this article, a character of a Lie group G means a (not necessarily unitary) continuous homomorphism of G to $\mathbb{C}^{\times}$.

For a matrix $X = (a_{ij})$, we denote by ${}^tX, \operatorname{tr} X$, and $\det X$ the transpose (a_{ji}) of X, the trace of X, and the determinant of X respectively.

We denote by $\mathfrak{S}_\ell$ the ℓth symmetric group.

For a complex Lie algebra $\mathfrak{g}$, we denote by $U(\mathfrak{g})$ its universal enveloping algebra.

For a Harish-Chandra module V, we denote by $[V]$ the corresponding distribution character. In this article, an irreducible Harish-Chandra module should be non-zero.

Ad means the adjoint action of a Lie group on its complexified Lie algebra.

1.2. Cohomological induction

Next, we fix the notations on the Vogan–Zuckerman cohomological induction of a Harish-Chandra module (cf. [36, 38, 39, 41], [44, 45], [16], [43], etc.).

Let G be a connected real reductive linear Lie group and let $\mathfrak{g}$ be its complexified Lie algebra. (Assuming connectedness simplifies definitions below a little bit.) We fix a maximal compact subgroup K of G and denote by θ the corresponding complexified Cartan involution of $\mathfrak{g}$. We denote by $\mathfrak{k}$ the complexified Lie algebra of K. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ with a Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ such that $\mathfrak{l}$ is stable under θ and the complex conjugations with respect to G. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ and a Weyl group invariant non-degenerate bilinear form $\langle\ ,\ \rangle$. Let L be the corresponding Levi subgroup in G to $\mathfrak{l}$. Let $\mathbb{C}_{\rho(\mathfrak{u})}$ be a 1-dimensional representation of the metaplectic double cover $L^{\sim}$ of L which is a square root of $x \mapsto \det(\mathrm{Ad}(x)|_{\mathfrak{u}})$ (cf. [41]). Let $(K \cap L)^{\sim}$ be the maximal compact subgroup of $L^{\sim}$ corresponding to $L \cap K$.

Let V be a Harish-Chandra $(\mathfrak{l}, (K \cap L)^{\sim})$-module such that $V \otimes \mathbb{C}_{\rho(\mathfrak{u})}$ is a Harish-Chandra $(\mathfrak{l}, K \cap L)$-module. Introducing the trivial action of $\mathfrak{u}$, we also regard $V \otimes \mathbb{C}_{\rho(\mathfrak{u})}$ as a $\mathfrak{q}$-module. Let λ be the infinitesimal character of V with respect to $\mathfrak{h}$. (It is well-defined up to the Weyl group action of $\mathfrak{l}$.)

Following [41], we introduce the following notions. We call V good (or λ is in the good range), if $\mathrm{Re}\langle\lambda, \alpha\rangle > 0$ holds for each root α of $\mathfrak{h}$ in $\mathfrak{u}$. We assume $[\mathfrak{l}, \mathfrak{l}]$ acts on V trivially. We denote by $\mathfrak{z}$ the center of $\mathfrak{l}$. We call V weakly fair (or λ is in the weakly fair range), if $\mathrm{Re}\langle\lambda|_{\mathfrak{z}}, \alpha\rangle \geqslant 0$ for each root α of $\mathfrak{h}$ in $\mathfrak{u}$.

For a non-negative integer i, we denote by $\mathcal{R}^i_{\mathfrak{q}}((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); V)$ the result of applying the ith Zuckerman functor (namely, the ith right derived functor of taking 'the K-finite part') to $\mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), V \otimes \mathbb{C}_{\rho(\mathfrak{u})})$, where Hom is defined using the left action of $\mathfrak{q}$ on $U(\mathfrak{g})$ and made into a left $\mathfrak{g}$-module using the right action. (For details, see [36], etc.) $\mathcal{R}^i_{\mathfrak{q}}((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); V)$ is zero or a Harish-Chandra $(\mathfrak{g}, K)$-module with a generalized infinitesimal character λ. (For example, we can prove this as follows. Using induction on $\|\ \|_{\mathrm{lambda}}$ (cf. [38] Corollary 3.25) and long exact sequences, we can reduce the problem to the case of standard modules, namely the case that $\mathfrak{q}$ is a Borel subalgebra. Then, we can apply the Hecht–Miličić–Schmid–Wolf duality theorem and can reduce the problem to the case of standard D-modules.) If $L \cap K$ is connected, we simply write $\mathcal{R}^i_{\mathfrak{q}}(\mathfrak{l} \uparrow \mathfrak{g}; V)$ for $\mathcal{R}^i_{\mathfrak{q}}((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); V)$. In fact, for $\mathrm{U}(m, n)$ this connectedness condition always holds.

The following three particular cases are important (in this article).

(1) (*Hyperbolic induction*). If $\mathfrak{q}$ is stable under the complex conjugation of $\mathfrak{g}$ with respect to G, there is a parabolic subgroup $Q = LU$ whose complexified Lie algebra is $\mathfrak{q}$ and whose nilradical is U. In this case, we have $\mathcal{R}^i_{\mathfrak{q}}((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); V) = 0$ for all $i > 0$ and $\mathcal{R}^0_{\mathfrak{q}}((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); V)$ coincides with the usual parabolic induction $\mathsf{Ind}(Q \uparrow G; V)$ (We also write $\mathsf{Ind}_Q^G(V)$.). Here $\mathsf{Ind}(Q \uparrow G; V)$ is the K-finite part of

$$\{f \in C^{\infty}(G) \otimes H \mid f(g\ell n) = \pi(\ell^{-1}) f(g) \ \ (g \in G, \ell \in L, n \in U)\}.$$

Here, (π, H) is any Hilbert globalization of $V \otimes \mathbb{C}_{\rho(\mathfrak{u})}$. If V is unitarizable, so is $\mathsf{Ind}(Q \uparrow G; V)$ (unitary induction). If L is connected and if $\mathbb{C}_{\lambda}$ is the one-dimensional L-representation with infinitesimal character λ, we simply write $\mathrm{Ind}_Q^G(\lambda)$ or $\mathrm{Ind}(Q \uparrow G; \lambda)$ for $\mathrm{Ind}_Q^G(\mathbb{C}_\lambda)$.

We have the following additive property.

Let $V_0, V_1, \ldots, V_k$ be Harish-Chandra $(\mathfrak{l}, (L \cap K)^\sim)$-modules such that $V_0 \otimes \mathbb{C}_{\rho(\mathfrak{u})}, \ldots, V_k \otimes \mathbb{C}_{\rho(\mathfrak{u})}$ are reduced to Harish-Chandra $(\mathfrak{l}, L \cap K)$-modules. Let $n_1, \ldots, n_k$ be integers. If we have a character identity $[V] = \Sigma_{i=1}^k n_i[V_i]$, we have $[\mathrm{Ind}(Q \uparrow G; V)] = \Sigma_{i=1}^k n_k[\mathrm{Ind}(Q \uparrow G; V_i)]$.

(2) (*Elliptic induction*). Assume $\mathfrak{q}$ is θ-stable and put $S = \dim(\mathfrak{u} \cap \mathfrak{k})$. If V is good, then $\mathcal{R}_\mathfrak{q}^i((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); V) = 0$ for $i \neq S$. If V is irreducible (resp. unitarizable) and good, $\mathcal{R}_\mathfrak{q}^S((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); V)$ is also irreducible (resp. unitarizable) ([38]).

Moreover, we assume $[\mathfrak{l}, \mathfrak{l}]$ acts on V trivially and V is irreducible and unitary. In this case, V is 1-dimensional and we denote it by $\mathbb{C}_\lambda$. Assume that λ is in the weakly fair range. Then, $\mathcal{R}_\mathfrak{q}^i((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); \mathbb{C}_\lambda) = 0$ for all $i \neq S$. We also have $\mathcal{R}_\mathfrak{q}^i((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); \mathbb{C}_\lambda)$ is either zero or non-zero unitarizable Harish-Chandra module ([38]). In this case, we put

$$\mathcal{A}(\mathfrak{q} \uparrow \mathfrak{g}; \lambda) = \mathcal{A}_\mathfrak{q}^\mathfrak{g}(\lambda) = \mathcal{R}_\mathfrak{q}^S((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); \mathbb{C}_\lambda).$$

We call it a derived functor module. (Our parameterization is different from that of [43].) In general, $\mathcal{A}(\mathfrak{q} \uparrow \mathfrak{g}; \lambda)$ may be reducible for some weakly fair λ. (An example occurs in $\mathrm{SO}(5, \mathbb{C})$ (K. Gross [10], Borho). Another example by D. A. Vogan is described in [4].) However, if $G = \mathrm{U}(m, n)$ (or other type A groups), the situation is much simpler. The moment map of cotangent bundle of any generalized flag variety of $\mathrm{GL}(n, \mathbb{C})$ is birational to its image and the closure of any nilpotent orbit of $\mathfrak{gl}(n, \mathbb{C})$ is normal. Hence the canonical homomorphism of the universal enveloping algebra to the ring of global sections of the sheaf of twisted differential operators on a generalized flag variety of $\mathrm{GL}(n, \mathbb{C})$ is always surjective (cf. [4] 6.2 Proposition (a)). Hence, if $G = \mathrm{U}(m, n)$, $\mathcal{A}(\mathfrak{q} \uparrow \mathfrak{g}; \lambda)$ is zero or irreducible for any weakly fair λ (cf. [41], [4]).

The additivity property of $\mathcal{R}_\mathfrak{q}^S$ in this case is described as follows. We fix an infinitesimal character λ in the good range with respect to $\mathfrak{q}$. Let $V_0, V_1, \ldots, V_k$ be Harish-Chandra $(\mathfrak{l}, (L \cap K)^\sim)$-modules with infinitesimal character λ and let $n_1, \ldots, n_k$ be integers. We also assume $V_i \otimes \mathbb{C}_{\rho(\mathfrak{u})}$ is reduced to a Harish-Chandra $(\mathfrak{l}, L \cap K)$-module for each $0 \leqslant i \leqslant k$. Moreover we assume a character identity $[V] = \Sigma_{i=1}^k n_i[V_i]$ holds. Then, we have $[\mathcal{R}_\mathfrak{q}^S((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); V)] = \Sigma_{i=1}^k n_k[\mathcal{R}_\mathfrak{q}^S((\mathfrak{l}, L \cap K) \uparrow (\mathfrak{g}, K); V_i)]$.

(3) (*Standard modules*). The case of $\mathfrak{q}$ = Borel. In this case, the induced modules are called standard representations. Such representations are precisely studied in [34], [36], [11], [31], [29, 30], [22], etc.

Finally, we recall the notion of induction-by-stage ([36] Corollary 6.3.10). Let $\mathfrak{q}_i$ $(i = 1, 2)$ be parabolic subalgebras of $\mathfrak{g}$ with Levi decompositions $\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i$ such that $\mathfrak{l}_i$ is stable under θ and the complex conjugation with respect to G. We denote by L_i the subgroup of G corresponding to $\mathfrak{l}_i$. Moreover, we assume $\mathfrak{q}_2 \subset \mathfrak{q}_1$. Then $\mathfrak{q}_3 = \mathfrak{q}_2 \cap \mathfrak{l}_1$ is a parabolic subalgebra of $\mathfrak{l}_1$ with a Levi decomposition $\mathfrak{q}_3 = \mathfrak{l}_2 + (\mathfrak{u}_2 \cap \mathfrak{l}_1)$. Put $\mathfrak{u}_3 = \mathfrak{u}_2 \cap \mathfrak{l}_1$. Let p be a non-negative integer and let V be a Harish-Chandra $(\mathfrak{l}_2, (L_2 \cap K)^{\sim})$-module such that $V \otimes \mathbb{C}_{\rho(\mathfrak{u}_2)}$ is reduced to a Harish-Chandra $(\mathfrak{l}_2, L_2 \cap K)$-module. Assume that $\mathcal{R}^i_{\mathfrak{q}_3}((\mathfrak{l}_2, L_2 \cap K) \uparrow (\mathfrak{l}_1, L_1 \cap K); V \otimes \mathbb{C}_{\rho(\mathfrak{u}_1)}) = 0$ for all $i \neq p$. Then, for all $q \geqslant 0$, we have

$$\begin{aligned} &\mathcal{R}^q_{\mathfrak{q}_1}((\mathfrak{l}_1, L_1 \cap K) \uparrow (\mathfrak{g}, K); \mathcal{R}^p_{\mathfrak{q}_3}((\mathfrak{l}_2, L_2 \cap K) \\ &\quad \uparrow (\mathfrak{l}_1, L_1 \cap K); V \otimes \mathbb{C}_{\rho(\mathfrak{u}_1)}) \otimes \mathbb{C}_{-\rho(\mathfrak{u}_1)}) \\ &\quad \cong \mathcal{R}^{p+q}_{\mathfrak{q}_2}((\mathfrak{l}_2, L_2 \cap K) \uparrow (\mathfrak{g}, K); V). \end{aligned}$$

If $\mathbb{C}_{\rho(\mathfrak{u}_1)}$ is reduced to a character of L_1, we can rewrite the above formula as follows.

$$\begin{aligned} &\mathcal{R}^q_{\mathfrak{q}_1}((\mathfrak{l}_1, L_1 \cap K) \uparrow (\mathfrak{g}, K); \mathcal{R}^p_{\mathfrak{q}_3}((\mathfrak{l}_2, L_2 \cap K) \uparrow (\mathfrak{l}_1, L_1 \cap K); V)) \\ &\quad \cong \mathcal{R}^{p+q}_{\mathfrak{q}_2}((\mathfrak{l}_2, L_2 \cap K) \uparrow (\mathfrak{g}, K); V). \end{aligned}$$

1.3. Notations for U(m, n)

Let ℓ be a positive integer. We denote by $M_\ell(\mathbb{C})$ the space of complex $\ell \times \ell$-matrices. We regard $\mathbb{C}^\ell$ as a space of vertical vectors. $M_\ell(\mathbb{C})$ and the general linear group $\mathrm{GL}(\ell, \mathbb{C})$ act on $\mathbb{C}^\ell$ by the left multiplication. We denote by $\mathfrak{gl}(\ell, \mathbb{C})$ the Lie algebra of $\mathrm{GL}(\ell, \mathbb{C})$ and identify it with $M_\ell(\mathbb{C})$. We denote by I_ℓ the identity matrix in $M_\ell(\mathbb{C})$. For $X, Y \in \mathfrak{gl}(\ell, \mathbb{C})$, we put

$$B(X, Y) = \operatorname{tr} XY.$$

Then, B is a non-degenerate symmetric bilinear form on $\mathfrak{gl}(\ell, \mathbb{C})$ which is invariant under the adjoint action. We denote by $E_{p,q}$ the matrix such that its (p, q)-entry is 1 and the other entries are 0. For $1 \leqslant k \leqslant \ell$, we denote by $e_k \in \mathbb{C}^\ell$ the vertical vector such that its i-th entry is 1 and the other entries are zero. We call $e_1, \ldots, e_\ell$ the standard basis of $\mathbb{C}^\ell$. For $1 \leqslant p \leqslant q \leqslant \ell$, we define

$$\mathbb{C}^\ell[p, q] = \sum_{k=p}^{q} \mathbb{C} e_k.$$

If $q < p$, we put $\mathbb{C}^\ell[p, q] = \{0\}$. We define a matrix J_ℓ in $M_\ell(\mathbb{C})$ as follows.

$$J_\ell = \sum_{i=1}^{\ell} E_{i, \ell - i + 1}.$$

We fix non-negative integers m and n such that $m+n>0$. We fix the following non-degenerate indefinite Hermitian forms on $\mathbb{C}^{m+n}$ as follows.

$$\left(\begin{pmatrix} x_1 \\ \vdots \\ x_{m+n} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{m+n} \end{pmatrix}\right)$$
$$= x_1\overline{y_1} + \cdots + x_m\overline{y_m} - x_{m+1}\overline{y_{m+1}} - \cdots - x_{m+n}\overline{y_{m+n}}$$
$$= (\overline{y_1}, \ldots, \overline{y_{m+n}}) \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{m+n} \end{pmatrix}.$$

We consider the following realization of the indefinite group

$$\mathrm{U}(m,n) = \{g \in \mathrm{GL}(m+n,\mathbb{C}) \mid (gx, gy) = (x,y)\ (x,y \in V)\}.$$

We denote by $\mathfrak{u}(m,n)$ the Lie algebra of $\mathrm{U}(m,n)$. Namely, we put

$$\mathfrak{u}(m,n) = \{X \in \mathfrak{gl}(m+n,\mathbb{C}) \mid (Xx,y) + (x,Xy) = 0\ \ (x,y \in V)\},$$

Then $\mathfrak{gl}(m+n,\mathbb{C})$ can be regarded as the complexification of $\mathfrak{u}(m,n)$. In order to make our notation systematic, we also denote $\mathfrak{gl}(m+n,\mathbb{C})$ by $\mathfrak{g}(m,n)$. We also mean by $\mathrm{U}(0,0)$ the trivial group $\{1\}$ and by $\mathfrak{u}(0,0)$ the trivial Lie algebra $\{0\}$.

For $a,b,c,d \in \mathbb{N}$ such that $0 \leqslant a \leqslant b \leqslant m$ and $0 \leqslant c \leqslant d \leqslant n$, we put

$$\mathrm{U}(m,n;a,b;c,d) = \{g \in \mathrm{U}(m,n) \mid g \text{ satisfies the following (1), (2).}\}.$$

(1) The restriction of g to $\mathbb{C}^{m+n}[1,a] + \mathbb{C}^{m+n}[b+1,m+c] + \mathbb{C}^{m+n}[m+d+1,m+n]$ is the identity.

(2) $g(\mathbb{C}^{m+n}[a+1,b] + \mathbb{C}^{m+n}[m+c+1,m+d]) \subseteq \mathbb{C}^{m+n}[a+1,b] + \mathbb{C}^{m+n}[m+c+1,m+d]$.

We consider an isomorphism ι of $\mathbb{C}^{m+n}[a+1,b] + \mathbb{C}^{m+n}[m+c+1,m+d]$ to $\mathbb{C}^{b+d-a-c}$ as follows.

$$\iota(e_i) = \begin{cases} e_{i-a} & (a+1 \leqslant i \leqslant b) \\ e_{(b-a)+(i-m-c)} & (m+c+1 \leqslant i \leqslant m+d) \end{cases}.$$

Using this ι, we can identify $\mathrm{U}(m,n;a,b;c,d)$ and $\mathrm{U}(b-a,d-c)$. We call such an identification the standard identification.

Let ℓ be a positive integer. Put

$$\mathbb{P}_\ell(m,n) = \left\{((m_1,\ldots,m_\ell),(n_1,\ldots,n_\ell)) \in \mathbb{N}^\ell \times \mathbb{N}^\ell \mid \sum_{i=1}^{\ell} m_i \right.$$
$$\left. = m, \sum_{i=1}^{\ell} n_i = n, m_j + n_j > 0 \text{ for all } 1 \leqslant j \leqslant \ell \right\},$$

We also put

$$\mathbb{P}(m,n) = \bigcup_{\ell>0} \mathbb{P}_\ell(m,n),$$
$$\mathbb{P}(0,0) = \mathbb{P}_0(0,0) = \{((\emptyset),(\emptyset))\}.$$

If $(\underline{\mathbf{m}},\underline{\mathbf{n}}) \in \mathbb{P}(m,n)$ satisfies $(\underline{\mathbf{m}},\underline{\mathbf{n}}) \in \mathbb{P}_\ell(m,n)$, we call ℓ the length of $(\underline{\mathbf{m}},\underline{\mathbf{n}})$.

For $(\underline{\mathbf{m}},\underline{\mathbf{n}}) = ((m_1,\dots,m_\ell),(n_1,\dots,n_\ell)) \in \mathbb{P}_\ell(m,n)$, we denote by $\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ the subgroup of $\mathrm{U}(m,n)$ generated by $\mathrm{U}(m,n;\Sigma_{i=1}^{k-1} m_i, \Sigma_{i=1}^{k} m_i; m+n-\Sigma_{i=1}^{k} n_i, m+n-\Sigma_{i=1}^{k-1} n_i)$ for all $1 \leqslant k \leqslant \ell$. Through the standard identification, we have the standard identifications of $\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ as follows.

$$\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}}) \cong \mathrm{U}(m_1,n_1) \times \mathrm{U}(m_2,n_2) \times \cdots \times \mathrm{U}(m_\ell,n_\ell).$$

If $m=n=0$, we put $\mathrm{U}((\emptyset),(\emptyset)) = \{1\}$.

We denote the Lie subalgebra of $\mathfrak{g}(m,n)$ corresponding to $\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ by $\mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}})$. Then, we have the following standard identifications:

$$\mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}}) \cong \mathfrak{g}(m_1,n_1) \times \mathfrak{g}(m_2,n_2) \times \cdots \times \mathfrak{g}(m_\ell,n_\ell).$$

We put $\mathrm{K}(m,n) = \mathrm{U}((m,0),(0,n)) \cong \mathrm{U}(m) \times \mathrm{U}(n)$ and $\mathfrak{k}(m,n) = \mathfrak{g}((m,0),(0,n))$. $\mathrm{K}(m,n)$ is a maximal compact subgroup of $\mathrm{U}(m,n)$. We denote by θ the complexified Cartan involution of $\mathfrak{g}(m,n)$ with respect to $\mathfrak{k}(m,n)$.

For $(\underline{\mathbf{m}},\underline{\mathbf{n}}) \in \mathbb{P}(m,n)$, $\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}})$, $\mathfrak{u}(\underline{\mathbf{m}},\underline{\mathbf{n}})$, and $\mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ are θ-stable. Put $\mathrm{K}(\underline{\mathbf{m}},\underline{\mathbf{n}}) = \mathrm{K}(m,n) \cap \mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ and denote by $\mathfrak{k}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ the $+1$-eigenspace with respect to θ in $\mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}})$. $\mathrm{K}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ is a maximal compact subgroup of $\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ and $\mathfrak{k}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ is the complexified Lie algebra of $\mathrm{K}(\underline{\mathbf{m}},\underline{\mathbf{n}})$.

For $(\underline{\mathbf{m}},\underline{\mathbf{n}}) = ((m_1,\dots,m_\ell),(n_1,\dots,n_\ell)) \in \mathbb{P}_\ell(m,n)$ and $\sigma \in \mathfrak{S}_\ell$, we put

$$(\underline{\mathbf{m}},\underline{\mathbf{n}})^\sigma = (\underline{\mathbf{m}}^\sigma,\underline{\mathbf{n}}^\sigma) = ((m_{\sigma(1)},\dots,m_{\sigma(\ell)}),(n_{\sigma(1)},\dots,n_{\sigma(\ell)})).$$

We define a group homomorphism $\phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\colon \mathfrak{S}_\ell \to \mathfrak{S}_{m+n}$ as follows.

$$\phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma)(i) = \begin{cases} i + \sum_{\sigma(j)<\sigma(i)} m_{\sigma(j)} - \sum_{k=1}^{h-1} m_k & \text{if } \sum_{k=1}^{h-1} m_k \leqslant i < \sum_{k=1}^{h} m_k \\ i + \sum_{\sigma(j)>\sigma(i)} n_{\sigma(j)} - \sum_{k=h+1}^{\ell} n_k & \text{if } \sum_{k=h+1}^{\ell} n_k \leqslant i-m < \sum_{k=h}^{\ell} n_k \end{cases}$$

Here, $\sigma \in \mathfrak{S}_\ell$, $1 \leqslant i \leqslant m+n$, and $1 \leqslant h \leqslant \ell$.

For $\sigma \in \mathfrak{S}_\ell$, we define $\Phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma) \in \mathrm{GL}(m+n,\mathbb{C})$ by $\Phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma)e_i = e_{\phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma)(i)}$ for $1 \leqslant i \leqslant m+n$. Immediately, we see $\Phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma) \in \mathrm{K}(m,n)$ and $\Phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma)\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}})\Phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma)^{-1} = \mathrm{U}(\underline{\mathbf{m}}^\sigma,\underline{\mathbf{n}}^\sigma)$ and $\mathrm{Ad}(\Phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma))\mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}}) = \mathfrak{g}(\underline{\mathbf{m}}^\sigma,\underline{\mathbf{n}}^\sigma)$.

For any subgroup H of $\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}})$, we denote by H^σ the subgroup $\Phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma) H \Phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma)^{-1}$ of $\mathrm{U}(\underline{\mathbf{m}}^\sigma, \underline{\mathbf{n}}^\sigma)$.

Let r be a non-negative integer such that $r \leqslant \min\{m, b\}$ and let d be a positive integer. We put

$$\mathbb{P}_d(r) = \{(r_1, \ldots, r_d) \in (\mathbb{N} - \{0\})^d \mid r_1 + \cdots + r_d = r\}.$$

We also put

$$\mathbb{P}(r) = \bigcup_{\ell > 0} \mathbb{P}_\ell(m, n),$$
$$\mathbb{P}(0) = \mathbb{P}_0(0) = \{(\emptyset)\}.$$

For $\underline{\mathbf{r}} \in \mathbb{P}_d(r)$, we regard $(\underline{\mathbf{r}}; \underline{\mathbf{r}}) \in \mathbb{P}_d(r, r)$.

For $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) = ((p_1, \ldots, p_\ell), (q_1, \ldots, q_\ell)) \in \mathbb{P}_\ell(m-r, n-r)$ and $\underline{\mathbf{r}} = (r_1, \ldots, r_d) \in \mathbb{P}_d(r)$, we define

$$\begin{aligned} \underline{\mathbf{r}} \star (\underline{\mathbf{p}}, \underline{\mathbf{q}}) &= (\underline{\mathbf{r}} \star \underline{\mathbf{p}}, \underline{\mathbf{r}} \star \underline{\mathbf{q}}) \\ &= ((r_1, \ldots, r_d, p_1, \ldots, p_\ell), (r_1, \ldots, r_d, q_1, \ldots, q_\ell)) \in \mathbb{P}_{\ell+d}(m, n). \end{aligned}$$

We also put $(\emptyset) \star (\underline{\mathbf{p}}, \underline{\mathbf{q}}) = (\underline{\mathbf{p}}, \underline{\mathbf{q}})$.

For $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}(m-r, n-r)$ and $\underline{\mathbf{r}} \in \mathbb{P}(r)$, we define

$$\mathrm{U}(\underline{\mathbf{r}}, \underline{\mathbf{r}}) \star \mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}}) = \mathrm{U}(\underline{\mathbf{r}} \star \underline{\mathbf{p}}, \underline{\mathbf{r}} \star \underline{\mathbf{q}})).$$

Comparing the standard identifications of $\mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$, $\mathrm{U}(\underline{\mathbf{r}}, \underline{\mathbf{r}})$, and $\mathrm{U}(\underline{\mathbf{r}}, \underline{\mathbf{r}}) \star \mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$, we obtain the standard identification $\mathrm{U}(\underline{\mathbf{r}}, \underline{\mathbf{r}}) \star \mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \cong \mathrm{U}(\underline{\mathbf{r}}, \underline{\mathbf{r}}) \times \mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$. For any subgroup H_1 of $\mathrm{U}(\underline{\mathbf{r}}, \underline{\mathbf{r}})$ and any subgroup H_2 of $\mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$, we denote by $H_1 \star H_2$ the subgroup of $\mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \star \mathrm{U}(\underline{\mathbf{r}}, \underline{\mathbf{r}})$ corresponding to $H_1 \times H_2$ under the standard identification.

We have $\mathrm{K}(\underline{\mathbf{r}}, \underline{\mathbf{r}}) \star \mathrm{K}(\underline{\mathbf{m}}, \underline{\mathbf{n}}) = \mathrm{K}(\underline{\mathbf{r}} \star \underline{\mathbf{m}}, \underline{\mathbf{r}} \star \underline{\mathbf{n}}))$. We use similar notations for the Lie algebras.

1.4. Cayley transforms and Cartan subalgebras

Let m, n be non-negative integers such that $m + n > 0$ as above. For $1 \leqslant i \leqslant \min\{m, n\}$, we put

$$\begin{aligned} C_i &= \exp[\tfrac{1}{4}\pi(-E_{i,m+n-i+1} + E_{m+n-i+1,i})] \\ &= \frac{1}{\sqrt{2}}(E_{i,i} - E_{i,m+n-i+1} + E_{m+n-i+1,i} + E_{m+n-i+1,m+n-i+1}) \\ &\quad + I_{m+n} - E_{ii} - E_{m+n-i+1,m+n-i+1}. \end{aligned}$$

Clearly, C_i's are commutative with each other and we have ${}^t\bar{C}_i = C_i^{-1}$.

For $1 \leqslant r \leqslant \min\{m,n\}$ and $X \in M_{m+n}(\mathbb{C})$, we put

$$c_r(X) = C_r C_{r-1} \dots C_1 X C_1^{-1} C_2^{-1} \dots C_r^{-1}.$$

We also put $c_0(X) = X$. We can regard c_r as an inner automorphism of both $\mathrm{GL}(m+n,\mathbb{C})$ and $\mathfrak{g}(m,n)$.

For a while, we are considering a special case $m = n$. In this case, $C_m C_{m-1} \dots C_1 \in \mathrm{GL}(2m,\mathbb{C})$ is written as follows.

$$C_m C_{m-1} \dots C_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} I_m & -J_m \\ J_m & I_m \end{pmatrix}.$$

We denote by $\mathbb{U}(m,m)$ the subgroup in $\mathrm{GL}(2m,\mathbb{C})$ such that $c_m(\mathbb{U}(m,m)) = \mathrm{U}(m,m)$. We call $\mathbb{U}(m,m)$ the twisted realization of $\mathrm{U}(m,m)$.

$\mathbb{U}(m,m)$ contains the following subgroup.

$$c_m^{-1}(\mathbb{GL}(m,\mathbb{C})) = \left\{ \begin{pmatrix} A & 0 \\ 0 & J_m {}^t\bar{A}^{-1} J_m \end{pmatrix} \middle| A \in \mathrm{GL}(m,\mathbb{C}) \right\}.$$

Hence $\mathbb{GL}(m,\mathbb{C})$ is a subgroup of $\mathrm{U}(m,m)$. We define the standard identification $\mathbb{GL}(m,\mathbb{C}) \cong \mathrm{GL}(m,\mathbb{C})$ by

$$c_m\left(\begin{pmatrix} A & 0 \\ 0 & J_m {}^t\bar{A}^{-1} J_m \end{pmatrix} \right) \leftrightarrow A.$$

For $\underline{\mathbf{m}} = (m_1, \dots, m_\ell) \in \mathbb{P}(m)$, we denote by $\mathbb{GL}(\underline{\mathbf{m}},\mathbb{C})$ the subgroup of $\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{m}})$ corresponding to $\mathbb{GL}(m_1,\mathbb{C}) \times \cdots \times \mathbb{GL}(m_\ell,\mathbb{C})$ via the standard identification of $\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{m}})$. Put $\mathbb{GL}((\emptyset),\mathbb{C}) = \{1\}$. We also consider the following Lie subalgebras in $\mathfrak{g}(m,m)$.

$$\mathfrak{gl}_L(m,\mathbb{C}) = c_m\left(\left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{gl}(m,\mathbb{C}) \right\} \right),$$

$$\mathfrak{gl}_R(m,\mathbb{C}) = c_m\left(\left\{ \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \middle| B \in \mathfrak{gl}(m,\mathbb{C}) \right\} \right).$$

Then $\mathfrak{gl}_L(m,\mathbb{C}) \times \mathfrak{gl}_R(m,\mathbb{C})$ is the complexified Lie algebra of $\mathbb{GL}(m,\mathbb{C})$.

We denote by $\mathfrak{b}(m,\mathbb{C})$ the Borel subalgebra of $\mathfrak{gl}(m,\mathbb{C})$ consisting of the upper triangular matrices. Put

$$\mathfrak{b}_L(m,\mathbb{C}) = c_m\left(\left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{b}(m,\mathbb{C}) \right\} \right),$$

$$\mathfrak{b}_R(m,\mathbb{C}) = c_m\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \,|B \in \mathfrak{b}(m,\mathbb{C})\right\}\right).$$

We put

$$S(m) = c_m\left(\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{U}(m,m)\right\}\right),$$

$$\mathfrak{s}(m) = c_m\left(\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathfrak{g}(m,m)\right\}\right).$$

We call $\mathrm{S}(m)$ (resp. $\mathfrak{s}(m)$) the Siegel parabolic subgroup (resp. subalgebra) of $\mathrm{U}(m,m)$ (resp. $\mathfrak{g}(m,m)$).

Next, we fix the complexification of a compact Cartan subalgebra of $\mathfrak{u}(m,n)$ as follows.

$$\mathfrak{h}(m,n;0) = \sum_{i=1}^{m+n} \mathbb{C}E_{i,i} \subseteq \mathfrak{g}(m,n).$$

For $1 \leqslant s \leqslant \min\{m,n\}$, we consider the following Cartan subalgebras:

$$\mathfrak{h}(m,n;s) = c_s(\mathfrak{h}(m,n;0)).$$

$\{\mathfrak{h}(m,n;0), \mathfrak{h}(m,n;1), \ldots, \mathfrak{h}(m,n;\min\{m,n\})\}$ forms a complete system of representatives of $\mathrm{K}(m,n)$ conjugacy class of Cartan subalgebras of $\mathfrak{g}(m,n)$ stable under θ and complex conjugation with respect to $\mathfrak{u}(m,n)$. We introduce a symmetric bilinear form $\langle\,,\rangle$ induced from $B(\,,)$ on $\mathfrak{h}(m,n;s)^*$. Of course, the Cayley transform c_s induces an isometry of $\mathfrak{h}(m,n;0)^*$ to $\mathfrak{h}(m,n;s)^*$. We denote by $\Delta(m,n;s)$ (resp. $\mathrm{W}(m,n;s)$) the root system (resp. the Weyl group) of $(\mathfrak{g}(m,n), \mathfrak{h}(m,n;s))$. We denote by $[m,n;s]_1, \ldots, [m,n;s]_{m+n}$ the dual basis of $\mathfrak{h}(m,n;s)^*$ to $c_s(E_{1,1}), \ldots, c_s(E_{m+n,m+n})$. We call $[m,n;s]_1, \ldots, [m,n;s]_{m+n}$ the standard basis of $\mathfrak{h}(m,n;s)^*$. Then we have:

$$\Delta(m,n;s) = \{[m,n;s]_i - [m,n;s]_j \mid 1 \leqslant i,j \leqslant m+n\}.$$

$\mathrm{W}(m,n;s)$ is nothing but the group consisting of all the permutations of $[m,n;s]_1$ $,\ldots, [m,n;s]_{m+n}$. We shall identify an element of the Weyl group $\mathrm{W}(m,n;s)$ and the corresponding permutation of $1,\ldots,m+n$ so that $w[m,n;s]_i = [m,n;s]_{w(i)}$ for all $w \in \mathrm{W}(m,n;s)$ and $1 \leqslant i \leqslant m+n$. Then $\langle\,,\rangle$ is a Weyl group invariant non-degenerate symmetric bilinear form on $\mathfrak{h}(m,n;s)^*$ such that $[m,n;s]_1,\ldots,$ $[m,n;s]_{m+n}$ form an orthonormal basis of $\mathfrak{h}(m,n;s)^*$. Put

$$\mathcal{P}(m,n;s) = \left\{ \sum_{i=1}^{m+n} \lambda_i[m,n;s]_i \in \mathfrak{h}(m,n;s)^* \mid \lambda_i + \lambda_{m+n-i+1} \in \mathbb{Z}\,(1 \leqslant i \leqslant s), \lambda_j \in \mathbb{Z}\,(s < j \leqslant m-s) \right\},$$

$$\mathcal{P}_u(m,n;s) = \left\{ \sum_{i=1}^{m+n} \lambda_i[m,n;s]_i \in \mathcal{P}(m,n;s) \mid \lambda_i - \lambda_{m+n-i+1} \in \sqrt{-1}\mathbb{R} \right\}.$$

$$\mathfrak{h}(m,n;s)^*_c = \left\{ \sum_{i=1}^{m+n} \lambda_i[m,n;s]_i \in \mathfrak{h}(m,n;s)^* \mid \operatorname{Re}\lambda_i \neq \operatorname{Re}\lambda_j \; (s < i < j \leqslant m-s) \right\}.$$

Let H$(m,n;s)$ be the Cartan subgroup of U(m,n) corresponding to $\mathfrak{h}(m,n;s)$. They are all connected. We easily see $\lambda \in \mathfrak{h}(m,n)^*$ can be exponentiated to a character (resp. a unitary character) of H$(m,n;s)$ if and only if $\lambda \in \mathcal{P}(m,n)$ (resp. $\lambda \in \mathcal{P}_u(m,n;s)$).

For $\lambda = \Sigma_{i=1}^{m+n} \lambda_i[m,n;s]_i \in \mathfrak{h}(m,n)^*_c$, we define a positive system of $\Delta(m,n;s)$ as follows.

$$\begin{aligned}\Delta^+(m,n;s;\lambda) = {} & \{[m,n;s]_i - [m,n;s]_j \mid 1 \leqslant i \leqslant s, i < j \leqslant m+n\} \\ & \cup\{[m,n;s]_i - [m,n;s]_j \mid m+n-s < j \\ & \leqslant m+n, i < j \leqslant m+n\} \\ & \cup\{[m,n;s]_i - [m,n;s]_j \mid s < i, j \leqslant m+n-s, \lambda_i > \lambda_j\}.\end{aligned}$$

For $\lambda \in \mathfrak{h}(m,n)^*_c$, we denote by $\mathfrak{n}(m,n;s;\lambda)$ the sum of all the root space in $\mathfrak{g}(m,n)$ corresponding to $\Delta^+(m,n;s;\lambda)$. Put $\mathfrak{b}(m,n;s;\lambda) = \mathfrak{h}(m,n;s) + \mathfrak{n}(m,n;s;\lambda)$. Then $\mathfrak{b}(m,n;s;\lambda)$ is a Borel subalgebra of $\mathfrak{g}(m,n)$ with the nilradical $\mathfrak{n}(m,n;s;\lambda)$. This Borel subalgebra is maximally real in the sense of [29,30].

We also put as follows.

$$\rho_+(m,n;s) = \sum_{i=1}^{m} \left(\frac{m+1}{2} - i\right)[m,n;s]_i,$$

$$\rho_-(m,n;s) = \sum_{i=1}^{n} \left(\frac{n+1}{2} - i\right)[m,n;s]_{m+i},$$

$$\rho(m,n;s) = \sum_{i=1}^{m+n} \left(\frac{m+n+1}{2} - i\right)[m,n;s]_i.$$

For $x \in \mathrm{W}(m,n;s)$, we put $\mathfrak{b}_x(m,n;s) = \mathfrak{b}(m,n;s;x\rho(m,n;s))$ and $\mathfrak{n}(m,n;s) = \mathfrak{n}(m,n;s;x\rho(m,n;s))$.

We denote by $\mathfrak{z}(m,n)$ the center of $\mathfrak{g}(m,n)$. Namely $\mathfrak{z}(m,n)$ is the one-dimensional subalgebra consisting of scalar matrices. Using $\langle\, ,\rangle$, we can regard $\mathfrak{z}(m,n)^*$ as a subspace of $\mathfrak{h}(m,n;s)^*$. If we put

$$[[m,n]] = \sum_{i=1}^{m+n} [m,n;s]_i,$$

then $[[m,n]]$ does not depend on s and we have $[[m,n]] \in \mathfrak{z}(m,n)^*$.

The Cartan involution θ acts on $\mathfrak{h}(m,n;s)$ as follows.

$$\theta([m,n;s]_i) = \begin{cases} [m,n;s]_{m+n-i+1} & \text{if } 1 \leqslant i \leqslant s \text{ or } m+n-s < i \leqslant m+n \\ [m,n;s]_i & \text{if } s < i \leqslant m+n-s \end{cases}.$$

Put

$$\begin{aligned} \Delta_i(m,n;s) &= \{\alpha \in \Delta(m,n;s) \mid \theta(\alpha) = \alpha\} \\ &= \{[m,n;s]_i - [m,n;s]_j \mid s < i,j < m-s+1\}, \\ \Delta_r(m,n;s) &= \{\alpha \in \Delta(m,n;s) \mid \theta(\alpha) = -\alpha\} \\ &= \{[m,n;s]_i - [m,n;s]_{m-i+1} \mid 0 < i \leqslant s \\ &\qquad \text{or } m-s < i \leqslant m\}, \\ \Delta_c(m,n;s) &= \{\alpha \in \Delta(m,n;s) \mid \theta(\alpha) \neq \pm\alpha\}. \end{aligned}$$

An element of $\Delta_i(m,n;s), \Delta_r(m,n;s)$, and $\Delta_c(m,n;r)$ are called imaginary, real, and complex, respectively.

Next, we fix $(\underline{\mathbf{m}},\underline{\mathbf{n}}) = ((m_1,\dots,m_\ell),(n_1,\dots,n_\ell)) \in \mathbb{P}_\ell(m,n)$. We put

$$\mathbb{S}(\underline{\mathbf{m}},\underline{\mathbf{n}}) = \{(s_1,\dots,s_\ell) \in \mathbb{N}^\ell \mid s_i \leqslant \min\{m_i,n_i\} \ \ (1 \leqslant i \leqslant \ell)\}.$$

We also put $\mathbb{S}((\emptyset),(\emptyset)) = (\emptyset)$. For $\underline{\mathbf{s}} = (s_1,\dots,s_\ell) \in \mathbb{S}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ and $\sigma \in \mathfrak{S}_\ell$, we put $\underline{\mathbf{s}}^\sigma = (s_{\sigma(1)},\dots,s_{\sigma(\ell)}) \in \mathbb{S}(\underline{\mathbf{m}}^\sigma,\underline{\mathbf{n}}^\sigma)$. For $\underline{\mathbf{s}} = (s_1,\dots,s_\ell) \in \mathbb{S}(\underline{\mathbf{m}},\underline{\mathbf{n}})$, we denote by $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$ the Cartan subalgebra of $\mathfrak{g}(m,n)$ corresponding to $\mathfrak{h}(m_1,n_1;s_1) \times \cdots \times \mathfrak{h}(m_\ell,n_\ell;s_\ell)$ via the standard identification of $\mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}})$. We also introduce a non-degenerate Weyl group invariant bilinear form $\langle\, ,\rangle$ on $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})^*$.

Let $\mathrm{H}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$ be the corresponding Cartan subgroup of $\mathrm{U}(m,n)$. We denote by $\mathfrak{z}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ the center of $\mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}})$. Using $\langle\, ,\rangle$, we regard $\mathfrak{z}(\underline{\mathbf{m}},\underline{\mathbf{n}})^*$ as a subspace of $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})^*$.

Let $\Delta(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$ (resp. $\underline{\Delta}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$) be the root system of $\mathfrak{g}(m,n)$ (resp. $\mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}})$) with respect to $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$. Let $\mathrm{W}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$ (resp. $\underline{W}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$) be the Weyl group of $\mathfrak{g}(m,n)$ (resp. $\mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}})$) with respect to $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$. We also define $\mathcal{P}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$, $\mathcal{P}_u(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$, $\Delta_i(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$, $\Delta_r(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$, and $\Delta_c(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$ as above. We have the following standard identifications.

$$\begin{aligned}
\underline{W}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}}) &\cong \underline{W}(m_1,n_1;s_1)\times\cdots\times\underline{W}(m_\ell,n_\ell;s_\ell),\\
\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})^* &\cong \mathfrak{h}(m_1,n_1;s_1)^*\times\cdots\times\mathfrak{h}(m_\ell,n_\ell;s_\ell)^*,\\
\mathcal{P}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}}) &\cong \mathcal{P}(m_1,n_1;s_1)\times\cdots\times\mathcal{P}(m_\ell,n_\ell;s_\ell), \quad \text{etc.}
\end{aligned}$$

$\lambda\in\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})^*$ can be exponentiated to a character of $\mathrm{H}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$ if and only if $\lambda\in\mathcal{P}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$. We denote by $\mathbb{C}_\lambda(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$ the one-dimensional representation of the metaplectic double cover of $\mathrm{H}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$ corresponding to $\lambda\in\frac{1}{2}\mathcal{P}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$. $\mathbb{C}_\lambda(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$ is unitarizable if and only if $\lambda\in\frac{1}{2}\mathcal{P}_u(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})$.

For $\lambda\in\mathfrak{h}(m_i,n_i;s_i)^*$ $(1\leqslant i\leqslant\ell)$, we denote by $\lambda^{(i)}$ the corresponding element in $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})^*$ with respect to the standard identification. For $1\leqslant i\leqslant\ell$ and $1\leqslant j\leqslant m_i+n_i$, we put $[\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}};i]_j=[m_i,n_i;s_i]_j^{(i)}$. Hence all the $[\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}};i]_j$ form a basis of $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})^*$.

If $\underline{\mathbf{m}}$, $\underline{\mathbf{n}}$, and $\underline{\mathbf{s}}$ are given and if there is no confusion, we simply write $[i]_j$ for $[\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}};i]_j$.

We denote by $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})^*_c$ the subset of $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})^*$ coressponding to $\mathfrak{h}(m_1,n_1;s_1)^*_c\times\cdots\times\mathfrak{h}(m_\ell,n_\ell;s_\ell)^*_c$ via the standard identification.

For $1\leqslant i\leqslant\ell$, we put

$$\begin{aligned}
[[\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}};i]]_+ &= \sum_{j=1}^{m_i}[\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}};i]_j,\\
[[\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}};i]]_- &= \sum_{j=m_i+1}^{m_i+n_i}[\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}};i]_j,\\
[[\underline{\mathbf{m}},\underline{\mathbf{n}};i]] &= [[m_i,n_i]]^{(i)}.
\end{aligned}$$

We easily see $[[\underline{\mathbf{m}},\underline{\mathbf{n}};i]]=[[\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}};i]]_++[[\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}};i]]_-$ for all i. We also see $[[\underline{\mathbf{m}},\underline{\mathbf{n}};1]],\ldots,[[\underline{\mathbf{m}},\underline{\mathbf{n}};\ell]]$ form a basis of $\mathfrak{z}(\underline{\mathbf{m}},\underline{\mathbf{n}})^*$. For $\lambda\in\mathfrak{z}(\underline{\mathbf{m}},\underline{\mathbf{n}})^*\cap\frac{1}{2}\mathcal{P}(\underline{\mathbf{m}},\underline{\mathbf{n}};0^\ell)$, we denote by $\mathbb{C}_\lambda(\underline{\mathbf{m}},\underline{\mathbf{n}})$ the corresponding one-dimensional representation of the metaplectic double cover of $\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}})$.

For the case $m=n$, for $\underline{\mathbf{r}}=(r_1,\ldots,r_d)\in\mathbb{P}(m)$, we denote by $\mathbb{GL}(\underline{\mathbf{r}},\mathbb{C})$ (resp. $S(\underline{\mathbf{r}})$) the sub-group of $\mathrm{U}(m,m)$ which corresponds to $\mathbb{GL}(r_1,\mathbb{C})\times\cdots\times\mathbb{GL}(r_d,\mathbb{C})$ (resp. $S(r_1)\times\cdots\times S(r_d)$) via the standard identification. We denote by $\mathfrak{gl}_{LR}(\underline{\mathbf{r}},\mathbb{C})$ (resp. $\mathfrak{b}_{LR}(\underline{\mathbf{r}},\mathbb{C})$) the Lie subalgebra of $\mathfrak{g}(m,m)$ corresponding to $(\mathfrak{gl}_L(r_1,\mathbb{C})\times\mathfrak{gl}_R(r_1,\mathbb{C}))\times\cdots\times(\mathfrak{gl}_L(r_d,\mathbb{C})\times\mathfrak{gl}_L(r_d,\mathbb{C}))$ (resp. $(\mathfrak{b}_L(r_1,\mathbb{C})\times\mathfrak{b}_R(r_1,\mathbb{C}))\times\cdots\times(\mathfrak{b}_L(r_d,\mathbb{C})\times\mathfrak{b}_R(r_d,\mathbb{C}))$) via the standard identification. We also define $\mathfrak{s}(\underline{\mathbf{r}})$ similarly. $\mathfrak{gl}_{LR}(\underline{\mathbf{r}},\mathbb{C})$ is the complexified Lie algebra of $\mathbb{GL}(\underline{\mathbf{r}},\mathbb{C})$.

We denote by $\mathrm{W}(r_i)$ the Weyl group for $(\mathfrak{gl}_L(r_i,\mathbb{C})\times\mathfrak{gl}_R(r_i,\mathbb{C}),\mathfrak{h}(r_i,r_i;r_i))$ for $1\leqslant i\leqslant d$. We also denote by $\underline{W}(\underline{\mathbf{r}})$ the Weyl group for $(\mathfrak{gl}_{LR}(\underline{\mathbf{r}},\mathbb{C}),\mathfrak{h}(\underline{\mathbf{r}},\underline{\mathbf{r}};\underline{\mathbf{r}}))$. Then we have the following standard identification.

$$\underline{W}(\underline{\mathbf{r}})=\mathrm{W}(r_1)\times\cdots\times\mathrm{W}(r_d).$$

2. Change of polarizations

2.1. Generalized unitary degenerate series

We fix non-negative integers m, n, and r such that $m+n>0$ and $0\leqslant r\leqslant\min\{m,n\}$. Put $p=m-r$ and $q=n-r$. We also fix $\underline{\mathbf{r}}=(r_1,\dots,r_d)\in\mathbb{P}_d(r)$ and $(\underline{\mathbf{p}},\underline{\mathbf{q}})=((p_1,\dots,p_h),(q_1,\dots,q_h))\in\mathbb{P}_h(p,q)$. Put $\ell=d+h$ and $(\underline{\mathbf{m}},\underline{\mathbf{n}})=((m_1,\dots,m_\ell),(n_1,\dots,n_\ell))=(\underline{\mathbf{r}}\star\underline{\mathbf{p}},\underline{\mathbf{r}}\star\underline{\mathbf{q}})\in\mathbb{P}_\ell(m,n)$.

For simplicity, we write $\mathfrak{l}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ and $L(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ for $\mathfrak{gl}_{LR}(\underline{\mathbf{r}},\mathbb{C})\star\mathfrak{g}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ and $\mathbb{GL}(\underline{\mathbf{r}},\mathbb{C})\star U(\underline{\mathbf{p}},\underline{\mathbf{q}})$ respectively.

For $\underline{\mathbf{t}}=(t_1,\dots,t_h)\in\mathbb{S}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ we have $\underline{\mathbf{r}}\star\underline{\mathbf{t}}=(r_1,\dots,r_d,t_1,\dots,t_h)\in\mathbb{S}(\underline{\mathbf{m}},\underline{\mathbf{n}})$. And $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})$ (resp. $H(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})$) is a Cartan subalgebra (resp. subgroup) of $\mathfrak{l}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ (resp. $L(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$). We denote by $\mathfrak{c}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ the center of $\mathfrak{l}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$. Using $\langle\ ,\ \rangle$, we regard $\mathfrak{c}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})^*$ as a subspace of $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})^*$.

For $1\leqslant i\leqslant 2d+h$, we put

$$[[\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};i]]=\begin{cases}[[\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}};i]]_+ & \text{if } 1\leqslant i\leqslant d\\ [[\underline{\mathbf{m}},\underline{\mathbf{n}};i]] & \text{if } d<i\leqslant d+h\\ [[\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}};2d-i+h+1]]_- & \text{if } d+h<i\leqslant 2d+h\end{cases}.$$

We immediately see $[[\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};1]],\dots,[[\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};2d+h]]$ form a basis of $\mathfrak{c}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})^*$ and they are independent of $\underline{\mathbf{t}}\in\mathbb{S}(\underline{\mathbf{p}},\underline{\mathbf{q}})$.

We define the following subset of $\Delta(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})$ for $\underline{\mathbf{t}}\in\mathbb{S}(\underline{\mathbf{p}},\underline{\mathbf{q}})$.

$$\begin{aligned}&\bar{\Delta}^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\underline{\mathbf{t}})\\ &=\{[i]_j-[k]_v\mid 1\leqslant i<d,1\leqslant j\leqslant r_i,d<k\leqslant\ell,1\leqslant v\leqslant p_{k-d}+q_{k-d}\}\\ &\quad\cup\{[i]_j-[k]_v\mid 1\leqslant k<d,r_k<v\leqslant 2r_k,d<i\leqslant\ell,1\leqslant j\leqslant p_{i-d}+q_{i-d}\}\\ &\quad\cup\{[i]_j-[k]_v\mid 1\leqslant i<k<d,1\leqslant i\leqslant r_i,1\leqslant v\leqslant r_k\}\\ &\quad\cup\{[i]_j-[k]_v\mid 1\leqslant k<i<d,r_i<i\leqslant 2r_i,r_k<v\leqslant 2r_k\}\\ &\quad\cup\{[i]_j-[k]_v\mid 1\leqslant i,k<d,1\leqslant i\leqslant r_i,r_i<v\leqslant 2r_i\}\\ &\quad\cup\{[i]_j-[k]_v\mid d<i<k\leqslant\ell,1\leqslant j\leqslant p_{i-d}+q_{i-d},1\leqslant v\leqslant p_{i-d}+q_{i-d}\}.\end{aligned}$$

We denote by $\mathfrak{v}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ the sum of the root spaces corresponding to $\bar{\Delta}^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\underline{\mathbf{t}})$. We easily see $\mathfrak{v}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ does not depend on the choice of $\underline{\mathbf{t}}$. Put $\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})=\mathfrak{l}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})+\mathfrak{v}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$. Then $\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ is a parabolic subalgebra of $\mathfrak{g}(m,n)$ with a Levi part (resp. the nilradical) $\mathfrak{l}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ (resp. $\mathfrak{v}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$).

Put $S_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}} = \dim(\mathfrak{v}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) \cap \mathfrak{k}(m,n))$ and

$$\bar{\rho}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) = \rho(\mathfrak{v}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})) = \frac{1}{2}\sum_{\alpha\in\bar{\Delta}^+(\underline{\mathbf{r}},\underline{\mathbf{p}},\underline{\mathbf{q}};\,\underline{\mathbf{t}})} \alpha \in \mathfrak{c}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})^*.$$

We also put

$$\begin{aligned}\overline{\mathcal{P}}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) &= \mathcal{P}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}}) \cap \mathfrak{c}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})^*,\\ \overline{\mathcal{P}}^\star(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) &= -\bar{\rho}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) + \overline{\mathcal{P}}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}),\\ \overline{\mathcal{P}}^\star_u(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) &= \overline{\mathcal{P}}^\star(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) \cap \tfrac{1}{2}\mathcal{P}_u(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}}).\end{aligned}$$

We see that $\lambda \in \mathfrak{c}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ can be exponentiated to a character of $L(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ if and only if $\lambda \in \overline{\mathcal{P}}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$. We denote by $\mathbb{C}_\lambda(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ the corresponding one-dimensional representation of the metaplectic double cover of $L(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ to $\lambda \in \overline{\mathcal{P}}^\star(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$. For $\lambda \in \overline{\mathcal{P}}^\star(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$, $\mathbb{C}_\lambda(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ is unitarizable if and only if $\lambda \in \overline{\mathcal{P}}^\star_u(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$.

Next, we consider a parabolic subgroup of U(m,n). We denote by $\mathfrak{m}(\underline{\mathbf{r}};p,q)$ the Lie subalgebra of $\mathfrak{g}((r_1,\ldots,r_d,p),(r_1,\ldots,r_d,q))$ corresponding to $\mathfrak{gl}_{LR}(r_1,\mathbb{C}) \times \cdots \times \mathfrak{gl}_{LR}(r_d,\mathbb{C}) \times \mathfrak{g}(p,q)$ via the standard identification. If $p = q = 0$, we put $\mathfrak{m}(\underline{\mathbf{r}};0,0) = \mathfrak{gl}_{LR}(\underline{\mathbf{r}},\mathbb{C})$. We denote by M$(\underline{\mathbf{r}};p,q)$ the subgroup of U(m,n) corresponding to $\mathbb{GL}(\underline{\mathbf{r}},\mathbb{C}) \times$ U(p,q) via the standard identification. We denote by $\underline{\Delta}(\underline{\mathbf{r}};p,q;\underline{\mathbf{t}})$ the root system for $(\mathfrak{m}(\underline{\mathbf{r}};p,q),\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}}))$. Put $\mathfrak{p}(\underline{\mathbf{r}};p,q) = \mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) + \mathfrak{m}(\underline{\mathbf{r}};p,q)$. Then, $\mathfrak{p}(\underline{\mathbf{r}};p,q)$ is stable under the complex conjugation of $\mathfrak{g}(m,n)$ with respect to $\mathfrak{u}(m,n)$. We denote by P$(\underline{\mathbf{r}};p,q)$ the corresponding parabolic subgroup of U(m,n) to $\mathfrak{p}(\underline{\mathbf{r}};p,q)$ with a Levi part M$(\underline{\mathbf{r}};p,q)$. We also denote by N$(\underline{\mathbf{r}};p,q)$ the nilradical of P$(\underline{\mathbf{r}};p,q)$. An arbitrary parabolic subgroup of U(m,n) is conjugate to some P$(\underline{\mathbf{r}},p,q)$.

Next, we put $\underline{\mathfrak{q}}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) = \mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) \cap \mathfrak{m}(\underline{\mathbf{r}};p,q)$. Then, $\underline{\mathfrak{q}}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ is a θ-stable parabolic subalgebra of $\mathfrak{m}(\underline{\mathbf{r}};p,q)$ with a Levi part $\mathfrak{l}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$. We denote by $\underline{\mathfrak{v}}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ the nilradical of $\underline{\mathfrak{q}}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$. We easily see $\dim(\underline{\mathfrak{v}}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) \cap (\mathfrak{m}(\underline{\mathbf{r}};p,q) \cap \mathfrak{k}(m,n))) = S_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}}$.

We put

$$\overline{\mathcal{P}}^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) = \left\{\lambda = \sum_{i=1}^{2d+h} \lambda_i[[\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};i]] \in \overline{\mathcal{P}}^\star(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) \mid \lambda_{d+1} \geqslant \lambda_{d+2} \geqslant \cdots \geqslant \lambda_\ell\right\},$$

$$\overline{\mathcal{P}}^+_u(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) = \overline{\mathcal{P}}^\star_u(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) \cap \overline{\mathcal{P}}^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}).$$

We immediately see:

LEMMA 2.1.1. *For $\lambda \in \overline{\mathcal{P}}^\star(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$, $\lambda \in \overline{\mathcal{P}}^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ if and only if the infinitesimal character of $\mathbb{C}_\lambda(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ is in the weakly fair range with respect to $\underline{\mathfrak{q}}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$.*

Hence, we can consider the corresponding derived functor module $\mathcal{A}(\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{m}(\underline{\mathbf{r}}; p, q); \lambda)$. As I mentioned in 1.2, $\mathcal{A}(\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{m}(\underline{\mathbf{r}}; p, q); \lambda)$ is zero or irreducible for any $\lambda \in \overline{\mathcal{P}}^{+}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$. Moreover, if $\lambda \in \overline{\mathcal{P}}_{u}^{+}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, it is unitarizable.

Using the fundamental results listed in 1.2, we easily see:

LEMMA 2.1.2. *Fix* $\lambda \in \overline{\mathcal{P}}^{+}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$.

(1) $\mathcal{R}^{i}_{\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})}(\mathfrak{l}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{g}(m, n); \mathbb{C}_{\lambda}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})) = 0$ *for all* $i \neq S_{\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}}$.

(2) *We have the following isomorphism of Harish-Chandra modules.*

$$\begin{aligned}&\mathsf{Ind}(\mathrm{P}(\underline{\mathbf{r}}; p, q) \uparrow \mathrm{U}(m, n); \mathcal{A}(\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{m}(\underline{\mathbf{r}}; p, q); \lambda))\\ &\cong \mathcal{R}^{S_{\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}}}_{\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})}(\mathfrak{l}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{g}(m, n); \mathbb{C}_{\lambda}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})).\end{aligned}$$

For $\lambda \in \overline{\mathcal{P}}^{+}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, we call $\mathsf{Ind}(\mathrm{P}(\underline{\mathbf{r}}; p, q) \uparrow \mathrm{U}(m, n); \mathcal{A}(\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{m}(\underline{\mathbf{r}}; p, q))$ a generalized (non-unitary) degeneratge series module.

If $\ell \leqslant 1$, we have $\mathfrak{p}(\underline{\mathbf{r}}; p, q) = \mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$. Hence, in this case, our modules are the hyperbolic induction from one-dimensional representations of a parabolic subgroup (namely, so-called 'degenerate series').

If $\lambda \in \overline{\mathcal{P}}_{u}^{+}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, the hyperbolic induction is a unitary induction and we call $\mathsf{Ind}(\mathrm{P}(\underline{\mathbf{r}}; p, q) \uparrow \mathrm{U}(m, n); \mathcal{A}(\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{m}(\underline{\mathbf{r}}; p, q); \lambda))$ a generalized unitary degenerate series module.

From a result in [9] (and the translation principle), we see any derived functor module of $\mathrm{GL}(n, \mathbb{C})$ is isomorphic to the hyperbolic induction form a one-dimensional unitary representation of a parabolic subalgebra. Hence, from the induction-by-stage, any hyperbolically unitarily induced module of $\mathrm{U}(m, n)$ from a derived functor module of any Levi subgroup is isomorphic to a module in generalized unitary degenerate series above.

For $\tau \in \mathfrak{S}_d$ and $\lambda = \Sigma_{i=1}^{2d+h} \lambda_i [[\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; i]] \in \mathfrak{c}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})^*$, we define

$$\begin{aligned}\tau(\lambda) = &\sum_{i=1}^{d} (\lambda_{\tau(i)} [[\underline{\mathbf{r}}^{\tau}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; i]] + \lambda_{2d+h-\tau(i)+1}\\ &\times [[\underline{\mathbf{r}}^{\tau}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; 2d + h - \tau(i) + 1]])\\ &+ \sum_{j=1}^{h} \lambda_{d+j} [[\underline{\mathbf{r}}^{\tau}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; d + j]].\end{aligned}$$

The following result is a special case of a classic result.

THEOREM 2.1.3. (Harish-Chandra, cf. [39] Theorem 3.19).
For all $\lambda \in \overline{\mathcal{P}}_{u}^{+}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$ *and* $\tau \in \mathfrak{S}_d$, *we have*

$$\mathsf{Ind}^{\mathrm{U}(m,n)}_{\mathrm{P}(\underline{\mathbf{r}}; p, q)}(\mathcal{A}^{\mathfrak{m}(\underline{\mathbf{r}}; p, q)}_{\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})}(\lambda)) \cong \mathsf{Ind}^{\mathrm{U}(m,n)}_{\mathrm{P}(\underline{\mathbf{r}}^{\tau}; p, q)}(\mathcal{A}^{\mathfrak{m}(\underline{\mathbf{r}}^{\tau}; p, q)}_{\mathfrak{q}(\underline{\mathbf{r}}^{\tau}; \underline{\mathbf{p}}, \underline{\mathbf{q}})}(\tau(\lambda))).$$

For later use, we introduce the following terminologies. For $\lambda = \Sigma_{i=1}^{2d+h}\lambda_i \times [[\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};i]] \in \mathfrak{c}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})^*$ and $1 \leqslant k \leqslant \ell$, we put

$$\overline{\overline{\lambda}}_{(k)} = \begin{cases} \max\{\operatorname{Re}\lambda_k, \operatorname{Re}\lambda_{2d+\ell-k+1}\} + \dfrac{r_k-1}{2} & (1 \leqslant k \leqslant d) \\ \operatorname{Re}\lambda_k + \dfrac{p_{k-d}+q_{k-d}-1}{2} & (d < k \leqslant \ell) \end{cases},$$

$$\underline{\underline{\lambda}}_{(k)} = \begin{cases} \min\{\operatorname{Re}\lambda_k, \operatorname{Re}\lambda_{2d+\ell-k+1}\} - \dfrac{r_k-1}{2} & (1 \leqslant k \leqslant d) \\ \operatorname{Re}\lambda_k - \dfrac{p_{k-d}+q_{k-d}-1}{2} & (d < k \leqslant \ell) \end{cases}.$$

We call $\lambda \in \overline{\mathcal{P}}^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ good enough if there exists some permutation $\sigma_\lambda \in \mathfrak{S}_\ell$ such that $\overline{\overline{\lambda}}_{(\sigma_\lambda(1))} \geqslant \underline{\underline{\lambda}}_{(\sigma_\lambda(1))} > \overline{\overline{\lambda}}_{(\sigma_\lambda(2))} \geqslant \underline{\underline{\lambda}}_{(\sigma_\lambda(2))} > \cdots > \overline{\overline{\lambda}}_{(\sigma_\lambda(\ell))} \geqslant \underline{\underline{\lambda}}_{(\sigma_\lambda(\ell))}$.

We denote by $\mathcal{G}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ the set of good enough characters in $\overline{\mathcal{P}}^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$.

Moreover, for $\lambda \in \mathcal{G}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$, $\sigma_\lambda \in \mathfrak{S}_\ell$ is uniquely determined, so hereafter σ_λ means this permutation.

We also easily see:

LEMMA 2.1.4. *For a good enough character $\lambda \in \mathcal{G}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$, the infinitesimal character of $\mathbb{C}_\lambda(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ is in the good range with respect to $\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$.*

If $\underline{\mathbf{r}} = (\emptyset)$, $\underline{\mathbf{p}} = \underline{\mathbf{m}}$, and $\underline{\mathbf{q}} = \underline{\mathbf{n}}$, then we simply write $\mathfrak{q}(\underline{\mathbf{m}},\underline{\mathbf{n}})$, $\mathfrak{v}(\underline{\mathbf{m}},\underline{\mathbf{n}})$, $\bar{\rho}(\underline{\mathbf{m}},\underline{\mathbf{n}})$, $\overline{\mathcal{P}}^+(\underline{\mathbf{m}},\underline{\mathbf{n}})$, and $\mathcal{G}(\underline{\mathbf{m}},\underline{\mathbf{n}})$ for $\mathfrak{q}((\emptyset);\underline{\mathbf{p}},\underline{\mathbf{q}})$, $\mathfrak{v}((\emptyset);\underline{\mathbf{p}},\underline{\mathbf{q}})$, $\bar{\rho}((\emptyset);\underline{\mathbf{p}},\underline{\mathbf{q}})$, $\overline{\mathcal{P}}^+((\emptyset);\underline{\mathbf{p}},\underline{\mathbf{q}})$, and $\mathcal{G}((\emptyset);\underline{\mathbf{p}},\underline{\mathbf{q}})$, respectively.

2.2. Canonical forms of generalized unitary degenerate series in the good enough range

As in 2.1, we fix m, n, r, p, q, $\underline{\mathbf{p}}$, $\underline{\mathbf{q}}$, $\underline{\mathbf{r}}$, $\underline{\mathbf{m}}$, and $\underline{\mathbf{n}}$. We also fix a good enough character $\lambda \in \mathcal{G}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$.

We can, roughly speaking, exchange the order of the elliptic induction and the hyperbolic induction in the definition of generalized unitarily degenerate series module for λ. Such a change of polarization enables us to reduce the decomposition of the module to the Siegel parabolic situation. In order to formulate this, we introduce the following notations.

First, we easily see $\mathfrak{s}(\underline{\mathbf{r}}) \star \mathfrak{g}(\underline{\mathbf{p}},\underline{\mathbf{q}}) = \mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}}) \cap \mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ and $\mathfrak{s}(\underline{\mathbf{r}}) \star \mathfrak{g}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ is stable under the complex conjugation with respect to $\mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}}) \cap \mathfrak{u}(m,n)$. $\mathfrak{l}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ is a Levi part of $\mathfrak{s}(\underline{\mathbf{r}}) \star \mathfrak{g}(\underline{\mathbf{p}},\underline{\mathbf{q}})$. $\mathfrak{s}(\underline{\mathbf{r}}) \star \mathfrak{g}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ corresponds to a parabolic subgroup $S(\underline{\mathbf{r}}) \star U(\underline{\mathbf{p}},\underline{\mathbf{q}})$ of $U(\underline{\mathbf{m}},\underline{\mathbf{n}})$.

We will consider the hyperbolic induction $\mathsf{Ind}(S(\underline{\mathbf{r}}) \star U(\underline{\mathbf{p}},\underline{\mathbf{q}}) \uparrow U(\underline{\mathbf{m}},\underline{\mathbf{n}}); \mathbb{C}_\lambda(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}))$, whose infinitesimal character is same as that of $\mathbb{C}_\lambda(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$.

We define the following subset of $\Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})$ for $\underline{\mathbf{t}} \in \mathbb{S}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$.

$$\bar{\Delta}_2^+(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \lambda) = \{[[i]_j - [k]_v \mid \underline{\underline{\lambda}}_{(i)} > \overline{\overline{\lambda}}_{(k)}, 1 \leqslant j \leqslant m_i + n_i, 1 \leqslant v \leqslant m_k + n_k\}.$$

We denote by $\mathfrak{w}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$ the sum of the root spaces corresponding to $\bar{\Delta}_2^+(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \lambda)$. We easily see $\mathfrak{w}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$ does not depend on $\underline{\mathbf{t}}$. We put $\mathfrak{f}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda) = \mathfrak{g}(\underline{\mathbf{m}}, \underline{\mathbf{n}}) + \mathfrak{w}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$. Then, $\mathfrak{f}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$ is a θ-stable parabolic subalgebra of $\mathfrak{g}(m,n)$ and $\mathfrak{g}(\underline{\mathbf{m}}; \underline{\mathbf{n}})$ (resp. $\mathfrak{w}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$) is a Levi part (resp. the nilradical) of $\mathfrak{f}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$. Put

$$\bar{\rho}_2(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda) = \rho(\mathfrak{w}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)) = \frac{1}{2} \sum_{\alpha \in \bar{\Delta}_2^+(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \lambda)} \alpha \in \mathfrak{z}(\underline{\mathbf{m}}, \underline{\mathbf{n}})^*.$$

We easily see:

LEMMA 2.1.1. *The infinitesimal character of* $\mathsf{Ind}(S(\underline{\mathbf{r}}) \star \mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathrm{U}(\underline{\mathbf{m}}, \underline{\mathbf{n}}); \lambda + \bar{\rho}_2(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)) \otimes \mathbb{C}_{-\bar{\rho}_2(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)}(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ *is in the good range with respect to* $\mathfrak{f}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$.

We also put $\mathfrak{q}_1(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda) = \mathfrak{w}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda) + (\mathfrak{s}(\underline{\mathbf{r}}) \star \mathfrak{g}(\underline{\mathbf{m}}, \underline{\mathbf{n}}))$. Then $\mathfrak{l}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$ is a Levi part of $\mathfrak{q}_1(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$. We denote by $\mathfrak{v}_1(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$ the nilradical of $\mathfrak{q}_1(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)$. We put $f_{\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda} = \dim(\mathfrak{v}_1(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda) \cap \mathfrak{k}(m,n)) = \dim(\mathfrak{w}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda) \cap \mathfrak{k}(m,n))$.

Using results listed in 1.2, we easily see:

LEMMA 2.2.2. *For* $\lambda \in \mathcal{G}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, *we have*

(1) *For* $i \geqslant 0, \mathcal{R}^i_{\mathfrak{q}_1(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)}(\mathfrak{l}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{g}(m,n); \mathbb{C}_\lambda(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})) \neq 0$ *if and only if* $i = f_{\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda}$.

(2) *We have an isomorphism of Harish-Chandra modules*:

$$\begin{aligned}&\mathcal{R}^{f_{\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda}}_{\mathfrak{f}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)}(\mathfrak{g}(\underline{\mathbf{m}}, \underline{\mathbf{n}}) \uparrow \mathfrak{g}(m,n); \mathsf{Ind}^{\mathrm{U}(\underline{\mathbf{m}}, \underline{\mathbf{n}})}_{\mathrm{S}(\underline{\mathbf{r}}) \star \mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}})}(\lambda + \bar{\rho}_2(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda))\\ &\quad \otimes \mathbb{C}_{-\bar{\rho}_2(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)}(\underline{\mathbf{m}}, \underline{\mathbf{n}}))\\ &\quad \cong \mathcal{R}^{f_{\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda}}_{\mathfrak{q}_1(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)}(\mathfrak{l}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{g}(m,n); \mathbb{C}_\lambda(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})).\end{aligned}$$

Now, we state the main result of Section 2.

THEOREM 2.2.3. *For* $\lambda \in \mathcal{G}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, *we have the following identity of distribution characters.*

$$\begin{aligned}&[\mathsf{Ind}(\mathrm{P}(\underline{\mathbf{r}}; p, q) \uparrow \mathrm{U}(m,n); \mathcal{A}(\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{m}(\underline{\mathbf{r}}, p, q); \lambda))]\\ &= [\mathcal{R}^{f_{\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda}}_{\mathfrak{f}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)}(\mathfrak{g}(\underline{\mathbf{m}}, \underline{\mathbf{n}}) \uparrow \mathfrak{g}(m,n); \mathsf{Ind}^{\mathrm{U}(\underline{\mathbf{m}}, \underline{\mathbf{n}})}_{\mathrm{S}(\underline{\mathbf{r}}) \star \mathrm{U}(\underline{\mathbf{p}}, \underline{\mathbf{q}})}(\lambda + \bar{\rho}_2(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda))\\ &\quad \otimes \mathbb{C}_{-\bar{\rho}_2(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)}(\underline{\mathbf{m}}, \underline{\mathbf{n}}))].\end{aligned}$$

In 2.3, we reduce this theorem to a result on standard modules (Proposition 2.3.4 below), which is proved in 2.4 using the Hecht–Miličić–Schmid–Wolf theory on standard modules.

From Lemma 2.2.1, and a result of Vogan in 1.2, we have:

COROLLARY 2.2.4. *Assume* $\lambda \in \mathcal{G}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$ *and* $\mathsf{Ind}^{\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}})}_{\mathrm{S}(\underline{\mathbf{r}})\star\mathrm{U}(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\lambda - \bar{\rho}_2(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda))$ *is irreducible. Then,* $\mathsf{Ind}(\mathrm{P}(\underline{\mathbf{r}}; p, q) \uparrow \mathrm{U}(m,n); \mathcal{A}(\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{m}(\underline{\mathbf{r}}, p, q); \lambda))$ *is also irreducible.*

For later use, we reformulate Theorem 2.2.3.

For $\sigma \in \mathfrak{S}_\ell$ and $\mu = \Sigma_{i=1}^{2d+h} \mu_i[[\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; i]] \in \mathfrak{c}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})^*$, we define $\sigma(\mu) \in \mathfrak{h}(\underline{\mathbf{m}}^\sigma, \underline{\mathbf{n}}^\sigma; (\underline{\mathbf{r}} \star \underline{\mathbf{t}})^\sigma)^*$ as follows.

$$\begin{aligned}\sigma(\mu) &= \sum_{i=1}^{d} (\mu_i[[\underline{\mathbf{m}}^\sigma, \underline{\mathbf{n}}^\sigma; (\underline{\mathbf{r}} \star \underline{\mathbf{t}})^\sigma; \sigma^{-1}(i)]]_+ \\ &\quad + \mu_{2d+h-i+1}[[\underline{\mathbf{m}}^\sigma, \underline{\mathbf{n}}^\sigma; (\underline{\mathbf{r}} \star \underline{\mathbf{t}})^\sigma; \sigma^{-1}(i)]]_-) \\ &\quad + \sum_{i=1}^{h} \mu_{d+i}[[\underline{\mathbf{m}}^\sigma, \underline{\mathbf{n}}^\sigma; \sigma^{-1}(d+i)]].\end{aligned}$$

We see that $\mathrm{Ad}(\Phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma_\lambda))(\mathfrak{f}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)) = \mathfrak{q}(\underline{\mathbf{m}}^{\sigma_\lambda}, \underline{\mathbf{n}}^{\sigma_\lambda})$ and $\sigma_\lambda(\bar{\rho}_2(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda)) = \bar{\rho}(\underline{\mathbf{m}}^{\sigma_\lambda}, \underline{\mathbf{n}}^{\sigma_\lambda})$. Since $\Phi_{\underline{\mathbf{m}},\underline{\mathbf{n}}}(\sigma_\lambda) \in \mathrm{K}(m,n)$, twisting by the corresponding inner automorphism does not change any Harish-Chandra $(\mathfrak{g}(m,n), \mathrm{K}(m,n))$-module up to isomorphisms. Twisting the right-hand side of the identity in Theorem 2.2.3, we can restate the theorem as follows.

COROLLARY 2.2.5. *For* $\lambda \in \mathcal{G}(\underline{\mathbf{r}};, \underline{\mathbf{p}}, \underline{\mathbf{q}})$, *we have the following identity of the distribution characters.*

$$\begin{aligned}&[\mathsf{Ind}(\mathrm{P}(\underline{\mathbf{r}}; p, q) \uparrow \mathrm{U}(m,n); \mathcal{A}(\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \uparrow \mathfrak{m}(\underline{\mathbf{r}}, p, q); \lambda))] \\ &= [\mathcal{R}^{f_{\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \lambda}}_{\mathfrak{q}(\underline{\mathbf{m}}^{\sigma_\lambda}, \underline{\mathbf{n}}^{\sigma_\lambda})}(\mathfrak{g}(\underline{\mathbf{m}}^{\sigma_\lambda}, \underline{\mathbf{n}}^{\sigma_\lambda}) \uparrow \mathfrak{g}(m,n); \mathsf{Ind}^{\mathrm{U}(\underline{\mathbf{m}}^{\sigma_\lambda}, \underline{\mathbf{n}}^{\sigma_\lambda})}_{(\mathrm{S}(\underline{\mathbf{r}})\star\mathrm{U}(\underline{\mathbf{p}},\underline{\mathbf{q}}))^{\sigma_\lambda}} \\ &\quad (\sigma_\lambda(\lambda) + \bar{\rho}(\underline{\mathbf{m}}^{\sigma_\lambda}, \underline{\mathbf{n}}^{\sigma_\lambda})) \otimes \mathbb{C}_{-\bar{\rho}(\underline{\mathbf{m}}^{\sigma_\lambda}, \underline{\mathbf{n}}^{\sigma_\lambda})}(\underline{\mathbf{m}}^{\sigma_\lambda}, \underline{\mathbf{n}}^{\sigma_\lambda}))].\end{aligned}$$

We call the module in the right-hand side the canonical form of the generalized degenerate series module in the left-hand side.

2.3. REDUCTION TO STANDARD MODULES

As in 2.1, we fix m, n, r, p, q, $\underline{\mathbf{p}}$, $\underline{\mathbf{q}}$, $\underline{\mathbf{r}}$, $\underline{\mathbf{m}}$, and $\underline{\mathbf{n}}$.

For a while, we also fix $\underline{\mathbf{t}} \in \mathbb{S}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$. We denote by $\underline{W}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}})$ (resp. $\underline{\Delta}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}})$) the Weyl group (the root system) for $(\mathfrak{l}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}), \mathfrak{h}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}}))$. Then, we have the following standard identification.

$$\begin{aligned}\underline{W}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}) &\cong \underline{W}(\underline{\mathbf{r}}) \times \underline{W}(\underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}) \\ &\cong \mathrm{W}(r_1) \times \cdots \times \mathrm{W}(r_d) \times \mathrm{W}(p_1, q_1; t_1) \\ &\quad \times \cdots \times \mathrm{W}(p_h, q_h; t_h).\end{aligned}$$

For $\eta = \Sigma_{i=1}^{2d+h}\Sigma_{j=1}^{u_i}\eta_{i,j}\{i\}_j \in \mathfrak{h}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})_c^*$, we define the following positive system of $\Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})$.

$$\begin{aligned}\underline{\Delta}^+(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta) = & \{[i]_j - [i]_v \mid 1 \leqslant i \leqslant d, 1 \leqslant j < v \leqslant r_i\} \\ & \cup\{[i]_j - [i]_v \mid d < i \leqslant \ell, 1 \leqslant j \leqslant t_{i-d}, j \\ & < v \leqslant p_{i-d} + q_{i-d}\} \\ & \cup\{[i]_j - [i]_v \mid d < i \leqslant \ell, \eta_{i,j} > \eta_{i,v}, t_{i-d} \\ & < j, v \leqslant p_{i-d} + q_{i-d} - t_{i-d}\} \\ & \cup\{[i]_j - [i]_v \mid d < i \leqslant \ell, p_{i-d} + q_{i-d} - t_{i-d} \\ & < v \leqslant p_{i-d} + q_{i-d}, 1 \leqslant j < v\} \\ & \cup\{[i]_j - [i]_v \mid 1 \leqslant i \leqslant d, r_i < j < v \leqslant 2r_i\}.\end{aligned}$$

Let $\underline{\mathfrak{n}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta)$ be the sum of the root spaces corresponding to $\underline{\Delta}^+(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta)$ and put $\underline{\mathfrak{b}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta) = \mathfrak{h}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}}) + \underline{\mathfrak{n}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta)$. Then $\underline{\mathfrak{b}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta)$ is a Borel subalgebra of $\mathfrak{l}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, whose nilradical is $\underline{\mathfrak{n}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta)$. We denote by $\underline{\Pi}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta)$ the corresponding base of $\underline{\Delta}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}}; \eta)$ to $\underline{\Delta}^+(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta)$.

Put $\underline{\mathbf{s}} = (s_1, \dots, s_\ell) = \underline{\mathbf{r}} \star \underline{\mathbf{t}} = (r_1, \dots, r_d, t_1, \dots, t_h)$. Let $\eta_i \in \mathfrak{h}(m_i, n_i, s_i)_c^*$ $(1 \leqslant i \leqslant \ell)$ and let $\eta = \Sigma_{i=1}^{\ell}\eta_i^{(i)} \in \mathfrak{h}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})_c^*$. Then, $\underline{\mathfrak{b}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; \eta)$ corresponds to $\mathfrak{b}_{LR}(r_1, \mathbb{C}) \times \cdots \times \mathfrak{b}_{LR}(r_d, \mathbb{C}) \times \mathfrak{b}(p_1, q_1; t_1; \eta_{d+1}) \times \cdots \times \mathfrak{b}(p_h, q_h; t_h; \eta_\ell)$ via the standard identification.

We put

$$\rho(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; i) = \begin{cases} \rho_+(r_k, r_k; r_k)^{(k)} + \rho_-(r_k, r_k; r_k)^{(k)} & \text{if } 1 \leqslant i \leqslant d \\ \rho(p_i, q_i; t_i)^{(d+i)} & \text{if } d < i \leqslant \ell \end{cases},$$

$$\underline{\rho}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}) = \sum_{i=1}^{\ell} \underline{\rho}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; i),$$

$$\rho(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}) = \underline{\rho}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}) + \bar{\rho}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}).$$

Hereafter, we shall consider various $\underline{\mathbf{t}} \in \mathbb{S}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$ at the same time, though we shall still fix $m, n, \dots$, etc.

For simplicity, for $\underline{\mathbf{t}} \in \mathbb{S}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$, we write $\underline{W}(;\underline{\mathbf{t}}) = \underline{W}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}})$ and $\underline{\rho}(;\underline{\mathbf{t}}) = \underline{\rho}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}})$. For $x \in \underline{W}(;\underline{\mathbf{t}})$, we write $\underline{\Delta}^+(;\underline{\mathbf{t}})_x, \underline{\Pi}(;\underline{\mathbf{t}})_x, \underline{\mathfrak{b}}(;\underline{\mathbf{t}})_x$ and $\underline{\mathfrak{n}}(;\underline{\mathbf{t}})_x$ for $\underline{\Delta}^+(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; x\underline{\rho}(;\underline{\mathbf{t}})), \underline{\Pi}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; x\underline{\rho}(;\underline{\mathbf{t}})), \underline{\mathfrak{b}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; x\underline{\rho}(;\underline{\mathbf{t}}))$ and $\underline{\mathfrak{n}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \underline{\mathbf{t}}; x\underline{\rho}(;\underline{\mathbf{t}}))$ respectively. Put $S_{\underline{\mathbf{t}},x} = \dim(\underline{\mathfrak{n}}(;\underline{\mathbf{t}})_x \cap \overline{\mathfrak{k}(\underline{\mathbf{m}}, \underline{\mathbf{n}})})$ for $x \in \underline{W}(;\underline{\mathbf{t}})$.

LEMMA 2.3.1.

(1) *For all* $\mu \in \overline{\mathcal{P}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, $\underline{\mathbf{t}} \in, \mathbb{S}(p, q)$, $x \in \underline{W}(;\underline{\mathbf{t}})$, *and* $i \geqslant 0$, $\mathcal{R}^i_{\underline{\mathfrak{b}}(;\underline{\mathbf{t}})_x}(\mathfrak{h}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}}) \uparrow \mathfrak{g}(\underline{\mathbf{m}}, \underline{\mathbf{n}}); \mathbb{C}_{x\underline{\rho}(;\underline{\mathbf{t}})+\mu}) \neq 0$ *if and only if* $i = S_{\underline{\mathbf{t}},x}$.

(2) *For all* $\underline{\mathbf{t}} \in \mathbb{S}(p, q)$ *and* $x \in \underline{W}(;\underline{\mathbf{t}})$, *there are integers* $n(\underline{\mathbf{t}}, x)$ *such that the following character identify holds for all* $\mu \in \overline{\mathcal{P}}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$.

$$[\mathbb{C}_\mu(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})] = \sum_{\underline{\mathbf{t}}\in\mathbb{S}(p,q)}\sum_{x\in\underline{W}(;\underline{\mathbf{t}})} n(\underline{\mathbf{t}},x)[\mathcal{R}^{S_{\underline{\mathbf{t}},x}}_{\underline{\mathfrak{b}}(\underline{\mathbf{t}})_x}(\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}}) \uparrow \mathfrak{l}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}});\mathbb{C}_{x\underline{\rho}(;\underline{\mathbf{t}})+\mu}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}}))].$$

Proof. (1) is easily deduced from results listed in 1.2. (Also see [29] (4.9).)

(2) is due to Zuckerman (cf. [36] Proposition 9.4.16). (In fact, all $n(\underline{\mathbf{t}},x)$ may be one of ±1.) □

For $\underline{\mathbf{t}}\in\mathbb{S}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ and $x\in\underline{W}(;\underline{\mathbf{t}})$, we put $\mathfrak{b}(;\underline{\mathbf{t}})_x=\underline{\mathfrak{b}}(;\underline{\mathbf{t}})_x+\mathfrak{v}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ and $\mathfrak{n}(;\underline{\mathbf{t}})_x=\underline{\mathfrak{n}}(;\underline{\mathbf{t}})_x+\mathfrak{v}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$. Then, $\mathfrak{b}(;\underline{\mathbf{t}})_x$ is a Borel subalgebra of $\mathfrak{g}(m,n)$ and $\mathfrak{n}(;\underline{\mathbf{t}})_x$ is the nilradical of $\mathfrak{b}(;\underline{\mathbf{t}})_x$. We also put $\Delta^+(;\underline{\mathbf{t}})_x=\underline{\Delta}^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\underline{\mathbf{t}};x\rho(;\underline{\mathbf{t}}))\cup\overline{\Delta}^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\underline{\mathbf{t}})$. We denote by $\Pi(;\underline{\mathbf{t}})_x$ the corresponding basis of $\Delta(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})$.

We have:

LEMMA 2.3.2. *For $\lambda\in\mathcal{G}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$, we have the following*:

(1) *For all $\underline{\mathbf{t}}\in\mathbb{S}(p,q)$, $x\in\underline{W}(;\underline{\mathbf{t}})$, and $i\geqslant 0$, $\mathcal{R}^i_{\mathfrak{b}(;\underline{\mathbf{t}})_x}(\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})\uparrow\mathfrak{g}(m,n);$ $\mathbb{C}_{x\underline{\rho}(;\underline{\mathbf{t}})+\lambda}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}}))\neq 0$ if and only if $i=S_{\underline{\mathbf{t}},x}+S_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}}$.*

(2) *We have*

$$[\mathsf{Ind}(P(\underline{\mathbf{r}};p,q)\uparrow \mathrm{U}(m,n);\mathcal{A}(\underline{\mathfrak{q}}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})\uparrow\mathfrak{m}(\underline{\mathbf{r}};p,q);\lambda))]$$
$$=\sum_{\underline{\mathbf{t}}\in\mathbb{S}(p,q)}\sum_{x\in\underline{W}(;\underline{\mathbf{t}})} n(\underline{\mathbf{t}},x)[\mathcal{R}^{S_{\underline{\mathbf{t}},x}+S_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}}}_{\underline{\mathfrak{b}}(;\underline{\mathbf{t}})_x}(\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}}) \uparrow\mathfrak{g}(m,n);\mathbb{C}_{x\underline{\rho}(;\underline{\mathbf{t}})+\lambda}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}}))].$$

Here, $n(\underline{\mathbf{t}},x)$'s are integers in Lemma 2.3.1 (2).

Proof. Lemma 2.1.3 means $x\underline{\rho}(;\underline{\mathbf{t}})+\lambda$ is in the good range with respect to $\underline{\mathfrak{q}}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ for all $\underline{\mathbf{t}}\in\mathbb{S}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ and $x\in\underline{W}(;\underline{\mathbf{t}})$. Hence we can easily reduce the lemma to Lemma 2.3.1 for $\mu=\lambda+\bar{\rho}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ using Lemma 2.1.2 and the results in 1.2. □

For $\underline{\mathbf{t}}\in\mathbb{S}(\underline{\mathbf{p}},\underline{\mathbf{q}})$, $x\in\underline{W}(;\underline{\mathbf{t}})$, and $\lambda\in\mathcal{G}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$, we define a Borel subalgebra $\mathfrak{b}(;\underline{\mathbf{t}};x,\lambda)$ of $\mathfrak{g}(m,n)$ as follows.

$$\mathfrak{b}_1(;\underline{\mathbf{t}};\lambda)_x=\underline{\mathfrak{b}}(;\underline{\mathbf{t}})_x+\mathfrak{v}_1(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\lambda).$$

We denote by $\Delta^+_1(;\underline{\mathbf{t}};\lambda)_x$ (resp. $\Pi_1(;\underline{\mathbf{t}};\lambda)$) the corresponding positive system (resp. basis) of $\Delta(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})$ to $\mathfrak{b}_1(;\underline{\mathbf{t}};\lambda)_x$.

We have:

LEMMA 2.3.3. *Take $\lambda\in\mathcal{G}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$. Then, we have*

(1) *Fix $\underline{\mathbf{t}}\in\mathbb{S}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ and $x\in\underline{W}(;\underline{\mathbf{t}})$. Then, $\mathcal{R}^i_{\mathfrak{b}_1(;\underline{\mathbf{t}};\lambda)_x}(\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})\uparrow\mathfrak{g}(m,n);$ $\mathbb{C}_{x\underline{\rho}(;\underline{\mathbf{t}})+\lambda}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}}))\neq 0$ if and only if $i=S_{\underline{\mathbf{t}},x}+f_{\underline{\mathbf{r}};\underline{\mathbf{q}},\underline{\mathbf{q}};\lambda}$.*

(2) *We have*

$$[\mathcal{R}^{f_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\lambda}}_{\mathfrak{f}(\underline{\mathbf{r}};\underline{\mathbf{p}};\underline{\mathbf{q}};\lambda)}(\mathfrak{g}(\underline{\mathbf{m}},\underline{\mathbf{n}})\uparrow\mathfrak{g}(m,n);\operatorname{Ind}^{\mathrm{U}(\underline{\mathbf{m}},\underline{\mathbf{n}})}_{\mathrm{S}(\underline{\mathbf{r}})\star\mathrm{U}(\underline{\mathbf{p}},\underline{\mathbf{q}})}$$
$$(\mathbb{C}_{\lambda+\bar{\rho}_2(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\lambda)}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}))\otimes\mathbb{C}_{-\bar{\rho}_2(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\lambda)}(\underline{\mathbf{m}},\underline{\mathbf{n}}))]$$
$$=\sum_{\underline{\mathbf{t}}\in\mathbb{S}(\underline{\mathbf{p}},\underline{\mathbf{q}})}\sum_{x\in\underline{W}(;\underline{\mathbf{t}})}n(\underline{\mathbf{t}},x)[\mathcal{R}^{S_{\underline{\mathbf{t}},x}+f_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\lambda}}_{\mathfrak{b}_1(;\underline{\mathbf{t}};\lambda)_x}$$
$$(\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})\uparrow\mathfrak{g}(m,n);\mathbb{C}_{x\underline{\rho}(;\underline{\mathbf{t}})+\lambda}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}}))].$$

Here, $n(\underline{\mathbf{t}},x)$*'s are integers in Lemma* 2.3.1 (2).

Proof. Lemma 2.2.1 means that $x\underline{\rho}(;\underline{\mathbf{t}})+\lambda$ is in the good range with respect to $\mathfrak{f}(;\underline{\mathbf{p}},\underline{\mathbf{q}};\lambda)$ for all $\underline{\mathbf{t}}\in\mathbb{S}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ and $x\in\underline{W}(;\underline{\mathbf{t}})$. Hence we can easily deduce the lemma from Lemma 2.3.1 for $\mu=\lambda+\rho(\mathfrak{v}_1(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\lambda))$ using Lemma 2.2.2 and the fundamental results listed in 1.2. □

From Lemma 2.3.2 and Lemma 2.3.3, we can reduce Theorem 2.2.3 to the following.

PROPOSITION 2.3.4. *For* $\lambda\in\mathcal{G}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$, $\underline{\mathbf{t}}\in\mathbb{S}(\underline{\mathbf{p}},\underline{\mathbf{q}})$, *and* $x\in\underline{W}(;\underline{\mathbf{t}})$, *we have*

$$\mathcal{R}^{S_{\underline{\mathbf{t}},x}+S_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}}}_{\mathfrak{b}(;\underline{\mathbf{t}})_x}(\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})\uparrow\mathfrak{g}(m,n);\dot{\mathbb{C}}_{x\underline{\rho}(;\underline{\mathbf{t}})+\lambda}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}}))$$
$$\cong\mathcal{R}^{S_{\underline{\mathbf{t}},x}+f_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\lambda}}_{\mathfrak{b}_1(;\underline{\mathbf{t}};\lambda)_x}(\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})\uparrow\mathfrak{g}(m,n);\mathbb{C}_{x\underline{\rho}(;\underline{\mathbf{t}})+\lambda}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})).$$

We prove Proposition 2.3.4 in 2.4.

2.4. CHANGE OF POLARIZATIONS FOR STANDARD MODULES

As in 2.1, we fix m, n, r, p, q, $\underline{\mathbf{p}}$, $\underline{\mathbf{q}}$, $\underline{\mathbf{t}}$, $\underline{\mathbf{m}}$, $\underline{\mathbf{n}}$. We also fix $\underline{\mathbf{t}}\in\mathbb{S}(\underline{\mathbf{p}},\underline{\mathbf{q}})$, $x\in\underline{W}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\underline{\mathbf{t}})=\underline{W}(;\underline{\mathbf{t}})$, and $\lambda=\Sigma_{i=1}^{2d+h}\lambda_i[[\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};i]]\in\mathcal{G}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$. For simplicity, we put $u_i=m_i+n_i$ for all $1\leqslant i\ell$. Throughout this section, we fix $\eta_{i,j}\in\mathbb{Q}$ for $1\leqslant i\leqslant\ell$ and $1\leqslant j\leqslant u_i$ such that $x\underline{\rho}(;\underline{\mathbf{t}})=\Sigma_{i=1}^{\ell}\Sigma_{j=1}^{u_i}\eta_{i,j}[i]_j$. For $d<i\leqslant\ell$, let $\tau_i\in\mathfrak{S}_{u_i-2t_{i-d}}$ be the unique permutation such that $\eta_{t_{i-d}+\tau_i(1)}>\eta_{t_{i-d}+\tau_i(2)}>\cdots>\eta_{t_{i-d}+\tau_i(u_i-2t_{i-d})}$.

Before going into the proof of Proposition 2.3.5, we introduce some terminologies.

First, we introduce another parametrization of the standard basis of $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}}\star\underline{\mathbf{t}})^*$ as follows.

$$\{i\}_j=\begin{cases}[i]_{t_{i-d}+\tau_i(j)} & \text{if } d<i\leqslant\ell \text{ and } t_{i-d}<j\leqslant p_{i-d}+q_{i-d}-t_{i-d}\\ [i]_j & \text{otherwise}\end{cases}.$$

Here, we describe the induced action of θ on $\mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{s}})^*$ as follows. The proof is easy.

LEMMA 2.4.1. *For* $1 \leqslant i \leqslant \ell$ *and* $1 \leqslant j \leqslant u_i$, *we have*

$$\theta(\{i\}_j) = \begin{cases} \{i\}_{2r_i-j+1} & \text{if } 1 \leqslant i \leqslant d \text{ and } 1 \leqslant j \leqslant 2r_i \\ \{i\}_j & \text{if } d < i \leqslant \ell \text{ and} \\ & t_{i-d} < i \leqslant p_{i-d} + q_{i-d} - t_{i-d} \\ \{i\}_{p_{i-d}+q_{i-d}-j+1} & \text{otherwise} \end{cases}.$$

We immediately see:

LEMMA 2.4.2.

(1)

$$\begin{aligned} \underline{\Pi}(;\underline{\mathbf{t}})_x &= \{\{i\}_j - \{i\}_{j+1} \mid 1 \leqslant i \leqslant d, 1 \leqslant j < r_i\} \\ &\cup \{\{i\}_j - \{i\}_{j+1} \mid 1 \leqslant i \leqslant d, r_i < j < 2r_i\} \\ &\cup \{\{i\}_j - \{i\}_{j+1} \mid d < i \leqslant \ell, 1 \leqslant j < p_{i-d} + q_{i-d}\}. \end{aligned}$$

(2)

$$\begin{aligned} \Pi(;\underline{\mathbf{t}})_x &= \underline{\Pi}(;\underline{\mathbf{t}})_x \\ &\cup \{\{i\}_{r_i} - \{i+1\}_1 \mid 1 \leqslant i \leqslant d\} \\ &\cup \{\{i\}_{p_{i-d}+q_{i-d}} - \{i+1\}_1 \mid d < i < \ell\} \\ &\cup \{\{\ell\}_{p_h+q_h} - \{d\}_{r_1+1}\} \\ &\cup \{\{i\}_{2r_i} - \{i-1\}_{r_{i-1}+1} \mid 1 < i \leqslant d\}. \end{aligned}$$

(3)

$$\begin{aligned} \Pi_1(;\underline{\mathbf{t}};\lambda)_x &= \underline{\Pi}(;\underline{\mathbf{t}})_x \\ &\mu\{\{i\}_{r_i} - \{i\}_{r_i+1} \mid 1 \leqslant i \leqslant d,\} \\ &\cup \{\{\sigma_\lambda(i)\}_{u_{\sigma_\lambda(i)}} - \{\sigma_\lambda(i+1)\}_1 \mid 1 \leqslant i < \ell\}. \end{aligned}$$

We put

$$\begin{aligned} A &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant i < k \leqslant d, \underline{\underline{\lambda}}_{(i)} > \overline{\overline{\lambda}}_{(k)}, 1 \leqslant j \leqslant 2r_i, 1 \leqslant v \leqslant 2r_i\}, \\ B &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant k < i \leqslant d, \underline{\underline{\lambda}}_{(i)} > \overline{\overline{\lambda}}_{(k)}, 1 \leqslant j \leqslant 2r_i, 1 \leqslant v \leqslant 2r_i\}, \\ C &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant i \leqslant d < k \leqslant \ell, \underline{\underline{\lambda}}_{(i)} > \overline{\overline{\lambda}}_{(k)}, 1 \leqslant j \leqslant 2r_i, 1 \leqslant v \leqslant u_k\}, \\ D &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant k \leqslant d < i \leqslant \ell, \underline{\underline{\lambda}}_{(i)} > \overline{\overline{\lambda}}_{(k)}, 1 \leqslant v \leqslant 2r_k, 1 \leqslant j \leqslant u_i\}. \end{aligned}$$

We consider $(\underline{\mathbf{r}} \star p, \underline{\mathbf{r}} \star q) = ((r_1, \ldots, r_d, p), (r_1, \ldots, r_d, q)) \in \mathbb{P}_{d+1}(m,n)$ and denote by $\underline{\Delta}(\underline{\mathbf{r}} \star p, \underline{\mathbf{r}} \star q; \underline{\mathbf{t}})$ the root system for $(\mathfrak{g}(\underline{\mathbf{r}} \star p, \underline{\mathbf{r}} \star q), \mathfrak{h}(\underline{\mathbf{m}},\underline{\mathbf{n}};\underline{\mathbf{r}} \star \underline{\mathbf{t}}))$.

The following lemma is easily deduced from the definitions and the fact $\lambda \in \overline{\mathcal{P}}^+(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$.

LEMMA 2.4.3. *Fix* $\underline{\mathbf{t}} \in \mathbb{S}(\underline{\mathbf{p}}, \underline{\mathbf{q}})$, $x \in \underline{W}(; \underline{\mathbf{t}})$, *and* $\lambda \in \mathcal{G}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$.

(1) *We have*

$$\begin{aligned}
&\Delta^+(; \underline{\mathbf{t}})_x \cap \underline{\Delta}(\underline{\mathbf{r}} \star p, \underline{\mathbf{r}} \star q; \underline{\mathbf{t}}) \\
&\quad = \Delta_1^+(; \underline{\mathbf{t}}; \lambda)_x \cap \underline{\Delta}(\underline{\mathbf{r}} \star p, \underline{\mathbf{r}} \star q; \underline{\mathbf{t}}) \\
&\quad = \{\{i\}_j - \{i\}_v \mid 1 \leqslant i \leqslant d, 1 \leqslant j < v \leqslant 2r_i\} \\
&\qquad \cup \{\{i\}_j - \{i\}_v \mid d < i \leqslant \ell, 1 \leqslant j < v \leqslant u_i\} \\
&\qquad \cup \{\{i\}_j - \{k\}_v \mid d < i < k \leqslant \ell, 1 \leqslant j \leqslant u_i, 1 \leqslant v \leqslant u_k\}.
\end{aligned}$$

(2) *We have*

$$\Delta_1^+(; \underline{\mathbf{t}}; \lambda)_x = (\Delta^+(; \underline{\mathbf{t}})_x \cap \underline{\Delta}(\underline{\mathbf{r}} \star p, \underline{\mathbf{r}} \star q; \underline{\mathbf{t}})) \cup A \cup B \cup C \cup D.$$

We also put

$$\begin{aligned}
A_0 &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant i < k \leqslant d, \underline{\underline{\lambda}}_{(i)} \\
&\qquad > \overline{\overline{\lambda}}_{(k)}, 1 \leqslant j \leqslant r_i, 1 \leqslant v \leqslant 2r_k\} \subseteq A, \\
B_0 &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant k < i \leqslant d, \underline{\underline{\lambda}}_{(i)} \\
&\qquad > \overline{\overline{\lambda}}_{(k)}, 1 \leqslant j \leqslant 2r_i, r_k < v \leqslant 2r_k\} \subseteq B, \\
C_0 &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant i \leqslant d < k \leqslant \ell, \underline{\underline{\lambda}}_{(i)} \\
&\qquad > \overline{\overline{\lambda}}_{(k)}, 1 \leqslant j \leqslant r_i, 1 \leqslant v \leqslant u_k\} \subseteq C, \\
D_0 &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant k \leqslant d < i \leqslant \ell, \underline{\underline{\lambda}}_{(i)} \\
&\qquad > \overline{\overline{\lambda}}_{(k)}, r_k < v \leqslant 2r_k, 1 \leqslant j \leqslant u_i\} \subseteq D,
\end{aligned}$$

$$\begin{aligned}
A_1 = A - A_0 &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant i < k \leqslant d, \underline{\underline{\lambda}}_{(i)} \\
&\qquad > \overline{\overline{\lambda}}_{(k)}, r_i < j \leqslant 2r_i, 1 \leqslant v \leqslant 2r_k\}, \\
B_1 = B - B_0 &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant k < i \leqslant d, \underline{\underline{\lambda}}_{(i)} \\
&\qquad > \overline{\overline{\lambda}}_{(k)}, 1 \leqslant j \leqslant 2r_i, 1 \leqslant v \leqslant r_k\}, \\
C_1 = C - C_0 &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant i \leqslant d < k \leqslant \ell, \underline{\underline{\lambda}}_{(i)} \\
&\qquad > \overline{\overline{\lambda}}_{(k)}, r_i < j \leqslant 2r_i, 1 \leqslant v \leqslant u_k\}, \\
D_1 = D - D_0 &= \{\{i\}_j - \{k\}_v \mid 1 \leqslant k \leqslant d < i \leqslant \ell, \underline{\underline{\lambda}}_{(i)} \\
&\qquad > \overline{\overline{\lambda}}_{(k)}, 1 \leqslant v \leqslant r_k, 1 \leqslant j \leqslant u_i\}.
\end{aligned}$$

From Lemma 2.4.1, we immediately see:

LEMMA 2.4.4. *Any root in* $A \cup B \cup C \cup D$ *is complex. Moreover, we have* $\theta(A_1) = A_0$, $\theta(B_1) = B_0$, $\theta(C_1) = C_0$, *and* $\theta(D_1) = D_0$.

We call a positive system Δ^+ of $\Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})$ admissible if Δ^+ satisfies the following two conditions.

(AD1) $\Delta^+ \cap \underline{\Delta}(\underline{\mathbf{r}} \star p, \underline{\mathbf{r}} \star q; \underline{\mathbf{t}}) = \Delta^+(;\underline{\mathbf{t}})_x \cap \underline{\Delta}(\underline{\mathbf{r}} \star p, \underline{\mathbf{r}} \star q; \underline{\mathbf{t}})$.

(AD2) $A_0 \cup B_0 \cup C_0 \cup D_0 \subseteq \Delta^+$.

For an admissible positive system Δ^+, we denote by $\mathrm{ht}(\Delta^+)$ the non-negative integer defined as follows.

$$\mathrm{ht}(\Delta^+) = \mathrm{card}(-\Delta^+ \cap (A_1 \cup B_1 \cup C_1 \cup D_1)).$$

The following immediately results from Lemma 2.4.3.

LEMMA 2.4.5.

(1) $\Delta^+(;\underline{\mathbf{t}})_x$ *is admissible.*

(2) $\Delta_1^+(;\underline{\mathbf{t}};\lambda)_x$ *is admissible and* $\mathrm{ht}(\Delta_1^+(;\underline{\mathbf{t}};\lambda)_x) = 0$.

(3) *Let* Δ^+ *be an admissible positive system such that* $\mathrm{ht}(\Delta^+) = 0$. *Then, we have* $\Delta^+ = \Delta_1^+(;\underline{\mathbf{t}};\lambda)_x$.

We also have

LEMMA 2.4.6. *Assume that* Δ^+ *is an admissible positive system such that* $\{k\}_j - \{k\}_{j+1}$ *is simple with respect to* Δ^+ *for any* $1 \leqslant k \leqslant d$ *and* $1 \leqslant j < 2r_k$. *Then, we have* $\Delta^+ = \Delta^+(;\underline{\mathbf{t}};\lambda)_x$.

Proof. We have only to show $A_1 \cup B_1 \cup C_1 \cup D_1 \subseteq \Delta^+$. We show $A_1 \subseteq \Delta^+$. (AD2) implies $\{i\}_{r_i} - \{k\}_v \in \Delta^+$ for all $1 \leqslant v \leqslant 2r_k$. Since $\{i\}_{r_i} - \{i\}_{r_i+1}, \{i\}_{r_i+1} - \{i\}_{r_i+2}, \ldots,$ are simple, we see $\{i\}_{r_i+1} - \{k\}_v, \{i\}_{r_i+2} - \{k\}_v, \ldots$ are in Δ^+ inductively. We can show $B_1, C_1, D_1 \subseteq \Delta^+$ just in the same way. □

For a root $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})$, we denote by s_α the corresponding reflection in $W(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})$.

Next we show:

LEMMA 2.4.7. *Let* Δ^+ *be an admissible positive system of* $\Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})$ *such that* $\mathrm{ht}(\Delta^+) > 0$. *Then there exists some complex simple root* $\alpha \in \Delta^+$ *which satisfies the following conditions.*

(S1) $\theta(\alpha) \notin \Delta^+$.

(S2) $\mathrm{Re}\langle x\underline{\rho} + \lambda, \alpha\rangle < 0$.

(S3) $s_\alpha(\Delta^+)$ *is admissible, and* $\mathrm{ht}(s_\alpha(\Delta^+)) = \mathrm{ht}(\Delta^+) - 1$.

Proof. From Lemma 2.4.6, there is some $1 \leqslant i \leqslant d$ and $1 \leqslant j < 2r_i$ such that $\beta = \{i\}_j - \{i\}_{j+1}$ is not simple. From (AD1) and Lemma 4.2.3 (1), we see

$\beta \in \Delta^+$. Hence, there exists a sequence of non-negative integers $a_0, a_1, \ldots, a_s$ and $b_0, b_1, \ldots, b_s$ such that $a_0 = a_s = i, b_0 = j, b_s = j+1$, and $\{a_v\}_{b_v} - \{a_{v+1}\}_{b_{v+1}}$ are simple for all $0 \leqslant v < s$. Next, we assume $\overline{\overline{\lambda}}_{(a_v)} \leqslant \underline{\underline{\lambda}}_{(a_{v+1})}$ for all $0 \leqslant v < s$. However $a_0 = a - s = i$ and $\lambda \in \mathcal{G}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$ imply $a_v = i$ for all $0 \leqslant v \leqslant s$. However, from (AD1), this contradicts Lemma 2.4.3 (1). Hence, there is some $0 \leqslant v < s$ such that $\overline{\overline{\lambda}}_{(a_v)} < \underline{\underline{\lambda}}_{(a_{v+1})}$ and we fix such v. We put $\alpha = \{a_v\}_{b_v} - \{a_{v+1}\}_{b_{v+1}}$ and show this α satisfies (S1)–(S3) above. We assume $d < a_v.a_{v+1} \leqslant \ell$. Then, $\lambda \in \mathcal{G}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$ implies $a_{v+1} < a_v$. But, from (AD1), it contradicts Lemma 2.4.3 (1). Hence, we have $-\alpha \in A \cup B \cup C \cup D$. From (AD2), we have $-\alpha \in A_1 \cup B_1 \cup C_1 \cup D_1$. From Lemma 2.4.4, we have $\theta(\alpha) \in -(A_0 \cup B_0 \cup c_0 \cup D_0)$. Hence, from (AD2) we have (S1). (S2) follows from $\overline{\overline{\lambda}}_{(a_v)} < \underline{\underline{\lambda}}_{(a_{v+1})}$. (S3) is clear. □

For an admissible positive system Δ^+, we denote by $\mathfrak{b}(\Delta^+)$ the corresponding Borel subalgebra of $\mathfrak{g}(m, n)$ containing $\mathfrak{h}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})$. The root spaces for Δ^+ are contained in the nilradical of $\mathfrak{b}(\Delta^+)$.

The following is a special case of a result of Hecht– Miličić–Schmid–Wolf and plays a crucial role in the proof.

THEOREM 2.4.8. ([29] (4.12)). *Let Δ^+ be an admissible positive system of* $\Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})$ *and let α be a simple root for Δ^+ satisfying the conditions* (S1) *and* (S2) *in Lemma* 2.4.7. *Then, for $i \geqslant 0$ we have the following isomorphism of Harish-Chandra modules.*

$$\begin{aligned}&\mathcal{R}^i_{\mathfrak{b}(\Delta^+)}(\mathfrak{h}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}}) \uparrow \mathfrak{g}(m, n); \mathbb{C}_{x\underline{\rho}(;\underline{\mathbf{t}})+\lambda}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}}))\\ &\cong \mathcal{R}^{i+1}_{\mathfrak{b}(s_\alpha(\Delta^+))}(\mathfrak{h}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}}) \uparrow \mathfrak{g}(m, n); \mathbb{C}_{x\underline{\rho}(;\underline{\mathbf{t}})+\lambda}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})).\end{aligned}$$

I stress here the choice on the positive system in [29] is opposite to our convention. There is also a difference in the convention on the parameter λ.

Proof of Proposition 2.3.4. Since $\Delta^+(;\underline{\mathbf{t}})_x$ is admissible, applying Lemma 2.4.7 and Theorem 2.4.8 successively, we have

$$\begin{aligned}&\mathcal{R}^{S_{\underline{\mathbf{t}},x}+S_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}}}_{\mathfrak{b}(;\underline{\mathbf{t}})_x}(\mathfrak{h}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}}) \uparrow \mathfrak{g}(m, n); \dot{\mathbb{C}}_{x\underline{\rho}(;\underline{\mathbf{t}})+\lambda}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}}))\\ &\cong \mathcal{R}^{S_{\underline{\mathbf{t}},x}+S_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}}+\mathrm{ht}(\Delta^+(;\underline{\mathbf{t}})_x)}_{\mathfrak{b}_1(;\underline{\mathbf{t}};\lambda)_x}(\mathfrak{h}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})\\ &\quad \uparrow \mathfrak{g}(m, n); \mathbb{C}_{x\underline{\rho}(;\underline{\mathbf{t}})+\lambda}(\underline{\mathbf{m}}, \underline{\mathbf{n}}; \underline{\mathbf{r}} \star \underline{\mathbf{t}})).\end{aligned}$$

However, Lemma 2.3.1 (1) and Lemma 2.3.3 (1) imply $S_{\underline{\mathbf{t}},x} + S_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}} + \mathrm{ht}(\Delta^+(;\underline{\mathbf{t}})_x) = S_{\underline{\mathbf{t}},x} + f_{\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\lambda}$. Hence we have Proposition 2.3.4. □

3. Decomposition formulas

3.1. Decomposition formula: Siegel parabolic case

We define for a non-negative integer r as follows.

$$\varepsilon(r) = \begin{cases} 0 & \text{if } r \text{ is even,} \\ \frac{1}{2} & \text{if } r \text{ is odd.} \end{cases}$$

Here, we consider $\mathrm{U}(r,r)$ and its Siegel parabolic subgroup $\mathrm{S}(r)$.
We put

$$\lambda(z;\mu) = (z+\sqrt{-1}\mu)[[r,r;r]]_+ + (z-\sqrt{-1}\mu)[[r,r;r]]_- \in \mathfrak{c}(r;0,0)^*.$$

Then we have

$$\overline{\mathcal{P}}_u^+(r;0,0) = \{\lambda(z;\mu) \mid z \in \tfrac{1}{2}\mathbb{Z}, \mu \in \mathbb{R}\}.$$

For $z \in \frac{1}{2}\mathbb{Z}$, $\lambda(z;0)$ can be regarded as an element of $\mathfrak{z}(r,r)^*$ and is extent to a character of the metaplectic double cover of $\mathrm{U}((i,r-i),(r-i,r))$ for all $0 \leqslant i \leqslant r$.

The study of degenerate series of $\mathrm{U}(r,r)$ with respect to the Siegel parabolic has a long history. We have the following description of the unitary degenerate series.

THEOREM 3.1.1. (Kashiwara–Vergne–Barbasch–Vogan–Johnson).
(1) *If $z+\varepsilon(r) \in \mathbb{Z}$, then we have the following decomposition formula.*

$$\begin{aligned} &\mathsf{Ind}(\mathrm{S}(r) \uparrow \mathrm{U}(r,r); \lambda(z;0)) \\ &\cong \bigoplus_{i=0}^{r} \mathcal{A}(\mathfrak{q}((i,r-i),(r-i,i)) \uparrow \mathfrak{g}(r,r); \lambda(z;0)). \end{aligned}$$

Here, $\lambda(z;0)$ is in the weakly fair range with respect to $\mathfrak{q}((i,r-i),(r-i,i))$ for $0 \leqslant i \leqslant r$. Each $\mathcal{A}(\mathfrak{q}((i,r-i),(r-i,i)) \uparrow \mathfrak{g}(r,r); \lambda(z;0))$ is a non-zero irreducible module, which is not isomorphic to other irreducible factors.

(2) *Assume $\lambda(z;\mu) \in \overline{\mathcal{P}}_u^+(r;0,0)$. If $\mu \neq 0$ or if $z+\varepsilon(r) \notin \mathbb{Z}$, then* $\mathsf{Ind}(\mathrm{S}(r) \uparrow \mathrm{U}(r,r); \lambda(z;\mu))$ *is irreducible.*

Several remarks are in order. First, in [15] the decomposition in (1) is studied. They proved the induced module is decomposed into a direct sum of $r+1$ distinct irreducible factors. They also described the irreducible factors in terms of the Weil representation. Second, the description of the irreducible factors in terms of derived functor modules is the matter of unipotent representations of $\mathrm{U}(r,r)$ and it is more or less known by Barbasch and Vogan. Actually, it rather easily follows from [2], Theorem 4.2. (For convenience of the readers, I shall give a proof in 3.4.) Third, (2) is proved in [14]. Actually, he treated all the non-unitary parameter for degenerate series with respect to a 'Siegel type' parabolic subgroup of $\mathrm{SU}(n,n)$, $\mathrm{Sp}(n,n)$, and $\mathrm{Spin}_0(n,n)$. Fourth, the structure of non-unitary degenerate series in Siegel situation is also studied by [33] and [25,26]. Speh treated the universal covering of $\mathrm{SU}(2,2)$ and Sahi worked on all the tube domain situation. Fifth, $\mathcal{A}(\mathfrak{q}((0,r),(r,0))\uparrow \mathfrak{g}(r,r);\lambda(z,0))$ and $\mathcal{A}(\mathfrak{q}((r,0),(0,r)\uparrow \mathfrak{g}(r,r);\lambda(z,0))$ correspond to the representations realized as Hardy spaces on the Siegel upper and lower half-planes. Their embeddings into degenerate series are nothing but the boundary value map to the Shilov boundary.

3.2. Decomposition Formula: Good Enough Case

As in 2.1, we fix m, n, r, p, q, $\underline{\mathbf{p}}$, $\underline{\mathbf{q}}$, $\underline{\mathbf{r}}$, $\underline{\mathbf{m}}$, and $\underline{\mathbf{n}}$. For $\underline{\mu} = (\mu_1,\dots,\mu_d)\in\mathbb{R}^d$ and $\underline{\mathbf{z}} = (z_1,\dots,z_d,z_{d+1},\dots,z_\ell)\in\frac{1}{2}\mathbb{Z}^\ell$, we put

$$\begin{aligned}\lambda(\underline{\mathbf{z}};\underline{\mu}) &= \sum_{i=1}^{d}((z_i+\sqrt{-1}\mu_i)[[\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};i]]\\ &\quad +(z_i-\sqrt{-1}\mu_i)[[\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};2d+h-i+1]])\\ &\quad +\sum_{i=1}^{h} z_{d+i}[[\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};d+i]].\end{aligned}$$

We easily see:

LEMMA 3.2.1.

$$\begin{aligned}&\overline{\mathcal{P}}_u^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})\\ &\quad = \{\lambda((z_1,\dots,z_\ell);\underline{\mu}))\mid z_1,\dots,z_d\in\tfrac{1}{2}\mathbb{Z},\underline{\mu}\in\mathbb{R}^d, z_{i+d}\\ &\qquad +\varepsilon(m+n-p_i-q_i)\in\mathbb{Z}\,(1\leqslant i\leqslant h), z_{d+1}\geqslant z_{d+2}\geqslant\cdots\geqslant z_\ell\}.\end{aligned}$$

We consider the following two extreme situations.

$$\begin{aligned}\mathcal{X}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) &= \{\lambda((z_1,\dots,z_\ell);(0,\dots,0))\in\overline{\mathcal{P}}_u^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})\mid z_i\\ &\qquad -\varepsilon(m+n-r_i)\in\mathbb{Z}\,(1\leqslant i\leqslant d)\},\end{aligned}$$

$$\begin{aligned}&\mathcal{Y}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})\\ &\quad = \{\lambda((z_1,\dots,z_\ell);(\mu_1,\dots,\mu_d))\in\overline{\mathcal{P}}_u^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})\mid \text{For all } 1\leqslant i\leqslant d,\\ &\quad\ \text{we have } \mu_i\neq 0 \text{ or } z_i+\varepsilon(m+n-r_i)\notin\mathbb{Z}\}.\end{aligned}$$

In order to state the result, we introduce some notations. For $\sigma \in \mathfrak{S}_\ell$ and $1 \leqslant i \leqslant \ell$, we put $e(\sigma; i) = \mathrm{card}\{j \mid 1 \leqslant j \leqslant d, \sigma(j) < \sigma(i)\}$. For $\sigma \in \mathfrak{S}_\ell$, we define $\hat{\sigma} \in \mathfrak{S}_{\ell+d}$ as follows.

$$\hat{\sigma}(i) = \begin{cases} \sigma\left(\frac{i+1}{2}\right) + e\left(\sigma; \frac{i+1}{2}\right) & \text{if } 1 \leqslant i \leqslant 2d \text{ and } i \text{ is odd,} \\ \sigma\left(\frac{i}{2}\right) + e\left(\sigma; \frac{i}{2}\right) + 1 & \text{if } 1 \leqslant i \leqslant 2d \text{ and } i \text{ is even,} \\ \sigma(i-d) + e(\sigma; i-d) & \text{if } 2d < i \leqslant \ell + d. \end{cases}$$

For $\underline{\mathbf{k}} = (k_1, \ldots, k_d) \in \mathbb{S}(\underline{\mathbf{r}}, \underline{\mathbf{r}})$, we define $\underline{\mathbf{r}} \mid \underline{\mathbf{k}} \in \mathbb{P}_{2d}(r, r)$ as follows.

$$\begin{aligned} \underline{\mathbf{r}} \mid \underline{\mathbf{k}} = & ((k_1, r_1 - k_1, k_2, r_2 - k_2, \ldots, k_d, r_d - k_d), \\ & (r_1 - k_1, k_1, r_2 - k_2, k_2, \ldots, r_d - k_d, k_d)). \end{aligned}$$

We remark that $\lambda \in \chi(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$ defines a character of the metaplectic double cover of $U((\underline{\mathbf{r}} \mid \underline{\mathbf{k}}) \star (\underline{\mathbf{p}}, \underline{\mathbf{q}}))$. Namely, we have:

$$\begin{aligned} \lambda(\underline{\mathbf{z}}, 0) = & \sum_{i=1}^{d} z_i([[(\underline{\mathbf{r}} \mid \underline{\mathbf{k}}) \star (\underline{\mathbf{p}}, \underline{\mathbf{q}}); 2i-1]] + [[(\underline{\mathbf{r}} \mid \underline{\mathbf{k}}) \star (\underline{\mathbf{p}}, \underline{\mathbf{q}}); 2i]]) \\ & + \sum_{j=1}^{h} z_{d+j}[[(\underline{\mathbf{r}} \mid \underline{\mathbf{k}}) \star (\underline{\mathbf{p}}, \underline{\mathbf{q}}); 2d+j]]. \end{aligned}$$

Now we can state:

THEOREM 3.2.2.

(1) *For* $\lambda \in \chi(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \cap \mathcal{G}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, *we have the following direct sum decomposition.*

$$\mathrm{Ind}_{P(\underline{\mathbf{r}}; p,q)}^{U(m,n)}(\mathcal{A}_{\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})}^{\mathfrak{m}(\underline{\mathbf{r}}; p,q)}(\lambda)) \cong \bigoplus_{\underline{\mathbf{k}} \in \mathbb{S}(\underline{\mathbf{r}}, \underline{\mathbf{r}})} \mathcal{A}_{\mathfrak{q}(((\underline{\mathbf{r}} \mid \underline{\mathbf{k}}) \star (\underline{\mathbf{p}}, \underline{\mathbf{q}}))^{\hat{\sigma}_\lambda})}^{\mathfrak{g}(m,n)}(\hat{\sigma}_\lambda(\lambda)).$$

Here, each direct summand in the right-hand side is a non-zero irreducible module.

(2) *For* $\lambda \in \mathcal{Y}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \cap \mathcal{G}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, $\mathrm{Ind}_{P(\underline{\mathbf{r}}; p,q)}^{U(m,n)}(\mathcal{A}_{\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})}^{\mathfrak{m}(\underline{\mathbf{r}}; p,q)}(\lambda))$ *is non-zero and irreducible.*

Proof. (1) is deduced from Corollary 2.2.5 and Theorem 3.1.1 (1) using the induction-by-stage. (2) follows from Corollary 2.2.4 and Theorem 3.1.1 (2). □

Remark. On the decomposition formula for a general good enough unitary character $\lambda(\underline{\mathbf{z}}; \underline{\mu}) \in \overline{\mathcal{P}_u^+}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}) \cap \mathcal{G}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, in view of Theorem 2.1.3, without loss of generality we can assume that there exists some $0 \leqslant f \leqslant d$ such that $\mu_i \neq 0$ or $z_i + \varepsilon(m+n-r-i) \notin \mathbb{Z}$ for all $1 \leqslant i \leqslant f$ and such that $\mu_j = 0$

and $z_j + \varepsilon(m+n-r_j) \in \mathbb{Z}$ for all $f < j \leqslant d$. Then, we can combine above (1) and (2) in Theorem 3.2.2 using the induction-by-stage, and easily obtain the decomposition formula of $\mathrm{Ind}_{\mathrm{P}(\underline{\mathbf{r}};p,q)}^{\mathrm{U}(m,n)}(\mathcal{A}_{\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})}^{\mathfrak{m}(\underline{\mathbf{r}};p,q)}(\lambda))$. Each direct summand of the formula is an irreducible module in generalized unitary degenerate series.

3.3. Decomposition formula: general case

We retain all the conventions and notations in 3.2.

For $\lambda = \Sigma_{i=1}^{2d+h}\lambda_i[[\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};i]] \in \overline{\mathcal{P}}_u^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\underline{\mathbf{t}})$, we denote by $C(\lambda)$ the set of permutations $\sigma \in \mathfrak{S}_\ell$ such that

$$\mathrm{Re}\,\lambda_{\sigma(1)} \geqslant \mathrm{Re}\,\lambda_{\sigma(2)} \geqslant \cdots \geqslant \mathrm{Re}\,\lambda_{\sigma(\ell)}.$$

Clearly $C(\lambda)$ is non-empty. In general, such σ is not unique. If $\lambda \in \mathcal{G}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$, we have $C(\lambda) = \{\sigma_\lambda\}$.

The following result comes from the main results of this article.

THEOREM 3.3.1.

(1) *For* $\lambda \in \mathcal{X}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ *and* $\sigma \in C(\lambda)$, *we have the following direct sum decomposition.*

$$\mathrm{Ind}_{\mathrm{P}(\underline{\mathbf{r}};p,q)}^{\mathrm{U}(m,n)}(\mathcal{A}_{\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})}^{\mathfrak{m}(\underline{\mathbf{r}};p,q)}(\lambda)) \cong \bigoplus_{\underline{\mathbf{k}}\in\mathbb{S}(\underline{\mathbf{r}},\underline{\mathbf{r}})} \mathcal{A}_{\mathfrak{q}(((\underline{\mathbf{r}}\,|\,\underline{\mathbf{k}})+(\underline{\mathbf{p}},\underline{\mathbf{q}}))^{\hat{\sigma}})}^{\mathfrak{g}(m,n)}(\hat{\sigma}(\lambda)).$$

Here, each direct summand in the right-hand side is either zero or irreducible.

(2) *For* $\lambda \in \mathcal{Y}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$, $\mathrm{Ind}_{\mathrm{P}(\underline{\mathbf{r}};p,q)}^{\mathrm{U}(m,n)}(\mathcal{A}_{\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})}^{\mathfrak{m}(\underline{\mathbf{r}};p,q)}(\lambda))$ *is irreducible or zero.*

Remark. By the same reasons as the remark just after Theorem 3.2.2, we can easily obtain the decomposition formula of $\mathrm{Ind}_{\mathrm{P}(\underline{\mathbf{r}};p,q)}^{\mathrm{U}(m,n)}(\mathcal{A}_{\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},bfq)}^{\mathfrak{m}(\underline{\mathbf{r}};p,q)}(\lambda))$ for general $\lambda \in \overline{\mathcal{P}}_u^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ combining the above (1) and (2) using the induction-by-stage. Each direct summand of the formula is either zero or an irreducible module in generalized unitary degenerate series.

Before going onto the proof of Theorem 3.3.1, we introduce the translation functor. Hereafter, we fix $\lambda = \lambda(\underline{\mathbf{z}},\underline{\mu}) \in \overline{\mathcal{P}}_u^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ and $\sigma \in C(\lambda)$.

First, we put

$$\mathcal{I}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) = \left\{\sum_{i=1}^{2d+h} \nu_i[[\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}]] \,\middle|\, \nu_i \in \mathbb{Z}\,(1 \leqslant i \leqslant 2d+h)\right\}.$$

We have:

LEMMA 3.3.2. *There exists some* $\eta \in \mathcal{G}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}) \cap \overline{\mathcal{P}}_u^+(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})$ *satisfying the following conditions.*

(E1) $\sigma_\eta = \sigma$.

(E2) $\eta - \lambda \in \mathcal{I}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$.

(E3) *There exists an irreducible finite dimensional* $\mathfrak{g}(m,n)$*-module* V *having a* $\mathfrak{q}_1(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \eta)$*-invariant line which* $\mathfrak{c}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$ *acts by* $\eta - \lambda$.

Proof. For example, we choose positive integers $y_1, \dots, y_\ell$ such that $y_i - y_{i+1} > m_{\sigma(i)} + n_{\sigma(i)} + m_{\sigma(i+1)} + n_{\sigma(i+1)} + 1$ and put $\eta = \lambda((z_1 + y_{\sigma^{-1}(1)}, \dots, z_\ell + y_{\sigma^{-1}(\ell)}), \underline{\mu})$. Then, we easily see η satisfies the above three conditions. □

Hereafter, we fix η and V as above. We define the translation functor T from the category of Harish-Chandra $(\mathfrak{g}(m,n), \mathrm{K}(m,n))$-modules with the same generalized infinitesimal character as $\mathbb{C}_\eta(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$ to the category of those with the same generalized infinitesimal character as $\mathbb{C}_\lambda(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$ as usual (For example see [41] p. 198). Namely, for a Harish-Chandra $(\mathfrak{g}(m,n), \mathrm{K}(m,n))$-module X with the same generalized infinitesimal character as $\mathbb{C}_\eta(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, we denote TX by the projection of $V \otimes_{\mathbb{C}} X$ on the generalized infinitesimal character same as $\mathbb{C}_\lambda(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$.
We have:

LEMMA 3.3.3. (Vogan).

(1) *We have* $T(\mathrm{Ind}_{\mathrm{P}(\underline{\mathbf{r}};p,q)}^{\mathrm{U}(m,n)}(\mathcal{A}_{\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})}^{\mathfrak{m}(\underline{\mathbf{r}};p,q)}(\eta))) \cong \mathrm{Ind}_{\mathrm{P}(\underline{\mathbf{r}};p,q)}^{\mathrm{U}(m,n)}(\mathcal{A}_{\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})}^{\mathfrak{m}(\underline{\mathbf{r}};p,q)}(\lambda)))$.

(2) *If* $\lambda \in \mathcal{X}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$, *then for all* $\underline{\mathbf{k}} \in \mathbb{S}(\underline{\mathbf{r}}, \underline{\mathbf{r}})$ *we have*

$$T(\mathcal{A}_{\mathfrak{q}(((\underline{\mathbf{r}}\,|\,\underline{\mathbf{k}})\star(\underline{\mathbf{p}},\underline{\mathbf{q}}))^{\hat{\sigma}})}^{\mathfrak{g}(m,n)}(\hat{\sigma}(\eta))) \cong \mathcal{A}_{\mathfrak{q}(((\underline{\mathbf{r}}\,|\,\underline{\mathbf{k}})\star(\underline{\mathbf{p}},\underline{\mathbf{q}}))^{\hat{\sigma}})}^{\mathfrak{g}(m,n)}(\hat{\sigma}(\lambda)).$$

Proof. The above (E1) implies that the infinitesimal character of $\mathbb{C}_\lambda(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$ is in the weakly fair range with respect to $\mathfrak{q}_1(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \eta)$. Hence, just the same argument as the proof of [41] Proposition 4.7 is also applicable to this situation. Namely, [36] Lemma 7.2.9(b), [41] Lemma 4.8, and Lemma 2.1.2 imply (1). Next, we show (2). Twisting by inner automorphisms, we easily see, for all $\underline{\mathbf{k}} \in \mathbb{S}(\underline{\mathbf{r}}, \underline{\mathbf{r}})$, $\hat{\sigma}(\lambda)$ is in the weakly fair range with respect to $\mathfrak{q}(((\underline{\mathbf{r}}\,|\,\underline{\mathbf{k}}) \star (\underline{\mathbf{p}}, \underline{\mathbf{p}}))^{\hat{\sigma}})$. Hence, (2) is nothing but [41] Proposition 4.7. □

Proof of Theorem 3.3.1. From Lemma 3.3.3 and Theorem 3.2.2 (1), we have the decomposition formula in Theorem 3.3.1 (1). In order to prove remaining part, first we remark that the assumption (i.e. $R_\lambda(\mathfrak{l}\colon \mathfrak{g})$ is a quotient of $R_\lambda(\mathfrak{g})$.) in [41] Corollary 5.9 always holds for $\mathrm{U}(m,n)$. Actually, the moment map of the cotangent bundle of the generalized flag variety is birational and the closure of any nilpotent orbit in $\mathfrak{gl}(m+n, \mathbb{C})$ is normal (Kraft-Procesi). Hence, from a result of Beilinson–Bernstein ([4], 6.2 Proposition (a)), the canonical homomorphism of the universal enveloping algebra to the ring of global sections of twisted differential operators is always surjective. However, $R_\lambda(\mathfrak{l}\colon \mathfrak{g})$ in [41] is isomorphic to the ring of global sections of twisted differential operators on a generalized flag variety of $\mathrm{GL}(m+n, \mathbb{C})$ (for example see [32]). The surjectivity of canonical homomorphism is nothing but the assumption in the Vogan's corollary in [41]. Hence, the remaining part of Theorem 3.3.1 (1) follows from this Corollary 5.9 in [41].

Theorem 3.3.1 (2) is also proved by the same idea. We denote by $\mathfrak{q}_1^{OP}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \eta)$ (resp. $\mathfrak{q}^{OP}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$) the opposite parabolic subalgebra to $\mathfrak{q}_1(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}}; \eta)$ (resp. $\mathfrak{q}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})$). Using [41] Proposition 5.7(a), [8] 5.1.7. Proposition, and the duality between induction and coinduction, we easily have

$$\begin{aligned}&\mathrm{Ann}_{\mathrm{U}(\mathfrak{g}(m,n))}(\mathsf{Ind}_{\mathrm{P}(\underline{\mathbf{r}};p,q)}^{\mathrm{U}(m,n)}(\mathcal{A}_{\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})}^{\mathfrak{m}(\underline{\mathbf{r}};p,q)}(\eta)))\\ &\quad\supseteq \mathrm{Ann}_{\mathrm{U}(\mathfrak{g}(m,n))}(\mathrm{U}(\mathfrak{g}(m,n)) \bigotimes_{\mathrm{U}(\mathfrak{q}^{OP}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}}))} \mathbb{C}_{\eta+\bar{\rho}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})).\end{aligned}$$

However, from [6] 4.10 Corollar (also see [13] 15.27 Corollar), we have

$$\begin{aligned}&\mathrm{Ann}_{\mathrm{U}(\mathfrak{g}(m,n))}(\mathsf{Ind}_{\mathrm{P}(\underline{\mathbf{r}};p,q)}^{\mathrm{U}(m,n)}(\mathcal{A}_{\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})}^{\mathfrak{m}(\underline{\mathbf{r}};p,q)}(\eta)))\\ &\quad\supseteq \mathrm{Ann}_{\mathrm{U}(\mathfrak{g}(m,n))}(\mathrm{U}(\mathfrak{g}(m,n)) \bigotimes_{\mathrm{U}(\mathfrak{q}_1^{OP}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\eta))} \mathbb{C}_{\eta+\rho(\mathfrak{v}_1(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}};\eta))}(\underline{\mathbf{r}}; \underline{\mathbf{p}}, \underline{\mathbf{q}})).\end{aligned}$$

Hence, we can regard $\mathsf{Ind}_{\mathrm{P}(\underline{\mathbf{r}};p,q)}^{\mathrm{U}(m,n)}(\mathcal{A}_{\mathfrak{q}(\underline{\mathbf{r}};\underline{\mathbf{p}},\underline{\mathbf{q}})}^{\mathfrak{m}(\underline{\mathbf{r}};p,q)}(\eta))$ as '$R_\eta(\mathfrak{l}:\mathfrak{g})$'-module in the terminology of [41]. As we remarked above, $R_\lambda(\mathfrak{l}:\mathfrak{g})$ is a quotient of $R_\lambda(\mathfrak{g})$. Hence, we can obtain Theorem 3.3.1 (2) by the same argument of the proof of [41]. □

Remark. In general, $C(\lambda)$ may have more than an element. This means we have more than one decomposition formula. Using the induction-by-stage, we can reduce the description of the relation between such decomposition formulas to the following result, which is more or less easy conclusion of [2] Theorem 4.2.

THEOREM 3.3.4. (Barbasch–Vogan). *Let* $0 \leqslant n \leqslant m, 0 \leqslant i \leqslant m,$ *and* $0 \leqslant j \leqslant n$.

(1) $\mathcal{A}(\mathfrak{q}((i, m-i), (j, n-j)) \uparrow \mathfrak{g}(m,n); \varepsilon(m+n)[[m,n]]) \neq 0$ *if and only if* $n \geqslant i+j$.

(2) *If* $n \geqslant i+j$, *then we have*

$$\begin{aligned}&\mathcal{A}(\mathfrak{q}((i, m-i), (j, n-j)) \uparrow \mathfrak{g}(m,n); \varepsilon(m+n)[[m,n]])\\ &\quad\cong \mathcal{A}(\mathfrak{q}((m-j, j), (n-i, i)) \uparrow \mathfrak{g}(m,n); \varepsilon(m+n)[[m,n]]).\end{aligned}$$

3.4. Appendix: a proof of Theorem 3.1.1 (1)

First, we remark that a result of Barbasch–Vogan can be restated as follows.

THEOREM 3.4.1. ([2] Theorem 4.2).

(1) *Let* V *be an irreducible Harish-Chandra* $(\mathfrak{g}(m,n), \mathrm{K}(m,n))$*-module with the same infinitesimal character as the trivial representation. Then its wave front set is the closure of single real nilpotent orbit.*

(2) *Let* I *be a primitive ideal of* $\mathrm{U}(\mathfrak{g}(m,n))$ *with the same infinitesimal character as the trivial representation. Let* $\mathcal{O}$ *be a real nilpotent orbit in* $\sqrt{-1}\mathfrak{u}(m,n)'$

whose complexification in $\mathfrak{g}(m,n)^*$ is open dense in the associated variety of I. Then, there exists one and only one, up to isomorphisms, Harish-Chandra $(\mathfrak{g}(m,n), \mathrm{K}(m,n))$-module whose annihilator in $\mathrm{U}(\mathfrak{g}(m,n))$ is I and whose wave front set is the closure of $\mathcal{O}$.

Hereafter we consider $\mathfrak{g} = \mathfrak{g}(r,r)$ and we denote by I the annihilator in $\mathrm{U}(\mathfrak{g})$ of the non-unitary degenerate series module M such that the trivial representation is its unique quotient. We easily see I is primitive using the duality between M and an irreducible generalized Verma module. The associated variety of I is the closure of the Richardson orbit with respect to the complexified Siegel parabolic subalgebra $\mathfrak{s}(r)$. There are just $r+1$ real nilpotent orbits whose complexification is the Richardson orbit of $\mathfrak{s}(r)$. Theorem 3.4.1 implies there are just $r+1$ irreducible Harish-Chandra modules $X_0;\ldots,X_r$ whose annihilators coincide with I, up to isomorphisms. From [14] or [25,25], M contains just $r+1$ irreducible submodules $Y_0,\ldots,Y_r$, which are not isomorphic to each other. The duality between the irreducible generalized Verma module and M implies that the annihilators of $Y_0,\ldots,Y_r$ are just I. Hence, we have $\{X_0,\ldots,X_r\} = \{Y_0,\ldots,Y_r\}$.

Fix $z\in\varepsilon(r)+\mathbb{Z}$. We put $M_0=\mathrm{Ind}_{\mathrm{S}(r)}^{\mathrm{U}(r,r)}(\lambda(z;0))$. We also put $A_i=\mathcal{A}_{\mathfrak{q}((i,r-i),(r-i,i))}^{\mathfrak{g}(r,r)}(\lambda(z;0))$, $B_i = \mathcal{A}_{\mathfrak{q}((i,r-i),(r-i,i))}^{\mathfrak{g}(r,r)}(\bar{\rho}((i,r-i),(r-i,i)))$ for $0 \leqslant i \leqslant r$. We denote by T the translation functor from the infinitesimal character of M to that of M_0. From [41] Proposition 4.7, we have $T(B_i) = A_i$ for $0 \leqslant i \leqslant r$. Similarly, we have $T(M) = M_0$. (This statement is easy consequence of the MacKey tensor product theorem of induced representation and Lemma 4.8 in [41].) Hence, in order that show $M_0 = \oplus_{i=0}^r A_i$, we have only to prove $\{X_0,\ldots,X_r\} = \{B_0,\ldots,B_r\}$. For example, from a result of Vogan ([24] Proposition 4.5), we see $B_0,\ldots,B_r$ are distinct from each other. Hence, it suffices to show $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(B_i)$ for $0 \leqslant i \leqslant r$. Since inner automorphisms preserve I, we have $I \subseteq \mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(B_i)$ for all $0 \leqslant i \leqslant r$. However, using results in [5] (See [41] Proposition 6.8), we can easily see the Gelfand–Kirillov dimension of B_i coincides with the dimension of the nilradical $\mathfrak{v}((i,r-i),(r-i,i))$ of $\mathfrak{q}((i,r-i),(r-i,i))$. (We have only to check that -1-eigenspace in $\mathfrak{g}(r,r)$ with respect to θ intersects the open orbit in $\mathfrak{v}((i,r-i),(r-i,i))$ with respect to $\mathrm{U}((i,r-i),(r-i,i))$, a Levi subgroup for $\mathfrak{q}((i,r-i),(r-i,i))$. It is fairly easy.) Hence, we have $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(B_i)$ for all $0 \leqslant i \leqslant r$. □

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