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# Harmonic analysis on quantum spheres associated with representations of $U_q(\mathfrak{so}_N)$ and $q$-Jacobi polynomials

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## Introduction

In this paper we carry out the $q$-analogue of harmonic analysis on spheres. Using quantum $R$-matrices of type $B$ or $D$, we first construct a quantum analogue of the algebra $\mathcal{D}$ of differential operators with polynomial coefficients on $A_q(V)$, the algebra of regular functions on the quantum vector space. This helps us to analyze the algebra $A_q(S^{N-1})$ of regular functions on quantum sphere $S_q^{N-1}$. This algebra $A_q(S^{N-1})$ has a structure of $U_q(\mathfrak{so}_N)$-module. To investigate the zonal spherical functions on $S_q^{N-1}$, we introduced two kinds of coideal $J_q$, corresponding to the left ideal $J = U(\mathfrak{so}_N) \cdot \mathfrak{k}$ of $U(\mathfrak{so}_N)$ where $\mathfrak{k} = \mathfrak{so}_{N-1} \subset \mathfrak{so}_N$. The zonal spherical functions on $S_q^{N-1}$ are defined as $J_q$-invariant functions in $A_q(S^{N-1})$.

They are expressed by two kinds of $q$-orthogonal polynomial associated with discrete and continuous measures, that is, big $q$-Jacobi polynomials $P_n^{(\alpha,\beta)}(X;q)$ and Rogers' continuous $q$-ultraspherical polynomials $C_n^{\lambda}(X;q)$, according to the choice of the coideals $J_q$. Furthermore, their orthogonality relations are also described by the invariant measure on $A_q(S^{N-1})$. We remark that big $q$-Jacobi polynomials will be considered only when $N = 2n+1 \geqslant 3$.

These results give a generalization of the works of [K1], [K2], and [NM1–4] to the higher dimensional quantum spheres, although we will only consider the zonal spherical functions.

Many authors discussed the differential calculus on quantum groups (cf. [W2], [P1], [NUW1], [WSW] ...). In this paper we use $R$-matrices (of type $B$ or $D$), to sew up $q$-analogues of commutation relations

$$\partial_i X_j - X_j \partial_i = \delta_{ij},$$

with "left $U_q(\mathfrak{so}_N)$-symmetry". The structure of the invariant subspace of this algebra of differential operators gives rise to the "oscillator representation" of

$U_q(\mathfrak{sl}_2)$. This fact is closely related to classical invariant theory (cf. [H], [HU] and [NUW2, 3]). U. C. Watamura et al. [WSW] also discussed a differential calculus on $A_q(V)$. They started with the exterior derivative $d$ on $A_q(V)$ with the usual nilpotency and Leibnitz Rule. It is a difference of our algebra $\mathcal{D}$ from their "algebra of differential operators" $\mathcal{D}'$ on $A_q(V)$ that we introduce a new generator $c$ corresponding to the group-like element of $U_q(\mathfrak{sl}_2)$, related to the oscillator representation (see Theorem 3.4). So our construction of the algebra $\mathcal{D}$ gives a more quantization of their algebra $\mathcal{D}'$, in fact their algebra $\mathcal{D}'$ is obtained by some specialization. Moreover, our approach conversely leads us to the "twisted Leibniz Rule" of the exterior derivative $d$ (more precisely, see comments after Theorem 2.7). We also remark that M. Noumi, T. Umeda and M. Wakayama recently studied the quantized spherical harmonics on the $q$-commutative polynomial ring "of type $A$", in the sense of a $U_q(\mathfrak{gl}_n)$-module ([NUW3]). They also obtained an explicit quantum analogue of Capelli identity related to the dual pair $(\mathfrak{sl}_2, \mathfrak{o}_n)$.

Throughout this paper we often use the following $q$-integers:

$$[n] = \frac{1-q^n}{1-q}, \quad \text{and} \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

## 1. Preliminaries on the quantized universal enveloping algebra $U_q(\mathrm{so}_N)$ and $R$-matrices

In this section we recall from [J1] and [RTF] about basic properties of quantum groups.

### 1.1. Quantized universal enveloping algebras

Let $P$ be a lattice of rank n with $\mathbb{Z}$-free basis $\{\varepsilon_j\}_{1\leqslant j\leqslant n}$:

$$P = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n. \tag{1.1}$$

We fix the symmetric bilinear form $(\ ,\ ): P \times P \to \mathbb{Z}$ such that $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. From now on we identify $P$ with its dual $P^* = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ by the symmetric bilinear form above. From Section 1 to Section 3, as the ground field we take the field $\mathbb{K} = \mathbb{Q}(q^{1/2})$ of rational functions in the indeterminate $q^{1/2}$, or the field $\mathbb{K} = \mathbb{C}$ of complex numbers assuming that $q$ is a real number with $q \neq 0, \pm 1$.

Recall that the simple Lie algebra $\mathfrak{so}_N$ of special orthogonal group corresponds to the root systems of $B_n$ and $D_n$, according as $N = 2n+1$ or $2n$. We take its simple roots as $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_n$ for $B_n$ series, and $\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_{n-1} + \varepsilon_n$ for $D_n$ series, respectively. The quantized universal enveloping algebra $U_q(\mathfrak{so}_N)$ is the associative $\mathbb{K}$-algebra generated by the elements $q^u (u \in \frac{1}{2}P^*)$ and $e_j, f_j$ $(1 \leqslant j \leqslant n)$ with the following fundamental relations:

$$\begin{aligned}
&(1) \quad q^0 = 1, \quad q^u \cdot q^v = q^{u+v} \quad (u, v \in \tfrac{1}{2}P^*),\\
&(2) \quad q^u e_j q^{-u} = q^{(u,\alpha_j)} e_j, \quad q^u f_j q^{-u} = q^{-(u,\alpha_j)} f_j \quad (u \in \tfrac{1}{2}P^*, 1 \leqslant j \leqslant n),\\
&(3) \quad e_i f_j - f_j e_i = \delta_{ij} \frac{q^{\alpha_j} - q^{-\alpha_j}}{q_j - q_j^{-1}},\\
&(4) \quad \sum_{\nu=0}^{m} \begin{bmatrix} m \\ \nu \end{bmatrix}_{q_i} (-1)^\nu e_i^{m-\nu} e_j e_i^\nu = 0 \quad (i \neq j),\\
&(5) \quad \sum_{\nu=0}^{m} \begin{bmatrix} m \\ \nu \end{bmatrix}_{q_i} (-1)^\nu f_i^{m-\nu} f_j f_i^\nu = 0 \quad (i \neq j),
\end{aligned} \tag{1.2}$$

where $q_j = q^{\frac{(\alpha_j,\alpha_j)}{2}}$, $m = 1 - \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ and $\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q \ldots [m-n+1]_q}{[n]_q \ldots [1]_q}$.

We will take the following Hopf algebra structure $U_q(\mathfrak{so}_N)$:

$$\begin{aligned}
&(1) \quad \Delta(q^u) = q^u \otimes q^u, \quad \varepsilon(q^u) = 1, \quad S(q^u) = q^{-u} \quad (u \in \tfrac{1}{2}P^*),\\
&(2) \quad \Delta(e_j) = q^{\alpha_j} \otimes e_j + e_j \otimes 1, \quad \varepsilon(e_j) = 0, \quad S(e_j) = -q^{-\alpha_j} e_j\\
&\qquad (1 \leqslant j \leqslant n),\\
&(3) \quad \Delta(f_j) = 1 \otimes f_j + f_j \otimes q^{-\alpha_j}, \quad \varepsilon(f_j) = 0, \quad S(f_j) = -f_j q^{\alpha_j}\\
&\qquad (1 \leqslant j \leqslant n),
\end{aligned} \tag{1.3}$$

where $\Delta, \varepsilon$ and $S$ denote the comultiplication, the counit and the antipode of $U_q(\mathfrak{so}_N)$ respectively. From now on we briefly write $U_q$ for $U_q(\mathfrak{so}_N)$.

REMARK 1. In what follows we introduce new symbol $e_n$ for $[2]_{q^{1/2}}^{-1} e_n$ (old) in the case of $B_n$-series to normalize the vector representations.

REMARK 2. We do not have a canonical embedding of $U_q(\mathfrak{so}_{N-1})$ into $U_q(\mathfrak{so}_N)$ because of the difference of their root systems.

Let $V$ be the $N$-dimensional $\mathbb{K}$-vector space with canonical basis $\{X_j\}_{1 \leqslant j \leqslant N}$:

$$V = \mathbb{K}X_1 \oplus \cdots \oplus \mathbb{K}X_N. \tag{1.4}$$

We consider the fundamental representation:

$$\rho_V : U_q(\mathfrak{so}_N) \to \mathrm{End}_{\mathbb{K}}(V). \tag{1.5}$$

For $B_n$ series we take $\rho_V$ as follows:

$$\rho_V(q^{\varepsilon_j}) = \sum_{i=1}^{N} E_{ii}q^{\delta_{ij}-\delta_{ij'}} \quad (1 \leqslant j \leqslant n),$$

$$\rho_V(e_j) = E_{jj+1} - E_{(j+1)'j'}, \quad \rho_V(f_j) = E_{j+1j} - E_{j'(j+1)'}$$
$$(1 \leqslant j \leqslant n-1), \tag{1.6}$$
$$\rho_V(e_n) = E_{nn+1}q^{1/2} - E_{n+1n'}, \quad \rho_V(f_n) = E_{n+1n}q^{-(1/2)} - E_{n'n+1},$$

where $j' = N - j + 1$ $(1 \leqslant j \leqslant N)$. For $D_n$ series the representations $\rho_V(q^{\varepsilon_j})$ $(1 \leqslant j \leqslant n)$ and $\rho_V(e_j), \rho_V(f_j)$ $(1 \leqslant j \leqslant n-1)$ are given by the preceding formulae and

$$\rho_V(e_n) = E_{n-1n'} - E_{n(n-1)'}, \quad \rho_V(f_n) = E_{n'n-1} - E_{(n-1)'n}. \tag{1.7}$$

Here $\{E_{ij}\}_{1 \leqslant i,j \leqslant N}$ are the linear operators on $V$ corresponding to the matrix units with respect to the basis $\{X_j\}$ such that $E_{ij} \cdot X_k = \delta_{jk}X_i$ and $E_{ij} \cdot E_{kl} = \delta_{jk}E_{il}$ for all $i, j, k, l$. Note that

$$1 < 2 < \cdots < n < n+1 < n' < \cdots < 2' < 1' \tag{1.8}$$

for $B_n$ series.

### 1.2. QUANTUM $R$-MATRICES

We use a quantum $R$-matrix $R \in \mathrm{End}_{\mathbb{K}}(V \otimes_{\mathbb{K}} V)$ associated with the quantized universal enveloping algebra $U_q(\mathfrak{so}_N)$. It is explicitly given by

$$R = R_q = \sum_{i,j=1}^{N} E_{ii} \otimes E_{jj}q^{\delta_{ij}-\delta_{ij'}}$$
$$+ (q - q^{-1}) \sum_{i>j} (E_{ij} \otimes E_{ji} - E_{ij} \otimes E_{i'j'}q^{\rho_i - \rho_j}) \tag{1.9}$$

where

$$(\rho_1, \ldots, \rho_N) = \begin{cases} (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -n + \frac{3}{2}, -n + \frac{1}{2}) \\ \quad \text{for } B_n \text{ series} \\ (n-1, n-2, \ldots, 1, 0, 0, -1, \ldots, -n+2, -n+1) \\ \quad \text{for } D_n \text{ series.} \end{cases}$$

This $R$-matrix satisfies the *Yang–Baxter equation*:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1.10}$$

in $\mathrm{End}_{\mathbb{K}}(V_1 \otimes V_2 \otimes V_3)$. Here $V_1 = V_2 = V_3 = V$, and as usual $R_{ab}$ denotes the action of $R$ on the $a$th and $b$th components of $V_1 \otimes V_2 \otimes V_3$ according to this order (e.g. $R_{12} = R \otimes \mathrm{id}$ and $R_{23} = \mathrm{id} \otimes R$). We remark that the Yang–Baxter equation (1.10) is equivalent to

$$(\check{R} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R})(\check{R} \otimes \mathrm{id}) = (\mathrm{id} \otimes \check{R})(\check{R} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R}), \tag{1.11}$$

where $\check{R} = PR$ and $P = \Sigma_{i,j} E_{ij} \otimes E_{ji}: u \otimes v \mapsto v \otimes u$ for $u, v \in V$. Moreover, this $R$-matrix has an another basic property below.

PROPOSITION 1.1. *The R-matrix $\check{R}$ is an intertwiner from $V \otimes_{\mathbb{K}} V$ to itself. Namely it is a $U_q(\mathfrak{so}_N)$-homomorphism from $V \otimes_{\mathbb{K}} V$ to itself.*

## 2. Differential calculus on the quantum vector space

In this section we will introduce the quantum vector spaace as in [RTF] and construct an algebra of differential operators on it.

### 2.1. THE ALGEBRAS $\mathcal{A} = A_q(V)$ AND $\hat{\mathcal{A}} = A_q(V^*)$

We keep the notations in Section 1. Recall that the tensor product $V \otimes_{\mathbb{K}} V$ is decomposed into the form

$$V \otimes_{\mathbb{K}} V = V_+ \oplus V_- \oplus V_\phi \tag{2.1}$$

as a $U_q$-module where $V_+, V_-$ and $V_\phi$ are the irreducible representations of highest weight $2\varepsilon_1, \varepsilon_1 + \varepsilon_2$ and 0 respectively. Accordingly the intertwiner $\check{R} = PR: V \otimes_{\mathbb{K}} V \to V \otimes_{\mathbb{K}} V$ has the spectral decomposition

$$\check{R} = qP^{(+)} - q^{-1}P^{(-)} + q^{1-N}P^{(\phi)}, \tag{2.2}$$

where $P^{(+)}, P^{(-)}$ and $P^{(\phi)}$ stand for the corresponding projections to each irreducible component. Note that the projection operator $P^{(-)}$ is explicitly given by

$$P^{(-)} = \frac{\check{R}^2 - (q + q^{1-N})\check{R} + q^{2-N}I}{(q + q^{-1})(q^{-1} + q^{1-N})} \tag{2.3}$$

$$= \frac{1}{(q + q^{-1})}\left(-\check{R} + qI - \frac{q - q^{-1}}{1 + q^{N-2}}J\right), \tag{2.4}$$

where $I = \Sigma_{i,j=1}^{N} E_{ii} \otimes E_{jj}$ and $J = \Sigma_{j=1}^{N} E_{jj} \otimes E_{j'j'}q^{\rho_j'}$.

Following [RTF] we introduce the algebra $\mathcal{A} = A_q(V)$ of regular functions on the quantum vector space defined by

$$\mathcal{A} = A_q(V) := T(V)/(V_-), \tag{2.5}$$

where $T(V)$ is the tensor algebra and $(V_-)$ denotes the two-sided ideal generated by the elements of $V_-$. In other words, the algebra $\mathcal{A}$ is the $\mathbb{K}$-algebra generated by $X_1, \ldots, X_N$ with fundamental relations:

$$\begin{aligned}
&(1) \quad X_l X_k + (q - q^{-1})\delta_{l>k} X_k X_l = q X_k X_l \quad \text{for} \quad k \neq l, l' \\
&(2) \quad X_{k'} X_k q^{\delta_{kk'}-1} + (q - q^{-1})\delta_{k'>k} X_k X_{k'} \\
&\qquad - (q - q^{-1}) \sum_{i>k} X_{i'} X_i q^{\rho_i - \rho_k} = q X_k X_{k'} - \frac{q - q^{-1}}{1 + q^{N-2}} Q q^{\rho_{k'}} \\
&\qquad \text{for all } k
\end{aligned} \tag{2.6}$$

where

$$Q = \sum_{j=1}^{N} X_j X_{j'} q^{\rho_{j'}} \quad \text{and} \quad \delta_{l>k} = \begin{cases} 1 & \text{if } l > k \\ 0 & \text{otherwise.} \end{cases}$$

We remark that $Q$ is the $U_q$-invariant element of $\mathcal{A}$, that is, $a \,.\, Q = \varepsilon(a) Q$ for all $a \in U_q$.

LEMMA 2.1.

(1) $X_k X_l = q X_l X_k$ *for* $k < 1$ *and* $k \neq l, l'$;

(2) $$X_{k'} X_k - X_k X_{k'} = \frac{q - q^{-1}}{q^{\rho_k - 1} + q^{-\rho_k + 1}} \sum_{j=k+1}^{(k+1)'} X_j X_{j'} q^{\rho_{j'}} \quad (1 \leqslant k \leqslant n-1);$$

(3) $X_{n'} X_n - X_n X_{n'} = (q^{1/2} - q^{-(1/2)}) X_{n+1}^2$ *for* $B_n$ *series,*
$X_{n'} X_n = X_n X_{n'}$ *for* $D_n$ *series;*

(4) $$Q = (1 + q^{N-2}) \left( \sum_{j=1}^{n} X_j X_{j'} q^{\rho_{j'}} + \frac{q}{q+1} X_{n+1}^2 \right) \quad \textit{for } B_n \textit{ series,}$$

$$Q = (1 + q^{N-2}) \left( \sum_{j=1}^{n} X_j X_{j'} q^{\rho_{j'}} \right) \quad \textit{for } D_n \textit{ series;}$$

(5) *The element* $X_{k'} X_k$ $(k' > k)$ *is expressed by a linear combination of the elements* $\{X_l X_{l'}\}$ *with* $l$ *such that* $k \leqslant l \leqslant l'$.

This proof is immediately obtained by (2.6).

We remark that the fundamental relations of (2.6) are equivalent to (1), (2) and (3) above.

PROPOSITION 2.2. *(1) The algebra $\mathcal{A}$ has a $\mathbb{K}$-basis $\{X^\nu = X_1^{\nu_1} \cdots X_N^{\nu_N}; \nu_j \in \mathbb{Z}_{\geqslant 0}$ for all $j\}$. (2) The center of $\mathcal{A}$ is generated by $Q$.*

The statement (1) is proved by using the *Diamond Lemma* ([B]). See also [NYM, Theorem 1.4].

Before proving Proposition 2.2–(2), we first introduce a total order on the set of monomials of $\mathcal{A}$. In what follows the symbol $X^\nu$ denotes the monomial

$$X^\nu = \begin{cases} X_1^{\nu_1} \cdots X_n^{\nu_n} X_{n+1}^{\nu_{n+1}} X_{n'}^{\nu_{n'}} \cdots X_{1'}^{\nu_{1'}} & \text{for } B_n \text{ series} \\ X_1^{\nu_1} \cdots X_n^{\nu_n} X_{n'}^{\nu_{n'}} \cdots X_{1'}^{\nu_{1'}} & \text{for } D_n \text{ series.} \end{cases} \tag{2.7}$$

Furthermore, $X^{\nu - m\varepsilon_j}$ denotes the element

$$X_1^{\nu_1} \cdots X_j^{\nu_j - m} \cdots X_{1'}^{\nu_{1'}}. \tag{2.8}$$

So the weight of $X^\nu$ is $\lambda := (\nu_1 - \nu_{1'})\varepsilon_1 + (\nu_2 - \nu_{2'})\varepsilon_2 + \cdots + (\nu_n - \nu_{n'})\varepsilon_n$, that is, $q^u \,.\, X^\nu = q^{(u,\nu)} X^\nu$ for all $u \in \frac{1}{2}P^*$.

To each monomial $X^\nu$ we associate a sequence $(\nu) := (|\nu|, \nu_1 - \nu_{1'}, \ldots, \nu_n - \nu_{n'}, \nu_1, \nu_2, \ldots, \nu_N)$ where $|\nu| = \Sigma_{j=1}^N \nu_j$. We define a total order $\succeq$ on the set of monomial basis $\{X^\nu\}$ of $\mathcal{A}$ by

$$X^\nu \succeq X^\mu \underset{\text{def}}{\Longleftrightarrow} (\nu) \succeq_{\text{lex}} (\mu) \tag{2.9}$$

where $\succeq_{\text{lex}}$ denotes the usual lexicographic order on $\mathbb{Z}_{\geqslant 0}^{N+n+1}$.

*Proof of Proposition 2.2–(2).* We use induction on the total order $\succeq$ above. Let $\varphi$ be a nonzero element which belongs to the center of $\mathcal{A}$. We can write $\varphi = d_0 X^\nu + d_1 X^{\nu^1} + \cdots + d_l X^{\nu^l}$ so that $X^\nu \succ X^{\nu^1} \succ \cdots \succ X^{\nu^l}$ and $d_j \in \mathbb{K}, d_j \neq 0$ for all $j$. Then using Lemma 2.1 we have for each $j$

$$\begin{aligned} \varphi X_j &\equiv d_0 X^\nu X_j \equiv q^{-\nu_{j+1} - \cdots - \widehat{\nu_{j'}} - \cdots - \nu_{1'}} d_0 X^{\nu + \varepsilon_j}, \\ X_j \varphi &\equiv d_0 X_j X^\nu \equiv q^{-(\nu_1 + \cdots + \nu_{j-1})} d_0 X^{\nu + \varepsilon_j} \\ &\qquad \text{modulo lower order terms,} \end{aligned} \tag{2.10}$$

where $\wedge$ indicates the part to be deleted. Since $q$ is not a root of unity, we have

$$\nu_1 + \cdots + \nu_{j-1} - (\nu_{j+1} \cdots + \widehat{\nu_{j'}} + \cdots + \nu_{1'}) = 0 \quad (1 \leqslant j \leqslant 1'). \tag{2.11}$$

Setting $j = 1$, we have $\nu_2 = \cdots = \nu_{2'} = 0$. Furthermore we have $\nu_1 = \nu_{1'}$ from (2.11) for the case of $j \neq 1$.

On the other hand, the leading term of $Q^m$ is $((1+q^{N-2})q^{-\rho_1})^m X_1^m X_{1'}^m$. So if we put $m=\nu_1=\nu_{1'}$ and $\psi=\varphi-d_0((1+q^{N-2})q^{-\rho_1})^{-m}Q^m$, then $\psi$ belongs to the center of $\mathcal{A}$ and $\varphi\succ\psi$. Hence by induction we complete the proof. □

Let $V^*$ be the dual space of $V$ with dual basis $\{\partial_j\}_{1\leqslant j\leqslant N}$ such that $\partial_j(X_k)=\delta_{jk}$ for all $j,k$. We endow $V^*$ with the following $U_q$-module structure:

$$(a\,.\,\xi)(v)=\xi(S(a)\,.\,v)\quad \text{for } a\in U_q, \xi\in V^* \quad \text{and} \quad v\in V, \tag{2.12}$$

where $S$ is the antipode of $U_q$. Then the *contragradient representation* $V^*$ is isomorphic to $V$ as left $U_q$-module through the map

$$\imath\colon V\xrightarrow{\sim}V^*,\quad X_j\mapsto\partial_{j'}q^{\rho_{j'}}\quad (1\leqslant j\leqslant N). \tag{2.13}$$

Here we also define the algebra $\hat{\mathcal{A}}=A_q(V^*)$ in a similar way as $\mathcal{A}$, that is,

$$\hat{\mathcal{A}}=A_q(V^*):=T(V^*)/(V_-^*) \tag{2.14}$$

where $T(V^*)$ is the tensor algebra related to $V^*$ and $V_-^*$ is the irreducible component of $V^*\otimes V^*$ of highest weight $\varepsilon_1+\varepsilon_2$, corresponding to $V_-$. It is clear that we can extend $\imath$ of (2.13) to the algebra isomorphism of $\mathcal{A}$ to $\hat{\mathcal{A}}$, and the quadratic element

$$\Delta=\sum_{j=1}^{N}\partial_j\,\partial_{j'}q^{\rho_j} \tag{2.15}$$

is the $U_q$-invariant element of $\hat{\mathcal{A}}$ corresponding to $Q$. The fundamental relations of $\hat{\mathcal{A}}$ are given in the next lemma.

LEMMA 2.3.

(1) $\partial_k\,\partial_l=q^{-1}\partial_l\,\partial_k$ *for* $k<l$ *and* $k\neq l,l'$;

(2) $$\partial_{k'}\,\partial_k-\partial_k\,\partial_{k'}=-\frac{q-q^{-1}}{q^{\rho_k-1}+q^{-\rho_k+1}}\sum_{j=k+1}^{(k+1)'}\partial_j\,\partial_{j'}q^{\rho_j}\quad(1\leqslant k\leqslant n-1);$$

(3) $\partial_{n'}\,\partial_n-\partial_n\,\partial_{n'}=-(q^{1/2}-q^{-(1/2)})\partial_{n+1}^2$ *for* $B_n$ *series,*
$\partial_{n'}\,\partial_n=\partial_n\,\partial_{n'}$ *for* $D_n$ *series.*

We remark that the projection operator of $V^*\otimes V^*$ to $V_-^*$ is expressed by a polynomial in $s^*=PR^t$ as in the case of $P^{(-)}$ of (2.4) (see Proposition 2.6).

PROPOSITION 2.4. *(1) The algebra* $\hat{\mathcal{A}}$ *has a* $\mathbb{K}$*-basis* $\{\partial^\mu=\partial_1^{\mu_1}\ldots\partial_N^{\mu_N};\mu_j\in\mathbb{Z}_{\geqslant 0}\}$. *(2) The center of* $\hat{\mathcal{A}}$ *is generated by* $\Delta$ *of (2.15).*

We also remark that the algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ become *algebras with left $U_q$-symmetry*. Here we call a $\mathbb{K}$-algebra $A$ *an algebra with left $U_q$-symmetry* in the sense that $A$ is the left $U_q$-module satisfying the following conditions:

for $\varphi, \psi \in A$ and $a \in U_q$

$$a\,.\,(\varphi\psi) = \sum_j (a_j^1\,.\,\varphi)(a_j^2\,.\,\psi) \quad \text{and} \quad a\,.\,1 = \varepsilon(a)1, \tag{2.16}$$

where $\Delta(a) = \Sigma_j a_j^1 \otimes a_j^2$, that is, the both multiplication $A \otimes A \to A$ and the unit homomorphism $\mathbb{K} \to A$ are homomorphisms of left $U_q$-modules.

For convenience we describe the action of generators $\{e_k\}, \{f_k\}$ of $U_q$ on $\mathcal{A}$.

LEMMA 2.5.

$$\begin{aligned}
B_n \textit{ series}:\ e_k\,.\,X^\nu &= X^{\nu+\varepsilon_k-\varepsilon_{k+1}}[\nu_{k+1}]_q\, q^{\nu_k-\nu_{k+1}+1} \\
&\quad - X^{\nu+\varepsilon_{(k+1)'}-\varepsilon_{k'}}[\nu_{k'}]_q\, q^{\nu_k-\nu_{k+1}+\nu_{(k+1)'}-\nu_{k'}+1} \\
&\quad (1 \leqslant k \leqslant n-1), \\
e_n\,.\,X^\nu &= X^{\nu+\varepsilon_n-\varepsilon_{n+1}}[\nu_{n+1}]\, q^{\nu_n-\nu_{n+1}+3/2} \\
&\quad - X^{\nu+\varepsilon_{n+1}-\varepsilon_{n'}}[\nu_{n'}]_q\, q^{\nu_n-\nu_{n'}+1}, \\
f_k\,.\,X^\nu &= X^{\nu-\varepsilon_k+\varepsilon_{k+1}}[\nu_k]_q\, q^{-\nu_k+\nu_{k+1}-\nu_{(k+1)'}+\nu_{k'}+1} \\
&\quad - X^{\nu-\varepsilon_{(k+1)'}+\varepsilon_{k'}}[\nu_{(k+1)'}]_q\, q^{-\nu_{(k+1)'}+\nu_{k'}+1} \\
&\quad (1 \leqslant k \leqslant n-1), \\
f_n\,.\,X^\nu &= X^{\nu-\varepsilon_n+\varepsilon_{n+1}}[\nu_n]_q\, q^{-\nu_n+\nu_{n'}+1/2} \\
&\quad - X^{\nu-\varepsilon_{n+1}+\varepsilon_{n'}}[\nu_{n+1}]\, q^{-\nu_{n+1}+\nu_{n'}+1};
\end{aligned} \tag{2.17}$$

$D_n$ *series*: *The action of* $e_k, f_k (k = 1, \ldots, n-1)$ *are as same as the above.*

$$\begin{aligned}
e_n\,.\,X^\nu &= X^{\nu+\varepsilon_{n-1}-\varepsilon_{n'}}[\nu_{n'}]_q\, q^{\nu_{n-1}-\nu_{n'}+1} \\
&\quad - X^{\nu+\varepsilon_n-\varepsilon_{(n-1)'}}[\nu_{(n-1)'}]_q q^{\nu_{n-1}+\nu_n-2\nu_{n'}-\nu_{(n-1)'}+1}, \\
f_n\,.\,X^\nu &= X^{\nu-\varepsilon_{n-1}+\varepsilon_{n'}}[\nu_{n-1}]_q\, q^{-\nu_{n-1}-2\nu_n+\nu_{n'}+\nu_{(n-1)'}+1} \\
&\quad - X^{\nu-\varepsilon_n+\varepsilon_{(n-1)'}}[\nu_n]_q\, q^{-\nu_n+\nu_{(n-1)'}+1}.
\end{aligned}$$

*Remark that we use the two kind of q-integers here.*

## 2.2. DIFFERENTIAL CALCULUS ON $\mathcal{A}$

In this subsection we construct an algebra of "differential operators" on $\mathcal{A}$.

PROPOSITION 2.6. *Put* $s = \check{R}, s^* = PR^t$ *and* $s_1 = P(R^{-1})^{t_1}$ *(*$t_1$ *denotes the transposition in the first component). Then we have the following commutative diagram of* $U_q$*-isomorphisms:*

$$\begin{array}{ccccc} V^* \otimes V & \xleftarrow{\imath \otimes \mathrm{id}} & V \otimes V & \xrightarrow{\imath \otimes \imath} & V^* \otimes V^* \\ \downarrow s_1 & & \downarrow s & & \downarrow s^* \\ V \otimes V^* & \xleftarrow{\mathrm{id} \otimes \imath} & V \otimes V & \xrightarrow{\imath \otimes \imath} & V^* \otimes V^* \end{array} \tag{2.18}$$

*where* $\imath$ *is the* $U_q$*-isomorphism of (2.13). Furthermore the following series of Yang–Baxter equations hold:*

$$(s \otimes \mathrm{id})(\mathrm{id} \otimes s_1)(s_1 \otimes \mathrm{id}) = (\mathrm{id} \otimes s_1)(s_1 \otimes \mathrm{id})(\mathrm{id} \otimes s)$$
$$\text{on} \quad V^* \otimes V \otimes V, \tag{2.19}$$
$$(s_1 \otimes \mathrm{id})(\mathrm{id} \otimes s_1)(s^* \otimes \mathrm{id}) = (\mathrm{id} \otimes s^*)(s_1 \otimes \mathrm{id})(\mathrm{id} \otimes s_1)$$
$$\text{on} \quad V^* \otimes V^* \otimes V. \tag{2.20}$$

*Proof.* The commutativity of the diagram above can be checked by direct calculations with $\imath = \Sigma_{j=1}^N E_{j'j} q^{\rho_{j'}}$ (Note that $R^{-1} = R_{q^{-1}}$). The equation (2.19) and (2.20) are equivalent to (1.10). □

REMARK. In general for any pair of representations $(\rho_{V_1}, V_1), (\rho_{V_2}, V_2)$, we can derive the fact that the matrices $PR_{V_1V_2} \in \mathrm{Hom}_{\mathbb{K}}(V_1 \otimes V_2, V_2 \otimes V_1), PR^t_{V_1V_2} \in \mathrm{Hom}_{\mathbb{K}}(V_1^* \otimes V_2^*, V_2^* \otimes V_1^*)$ and $P(R^{-1}_{V_1V_2})^{t_1} \in \mathrm{Hom}_{\mathbb{K}}(V_1^* \otimes V_2, V_2 \otimes V_1^*)$ are actually intertwiners, where $R_{V_1V_2} := (\rho_{V_1} \otimes \rho_{V_2})(\mathcal{R})$ and $\mathcal{R}$ is the universal $R$-matrix in $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$.

Now, let $c$ be an indeterminate over $\mathbb{K}$ with $U_q$-invariance: $a \,.\, c = \varepsilon(a)c$ for all $a \in U_q$. We want to sew up $q$-analogues of *Heisenberg's commutation relations*:

$$\partial_i X_j - X_j \partial_i = \delta_{ij} \tag{2.21}$$

in a $U_q$-module $\mathcal{A} \otimes_{\mathbb{K}} \hat{\mathcal{A}} \otimes_{\mathbb{K}} \mathcal{L}$ where $\mathcal{L} = \mathbb{K}[c, c^{-1}]$.

We first consider the following intertwiners:

$$s_2 \colon V^* \otimes V \to V \otimes V^* \oplus \mathbb{K}c,$$
$$s_3 \colon \mathbb{K}c \otimes V \to V \otimes \mathbb{K}c, \tag{2.22}$$
$$s_4 \colon \mathbb{K}c \otimes V^* \to V^* \otimes \mathbb{K}c,$$

such that

$$s_2\colon \partial_i \otimes X_j \mapsto s_1(\partial_i \otimes X_j) + \delta_{ij}c,$$
$$s_3\colon c \otimes X_j \mapsto qX_j \otimes c \quad (1 \leqslant j \leqslant N), \tag{2.23}$$
$$s_4\colon c \otimes \partial_j \mapsto q^{-1}\,\partial_j \otimes c \quad (1 \leqslant j \leqslant N).$$

We set a $\mathbb{K}$-vector space

$$W := V \oplus V^* \oplus \mathbb{K}c \oplus \mathbb{K}c^{-1}. \tag{2.24}$$

Furthermore, we set a $\mathbb{K}$-vector subspace $F$ in the tensor algebra $T(W)$ as follows:

$$\begin{aligned} F := {} & V_- \oplus V_-^* \oplus \mathbb{K}(c \cdot c^{-1} - 1) \oplus \mathbb{K}(c^{-1} \cdot c - 1) \\ & \oplus \left( \bigoplus_{i,j=1}^{N} \mathbb{K}(\mathrm{id} - s_2)(\partial_i \otimes X_j) \right) \\ & \oplus \left( \bigoplus_{j=1}^{N} \mathbb{K}(\mathrm{id} - s_3)(c \otimes X_j) \right) \oplus \left( \bigoplus_{j=1}^{N} \mathbb{K}(\mathrm{id} - s_4)(c \otimes \partial_j) \right) \\ & \quad \text{(see (2.5), (2.14))}. \end{aligned} \tag{2.25}$$

Then we define "the algebra of differential operators" $\mathcal{D}$ by

$$\mathcal{D} := T(W)/(F), \tag{2.26}$$

where $(F)$ denotes the two-sided ideal in $T(W)$. In other words, the algebra $\mathcal{D}$ is the $\mathbb{K}$-algebra generated by $X_1, \ldots, X_N, \partial_1, \ldots, \partial_N$ and $c, c^{-1}$ with following fundamental relations:

$$(1), (2) \text{ and } (3) \text{ of Lemma 2.1}, \tag{2.27}$$

$$(1), (2) \text{ and } (3) \text{ of Lemma 2.3}, \tag{2.28}$$

$$c \cdot c^{-1} = 1 = c^{-1} \cdot c, \tag{2.29}$$

$$\begin{aligned} \partial_k X_k = {} & X_k\,\partial_k q^{\delta_{kk'}-1} - (q - q^{-1}) \sum_{j<k} X_j\,\partial_j \\ & + (q - q^{-1})\delta_{k>k'} X_{k'}\,\partial_{k'} q^{2\rho_{k'}} + c \quad (1 \leqslant k \leqslant N), \end{aligned} \tag{2.30}$$

$$\partial_k X_j = X_j \partial_k + (q - q^{-1})\delta_{k>j'} X_{k'} \partial_{j'} q^{-\rho_k + \rho_{j'}} \quad (k \neq j, j'), \tag{2.31}$$

$$\partial_k X_{k'} = q X_{k'} \partial_k \quad (k \neq k'), \tag{2.32}$$

$$c \cdot X_j = q X_j \cdot c, \quad c \cdot \partial_j = q^{-1} \partial_j \cdot c. \tag{2.33}$$

We remark that the relations (2.30)–(2.32) are due to $(\mathrm{id} - s_2)(\partial_i \otimes X_j) = 0$.

THEOREM 2.7. *The $\mathbb{K}$-algebra $\mathcal{D}$ has $\mathbb{K}$-basis $\{X^\nu \partial^\mu c^l = X_1^{\nu_1} \cdots X_N^{\nu_N} \partial_1^{\mu_1} \cdots \partial_N^{\mu_N} c^l; \nu_j, \mu_j \in \mathbb{Z}_{\geqslant 0}, l \in \mathbb{Z}\}$ and has a structure of left $U_q$-symmetry (see the remarks after Proposition 2.4). Namely, there exists a canonical $U_q$-isomorphism of $\mathcal{A} \otimes_{\mathbb{K}} \hat{\mathcal{A}} \otimes_{\mathbb{K}} \mathcal{L}$ onto $\mathcal{D}$.*

*Proof.* By using the fundamental relations in $\mathcal{D}$, any element of $\mathcal{D}$ can be expressed in a linear combination of *normally ordered monomials* of the form $X^\nu \partial^\mu c^l = X_1^{\nu_1} \cdots X_N^{\nu_N} \partial_1^{\mu_1} \cdots \partial_N^{\mu_N} c^l$. We call this procedure *normal reduction.* As we know from the way of construction of $\mathcal{D}$, it is clear that the embeddings $\mathcal{A} \hookrightarrow \mathcal{D}, \hat{\mathcal{A}} \hookrightarrow \mathcal{D}$ and $\mathcal{L} \hookrightarrow \mathcal{D}$ are $\mathbb{K}$-algebra homomorphisms. It is also clear that there is a canonical $U_q$-homomorphism of $\mathcal{A} \otimes_{\mathbb{K}} \hat{\mathcal{A}} \otimes_{\mathbb{K}} \mathcal{L}$ onto $\mathcal{D}$. To complete the proof, we will show the independence of the monomials $X^\nu \partial^\mu c^l$ in the statement above.

Let $\mathcal{D}'$ be the subalgebra of $\mathcal{D}$ generated by $\{X_j\}, \{\partial_j\}$ and $c$ with fundamental relations (2.27)–(2.23) except (2.29). We will first show that $\mathcal{D}'$ has $\mathbb{K}$-basis $\{X^\nu \partial^\mu c^l = X_1^{\nu_1} \cdots X_N^{\nu_N} \partial_1^{\mu_1} \cdots \partial_N^{\mu_N} c^l; \nu_j, \mu_j, l \in \mathbb{Z}_{\geqslant 0}\}$. Then one can easily show that the algebra $\mathcal{D}$ has desired bases. Owing to the Diamond Lemma ([B]), we have enough to show that the normal reduction of the monomials $\partial_i X_j X_k$ $(j > k)$ and $\partial_i \partial_j X_k$ $(i > j)$ are compatible with the relations of $\mathcal{A}$ and $\hat{\mathcal{A}}$ (Other cases are trivial). In other words the normal reduction has no ambiguities (see [B, Theorem 1.2]). Let

$$\partial_i X_j = \sum_{\alpha,\beta} r_{\alpha\beta}^{ij} X_\alpha \partial_\beta + \delta_{ij} c \quad \text{and} \quad X_j X_k = f(X) = \sum_{\alpha \leqslant \beta} m_{\alpha\beta}^{jk} X_\alpha X_\beta$$

$$(\text{we put } s_1 = P(R^{-1})^{t_1} = \sum E_{\alpha i} \otimes E_{\beta j} r_{\alpha\beta}^{ij}, \quad \text{and}$$

$$r_{\alpha\beta}^{ij}, m_{\alpha\beta}^{jk} \in \mathbb{K}) \tag{2.34}$$

be the results of normal reductions of the monomials $\partial_i X_j$ and $X_j X_k$ respectively. Then one can consider the following two ways of reductions ($\smile$ indicates the part to be reduced):

$$\partial_i X_j X_k = \begin{cases} \underset{\smile}{\partial_i X_j} X_k = (\sum_{\alpha,\beta} r^{ij}_{\alpha\beta} X_\alpha\, \partial_\beta + \delta_{ij} c) X_k \\ \qquad = \sum_{\alpha,\beta} r^{ij}_{\alpha\beta} X_\alpha\, \underset{\smile}{\partial_\beta X_k} + \delta_{ij} \underset{\smile}{c \cdot X_k} = \cdots \\ \qquad = \sum_{l=1}^{N} g_l(X)\, \partial_l + g_0(X) \cdot c \\ \partial_i \underset{\smile}{X_j X_k} = \partial_i f(X) = \partial_i (\sum_{\alpha \leqslant \beta} m^{jk}_{\alpha\beta} X_\alpha X_\beta) \\ \qquad = \sum_{\alpha \leqslant \beta} m^{jk}_{\alpha\beta}\, \underset{\smile}{\partial_i X_\alpha} X_\beta \\ \qquad = \sum_{\alpha \leqslant \beta} m^{jk}_{\alpha\beta} (\sum_{\gamma,\delta} r^{i\alpha}_{\gamma\delta} X_\gamma\, \underset{\smile}{\partial_\delta X_\beta} + \delta_{i\alpha} \underset{\smile}{c \cdot X_\beta}) \\ \qquad = \cdots \\ \qquad = \sum_{l=1}^{N} h_l(X)\, \partial_l + h_0(X) \cdot c, \end{cases} \tag{2.35}$$

where $g_l(X), h_l(X)\, (0 \leqslant l \leqslant)$ are polynomials in $\check{R}$, determined by the normal reductions above. So, we have to show that $g_l(X) = h_l(X)$ for all $l$. Since the projection $P^{(-)}$ is a polynomial in $\check{R}$ (see (2.3)), one gets

$$(P^{(-)} \otimes \mathrm{id})(\mathrm{id} \otimes s_1)(s_1 \otimes \mathrm{id}) = (\mathrm{id} \otimes s_1)(s_1 \otimes \mathrm{id})(\mathrm{id} \otimes P^{(-)}) \tag{2.36}$$

by using (2.19) successively. From (2.36) and the definition of $s_2$ one gets

$$\partial_j V_- \subset \sum_{l=1}^{N} V_-\, \partial_l + \sum_{l=1}^{N} \mathbb{K} X_l \cdot c. \tag{2.37}$$

Hence we have $g_l(X) = h_l(X)$ in $\mathcal{A}$ for $l = 1, \ldots, N$, since $X_j X_k - f(X) \in V_-$. To show that $g_0(X) = h_0(X)$, we have to investigate case by case. For example, for any $k > j, k \neq j, j'$ and $k \neq k'$ we have

$$\begin{aligned} \underset{\smile}{\partial_k X_k} X_j = \Big\{ & X_k\, \partial_k q^{-1} - (q - q^{-1}) \sum_{l<k} X_l\, \partial_l \\ & + (q - q^{-1}) \delta_{k>k'} X_{k'}\, \partial_{k'} q^{2\rho_{k'}} + c \Big\} \cdot X_j \\ \equiv {} & -(q - q^{-1}) X_j\, \partial_j X_j + c \cdot X_j \bmod \mathcal{A} \otimes \hat{\mathcal{A}} \\ \equiv {} & -(q - q^{-1}) X_j \cdot c + q X_j \cdot c \bmod \mathcal{A} \otimes \hat{\mathcal{A}} \\ = {} & q^{-1} X_j \cdot c. \end{aligned} \tag{2.38}$$

On the other hand,

$$\begin{aligned}\partial_k \underbrace{X_k X_j} &= \partial_k X_j X_k q^{-1} \\ &= X_j\, \partial_k X_k q^{-1} \\ &\equiv q^{-1} X_j \cdot c \bmod \mathcal{A} \otimes \hat{\mathcal{A}}. \end{aligned} \tag{2.39}$$

We can check that $g_0(X) = h_0(X)$ about all other cases in the same way. So we have proved that the fundamental relations in $\mathcal{A}$ are compatible with the multiplication in $\mathcal{D}$. As to the monomial $\partial_i\, \partial_j X_k$, we can prove it in the same way. □

Here we must refer to the work of U. C. Watamura, M. Schlieker and S. Watamura [WSW]. As mentioned in the introduction, they also constructed "the algebra of differential operators" $\mathcal{D}'$ starting from introducing the exterior derivative with left $A_q(\mathrm{SO}(N))$-covariance where $A_q(\mathrm{SO}(N))$ is the coordinate ring of quantum group $\mathrm{SO}_q(N)$ (see [RTF]). Their algebra $\mathcal{D}' = \mathbb{K}[x^1, \ldots, x^N, \partial_1, \ldots, \partial_N]$ in [WSW] should have the "right" $U_q$-module structure and the "right" $U_q$-symmetry. But these do not seem clear from their construction.

To clarify the difference between their algebra $\mathcal{D}'$ and our algebra $\mathcal{D}$, we will first construct a "right $U_q$-symmetry" version of $\mathcal{D}$. Let $\mathcal{D}''$ be the algebra obtained by replacing $s, s_1$ and $s^*$ by $PR^{-1}, PR^{t_2}$ and $P(R^{-1})^t$, moreover $s_3$ and $s_4$ by $c \otimes X_j \to q^{-1} X_j \otimes c$ and $c \otimes \partial_j \to q\, \partial_j \otimes c$. Then the algebra $\mathcal{D}''$ has the same properties of $\mathcal{D}$ with right $U_q$-symmetry and the algebra $\mathcal{D}'$ in [WSW] is obtained by resetting $c \to q^{-1}c, \partial_j c^{-1} \to \partial_j$ and $X_j \to x^j$. Here remark that our matrix $\check{R} = PR$ coincides with $\hat{R}$ in [WSW].

Conversely the structure of our algebra $\mathcal{D}''$ leads us to "the twisted Leibniz Rule", that is, for $f, g \in A_q(V)$ we have $d(fg) = (df)c(g) + f(dg)\,(= (df)gq^{-\deg g} + f(dg)$ if $g$ is homogeneous). In fact the calculations of (II.19) and (II.26) in [WSW], by using this twisted exterior derivative $d$ and the derivatives $\partial_j$ such that $d = \Sigma_j dX_j\, \partial_j$, determines the same structure of $\mathcal{D}''$. As we will know later (Theorem 3.4), our generator $c$ is essentially corresponding to a group-like element of $U_q(\mathfrak{sl}_2)$ related to the oscillator representation.

We now consider a canonical map

$$\mathcal{D} \mapsto \mathcal{D} \Big/ \left( \sum_{j=1}^{N} \mathcal{D}\, \partial_j + \mathcal{D}(c-1) + \mathcal{D}(c^{-1}-1) \right) \simeq \mathcal{A}. \tag{2.40}$$

We denote by $\partial(\varphi)$ the canonical image of $\partial \otimes \varphi$ for $\partial \in \mathcal{A} \otimes \hat{\mathcal{A}} \otimes \mathbb{K}[c, c^{-1}]$ and $\varphi \in \mathcal{A}$. Then we can directly calculate the action of $\partial_k$ on the monomial basis in $\mathcal{A}$.

PROPOSITION 2.8.

$$\partial_k(X^\nu) = X^{\nu-\varepsilon_k}[\nu_k]_q\, q^{\nu_{k+1}+\cdots\nu_{1'}} \quad (1 \leqslant k \leqslant n)$$

$$\partial_{n+1}(X^\nu) = X^{\nu-\varepsilon_{n+1}}[\nu_{n+1}]\, q^{\nu_{n'}+\cdots+\nu_{1'}} \quad (B_n \textit{ series only})$$

$$\begin{aligned}\partial_{k'}(X^\nu) = {} & X^{\nu-\varepsilon_{k'}}[\nu_{k'}]_q\, q^{\nu_{k'}+\nu_{(k-1)'}+\cdots+\nu_{1'}} \\ & + \sum_{j=k+1}^{n} X^{\nu+\varepsilon_k-\varepsilon_j-\varepsilon_{j'}}[\nu_j]_q[\nu_{j'}]_q\,(q-q^{-1})q^{\rho_k-\rho_j} \\ & \times q^{\nu_k+\cdots+\nu_{j-1}+\nu_{(j-1)'}+\cdots+\nu_{1'}} \\ & + X^{\nu+\varepsilon_k-2\varepsilon_{n+1}}\frac{q-q^{-1}}{1+q}[\nu_{n+1}-1][\nu_{n+1}]q^{\rho_k+2} \\ & \times q^{(\nu_k+\cdots+\nu_{1'})-2\nu_{n+1}} \quad (1 \leqslant k \leqslant n). \end{aligned} \tag{2.41}$$

REMARK. In the notations above we distinguish $\nu_{n+1}$ of $B_n$ series and $\nu_{n'}$ of $D_n$ series, so the last term of the third equation does not appear for $D_n$ series.

### 2.3. Some fundamental identities in $\mathcal{D}$

In this subsection we investigate the structure of $\mathcal{D}$ related to the *oscillator representation* of $U_q(\mathfrak{sl}_2)$ (see Theorem 3.4).

PROPOSITION 2.9. *For any $j$ the following relations hold in $\mathcal{D}$:*

(1) $$EX_j = q^{-1}X_jE + \frac{q-q^{-1}}{1+q^{N-2}}q^{N-2-\rho_j}Q\,\partial_{j'} + X_j\cdot c;$$

$$\textit{where } E = \sum_{k=1}^{N} X_k\,\partial_k;$$

(2) $$\Delta X_j = X_j\Delta + (1+q^{N-2})q^{-\rho_j}\,\partial_{j'}\cdot c;$$

(3) $$\tilde{E}X_j = q^{-1}X_j\tilde{E} + X_j\cdot c;$$

$$\textit{where } \tilde{E} = E - \frac{q-q^{-1}}{(1+q^{N-2})^2}q^{N-1}Q\Delta\cdot c^{-1},$$

(4) $$\partial_jQ = Q\,\partial_j + (1+q^{N-2})q^{-\rho_j+1}X_{j'}\cdot c;$$

(5) $$\partial_jE = q^{-1}E\,\partial_j + \frac{q-q^{-1}}{1+q^{N-2}}q^{N-2-\rho_j}X_{j'}\Delta + q^{-1}\,\partial_j\cdot c;$$

(6) $EQ = QE + (1 + q^{N-2})qQ \cdot c;$

(7) $\Delta E = E\Delta + (1+q^{N-2})q^{-1}\Delta \cdot c.$ (2.42)

We remark that $E$ is the trivial element of $V \otimes V^*$. From Proposition 2.9–(3), we have

$$\tilde{E}(X^\nu) = [\nu_1 + \cdots + \nu_{1'}]_q X^\nu. \tag{2.43}$$

Hence for any $\varphi \in \mathcal{A}$ we have

$$\tilde{E}(\varphi) = \frac{c - c^{-1}}{q - q^{-1}}(\varphi). \tag{2.44}$$

Now we shall write $q^\varepsilon$ for $c$ conveniently, so we have

$$\tilde{E} = \frac{q^\varepsilon - q^{-\varepsilon}}{q - q^{-1}} = [\varepsilon]_q \tag{2.45}$$

as an operator on $\mathcal{A}$. So it is convenient to use $\tilde{E}$ for $E$.

We can show the following most important relations in $\mathcal{D}$.

PROPOSITION 2.10. *There exists a following identity between Laplacian $\Delta$ and length $Q$:*

$$\Delta Q = Q\Delta + (1 + q^{N-2})^2 q^{-N+2} E \cdot c + \frac{(1 + q^{N-2})^2}{1 + q} q^{-N+3}[N]c^2. \tag{2.46}$$

*Furthermore, for any $s \geqslant 1$ we have using $\tilde{E}$,*

$$\Delta Q^s = q^{2s}Q^s\Delta + \frac{(1 + q^{N-2})^2}{1 + q} q^{-N+2}[2s]Q^{s-1}\tilde{E} \cdot c + \frac{(1 + q^{N-2})^2}{(1 + q)^2} q^{-N+3}[2s][N + 2s - 2]Q^{s-1} \cdot c^2. \tag{2.47}$$

COROLLARY 2.11. *As an operator on $\mathcal{A}$ one has*

$$\Delta Q^s = q^{2s}Q^s\Delta + Q^{s-1}\frac{(1 + q^{N-2})^2}{(1 + q)^2} q^{-N+3}[2s][N + 2s - 2 + 2\varepsilon]. \tag{2.48}$$

*In particular we have*

$$\Delta(Q^s) = Q^{s-1}\frac{(1 + q^{N-2})^2}{(1 + q)^2} q^{-N+3}[2s][N + 2s - 2]. \tag{2.49}$$

Proposition 2.10 can be shown by direct calculation using next lemma.

LEMMA 2.12. *The following nontrivial relations hold in $\mathcal{A}$ and $\hat{\mathcal{A}}$:*

$$X_1 X_N\, q^{-1} = X_N X_1\, q - \frac{q-q^{-1}}{1+q^{N-2}} Q q^{\rho_1}, \tag{2.50}$$

$$\partial_N\, \partial_1\, q^{-1} = \partial_1\, \partial_N\, q^{-}\frac{q-q^{-1}}{1+q^{N-2}} \Delta q^{\rho_1}. \tag{2.51}$$

*Proof.* This immediately follows from (2.6)–(2) and the algebra isomorphism $\imath$ of (2.13). □

Finally we describe the action of $\Delta$ to $\mathcal{A}$.

PROPOSITION 2.13. *The action of Laplacian $\Delta$ to the monomial basis of $\mathcal{A}$ is given by*

$$\Delta(X^{\nu}) = (1+q^{N-2}) q^{\nu_1+\cdots+\nu_{1'}-1} \times \left\{ \sum_{j=1}^{n} X^{\nu-\varepsilon_j-\varepsilon_{j'}} [\nu_j]_q [\nu_{j'}]_q\, q^{-\rho_j} \times q^{\nu_1+\cdots+\nu_{j-1}+\nu_{(j-1)'}+\cdots+\nu_{1'}} + X^{\nu-2\varepsilon_{n+1}} \frac{1}{1+q} [\nu_{n+1}-1][\nu_{n+1}] q^{\nu_1+\cdots+\nu_{1'}-2\nu_{n+1}+2} \right\}. \tag{2.52}$$

REMARK. For $D_n$ series we put $\nu_{n+1} = 0$ (see Proposition 2.10).

## 3. Quantum spheres and the space of harmonic polynomials

### 3.1. QUANTIZED HARMONICS

We will first study the irreducible decomposition of the algebra $\mathcal{A} = \mathcal{A}_q(V)$.

From Proposition 2.2–(1) we immediately get the homogeneous decomposition of $\mathcal{A}$:

$$\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}_k, \tag{3.1}$$

where $\mathcal{A}_k$ denotes the subspace of homogeneous polynomials of degree $k$. Let $H_k$ be *the space of harmonic polynomials* of degree $k$ defined by

$$H_k := \{\varphi \in \mathcal{A}_k; \Delta(\varphi) = 0\}. \tag{3.2}$$

THEOREM 3.1. *The space $\mathcal{A}_k$ is decomposed as follows:*

$$\mathcal{A}_k = \begin{cases} H_k & (k = 0, 1) \\ H_k \oplus Q\mathcal{A}_{k-2} & (k \geqslant 2). \end{cases} \tag{3.3}$$

*In particular*

$$\mathcal{A}_k = \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} Q^j H_{k-2j}, \tag{3.4}$$

*where $\lfloor p \rfloor$ denotes the maximum integer less than or equals to p.*

*Proof.* (Step 1) This is clear when $k = 0, 1$. Suppose that $k \geqslant 2$, and we will show that $H_k \cap Q\mathcal{A}_{k-2} = 0$ in $\mathcal{A}_k$. Let $F$ be a nonzero element of $H_k \cap Q\mathcal{A}_{k-2}$. We can take the maximum integer $j \geqslant 1$ such that $F = Q^j G$ for some nonzero element of $\mathcal{A}_{k-2j}$. Then from (2.48) we have

$$0 = \Delta(F) = \Delta(Q^j G) = q^{2j} Q^j \Delta(G) + Q^{j-1} \frac{(1+q^{N-2})^2}{(1+q)^2} q^{-N+3} \times [2j][N + 2j - 2 + 2(k - 2j)]G. \tag{3.5}$$

Hence we have

$$F = Q^j G = \frac{(1+q)^2}{(1+q^{N-2})^2} q^{N-3+2j} \times \frac{(-1)}{[2j][N - 2j - 2 + 2k]} Q^{j+1} \Delta(G). \tag{3.6}$$

Here $\Delta(G)$ and the denominator in the right-hand side are not zero, so we have contradiction about the maximality of $j$.

(Step 2) We put $d_k = \dim_{\mathbb{K}} \mathcal{A}_k$ and $h_k = \dim_{\mathbb{K}} H_k$, then we have $h_k + d_{k-2} \leqslant d_k$ from (Step 1). On the other hand, the kernel of $\Delta\colon \mathcal{A}_k \to \mathcal{A}_{k-2}$ is just $H_k$, so we have $d_k - h_k \leqslant d_{k-2}$. Hence $h_k + d_{k-2} = d_k$. □

THEOREM 3.2. *Suppose $N \geqslant 3$, then the spaces $H_k\,(k \geqslant 0)$ are irreducible $U_q$-modules with highest weight vector $X_1^k$.*

Before proving Theorem 3.2, we remark the general results by Lusztig [L].

Let $P'$ be a $\mathbb{Z}$-lattice $\Sigma_{j=1}^{n} \mathbb{Z}\Lambda_j$ where $\Lambda_j$ are the fundamental weights associated with a simple Lie algebra $\mathfrak{g}$ of rank $n$, and $P^+$ be the set of all dominant integral weights in $P'$:

$$P^+ := \left\{ \lambda \in P'; \frac{2(\lambda, \alpha_j)}{(\alpha_j, \alpha_j)} \in \mathbb{Z}_{\geqslant 0} \quad \text{for all } j \right\}. \tag{3.7}$$

For each $\lambda \in P^+$ we denote by $V(\lambda)$ the unique irreducible $U_q(\mathfrak{g})$-module with highest weight $\lambda$. Lusztig ([L]) showed that every finite dimensional irreducible "$P'$-weighted" $U_q(\mathfrak{g})$-module is isomorphic to $V(\lambda)$ for some $\lambda \in P^+$. Here "$P'$-weighted" module means that it has a $\mathbb{K}$-basis consisting of weight vectors with weights in $P'$. Furthermore, for each $V(\lambda)(\lambda \in P^+)$ the analogue of the Weyl's character formula holds. So one sees that $V(\lambda)$ has the same degree as the classical one.

LEMMA 3.3. *Let $\mathcal{D}^{U_q}$ be the set of all left $U_q$-invariant element in $\mathcal{D}$:*

$$\mathcal{D}^{U_q} := \{\eta \in \mathcal{D}; a \,.\, \eta = \varepsilon(a)\eta \quad \text{for all } a \in U_q\}. \tag{3.8}$$

*Then the action of $\mathcal{D}^{U_q}$ and $U_q$ on $\mathcal{A}$ are commuting with each other.*

*Proof.* For each $a \in U_q, \eta \in \mathcal{D}^{U_q}$ and $\varphi \in \mathcal{A}$, we have

$$\begin{aligned} a \,.\, (\eta \otimes \varphi) &= \sum_j (a_j^1 \,.\, \eta) \otimes (a_j^2 \,.\, \varphi) = \sum_j \varepsilon(a_j^1)\eta \otimes (a_j^2 \,.\, \varphi) \\ &= \eta \otimes (((\varepsilon \otimes \mathrm{id})\Delta)(a)) \,.\, \varphi \\ &= \eta \otimes (a \,.\, \varphi) \end{aligned} \tag{3.9}$$

where $\Delta(a) = \Sigma_j a_j^1 \otimes a_j^2$. Then we have $a \,.\, (\eta(\varphi)) = \eta(a \,.\, \varphi)$. □

*Proof of Theorem 3.2.* From Lemma 3.3 and Proposition 2.13, we see that $H_k\, (k \geqslant 0)$ are left $U_q$-modules and $X_1^k$ is a highest weight vector of $H_k$ of weight $k\varepsilon_1$ for all $k$. Therefore there is a $U_q$-isomorphism of $V(k\varepsilon_1)$ into $H_k$. On the other hand we can see that $\dim_{\mathbb{K}} H_k = \begin{pmatrix} N+k-1 \\ k \end{pmatrix} - \begin{pmatrix} N+k-3 \\ k-2 \end{pmatrix}$ from Proposition 2.2 and Theorem 3.1, which coincides with that of $V(k\varepsilon_1)$ where $\begin{pmatrix} n \\ m \end{pmatrix} := \frac{n(n-1)\cdots(n-m+1)}{m!}$. So we have $H_k \simeq V(k\varepsilon_1)$ for all $k$. □

The canonical map of (2.40) induces a $\mathbb{K}$-algebra homomorphism

$$\rho \colon \mathcal{D} \to \mathrm{End}_{\mathbb{K}}(\mathcal{A}) \tag{3.10}$$

such that $\rho(\eta)(\varphi) = \eta(\varphi)$ for $\eta \in \mathcal{D}$ and $\varphi \in \mathcal{A}$. Then we have the next statement.

THEOREM 3.4. *The space $\mathcal{D}^{U_q}$ of $\mathcal{D}$ becomes an algebra and is generated by $Q, \Delta, E$ and $c, c^{-1}$ over $\mathbb{K}$. Furthermore, the image $\rho(\mathcal{D}^{U_q})$ gives rise to a representation of $U_q(\mathfrak{sl}_2)$ on $\mathcal{A}$ (there is a $\mathbb{K}$-algebra homomorphism of $U_q(\mathfrak{sl}_2)$ onto $\rho(\mathcal{D}^{U_q})$).*

*Proof.* Let $\hat{\mathcal{A}}_k$ be the homogeneous subspace of degree $k$ in $\hat{\mathcal{A}}$, and $\hat{H}_k$ be a left $U_q$-module generated by $\partial_{\bar{1}}^k$. Then the module $\hat{H}_k$ is an irreducible $U_q$-module with highest weight vector $\partial_{\bar{1}}^k$, isomorphic to $V(k\varepsilon_1)$ through the algebra isomorphism $\imath$ of (2.13). Moreover we have

$$\hat{\mathcal{A}}_k = \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} \hat{H}_{k-2j}\Delta^j. \tag{3.11}$$

Hence we have

$$\mathcal{A} \otimes \hat{\mathcal{A}} = \bigoplus_{k=0}^{\infty} \left[ \bigoplus_{l=0}^{k} (\mathcal{A}_l \otimes \hat{\mathcal{A}}_{k-l}) \right] \tag{3.12}$$

and

$$\mathcal{A}_l \otimes \hat{\mathcal{A}}_{k-l} = \bigoplus_{\substack{0 \leqslant s \leqslant \lfloor \frac{l}{2} \rfloor \\ 0 \leqslant t \leqslant \lfloor \frac{k-l}{2} \rfloor}} Q^s H_{l-2s} \otimes \hat{H}_{k-l-2t}\Delta^t. \tag{3.13}$$

So we have enough to investigate the $U_q$-invariant subspace of $H_{r_1} \otimes \hat{H}_{r_2}$.

Recall that $V(r_1\varepsilon_1) \otimes V(r_2\varepsilon_1)$ has trivial representation with multiplicity one if $r = r_1 = r_2$ and otherwise it has no trivial representation, since the dual of $V(r_1\varepsilon_1)$ is isomorphic to itself in this case. Therefore we have to show that the $U_q$-invariant element of $H_r \otimes \hat{H}_r$ is expressed by a polynomial in $Q, \Delta, E$ and $c, c^{-1}$. We will prove this by induction on $r$.

We can easily see that a $U_q$-invariant element $E^r$ has a nonzero term $X_1^r\, \partial_{\bar{1}}^r\, q^{-r}$ when we reduce $E^r$ to the normal order in $\mathcal{D}$ (see the proof of Theorem 2.7). Hence it is clear that the image $\varphi$ of the projection $E^r$ to the trivial representation of $H_r \otimes \hat{H}_r \simeq V(r\varepsilon_1) \otimes V(r\varepsilon_1)$ does not disappear. We remark that this $\varphi$ is the unique $U_q$-invariant element of $H_r \otimes \hat{H}_r$ up to constant multiple. Hence, from the decomposition of (3.13) with $l = k - l = r$ and by induction on $r$, $\varphi$ can be expressed by a polynomial in $E, Q, \Delta$ and $c, c^{-1}$.

From first statement and the definition of $\tilde{E}$, we can say that the algebra $\mathcal{D}^{U_q}$ is generated by $Q, \Delta, \tilde{E}$ and $c, c^{-1}$. Furthermore, from (2.44) and (2.45), the image $\rho(\mathcal{D}^{U_q})$ is generated by $Q, \Delta$ and $c, c^{-1}$. Let

$$\tilde{\Delta} := \Delta c^{-1} \frac{(-1)q^{N/2}}{(1+q^{N-2})^2},$$
$$\tilde{c} := q^{N/2+\varepsilon} = q^{N/2} \cdot c, \tag{3.14}$$

then we have from (2.48)

$$Q\tilde{\Delta} - \tilde{\Delta}Q = \frac{\tilde{c} - \tilde{c}^{-1}}{q - q^{-1}}$$

$$\tilde{c} \cdot Q = q^2 Q \cdot \tilde{c}, \quad \tilde{c} \cdot \Delta = q^{-2} \Delta \cdot \tilde{c}. \tag{3.15}$$

This completes the proof. □

REMARK. This theorem inspire us with an analogue of classical Capelli identity. In fact for lower dimensions (e.g. $N = 3, 5$) we can find central elements of $U_q(\mathfrak{so}_N)$ which coincides with the Casimir element of $U_q(\mathfrak{sl}_2)$ on $\mathrm{End}_{\mathbb{K}}(\mathcal{A})$. But we have not yet found the general expression of the central element of $U_q(\mathfrak{so}_N)$ for the Capelli identity, although a class of central elements are obtained in [RTF].

### 3.2. QUANTUM SPHERES

Here we will introduce *a quantum sphere* $S_q^{N-1}$ following [RTF]. We define the quotient algebra

$$A_q(S^{N-1}) := \mathcal{A}/(Q - 1), \tag{3.16}$$

where $(Q - 1)$ denotes the two-sided ideal in $\mathcal{A}$ generated by $Q - 1$. The algebra $A_q(S^{N-1})$ is regarded as a ring of regular functions on the quantum complex $(N - 1)$-dimensional sphere.

PROPOSITION 3.5. *The algebra $A_q(S^{N-1})$ is a left $U_q$-module and is decomposed as follows:*

$$A_q(S^{N-1}) = \bigoplus_{k=0}^{\infty} \tilde{H}_k \tag{3.17}$$

*where $\tilde{H}_k$ is an irreducible $U_q$-module isomorphic to $H_k$.*

*Proof.* Since $Q$ is a trivial element, it is clear that $A_q(S^{N-1})$ is a left $U_q$-module. Let $\tilde{H}_k$ be the canonical image of the projection of $H_k$ to $A_q(S^{N-1})$. Then it is also clear that $\tilde{H}_k$ is a left $U_q$-module with highest weight vector $X_1^k$. So we have $\tilde{H}_k \simeq H_k \simeq V(k\varepsilon_1)$. From Theorem 3.1, we have

$$A_q(S^{N-1}) = \sum_{k=0}^{\infty} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \tilde{H}_{k-2j} = \sum_{k=0}^{\infty} \tilde{H}_k = \bigoplus_{k=0}^{\infty} \tilde{H}_k, \tag{3.18}$$

as desired. □

## 4. The $q$-orthogonal polynomials as zonal spherical functions

In Sections 4 and 5 we take the field $\mathbb{K} = \mathbb{C}$ of complex numbers assuming that $q$ as a real number with $q \neq 0, \pm 1$. We will first introduce the coideals in

$U_q = U_q(\mathfrak{so}_N)$, corresponding to the left ideal $J = U(\mathfrak{so}_N) \cdot \mathfrak{k}$ where $\mathfrak{k}$ is the Lie subalgebra $\mathfrak{k} = \mathfrak{so}_{N-1} \subset \mathfrak{so}_N$. Here coideal $J_q$ in $U_q$ means a $\mathbb{K}$-linear subspace of $U_q$ such that

$$\Delta(J_q) \subset J_q \otimes U_q + U_q \otimes J_q, \quad \text{and} \quad \varepsilon(J_q) = 0. \tag{4.1}$$

The subgroup $\mathrm{SO}(N-1)$ of $\mathrm{SO}(N)$ is realized as the stabilizer of a fixed vector of $V$. We will define two types of left ideal as follows:

$$\text{Type I} \quad J_q := \left\langle e_1, \dots, e_{n-1}, f_1, \dots, f_{n-1}, \frac{q^{\varepsilon_1}-1}{q-1}, \dots, \frac{q^{\varepsilon_n}-1}{q-1} \right\rangle$$

for $B_n$ series only

$$\text{Type II} \quad J_q := \begin{cases} \left\langle e_2, \dots, e_n, f_2, \dots, f_n, \theta_1, \theta_2, \dfrac{q^{\varepsilon_2}-1}{q-1}, \dots, \dfrac{q^{\varepsilon_n}-1}{q-1} \right\rangle \\ \quad \text{for } B_n (n>1) \text{ and } D_n (n>2) \text{ series} \\ \langle \theta_1 \rangle \quad (N=3) \\ \left\langle \theta_1, \theta_2, \dfrac{q^{\varepsilon_2}-1}{q-1} \right\rangle \quad (N=4), \end{cases} \tag{4.2}$$

where

$$\theta_1 := \begin{cases} s \cdot e_1 + (-1)^{n-1} t \cdot q^{1/2} q^{\varepsilon_1} f_2 \cdots f_n f_n \cdots f_2 f_1 \\ \quad \text{for } B_n(n>1) \text{ series}, \\ s \cdot e_1 + (-1)^{n-2} t \cdot q^{\varepsilon_1} f_2 \cdots f_{n-1} f_n f_{n-2} \cdots f_2 f_1 \\ \quad \text{for } D_n(n>2) \text{ series}, \\ s \cdot e_1 + t \cdot q^{1/2} q^{\varepsilon_1} f_1 \quad (N=3), \\ s \cdot e_1 + t \cdot q^{\varepsilon_1} f_2 \quad (N=4), \end{cases} \tag{4.3}$$

$$\theta_2 := \begin{cases} t \cdot q^{1/2} q^{\varepsilon_1} f_1 + (-1)^{n-1} s \cdot e_2 \cdots e_n e_n \cdots e_2 e_1 \\ \quad \text{for } B_n(n>1) \text{ series}, \\ t \cdot q^{\varepsilon_1} f_1 + (-1)^{n-2} s \cdot e_2 \cdots e_{n-1} e_n e_{n-2} \cdots e_2 e_1 \\ \quad \text{for } D_n(n>2) \text{ series}, \\ t \cdot q^{\varepsilon_1} f_1 + s \cdot e_2 \quad (N=4), \end{cases} \tag{4.4}$$

for $s, t \in \mathbb{R}\,(s \neq 0, t \neq 0)$, and $\langle a_1, \dots, a_r \rangle\,(a_j \in U_q)$ means the left ideal in $U_q$ generated by $a_1, \dots, a_r$. Note that

$$\Delta\left(\frac{q^{\varepsilon_j}-1}{q-1}\right) = q^{\varepsilon_j} \otimes \frac{q^{\varepsilon_j}-1}{q-1} + \frac{q^{\varepsilon_j}-1}{q-1} \otimes 1. \tag{4.5}$$

PROPOSITION 4.1. *The left ideals defined above become coideals in $U_q$.*

Our coideals defined above are to be regarded as $q$-analogues of $J = U(\mathfrak{so}_N)\cdot\mathfrak{k}$ by the next proposition.

PROPOSITION 4.2. *The $J_q$-invariant subspace of $\mathcal{A}$ is a commutative ring generated by $Q$ and $\zeta$, where $\zeta = X_{n+1}$ for type I and $\zeta = s\cdot X_1 + t\cdot X_{1'}$ for type II.*

*Proof.* We only prove the case of Type II, because it is more complicated than the case of Type I. We use the induction on the total order $\succeq$ in $\mathcal{A}$ of (2.9).

One can directly check that the $J_q$-invariant element of degree less than three is expressed by a polynomial in $Q$ and $\zeta$. Let $\varphi$ be a $J_q$-invariant element of $\mathcal{A}$. We take

$$\varphi = d_0X^{\nu} + d_1X^{\nu^1} + \cdots + d_lX^{\nu^l} \quad (d_j \in \mathbb{K}, d_j \neq 0) \tag{4.6}$$

so that $|\nu| = |\nu^1| = \cdots = |\nu^l| = k > 2$ and $X^{\nu} \succ X^{\nu^1} \succ \cdots \succ X^{\nu^l}$. One can show that the leading term $X^{\nu}$ equals $X_1^{\nu_1}X_{1'}^{\nu_{1'}}$ by the conditions:

$$e_j\,.\,\varphi = 0 \quad (2 \leqslant j \leqslant n), \qquad \frac{q^{\varepsilon_j}-1}{q-1}\cdot\varphi = 0\,(2 \leqslant j \leqslant n) \quad \text{and}$$

$$\theta_1\,.\,\varphi = 0. \tag{4.7}$$

We remark that the leading term of $Q^m$ is $\{(1+q^{N-2})q^{\rho_{1'}}\}^m X_1^m X_{1'}^m$. If $\nu_1 \geqslant v_{1'}$, then we have

$$\varphi \succ \psi := \varphi - d_0Q^{\nu_{1'}}\zeta^{\nu_1-\nu_{1'}}\cdot s^{-\nu_1+\nu_{1'}}\{(1+q^{N-2})q^{\rho_{1'}}\}^{-\nu_{1'}}. \tag{4.8}$$

Hence $\psi$ is a polynomial of $Q$ and $\zeta$ by induction, so is $\varphi$. To complete the proof, we will show that the case $\nu_1 < \nu_{1'}$ does not happen. Suppose $\nu_1 < \nu_{1'}$ and let $m$ be the maximum number such that

$$\nu_1 - \nu_{1'} = \nu_1^1 - \nu_{1'}^1 = \cdots = \nu_1^m - \nu_{1'}^m. \tag{4.9}$$

Then we have

$$\varphi = d_0X^{\nu} + d_1X^{\nu^1} + \cdots + d_mX^{\nu^m} + \text{lower order terms}. \tag{4.10}$$

Since $\nu_{1'}^m > \nu_1^m \geqslant 0$, the term $X^{\nu^m+\varepsilon_{2'}-\varepsilon_{1'}}$ does not disappear in $\theta_1\,.\,X^{\nu^m}$. So $\varphi$ must have the term $X^{\nu^m+\varepsilon_1-\varepsilon_{1'}}$ by the condition $\theta_1\,.\,\varphi = 0$ (Note that $\varphi$ does not have the term $X^{\nu^m-\varepsilon_1+\varepsilon_2+\varepsilon_{2'}-\varepsilon_{1'}}$ by the maximality of $m$). But the weight of $X^{\nu^m+\varepsilon_1-\varepsilon_{1'}}$ is higher than that of $X^{\nu}$. This is contradiction. □

We call $\varphi \in A_q(S^{N-1})$ the *zonal spherical function* associated with the irreducible representation $\tilde{H}_k$ if and only if $\varphi \in \tilde{H}_k$ and $J_q \,.\, \varphi = 0$. We denote by $\mathcal{H}$ the $J_q$-invariant subspace of $A_q(S^{N-1})$.

LEMMA 4.3. *For each $k$, let $H_k^{J_q}$ and $\tilde{H}_k^{J_q}$ be the $J_q$-invariant subspace of $H_k$ and $\tilde{H}_q$ respectively. Then*

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \tilde{H}_k^{J_q} \quad \text{and} \quad \dim_{\mathbb{K}} H_k^{J_q} = \dim_{\mathbb{K}} \tilde{H}_k^{J_q} = 1 \quad \text{for all } k \geqslant 0. \tag{4.11}$$

*Proof.* The Littlewood–Richardson Rule ([Na]) shows the decomposition

$$H_k \otimes H_1 \simeq \tilde{H}_k \otimes \tilde{H}_1 \simeq$$
$$\simeq \begin{cases} V((k+1)\varepsilon_1) \oplus V(k\varepsilon_1) \oplus V((k-1)\varepsilon_1) & (N=3), \\ V((k+1)\varepsilon_1) \oplus V((k-1)\varepsilon_1) \oplus V(k\varepsilon_1+\varepsilon_2) & \\ \qquad \oplus V(k\varepsilon_1 - \varepsilon_2) & (N=4), \\ V((k+1)\varepsilon_1) \oplus V((k-1)\varepsilon_1) \oplus V(k\varepsilon_1+\varepsilon_2) & (N \geqslant 5). \end{cases} \tag{4.12}$$

Let $P_k$ be a nonzero $J_q$-invariant polynomial in $H_k$. From Proposition 4.2 we may write

$$P_k = a_{k,0}\zeta^k + a_{k,1}\zeta^{k-2}Q + a_{k,2}\zeta^{k-4}Q^2 + \cdots, \tag{4.13}$$

where $a_{k,j} \in \mathbb{K}$ for all $j$. Since $\mathcal{A}_{k+1} = \oplus_{j=0}^{\lfloor\frac{k+1}{2}\rfloor} Q^j H_{k+1-2j}$, from (4.12) we have

$$P_k\zeta = \zeta P_k \in H_{k+1}^{J_q} \oplus Q H_{k-1}^{J_q}. \tag{4.14}$$

From this one can inductively show that $a_{k,0} \neq 0$ for all $k$ and that the projection of $\zeta P_k$ to $H_{k+1}^{J_q}$ is not zero. So we have $\dim_{\mathbb{K}} H_k^{J_q} \geqslant 1$. On the other hand, let $P_k' = \Sigma_{j=0}^{\lfloor\frac{k}{2}\rfloor} a_{k,j}'\zeta^{k-2j}Q^j$ be an another polynomial in $H_k^{J_q}$. Then we have $P_k' - a_{k,0}^{-1}a_{k,0}' \times P_k \in \oplus_{j\geqslant 1}^{\lfloor\frac{k}{2}\rfloor} H_{k-2j}^{J_q} \cdot Q^j$. Again by the argument above, it must be zero. Hence $\dim_{\mathbb{K}} H_k^{J_q} = 1$ for all $k$. The similar argument shows that $\dim_{\mathbb{K}} \tilde{H}_k^{J_q} = 1$. □

REMARK. From this lemma it is clear that the $J_q$-invariant space $\tilde{H}_k^{J_q}$ is generated by the canonical image of a nonzero $J_q$-invariant polynomial in $H_k$.

To describe the zonal spherical functions we shall introduce some $q$-orthogonal polynomials.

The *big q-Jacobi polynomials* are defined by

$$P_n^{(\alpha,\beta)}(z;c,d:q) := {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, zq^{\alpha+1}/c \\ q^{\alpha+1}, -q^{\alpha+1}d/c \end{matrix} ; q, q\right], \tag{4.15}$$

where

$${}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix} ; q, x\right] := \sum_{j=0}^{\infty} \frac{(a_0;q)_j(a_1;q)_j\cdots(a_r;q)_j}{(q;q)_j(b_1;q)_j\cdots(b_r;q)_j} x^j, \tag{4.16}$$

and

$$(a;q)_n := \begin{cases} 1 & \text{if } n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{if } n \geqslant 1. \end{cases} \tag{4.17}$$

We also use the notation

$$(a_1, a_2, \ldots, a_r; q)_n := (a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n. \tag{4.18}$$

Our parametrization follows [NM2] (see also [GR]). The big $q$-Jacobi polynomials $P_n^{(\alpha,\beta)}(z;c,d:q)$ satisfy the following $q$-difference equation (see [NM2]).

$$\begin{aligned} &\{(c - zq^{\alpha+1})(d + zq^{\beta+1})T_{q,z} - (1+q)cd - q(c(1+q^\beta) - d(1+q^\alpha))z \\ &\quad + q^{-n+1}(1 + q^{\alpha+\beta+2n+1})z^2 \\ &\quad + q(c-z)(d+z)T_{q,z}^{-1}\}P_n^{(\alpha,\beta)}(z;c,d:q) = 0 \end{aligned} \tag{4.19}$$

where $T_{q,z}$ is *q-shift operator* defined by

$$T_{q,z} \,.\, z^n = q^n z^n \quad \text{for all } n \in \mathbb{Z}. \tag{4.20}$$

Another $q$-orthogonal polynomial is *Rogers' continuous q-ultraspherical polynomial* defined by (see pp. 168–172 in [GR])

$$C_n^\lambda(X;q) := \frac{(q^{2\lambda};q)_n}{(q;q)_n} q^{-(n\lambda/2)} \, {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{2\lambda+n}, zq^{\lambda/2}, z^{-1}q^{\lambda/2} \\ q^{\lambda+1/2}, -q^{\lambda+1/2}, -q^\lambda \end{matrix} ; q, q\right], \tag{4.21}$$

where $X = (z + z^{-1})/2$. This satisfies the following recurrence relation:

$$2XC_n^\lambda(X;q) = F_n C_{n+1}^\lambda(X;q) + G_n C_{n-1}^\lambda(X;q) \tag{4.22}$$

with $C_{-1}^{\lambda}(X;q) \equiv 0, C_0^{\lambda}(X;q) \equiv 1$, where

$$F_n = \frac{1-q^{n+1}}{1-q^{\lambda+n}} \quad \text{and} \quad G_n = \frac{1-q^{2\lambda+n-1}}{1-q^{\lambda+n}}. \tag{4.23}$$

REMARK. $C_n^{\lambda}(X;q) = 2^n(F_0F_1\cdots F_{n-1})^{-1}X^n +$ lower terms.

THEOREM 4.4. *If we take $J_q$ of type I, then for each $k \geqslant 0$ the zonal spherical function $\varphi_k$ associated with $\tilde{H}_k$ is expressed by big q-Jacobi polynomial up to constant multiple:*

$$\varphi_k = P_k^{(N-3)/2,(N-3)/2}(z;1,1:q) \tag{4.24}$$

*where*

$$z = L_1^{1/2}\zeta = L_1^{1/2}X_{n+1} \quad \textit{and} \quad L_1 = \frac{1+q^{N-2}}{(1+q)q^{N-2}}. \tag{4.25}$$

REMARK. The leading coefficient of $\zeta^k$ in $\varphi_k$ is $\frac{(q^{N+k-2};q)_k}{(q^{N-1};q^2)_k}(L_1^{1/2}q^{(N-1)/2})^k$.

LEMMA 4.5. *We keep the notations in Theorem 4.4. We define a q-difference operator $D_k$ on $\mathcal{H}$ by*

$$D_k = \frac{(1+q^{N-2})q^k}{(1+q)(1-q)^2}\zeta^{-2} \times [(1-q^{N-1}z^2)T_{q,z} + q(1-z^2)T_{q,z}^{-1}$$
$$- (1+q) + (1+q^{N+2k-2})q^{-k+1}z^2]. \tag{4.26}$$

*Then $D_k$ satisfies the following commutative diagram:*

$$\begin{array}{ccccc} \mathbb{K}[Q,\zeta]_k & \xrightarrow{\Delta} & \mathbb{K}[Q,\zeta]_{k-2} & \xrightarrow{Q} & \mathbb{K}[Q,\zeta]_k \\ \downarrow{\scriptstyle Q\to 1} & & \downarrow{\scriptstyle Q\to 1} & & \downarrow{\scriptstyle Q\to 1} \\ \mathbb{K}[\zeta] & \xrightarrow{D_k} & \mathbb{K}[\zeta] & \xrightarrow{\text{id}} & \mathbb{K}[\zeta] \end{array}, \tag{4.27}$$

*where $\mathbb{K}[Q,\zeta]_k$ is the homogeneous subspace of degree $k$ in $\mathbb{K}[Q,\zeta]$.*

*Proof.* The action of the Laplace operator $\Delta$ to the basis $Q^jX_{n+1}^{k-2j}$ $(0 \leqslant j \leqslant \lfloor \frac{k}{2} \rfloor)$ of $\mathbb{K}[Q,\zeta]_k$ is described as

$$\Delta(Q^jX_{n+1}^{k-2j}) = q^{2j}Q^j\frac{(1+q^{N-2})q}{1+q}[k-2j-1][k-2j]X_{n+1}^{k-2j-2} +$$

$$+ \; Q^{j-1}\frac{(1+q^{N-2})^2}{(1+q)^2}q^{-N+3}[2j]$$

$$\times\; [N-2j-2+2k]X_{n+1}^{k-2j} \tag{4.28}$$

using (2.48) and (2.52). Taking $Q \to 1$, we rewrite the right-hand side of (4.28) by using $q$-shift operators, and we obtain the expression of (4.26). □

*Proof of Theorem 4.4.* Ley $\Phi_k$ be a nonzero $J_q$-invariant polynomial in $H_k$. Then the image of canonical limit $Q \to 1$ of $\Phi_k$ is a nonzero zonal spherical function belonging to $\tilde{H}_k$. From Lemma 4.4, we have $D_k \,.\, \varphi_k = 0$ since $\Delta(\Phi_k) = 0$. Comparing (4.19) with this, we have the expression of $\varphi_k$ as desired. □

THEOREM 4.6. *If we take $J_q$ of type II, then for each $k \geqslant 0$ the zonal spherical function $\varphi_k$ associated with $\tilde{H}_k$ is expressed by Rogers' continuous q-ultraspherical polynomial up to constant multiple:*

$$\varphi_k(Y) = C_k^{(N-2)/2}(Y; q^2), \tag{4.29}$$

*where* $2L^{-1}Y = \zeta$ *and* $L = \sqrt{\frac{(1+q^{N-2})}{st}}q^{-(N-2)/2}$.

REMARK. The leading coefficient of $\zeta^k$ in $\varphi_k$ is $L^k(F_0F_1\cdots F_{k-1})^{-1}$ (see (4.22)).

*Proof.* Let $\Phi_k$ be the nonzero $J_q$-invariant polynomial in the form:

$$\Phi_k = \zeta^k + a_1^{(k)}Q\zeta^{k-2} + a_2^{(k)}Q^2\zeta^{k-4} + \cdots. \tag{4.30}$$

From Lemma 3.4, we can write

$$\Delta(\zeta^k) = b_0^{(k)}\zeta^{k-2} + b_1^{(k)}Q\zeta^{k-4} + \cdots. \tag{4.31}$$

So we have

$$\begin{aligned}\Delta(\Phi_k) &= b_0^{(k)}\zeta^{k-2} + b_1^{(k)}Q\zeta^{k-4} + \cdots \\ &\quad + a_1^{(k)}\Big(q^2Q\Delta + \frac{(1+q^{N-2})^2}{(1+q)^2}q^{-N+3}[2][N+2\varepsilon]\Big)(\zeta^{k-2}) \\ &\quad + \cdots \\ &= 0.\end{aligned} \tag{4.32}$$

Noting the coefficient of $\zeta^{k-2}$ in (4.32), we have

$$a_1^{(k)} = \frac{(1+q)^2}{(1+q^{N-2})^2} \times \frac{-q^{N-3}}{[2][N+2k-4]}b_0^{(k)}. \tag{4.33}$$

From Lemma 4.3, we have

$$\zeta \cdot \Phi_k - \Phi_{k+1} = (a_1^{(k)} - a_1^{(k+1)})Q\Phi_{k-1}. \tag{4.34}$$

Thus we will obtain the three-term recurrence relation of $\Phi_k$ by calculating $b_0^{(k)}$.

We set $2L^{-1}Y = \zeta$ and $\varphi_k(Y) := L^k(F_0 \cdots F_{k-1})^{-1}\Phi_k|_{Q\to 1}$ where $L \in \mathbb{K}, L \neq 0$. Of course, $\varphi_k(Y)$ is the zonal spherical function associated with $\tilde{H}_k$. Thus the recurrence relation (4.34) is reduced to the following form:

$$2Y\varphi_k = F_k\varphi_{k+1} + L^2(a_1^{(k)} - a_1^{(k+1)})F_{k-1}^{-1}\varphi_{k-1}. \tag{4.35}$$

Carrying out the calculation of $\Delta((s \cdot X_1 + t \cdot X_{1'})^k)$ with noting the coefficient of the lowest weight term $X_{1'}^{k-2}$, we have

$$b_0^{(k)} = st(1 + q^{N-2})\sum_{j=1}^{k-1}([j]_q q^j q^{\rho_{1'}} + j \cdot q^{2j-1}q^{\rho_1}). \tag{4.36}$$

From (4.34) and (4.37), we have

$$\begin{aligned} a_1^{(k)} - a_1^{(k+1)} &= st\frac{q^{(N-2)/2}}{1 + q^{N-2}} \times \frac{(1 - q^k)(1 - q^{2N+2k-6})(1 + q^k)}{(1 - q^{N+2k-4})(1 - q^{N+2k-2})} \\ &= L^{-2}F_{k-1}G_k, \end{aligned} \tag{4.37}$$

with $\lambda = \frac{N-2}{2}$ and $q^2$-base. Hence by comparing (4.35) with (4.22), we have Theorem 4.6. □

## 5. Invariant measure and orthogonality

In this section we will show that the orthogonality relations of zonal spherical functions in the previous section are expressed by the invariant functional on $A_q(S^{N-1})$. Here we keep the notations in Section 4.

PROPOSITION 5.1. *There is a unique left $U_q$-invariant functional (intertwiner)*

$$h: A_q(S^{N-1}) \to \mathbb{K} \tag{5.1}$$

*with $h(1) = 1$. The value of $h$ on the elements $\{X^\nu\}$ is given by*

$$h(X^\nu) =$$

$$= \begin{cases} \dfrac{(q^{-2};q^{-2})_{\nu_1}\cdots(q^{-2};q^{-2})_{\nu_n}(q^{-1};q^{-2})_m}{(q^{-N};q^{-2})_{\nu_1+\cdots+\nu_n+m}} q^{-(\rho_1\nu_1+\cdots+\rho_n\nu_n)-m} \\ \times \dfrac{(1+q)^m}{(1+q^{N-2})^{\nu_1+\cdots+\nu_n+m}} \\ \textit{if } \nu_1=\nu_{1'},\ldots,\nu_n=\nu_{n'} \textit{ and } \nu_{n+1}=2m\in 2\mathbb{Z}_{\geqslant 0} \\ (\textit{for } D_n \textit{ series we set } m=0) \\ 0 \quad \textit{otherwise.} \end{cases} \tag{5.2}$$

The proof is carried out by the similar arguments in [NYM, Proposition 4.5].

We now introduce involutive algebra anti-automorphisms ($*$-*operations*) on $A_q(S^{N-1})$ and $U_q(\mathfrak{so}_N)$ as follows:

$$X_j^* = X_{j'}q^{\rho_{j'}} \quad \text{in} \quad A_q(S^{N-1}) \quad (1 \leqslant j \leqslant N) \tag{5.3}$$

and

$$(q^u)^* = q^u \quad (u \in P^*), \quad e_j^* = q_j^{-1} f_j q^{\alpha_j},$$

$$f_j^* = q_j q^{-\alpha_j} e_j \quad (1 \leqslant j \leqslant n). \tag{5.4}$$

Then $U_q(\mathfrak{so}_N)$ becomes a Hopf $*$-algebra with this $*$-operation. These $*$-operations on $A_q(S^{N-1})$ and $U_q(\mathfrak{so}_n)$ are compatible in the sense that

$$(a\,.\,\varphi)^* = S(a)^*\,.\,\varphi^* \quad \text{for } a \in U_q \quad \text{and} \quad \varphi \in A_q(S^{N-1}). \tag{5.5}$$

This fact can be checked by direct calculations. We now define a *hermitien form* $\langle\ ,\ \rangle$ on $A_q(S^{N-1})$ by the formula

$$\langle \varphi, \psi \rangle := h(\varphi^* \psi) \quad \text{for } \varphi, \psi \in A_q(S^{N-1}). \tag{5.6}$$

This form satisfies the following invariance

$$\langle \varphi, a\,.\,\psi \rangle = \langle a^*\,.\,\varphi, \psi \rangle \tag{5.7}$$

for any $a \in U_q$ and $\varphi, \psi \in A_q(S^{N-1})$. As to the detail arguments, we can refer to [N1, Sections 1 and 6], [RTF] and [W1].

We denote by $\langle\ ,\ \rangle_{\mathcal{H}}$ the restricted form of $\langle\ ,\ \rangle$ to $\mathcal{H} = \mathbb{K}[\zeta]$. In the following we use the *q-integral*:

$$\int_0^a F(z)d_qz := a(1-q)\sum_{n=0}^{\infty} F(aq^n)q^n, \quad \text{and}$$

$$\int_b^a d_qz := \int_0^a d_qz - \int_0^b d_qz. \tag{5.8}$$

PROPOSITION 5.2. *If we take $J_q$ of type I, then we have*

$$h(\varphi) = M_1^{-1} \int_{-1}^{1} \varphi(z) w_1(z;q) d_q z \quad \text{for } \varphi = \varphi(z) \in \mathcal{H}, \tag{5.9}$$

*where*

$$w_1(z;q) = (q^2 z^2; q^2)_{(N-3)/2},$$

$$z = L_1^{1/2} \zeta = \left( \frac{1+q^{N-2}}{(1+q)q^{N-2}} \right)^{1/2} X_{n+1} \ (\text{see } (4.25)), \tag{5.10}$$

$$M_1 = \int_{-1}^{1} w_1(z;q) d_q z = 2(1-q) \frac{(q^2;q^2)_{(N-3)/2}}{(q;q^2)_{(N-1)/2}} = 2 \frac{[N-3]!!}{[N-2]!!}, \tag{5.11}$$

*and* $[2m+1]!! = [2m+1][2m-1]\cdots[1], [2m]!! = [2m][2m-2]\cdots[2]$.

From Proposition 5.1 we have

$$h(\zeta^{2m}) = h(X_{n+1}^{2m}) = L_1^{-m} \frac{(q;q^2)_m}{(q^N;q^2)_m},$$

$$h(\zeta^{2m+1}) = 0 \quad (m \in \mathbb{Z}_{\geqslant 0}). \tag{5.12}$$

On the other hand, we have a kind of *q-beta integral*

$$\int_0^1 z^\alpha (q^2 z^2; q^2)_\beta d_q z = \frac{[\alpha-1]!![2\beta]!!}{[2\beta+\alpha+1]!!} \quad (\alpha, \beta \in \mathbb{Z}_{\geqslant 0}) \tag{5.13}$$

Then Proposition 5.2 immediately follows from (5.12) and (5.13).

REMARK. We have

$$\langle \varphi_m, \varphi_n \rangle_{\mathcal{H}} = \frac{\delta_{m,n}}{M_1} \frac{(q;q)_m (1-q^{N-2})}{(q^{N-2};q)_m (1-q^{N+2m-2})}, \tag{5.14}$$

from the following orthogonality relations of big $q$-Jacobi polynomials;

$$\int_{-d}^{c} P_n^{(\alpha,\beta)} P_m^{(\alpha,\beta)} \times (qz/c;q)_\alpha (-qz/d;q)_\beta d_q z$$

$$= \frac{\delta_{m,n}}{M} \frac{(q;q)_m (1-q^{\alpha+\beta+1})(q^{\beta+1}, -q^{\beta+1}c/d;q)_m}{(q^{\alpha+\beta+1};q)_m (1-q^{\alpha+\beta+2m+1})(q^{\alpha+1}, -q^{\alpha+1}d/c;q)_m}, \tag{5.15}$$

where

$$M = \int_{-d}^{c} (qz/c;q)_\alpha (-qz/d;q)_\beta d_q z =$$

$$= c\frac{(1-q)(q;q)_\alpha(-d/c;q)_{\alpha+1}(-qc/d;q)_\beta}{(q^{\beta+1};q)_{\alpha+1}}. \tag{5.16}$$

We also remark that our big $q$-Jacobi polynomials $P_n^{(\alpha,\beta)}(z;c,d:q)$ and their orthogonalities are obtained by transforming $x \mapsto q^{\alpha+1}z/c, a \mapsto q^\alpha, b \mapsto q^\beta$ and $c \mapsto -q^\alpha d/c$ of $P_n(X;a,b,c:q)$ in [GR, pp. 166–168].

PROPOSITION 5.4. *We take $J_q$ of type II, keeping the notations of Theorem 4.5 with fixing $s = q^{(1/2)\rho_1}$ and $t = q^{(1/2)\rho_{1'}}$. Then we have*

$$\langle \varphi_m, \varphi_n \rangle_{\mathcal{H}} = \delta_{m,n}\frac{(1-q^{N-2})(q^{2N-4};q^2)_m}{(1-q^{N+2m-2})(q^2;q^2)_m}. \tag{5.17}$$

COROLLARY 5.5.

$$h(\varphi(Y)) = M_2^{-1}\int_{-1}^{1}\varphi(Y)W_{(N-2)/2}(Y;q^2)\,\mathrm{d}Y \quad (\varphi(Y)\in\mathcal{H}), \tag{5.18}$$

*where*

$$W_\lambda(Y;q) := \frac{\prod_{k=0}^{\infty}(1-2q^k(2Y^2-1)+q^{2k})}{\prod_{k=0}^{\infty}(1-2q^{\lambda+k}(2Y^2-1)+q^{2\lambda+2k})} \tag{5.19}$$

*and*

$$M_2 = \int_{-1}^{1} W_{(N-2)/2}(Y;q^2)\,\mathrm{d}Y = \frac{2\pi(q^{N-2},q^N;q^2)_\infty}{(q^2,q^{2N-4};q^2)_\infty}. \tag{5.20}$$

*Proof of Proposition 5.4.* Since $\tilde{H}_k \otimes \tilde{H}_l$ has the trivial representation if and only if $k = l$, subspaces $\tilde{H}_k$ $(k \geqslant 0)$ of $A_q(S^{N-1})$ are orthogonal to each other with respect to the hermitien form $\langle\ ,\ \rangle$. Hence we have $\langle \varphi_m, \varphi_n\rangle = 0$ if $m \neq n$. From (4.23) and (4.30) we have

$$2^2Y^2\varphi_k = F_k(F_{k+1}\varphi_{k+2} + G_{k+1}\varphi_k) + G_k(F_{k-1}\varphi_k + G_{k-1}\varphi_{k-2})$$

$$\cdots$$

$$2^kY^k\varphi_k = G_kG_{k-1}\cdots G_1\varphi_0 + \sum_{l=1}^{2k} c_l\varphi_l \quad \text{for some } c_l \in \mathbb{K}. \tag{5.21}$$

Then we have

$$h(Y^j\varphi_k) = \begin{cases} 0 & \text{if } 0 \leqslant j \leqslant k-1 \\ 2^{-k}G_1\cdots G_k & \text{if } j = k. \end{cases} \tag{5.22}$$

Since the leading coefficient of $Y^k$ in $\varphi_k$ is $2^k(F_0 \cdots F_{k-1})^{-1}$, we have from (5.22)

$$\begin{aligned}\langle \varphi_k, \varphi_k \rangle_{\mathcal{H}} &= h(\varphi_k \varphi_k) \quad (\because Y^* = Y) \\ &= \frac{G_1 \cdots G_k}{F_0 \cdots F_{k-1}} = \frac{(1-q^{N-2})(q^{2N-4};q^2)_k}{(1-q^{N+2k-2})(q^2;q^2)_k} \end{aligned} \tag{5.23}$$

as desired. □

Corollary 5.5 is directly obtained by comparing Proposition 5.4 with the orthogonality relations of $C_n^\lambda(Y;q)$ in [GR, pp. 171–172].

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