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The residual spectrum of Sp_4

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Abstract. We completely determine the residual spectrum of Sp_4 which is the orthogonal complement of the cuspidal spectrum in $L^2_d(G(F)\backslash G(\mathbb{A}))$, the discrete spectrum of Sp_4. They are spaces of residues of Eisenstein series associated to the cuspidal representations of the Levi factors of parabolic subgroups. We follow Langlands to analyze intertwining operators and L-functions in the constant terms of Eisenstein series.

1. Introduction

Let $G = Sp_4$ be the symplectic group of degree two defined over a number field F and $G(\mathbb{A})$ be its adele group. By the general theory of Eisenstein series (Langlands [18]), one knows that the Hilbert space $L^2(G(F)\backslash G(\mathbb{A}))$ has an orthogonal decomposition of the form

$$L^2(G(F)\backslash G(\mathbb{A})) = L^2(G) \oplus L^2(B) \oplus L^2(P_1) \oplus L^2(P_2),$$

where B is a Borel subgroup and P_i are standard maximal parabolic subgroups in G for $i = 1, 2$. The purpose of this note is to describe explicitly the spaces $L^2_d(P_i)$ and $L^2_d(B)$ associated to the discrete spectrum in $L^2(P_i)$, $i = 1, 2$, and $L^2(B)$ (Theorems 3.3, 4.1 and 5.4). They are the non-cuspidal discrete spectrum, called the residual spectrum. They are spaces of residues of Eisenstein series associated to the cuspidal representations of the Levi factors of parabolic subgroups. In order to obtain the residues of Eisenstein series, we follow Langlands [18], that is, we use the fact that the constant terms of Eisenstein series in the Fourier expansion determine the analytic behavior of Eisenstein series themselves, such as poles and square integrability. The constant terms of Eisenstein series are a sum of intertwining operators which can be normalized by certain L-functions attached to the cuspidal representations of the Levi factors of parabolic subgroups (cf. [17] and [27]).

For $L^2_d(P)$ with P Siegel parabolic subgroup, the Levi factor is $M = GL_2$. Jacquet–Langlands theory tells us the discrete spectrum and the corresponding L-functions of GL_2. We have to look at Eisenstein series associated to cuspidal representations of GL_2. The L-function in the constant terms of Eisenstein series is exactly the Jacquet–Langlands L-function and Hecke L-function. The poles and irreducibility of the images of local intertwining operators associated to tempered representations are well known. We prove the same results for non-tempered cases by inducing from Borel subgroups. Here we need an observation due to Shahidi concerning the effect of intertwining operators when induced from Borel subgroups because the intertwining operator associated to the longest element of Weyl group of the split component of the Levi subgroup of Siegel parabolic subgroup, is not any more associated to the longest element of the Weyl group.

So we cannot use Langlands' classification theorem directly. In this way we obtain a decomposition of $L_d^2(P)$, P Siegel parabolic, parametrized by cuspidal representations π of GL_2 with trivial central characters and $L(\frac{1}{2}, \pi) \neq 0$. However we were unable to prove multiplicity one results. This requires further research.

For $L_d^2(P)$, P non-Siegel parabolic subgroup, the Levi factor is $SL_2 \times GL_1$. The L-functions in the constant terms of the Eisenstein series associated to cuspidal representations of $SL_2 \times GL_1$ are exactly Gelbart–Jacquet L-functions ([5]). We analyze the intertwining operators in the same way as in the Siegel case. Consequently we obtain a decomposition of $L_d^2(P)$ with P non-Siegel parabolic, parametrized by monomial representations of GL_2. Recall that a cuspidal representation σ of GL_2 is called monomial if $\sigma \simeq \sigma \otimes \eta$ for a quadratic grössencharacter η of F. In this case also, multiplicity one results require further research. We note that Schwermer [23] obtained similar results for GSp_4 for the above maximal parabolic cases when $F = \mathbb{Q}$.

The most interesting and surprising of all is the analysis of $L_d^2(B)$ since not every element of the global L-packet appears. In fact only those for which a certain parity sign, determined by certain subtle identities of Labesse–Langlands [16] satisfied by local standard intertwining operators for SL_2, is positive will appear.

Let us describe our result more precisely. For μ, ν grössencharacters of F, we define a character $\chi = \chi(\mu, \nu)$ of B. Then only the quadratic characters $\chi = \chi(\mu, \mu)$, $\mu^2 = 1$, contribute to $L_d^2(B)$. The trivial character gives only constants. For μ a non-trivial character, the pole of the Eisenstein series only at $\beta_1 = \alpha_1 + \frac{1}{2}\alpha_2$ contributes to $L_d^2(B)$, where α_1 is the short root and α_2 the long one. Let σ (respectively τ) be the simple reflection in the hyperplane orthogonal to α_1 (respespectively α_2). For $f \in I(\beta_1, \chi) = \mathrm{Ind}_B^G(\chi \otimes e^{\langle \beta_1, H_B(\ )\rangle})$, let $H(f, \chi) = \mathrm{Res}_{\beta_1}\mathrm{Res}_{\langle \Lambda, \alpha_1^\vee\rangle = 1} E(g, f, \Lambda)$ be the iterated residue of the Eisenstein series and $B(\mu)$ be the space spanned by $H(f, \chi)$. Then $B(\mu) \subset L_d^2(B)$. The constant term of $H(f, \chi)$ is a sum of two normalized intertwining operators $R(\sigma\tau\sigma\tau, \beta_1, \chi)$ and $R(\sigma\tau\sigma, \beta_1, \chi)$ from $I(\beta_1, \chi)$ to $I(-\beta_1, \chi)$. In order to analyze the image of the intertwining operators, we look at the local intertwining operators $R(\sigma\tau\sigma\tau, \beta_1, \chi_v)$ and $R(\sigma\tau\sigma, \beta_1, \chi_v)$ from $I(\beta_1, \chi_v)$ to $I(-\beta_1, \chi_v)$, where $\chi_v = \chi(\mu_v, \mu_v)$. Here we use Shahidi's idea of inducing in stages via the non-Siegel parabolic subgroup, that is, use the fact that $I(\beta_1, \chi_v) = \mathrm{Ind}_P^G(|\ |_v \mu_v \times \mathrm{Ind}_{B_0}^{SL_2}(\mu_v))$, where B_0 is the corresponding Borel subgroup of SL_2. Suppose μ_v is not trivial. Then $\mathrm{Ind}_{B_0}^{SL_2}(\mu_v)$ is reducible. Fix a non-trivial additive character $\psi = \otimes_v \psi_v$ of $\mathbb{A}/F$. Let $\mathrm{Ind}_{B_0}^{SL_2}(\mu_v) = \pi_+(\mu_v) \oplus \pi_-(\mu_v)$, as in [16], i.e., with $\pi_+(\mu_v)$ generic with respect to ψ_v. Let $\epsilon(\pi_+(\mu_v)) = 1$ and $\epsilon(\pi_-(\mu_v)) = -1$. Observe that for almost all v, $\pi_+(\mu_v)$ is spherical. Then the common image of $R(\sigma\tau\sigma\tau, \beta_1, \chi_v)$ and $R(\sigma\tau\sigma, \beta_1, \chi_v)$ is a sum of two Langlands' quotients $J_\pm(\mu_v)$ of $\mathrm{Ind}_P^G(|\ |_v \mu_v \times \pi_\pm(\mu_v))$, respectively. If μ_v is trivial, $\mathrm{Ind}_{B_0}^{SL_2}(\mu_v)$ is irreducible. In this case, we take $\pi_-(\mu_v) = 0$. Let $\pi(\mu_v) = \{\pi_+(\mu_v), \pi_-(\mu_v)\}$ and if $\pi_v \in \pi(\mu_v)$, let $\epsilon(\pi_v)$ be the corresponding sign. Let $J(\mu_v) = \{J_+(\mu_v), J_-(\mu_v)\}$. Then we define $J(\mu)$ to be the collection

$$J(\mu) = \left\{ \Pi = \otimes \Pi_v | \Pi \in J(\mu_v) \text{ for all } v, \right.$$

$$\left. \Pi_v = J_+(\mu_v) \text{ for almost all } v, \quad \prod_v \epsilon(\pi_v) = 1 \right\}.$$

We note that $\prod_v \epsilon(\pi_v)$ is well defined and independent of the choice of ψ. Then we obtain irreducible decomposition of $L^2_d(B)$ as follows:

$$L^2_d(B) = \oplus_\mu B(\mu) \oplus B(\mu_0),$$

where μ runs through all non-trivial quadratic grössencharacters of F and $B(\mu_0)$ is the space of constant functions. For each μ, the constant term map gives rise to an isomorphism from $B(\mu)$ to $J(\mu)$. Here the condition $\prod_v \epsilon(\pi_v) = 1$ comes from subtle analysis of the normalized intertwining operator $R(\tau, \beta_1, \chi_v)$ which is the intertwining operator for $\mathrm{Ind}_{B_0}^{SL_2}(\mu_v)$, as in [16]. For $\prod_v \epsilon(\pi_v) = -1$, there is a cancellation between the two intertwining operators $R(\sigma\tau\sigma\tau, \beta_1, \chi_v)$ and $R(\sigma\tau\sigma, \beta_1, \chi_v)$.

In a separate paper [35], we give the Arthur parameter for the representations in $J(\mu)$ and verify Arthur's conjecture on the multiplicity formula (See [1] for Arthur's conjecture). It turns out that the parity sign $\epsilon(\pi_v)$ enters into the multiplicity formula, as anticipated by Arthur [1]. Keys–Shahidi [12] generalized the parity sign to general quasi-split groups. Using their result, we can generalize our result, at least, to split classical groups.

Watanabe [31] studied a subspace $L^2_d(B, K_S)$ consisting of K_S-invariant elements of $L^2_d(B)$, where $K_S = K_\infty \times \prod_{v \notin S} K_v \times \prod_{v \in S} \mathrm{Ker}(r_v)$, is a compact subgroup of $G(\mathbb{A})$, where $r_v \colon K_v \mapsto G(\mathbf{k}_v)$ is the reduction homomorphism and S is a finite set of places of F (See Section 2 below for notations). In particular, he obtained the result that only the quadratic characters $\chi(\mu, \mu)$, $\mu^2 = 1$ of B, contribute to $L^2_d(B, K_S)$ and that the trivial character gives the constants. As explained above, our results reaffirm his description of the subspace $L^2_d(B, K_S)$ of $L^2_d(B)$. He also made a conjectural description of $L^2_d(B)$ based on his calculation, conjecturing that $B(\mu)$ is irreducible. As explained above, we see that his conjecture is not true.

After this paper was accepted for publication, the author learned that T. Kon-No announced a similar result. But his results are over a totally real number field.

2. Preliminaries

Let F be a number field and $G = Sp_4$ be the symplectic group of degree 2, that is

$$G(F) = \left\{ g \in GL_4(F) \mid g \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let T and U be a maximal split torus and a maximal unipotent subgroup of G, respectively, as follows:

$$\begin{aligned} T(F) &= \{t(a,b) = \mathrm{diag}(a, b, a^{-1}, b^{-1}) \in G(F)\} \\ U(F) &= \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \in G(F) \mid A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad S = {}^t S \right\} \end{aligned}$$

Then $B = TU$ is a Borel subgroup in G.

Let $X(T)$ (respectively $X^*(T)$) be the character (respectively cocharacter) group of T. There is a natural pairing $\langle\, , \rangle \colon X(T) \times X^*(T) \mapsto \mathbb{Z}$. We take $\alpha_1, \alpha_2 \in X(T)$ such that $\alpha_1(t(a,b)) = ab^{-1}$ and $\alpha_2(t(a,b)) = b^2$. Then $\Delta = \{\alpha_1, \alpha_2\}$ is a set of simple roots and $\psi^+ = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = 2\alpha_1 + \alpha_2\}$ is a set of positive roots. Further, $\beta_1 = \frac{\alpha_4}{2}$

and $\beta_2 = \alpha_3$ are the fundamental weights of G with respect to (B,T). The coroot corresponding to α_i is denoted by $\alpha_i^\vee$ for $1 \le i \le 4$. Here $\alpha_1^\vee(t) = \mathrm{diag}(t,t^{-1},t^{-1},t)$, $\alpha_2^\vee(t) = \mathrm{diag}(1,t,1,t^{-1})$. $\alpha_3^\vee = \alpha_1^\vee + 2\alpha_2^\vee$, $\alpha_4^\vee = \alpha_1^\vee + \alpha_2^\vee$. We have $\langle \alpha_i, \alpha_i^\vee \rangle = 2$, for $i = 1,2$. $\langle \alpha_1, \alpha_2^\vee \rangle = -1$, $\langle \alpha_2, \alpha_1^\vee \rangle = -2$.

Since G is simply connected, $X(T) = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$ and $X^*(T) = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee$. Set $\mathfrak{a}^* = X(T) \otimes \mathbb{R}$, $\mathfrak{a}_\mathbb{C}^* = X(T) \otimes \mathbb{C}$, and $\mathfrak{a} = X^*(T) \otimes \mathbb{R} = \mathrm{Hom}(X(T), \mathbb{R})$, $\mathfrak{a}_\mathbb{C} = X^*(T) \otimes \mathbb{C}$. Then $\{\beta_1, \beta_2\}$ and $\{\alpha_1^\vee, \alpha_2^\vee\}$ are the pair of dual bases for $\mathfrak{a}^*$ and $\mathfrak{a}$. The positive Weyl chamber in $\mathfrak{a}^*$ is

$$\begin{aligned} C^+ &= \{\Lambda \in \mathfrak{a}^* \mid \langle \Lambda, \alpha^\vee \rangle > 0, \quad \text{for all } \alpha \text{ positive roots}\} \\ &= \{a\beta_1 + b\beta_2 \mid a, b > 0\}. \end{aligned}$$

Let σ (respectively τ) be the simple reflection in the hyperplane orthogonal to α_1 (α_2 respectively). Then the Weyl group is given by

$$W = \{1, \sigma, \tau, \tau\sigma, \sigma\tau, \sigma\tau\sigma, \tau\sigma\tau, \sigma\tau\sigma\tau\}.$$

Let K_∞ be the standard maximal compact subgroup in $G(\mathbb{A}_\infty)$ and $K_v = G(\mathcal{O}_v)$ for finite v. The product $K = K_\infty \times \prod K_v$ is a maximal compact subgroup in $G(\mathbb{A})$.

Let P_1 be the Siegel parabolic subgroup generated by the short root α_1 and P_2 be the non-Siegel parabolic subgroup generated by the long root α_2.

We know that the discrete spectrum of GL_2 is, for ω a grössencharacter of F, trivial on F_∞^+ (F_∞^+ is defined through the identification of $\mathbb{A}_F^\times = (\mathbb{A}_F^\times)^1 \times F_\infty^+$),

$$L_d^2(GL_2, \omega) = L_0^2(GL_2, \omega) \oplus L_{sp}^2,$$

where $L_0^2(GL_2, \omega)$ is the space of cuspidal representations with the central character ω and L_{sp}^2 is the space spanned by the functions $\chi(\det g)$, $\chi^2 = \omega$. (See, for example, [4].)

3. Decomposition of $L_d^2(P)$ for the Siegel parabolic subgroup

We have a Levi decomposition $P = P_1 = MN$,

$$M(F) = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \mid A \in GL_2(F) \right\}, \quad N(F) = \left\{ \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \mid S = {}^tS \right\}.$$

Let $\mathfrak{a}_P^* = X(M) \otimes \mathbb{R} = \mathbb{R}\beta_2$, $\mathfrak{a}_P = \mathbb{R}\beta_2^\vee$ and ρ_P be the half sum of roots generating N, i.e., $\rho_P = \frac{3}{2}\beta_2$. Let $\tilde{\alpha} = \beta_2$ and identify $s \in \mathbb{C}$ with $s\tilde{\alpha} \in \mathfrak{a}_\mathbb{C}^*$. Here we follow the conventions of Shahidi [27]. Let $\pi = \otimes\pi_v$ be a cuspidal representation of $M = GL_2$. Given a K-finite function φ in the space of π, we shall extend φ to a function $\tilde{\varphi}$ on $G(\mathbb{A})$ and set $\Phi_s(g) = \tilde{\varphi}(g)\exp\langle s + \rho_P, H_P(g) \rangle$, where H_P is the Harish–Chandra homomorphism (see Shahidi [27, p. 551]). Define an Eisenstein series

$$E(s, \tilde{\varphi}, g, P) = \sum_{\gamma \in P(F)\backslash G(F)} \Phi_s(\gamma g).$$

It is known (Langlands [18]) that $E(s, \tilde{\varphi}, g, P)$ converges for $\mathrm{Re}\, s \gg 0$ and extends to a meromorphic function of s in $\mathbb{C}$, with only a finite number of poles in the plane $\mathrm{Re}\, s \geqslant 0$, all simple and on the real axis since the central character ω of π is trivial on F_∞^+. It is also known (Langlands [18]) that $L_d^2(P)$ is spanned by the residues of the Eisenstein series for $\operatorname{Re} s > 0$. We also know ([18]) that the poles of the Eisenstein series coincide with those of its constant terms. It is enough to consider the constant term along P. The constant term of $E(s, \tilde\varphi, g, P)$ along P is

$$E_0(s, \tilde\varphi, g, P) = \sum_{w \in \Omega_1} M(s, \pi, w)\Phi_s(g),$$

where $\Omega_1 = \{1, \tau\sigma\tau\}$ and for $f \in I(s, \pi) = \operatorname{Ind}_P^G \pi \otimes \exp(s, H_P(\ ))$, and $\operatorname{Re} s \gg 0$,

$$M(s, \pi, w)f(g) = \int_{N_w(\mathbb{A})} f(w^{-1}ng)\, dn,$$

where $N_w = U \cap w\overline{N}w^{-1}$, where $\overline{N}$ is the unipotent radical opposed to N (i.e., generated by negative roots in N). We note that for each s, the representation of $G(\mathbb{A})$ on the space of Φ_s is equivalent to $I(s, \pi)$. We know that

$$M(s, \pi, w) = \otimes A(s, \pi_v, w), \quad A(s, \pi_v, w) f_v(g) = \int_{N_w(F_v)} f_v(w^{-1}ng)\, dn,$$

where $f = \otimes f_v$, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$ for almost all v.

Let ${}^L M = GL_2(\mathbb{C})$ be the L-group of M. Denote by r the adjoint action of ${}^L M$ on the Lie algebra ${}^L\mathfrak{n}$ of ${}^L N$, the L-group of N. Then $r = r_1 \oplus r_2$, $r_1 = \rho_2$, $r_2 = \wedge^2 \rho_2$, where ρ_2 is the standard representation of $GL_2(\mathbb{C})$ (see Shahidi [28, p. 287]). Since $M(s, \pi, w)$ is the identity for $w = 1$, it is enough to consider $M(s, \pi, w)$ for w the longest element in Ω_1, i.e., $w = \tau\sigma\tau$. Then it is well known (Shahidi [27, p. 554]) that

$$M(s, \pi, w) f = \bigotimes_{v \in S} A(s, \pi_v, w) f_v \otimes \bigotimes_{v \notin S} \times \frac{L_S(s, \pi, \tilde r_1)}{L_S(1+s, \pi, \tilde r_1)} \frac{L_S(2s, \pi, \tilde r_2)}{L_S(1+2s, \pi, \tilde r_2)},$$

where S is a finite set of places of F, including all the archimedean places such that for every $v \notin S$, π_v is a class 1 representation and if $f = \otimes f_v$, for $v \notin S$, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$. Here $\tilde r_i$ denotes the contragredient of r_i for each i and $\tilde f_v$ is the K_v-fixed function in the space of $I(-s, w(\pi_v))$. Finally $L_S(s, \pi, \tilde r_i) = \prod_{v \notin S} L(s, \pi_v, \tilde r_i)$, where $L(s, \pi_v, \tilde r_i)$ is the local Langlands' L-function attached to π_v and r_i (see Shahidi [27, p. 554]).

(A) Analysis of $L_S(s, \pi, \tilde r_1)$.

For $v \notin S$, $\pi_v = \pi_{\mu_1, \mu_2}$,

$$L(s, \pi_v, \tilde r_1) = (1 - \mu_1(\varpi) q_v^{-s})^{-1} (1 - \mu_2(\varpi) q_v^{-s})^{-1} = L(s, \pi_v),$$

where ϖ is a uniformizing parameter. So $L_S(s, \pi, \tilde r_1)$ is exactly the (partial) Jacquet–Langlands L-function. We know ([11]) that it is absolutely convergent for $\operatorname{Re} s > 1$ and hence it has no zero there. We also know ([11]) that the completed L-function $L(s, \pi, \tilde r_1)$ can be continued to an entire function in s and so it has no pole for $\operatorname{Re} s > 0$.

(B) Analysis of $L_S(s,\pi,\tilde{r}_2)$.

For $v \notin S$,

$$L(s,\pi_v,\tilde{r}_2) = (1-\omega_{\pi_v}(\varpi)q_v^{-s})^{-1} = L(s,\omega_{\pi_v}).$$

So $L_S(s,\pi,\tilde{r}_2)$ is the (partial) Hecke L-function. We know that it has no zero for $\operatorname{Re} s > 1$. We also know ([34]) that the completed L-function $L(s,\pi,\tilde{r}_2)$ has a pole for $\operatorname{Re} s > 0$ if and only if $s = 1$ and $\omega_\pi = 1$.

(C) Analysis of $A(s,\pi_v,w)$ for $v \in S$.

We will show

PROPOSITION 3.1. *For each* $v \in S$,

$$L(s,\pi_v)^{-1}L(2s,\omega_\pi)^{-1}A(s,\pi_v,w)$$

can be continued to holomorphic function for $\operatorname{Re} s > 0$.

If π_v is a tempered representation, we know that $A(s,\pi_v,w)$ is holomorphic for $\operatorname{Re} s > 0$ due to Harish–Chandra (Theorems 5.3 and 5.4 in [29]). In this case, we can see easily from Jacquet–Langlands theory ([4, p. 113]) that $L(s,\pi_v)^{-1}L(2s,\omega_v)^{-1}$ is holomorphic and non-zero for $\operatorname{Re} s > 0$. This proves the proposition in the tempered case. So it is enough to consider the case where π_v is complementary series representations. Let $\pi_v = \pi_{\mu_1,\mu_2}$ with μ_1,μ_2 characters of $F^\times$. Let $\chi(\mu_1,\mu_2)$ be the character of T defined by $\chi(\mu_1,\mu_2)(t(a,b)) = \mu_1(a)\mu_2(b)$. Then we have

$$\operatorname{Ind}_{P_v}^{G_v}\pi_v \otimes \exp(s,H_P(\ )) = \operatorname{Ind}_{B_v}^{G_v}\chi(\mu_1,\mu_2)\otimes\exp(s\tilde{\alpha},H_B(\ )). \tag{3.1}$$

Let $\mu_1 = c_1|\ |_v^{s_1}$ and $\mu_2 = c_2|\ |_v^{s_2}$, where c_1, c_2 are unitary characters of $\mathcal{O}_v^\times$ and $0 \leqslant \operatorname{Im} s_i < \frac{2\pi i}{\log q_v}$. Let $\eta(t(a,b)) = c_1(a)c_2(b)$, $\nu(t(a,b)) = |a|_v^{s_1}|b|_v^{s_2}|ab|_v^{s}$. Then η is a character of $T(F_v)\cap K_v$. Then it is enough to consider the intertwining operator $A(\nu,\eta;\tau\sigma\tau)$. We have

LEMMA. $A(\nu,\eta;\tau\sigma\tau) = A(\sigma\tau\nu,\sigma\tau\eta;\tau)A(\tau\nu,\tau\eta;\sigma)A(\nu,\eta;\tau)$.

By Winarsky [32, p. 952], we know that if w_α is a simple reflection, $(1-q_v^{-s_\alpha})A(\nu,\eta;w_\alpha)$ is entire for η_α trivial and $A(\nu,\eta;w_\alpha)$ is entire for η_α non-trivial, where $\eta_\alpha(t) = \eta\circ\alpha^\vee(t)$, $\nu_\alpha(t) = \nu\circ\alpha^\vee(t) = |t|_v^{s_\alpha}$. Now we can see that $\nu_{\alpha_2}(t) = |t|_v^{s+s_2}$, $\eta_{\alpha_2}(t) = c_2(t)$; $(\tau\nu)_{\alpha_1}(t) = |t|_v^{2s+s_1+s_2}$, $(\tau\eta)_{\alpha_1}(t) = c_1(t)c_2(t)$; $(\sigma\tau\nu)_{\alpha_2}(t) = |t|_v^{s+s_1}$, $(\sigma\tau\eta)_{\alpha_2}(t) = c_1(t)$. From these, we can see at once our assertion for p-adic cases.

For real places, Shahidi [26, p. 110] showed that $\Gamma(\frac{1}{2}(\nu_\alpha+\epsilon_\alpha))A(\nu,\eta;w_\alpha)$ is entire, where $\eta_\alpha(t) = \operatorname{sgn}(t)^{\epsilon_\alpha}$, $\epsilon_\alpha = 0,1$. The complex places go in the same way. Therefore again these prove Proposition 3.1.

(D) Conclusion.

In conclusion, $E(s,\tilde{\varphi},g,P)$ has a pole in the half plane $\operatorname{Re} s > 0$ if and only if $\omega_\pi = 1$ and $s = \frac{1}{2}$ and $L(\frac{1}{2},\pi) \neq 0$. We know the following

PROPOSITION 3.2. *For each* v, *the image of* $A(\frac{1}{2},\pi_v,\tau\sigma\tau)$ *is irreducible.*

Proof. If π_v is tempered this is well known by Langlands' Classfication Theorem ([2, Lemma 4.5]) since $\tau\sigma\tau$ is the longest element in the Weyl group of the split component of M in G modulo that in M. If $\pi_v = \pi_{\mu_1,\mu_2}$ is a complementary series representation, then $|\mu_1(x)| = |x|^r$, $|\mu_2(x)| = |x|^{-r}$, $0 < r < \frac{1}{2}$. By (3.1),

$$A\left(\frac{1}{2}, \pi_v, w\right) = A(\nu, \eta; \tau\sigma\tau),$$

where $\nu = \exp(\Lambda, H_B(\ ))$ with $\Lambda = (s_1 - s_2)\beta_1 + (\frac{1}{2} - s_2)\beta_2$, $\mathrm{Re}\,(s_1 - s_2) = 2r$, $\mathrm{Re}\,(\frac{1}{2} - s_2) = \frac{1}{2} - r$. So $\mathrm{Re}\,\Lambda$ is in the positive Weyl chamber.

It is enough to consider the image of $A(\nu, \eta; \tau\sigma\tau)$, which is an intertwining operator associated to the Borel subgroup. Here $\tau\sigma\tau$ is not any more the longest element of the Weyl group associated to the Borel subgroup. We need the following observation due to Shahidi.

Observation. Let G be a reductice group and $P = MN$ be a parabolic subgroup. Let $P_0 = M_0N_0 \subset P$ be an another parabolic subgroup with $M_0 \subset M$. Let $I(\Lambda, \pi) = \mathrm{Ind}_P^G\left(\pi \otimes q^{\langle \Lambda, H_P(\,)\rangle}\right)$ and $\pi = I_{M_0}(\Lambda_0, \pi_0) = \mathrm{Ind}_{M_0}^M\left(\pi_0 \otimes q^{\langle \Lambda_0, H_{P_0}^M(\,)\rangle}\right)$ be an irreducible representation of M, where π_0 is a tempered representation of M_0 and Λ_0 is in the corresponding positive Weyl chamber. Then $I(\Lambda, \pi) = I(\tilde{\Lambda} + \Lambda_0, \pi_0)$, where $\tilde{\Lambda}$ extends Λ to $(\mathfrak{a}_0)_{\mathbb{C}}^*$, i.e., $\langle \tilde{\Lambda}, H_{P_0}(a)\rangle = \langle \Lambda, H_P(a)\rangle$ for all $a \in A_{P_0}$. By inducing in stages and the factorization property of intertwining operators, we have

$$A(\tilde{\Lambda} + \Lambda_0, \pi_0, \tilde{w}) = A_{M_0}(\Lambda_0, \pi_0, w_0)A(\Lambda, \pi, w),$$

where w is the longest element of the Weyl group of the split component of M in G and $\tilde{w}$ is that of M_0 in G and $\tilde{w} = w_0 w$ with w_0 is the longest element of the Weyl group of the split component of M_0 in M. Here the operator $A_{M_0}(\Lambda_0, \pi_0, w_0)$: $I_{M_0}(\Lambda_0, \pi_0) \mapsto I_{M_0}(w\Lambda_0, w\pi_0)$ establishes an isomorphism since $I_{M_0}(\Lambda_0, \pi_0)$ is irreducible, and is identified with its induced map. Therefore, if $\tilde{\Lambda} + \Lambda_0$ is in the positive Weyl chamber, then the image of $A(\Lambda, \pi, w)$ is irreducible since the image of $A(\tilde{\Lambda} + \Lambda_0, \pi_0, \tilde{w})$ is irreducible by Langlands' classification theorem ([2, Lemma 4.5]).

In our case, from (3.1), $\pi = \pi_{\mu_1,\mu_2} = \mathrm{Ind}_{B_0}^{GL_2}\left(\pi_0 \otimes q^{\langle \Lambda_0, H_{B_0}(\,)\rangle}\right)$, where $\Lambda_0 = r\alpha_1$. Also $\tilde{\Lambda} = \frac{1}{2}\tilde{\alpha} = \frac{1}{2}\beta_2$. So $\tilde{\Lambda} + \Lambda_0 = 2r\beta_1 + (\frac{1}{2} - r)\beta_2$ belongs to the positive Weyl chamber. This proves Proposition 3.2.

The residue of $E(s, \tilde{\varphi}, g, P)$ at $s = \frac{1}{2}$ is a certain automorphic form $H(g, \tilde{\varphi})$, concentrated on the class of P (see Jacquet [10, p. 187] for the definition). In order to prove that $H(g, \tilde{\varphi})$ is square integrable, we write the intertwining operators $M(s, \pi, w)$ as follows:

$$M(s, \pi, w)\Phi_s(g) = T(s, \pi, w)\tilde{\varphi}(g) \exp\left\{\langle w(s\beta_2) + \rho_P, H_P(g)\rangle\right\}.$$

We normalize $T(s, \pi_v, w)$ by

$$\begin{aligned} T(s, \pi_v, w) &= N(s, \pi_v, w) r_v(w), \\ r_v(\tau\sigma\tau) &= \frac{L(s, \pi_v, \tilde{r}_1)}{L(1+s, \pi_v, \tilde{r}_1)} \frac{L(2s, \pi_v, \tilde{r}_2)}{L(1+2s, \pi_v, \tilde{r}_2)}. \end{aligned}$$

Let $T(s,\pi,w)=\bigotimes_v T(s,\pi_v,w)$. Then the constant term of $H(g,\tilde{\varphi})$ is given by

$$(\text{constant})\left[T\left(\frac{1}{2},\pi,\tau\sigma\tau\right)\tilde{\varphi}\right](g)\exp\left\{\left\langle\tau\sigma\tau\frac{1}{2}\beta_2+\rho_P,H_P(g)\right\rangle\right\}.$$

Since $\tau\sigma\tau\beta_2=-\beta_2$, Langlands' square integrability criterion shows that $H(g,\tilde{\varphi})$ is square integrable (see [10, p. 187] or [18, p. 104]). Furthermore, the space spanned by the constant terms is irreducible. So the same is true of the space spanned by the $H(g,\tilde{\varphi})$. Denote it by $B(\pi)$. Then we have proved

THEOREM 3.3. *For P Siegel parabolic,*

$$L_d^2(P)=\bigoplus_{\pi}B(\pi),$$

where π runs over cuspidal representations of GL_2 with trivial central characters and $L(\frac{1}{2},\pi)\neq 0$.

Remark 3.1. We don't know whether the multiplicity of $B(\pi)$ in $L_d^2(P)$ is one.

Remark 3.2. Let's look at Fourier coefficients of the highest rank terms. By Shahidi [27], for χ a non-degenerate character of $U(F)\backslash U(\mathbb{A})$, the highest rank terms of the Fourier coefficients are given by

$$E_\chi(s,\tilde{\varphi},e,P)=\prod_{v\in S}W_v(e_v)L_S(1+s,\pi,\tilde{r}_1)^{-1}L_S(1+2s,\pi,\tilde{r}_2)^{-1}.$$

They are holomorphic for $\operatorname{Re}s>0$. Therefore, $B(\pi)$ is not generic in the sense of [27, p. 555].

4. Decomposition of $L_d^2(P)$ for the Non-Siegel Maximal Parabolic Subgroup

We have a Levi decomposition $P=MN$;

$$M=\left\{\begin{pmatrix} x & 0 & 0 & 0\\ 0 & a & 0 & b\\ 0 & 0 & x^{-1} & 0\\ 0 & c & 0 & d\end{pmatrix}\,\middle|\, x\in F^\times,\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(F)\right\},$$

i.e., $M=GL_1\times SL_2$. In this case, $\mathfrak{a}_P^*=X(M)\otimes\mathbb{R}=\mathbb{R}\beta_1$, $\mathfrak{a}_P=\mathbb{R}(\beta_1{}^\vee)$ and $\rho_P=$the half sum of roots generating N, i.e., $\rho_P=2\beta_1$. Also $\tilde{\alpha}=\beta_1$ and we identify $s\in\mathbb{C}$ with $s\tilde{\alpha}\in\mathfrak{a}_{\mathbb{C}}^*$.

We know that the discrete spectrum of SL_2 is

$$L_d^2(SL_2)=L_0^2(SL_2)\oplus L_{co}^2,$$

where $L_0^2(SL_2)$ is the space of cuspidal representations and L_{co}^2 is the space of constant functions.

For π a cuspidal representation of $M = GL_1 \times SL_2$, $\pi = \theta \otimes \sigma$, where σ is a cuspidal representation of SL_2 and θ is a grössencharacter of F. We define the Eisenstein series exactly the same way as in the Siegel parabolic case. In this case, the constant term along P is given by

$$E_0(s, \varphi, g, P) = \sum_{w \in \Omega_2} M(s, \pi, w) f(g),$$

where $\Omega_2 = \{1, \sigma\tau\sigma\}$. Since $M = GL_1 \times SL_2$, ${}^L M = GL_1(\mathbb{C}) \times SO_3(\mathbb{C})$ and $r_1 = \rho_1 \otimes \tau_0$, where ρ_1 is the standard representation of $GL_1(\mathbb{C})$ and τ_0 is the standard representation of $SO_3(\mathbb{C})$. Then for $w = \sigma\tau\sigma$,

$$M(s, \pi, w) f = \bigotimes_{v \in S} A(s, \pi_v, w) f_v \otimes \bigotimes v \notin S \tilde{f}_v \times \frac{L_S(s, \pi, \tilde{r}_1)}{L_S(1 + s, \pi, \tilde{r}_1)},$$

where S is a finite set of places F, including all the archimedean places, such that for every $v \notin S$, π_v is a class 1 representation and if $f = \otimes f_v$, for $v \notin S$, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$. Here $L_S(s, \pi, \tilde{r}_1) = \prod_{v \notin S} L(s, \pi_v, \tilde{r}_1)$, where $L(s, \pi_v, \tilde{r}_1)$ is the local Langlands' L-function attached to π_v and r_1 (see Shahidi [27, p. 554]).

(A) Analysis of $L_S(s, \pi, \tilde{r}_1)$. For $v \notin S$, σ_v is a class 1 representation of $SL_2(F_v)$. Then $\sigma_v \subset \tilde{\sigma}_v|_{SL_2(F_v)}$, where $\tilde{\sigma}_v = \pi_{\mu_v, \nu_v}$, is a class 1 representation of $GL_2(F_v)$. Then

$$\begin{aligned} L(s, \pi_v, \tilde{r}_1) &= \left(1 - \mu_v \nu_v^{-1} \theta_v(\varpi) q_v^{-s}\right)^{-1} \left(1 - \theta_v(\varpi) q_v^{-s}\right)^{-1} \left(1 - \mu_v^{-1} \nu_v \theta_v(\varpi) q_v^{-s}\right)^{-1} \\ &= L_2(s, \tilde{\sigma}_v, \theta_v) = L(s, \Sigma_v \otimes \theta_v), \end{aligned}$$

where $L_2(s, \tilde{\sigma}_v, \theta_v)$ is the Adjoint square L-function and Σ_v is the Gelbart–Jacquet lift of $\tilde{\sigma}_v$ to GL_3 (see [5]). Here the L-function does not depend on the choice of $\tilde{\sigma}$ due to the following lemma ([3, Lemma 1.9.2]).

LEMMA. *Every automorphic representation σ of SL_2 is contained in an automorphic representation $\tilde{\sigma}$ of GL_2. If $\tilde{\sigma}$ and $\tilde{\sigma}_1$ contain σ, then $\tilde{\sigma}_1 = \tilde{\sigma} \otimes \omega$ for a character ω of $\mathbb{A}_F^{\times}$.*

It is well known that $L_S(s, \pi, \tilde{r}_1) = L_S(s, \Sigma \otimes \theta)$ converges absolutely for $\operatorname{Re} s > 1$ and therefore it has no zero there (see, for example, [6, p. 69]).

By Gelbart–Jacquet [5, Theorem 9.3], we know that $L_2(s, \tilde{\sigma}, \theta)$ is entire for any θ, if σ is not monomial. We say that a cuspidal representation σ of SL_2 is monomial if a cuspidal representation $\tilde{\sigma}$ of GL_2 is monomial, where $\sigma \subset \tilde{\sigma}|_{SL_2}$. Recall that a cuspidal representation $\tilde{\sigma}$ on GL_2 is called monomial if $\tilde{\sigma} \simeq \tilde{\sigma} \otimes \eta$ for a grössencharacter η of F, $\eta^2 = 1$, $\eta \neq 1$. In this case, η determines a quadratic extension E of F. Then there exists a grössencharacter Ω of E such that $\tilde{\sigma}$ is the automorphic representation $\pi(\Omega)$ of GL_2 attached to Ω (see [5, p. 491] for more details). Let Ω' be the conjugate of Ω. Then Ω satisfies $\Omega \neq \Omega'$, because $\pi(\Omega)$ is not cuspidal if $\Omega = \Omega'$. The Gelbart-Jacquet lift Σ of $\tilde{\sigma}$ is

$$\Sigma = \operatorname{Ind}_P^{GL_3} \left(\pi(\Omega \Omega'^{-1}) \otimes \eta\right),$$

where P is the standard maximal parabolic subgroup of GL_3 of type $(2,1)$. If $\Omega\Omega'^{-1} = \tilde{\eta}\circ N_{E/F}$, where $\tilde{\eta}$ is a grössencharacter of F, then $\pi(\Omega\Omega'^{-1})$ is not cuspidal and

$$L(s,\Sigma\otimes\theta) = L(s,\tilde{\eta}\theta)L(s,\tilde{\eta}\theta\eta)L(s,\theta\eta).$$

Here we note that $\Omega\Omega'^{-1}$ is of order two and so $\tilde{\eta}$ is a quadratic character. By [16, p. 774], there are three different pairs (η,Ω), (η_1,Ω_1) and (η_2,Ω_2) such that $\tilde{\sigma} = \pi(\Omega) = \pi(\Omega_1) = \pi(\Omega_2)$.* But by [11, p. 397], $\tilde{\sigma} = \tilde{\sigma}\otimes\tilde{\eta}$. Therefore, η, $\tilde{\eta}$ and $\eta\tilde{\eta}$ are exactly such quadratic characters. So $L(s,\Sigma\otimes\theta)$ has a simple pole at $s=1$ if θ is one of the above three characters and $L(s,\Sigma\otimes\theta)$ is independent of the choice of data (η,Ω).

Otherwise, $\pi(\Omega\Omega'^{-1})$ is a cuspidal representation and

$$L(s,\Sigma\otimes\theta) = L(s,\pi(\Omega\Omega'^{-1})\otimes\theta)L(s,\theta\eta).$$

Here we note that in this case η is uniquely determined by σ ([16, p. 774]). So, $L(s,\Sigma\otimes\theta)$ has a simple pole at $s=1$ if $\theta=\eta$.

(B) Analysis of $A(s,\pi_v,w)$ for $v\in S$. If π_v is a tempered representation, we know that $A(s,\pi_v,w)$ is holomorphic for $Re\, s>0$. In this case, we can see from Gelbart–Jacquet [5] that $L(s,\pi_v,\tilde{r}_1)$ is holomorphic and non-zero for $\mathrm{Re}\, s>0$. Therefore

$$L(s,\pi_v,\tilde{r}_1)^{-1}A(s,\pi_v,w)$$

is holomorphic for $\mathrm{Re}\, s>0$.

Let's consider the case π_v non-tempered. Then $\pi_v = \theta_v\otimes\sigma_v$, $\sigma_v\subset\tilde{\sigma}\mid_{GL_2(F_v)}$, where $\tilde{\sigma} = \pi_{\mu_1,\mu_2}$ is a complementary series representation of $GL_2(F_v)$. Then

$$\mathrm{Ind}_{P_v}^{G_v}\pi_v\otimes\exp(s,H_P(\,)) = \mathrm{Ind}_{B_v}^{G_v}\chi(\theta_v,\mu_1\mu_2^{-1})\otimes\exp(s\tilde{\alpha},H_B(\,)).$$

So in the same way as in Siegel parabolic case,

$$L(s,\pi_v,\tilde{r}_1)^{-1}A(s,\pi_v,w)$$

is holomorphic for $\mathrm{Re}\, s>0$.

(C) Conclusion. $E(s,\varphi,g,P)$ has a pole in the plane $\mathrm{Re}\, s>0$ if and only if $s=1$ and $\pi=\theta\otimes\sigma$ where σ is a monomial representation and θ is determined by σ as in Theorem 4.1 below. In that case, as in the Siegel parabolic case, the residues at $s=1$ form an irreducible component of $L_d^2(P)$. Denote it by $B(\pi)$. Then we have proved

THEOREM 4.1. *For P non-Siegel maximal parabolic,*

$$L_d^2(P) = \bigoplus_{\pi} B(\pi),$$

where $\pi=\theta\otimes\sigma$, σ runs over monomial representations and θ is determined by σ, i.e., if $\pi(\Omega\Omega'^{-1})$ is a cuspidal representation, then $\theta=\eta$; if $\Omega\Omega'^{-1}=\tilde{\eta}\circ N_{E/F}$, then $\theta=\eta$, $\tilde{\eta}$ or $\eta\tilde{\eta}$ (the three quadratic characters determined by $\tilde{\sigma}$).

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* Thanks are due to the referee who pointed this out.

Remark 4.1. We do not know whether the multiplicity of $B(\pi)$ in $L^2_d(P)$ is one.

Remark 4.2. As in the Siegel parabolic case, $B(\pi)$ is not generic in the sense of [27, p. 555].

5. Decomposition of $L^2_d(B)$ for the Borel Subgroup B

We fix an additive character $\psi = \otimes_v \psi_v$ of $\mathbb{A}/F$ and let $\xi(z,\mu)$ be the Hecke L-function with the ordinary Γ-factor so that it satifies the functional equation $\xi(z,\mu) = \epsilon(z,\mu)\xi(1-z,\mu^{-1})$, where $\epsilon(z,\mu) = \prod_v \epsilon(z,\mu_v,\psi_v)$ is the usual ϵ-factor (see, for example, [34, p. 158]). If μ is the trivial character μ_0, then we write simply $\xi(z)$ for $\xi(z,\mu_0)$. We have the Laurent expansion of $\xi(z)$ at $z=1$:

$$\xi(z) = \frac{c(F)}{z-1} + a + \cdots. \tag{5.1}$$

For grössencharacters μ, ν of F, we define a character $\chi = \chi(\mu,\nu)$ of $T(\mathbb{A})$ by $\chi(\mu,\nu)(t(a,b)) = \mu(a)\nu(b)$. Let $I(\chi)$ be the space of functions Φ on $G(\mathbb{A})$ satisfying $\Phi(ntg) = \chi(t)\Phi(g)$ for any $u \in U(\mathbb{A})$, $t \in T(\mathbb{A})$ and $g \in G(\mathbb{A})$. Then for each $\Lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the representation of $G(\mathbb{A})$ on the space of functions of the form

$$g \mapsto \Phi(g)\exp\langle \Lambda + \rho_B, H_B(g)\rangle, \quad \Phi \in I(\chi),$$

is equivalent to $I(\Lambda,\chi) = \mathrm{Ind}_B^G \chi \otimes \exp(\Lambda, H_B())$, where ρ_B is the half-sum of positive roots, i.e., $\rho_B = \beta_1 + \beta_2$. We form the Eisenstein series:

$$E(g,f,\Lambda) = \sum_{\gamma \in B(F)\backslash G(F)} f(\gamma g),$$

where $f = \Phi\, \mathrm{e}^{\langle \Lambda + \rho_B, H_B()\rangle} \in I(\Lambda,\chi)$. The Eisenstein series converges absolutely for $\mathrm{Re}\,\Lambda \in C^+ + \rho_B$ and extends to a meromorphic function of Λ. It is an automorphic form and the constant term of $E(g,f,\Lambda)$ along B is given by

$$E_0(g,f,\Lambda) = \int_{U(F)\backslash U(\mathbb{A})} E(ug,f,\Lambda)\,\mathrm{d}u = \sum_{w \in W} M(w,\Lambda,\chi)f(g),$$

where W is the Weyl group and for sufficiently regular Λ,

$$M(w,\Lambda,\chi)f(g) = \int_{U_w(\mathbb{A})} f(w^{-1}ug)\,\mathrm{d}u,$$

where $U_w = U \cap w\overline{U}w^{-1}$, $\overline{U}$ is the unipotent radical opposed to U. Then $M(w,\Lambda,\chi)$ defines a linear map from $I(\Lambda,\chi)$ to $I(w\Lambda,w\chi)$ and satisfies the functional equation of the form

$$M(w_1w_2,\Lambda,\chi) = M(w_1,w_2\Lambda,w_2\chi)M(w_2,\Lambda,\chi).$$

And the Eisenstein series satisfies the functional equation

$$E(g,M(w,\Lambda,\chi)f,w\Lambda) = E(g,f,\Lambda). \tag{5.2}$$

Let S be a finite set of places of F, including all the archimedean places such that for every $v \notin S$, χ_v, ψ_v are unramified and if $f = \otimes f_v$ for $v \notin S$, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$. We have

$$M(w, \Lambda, \chi) = \otimes_v A(w, \Lambda, \chi_v).$$

Then by applying Gindikin–Karpelevic method (Langlands [17]), we can see that for $v \notin S$,

$$A(w, \Lambda, \chi_v) f_v = \prod_{\alpha>0, w\alpha<0} \frac{L(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee)}{L(\langle \Lambda, \alpha^\vee \rangle + 1, \chi_v \circ \alpha^\vee)} \tilde{f}_v,$$

where $L(s, \eta_v)$ is the local Hecke L-function attached to a character η_v of $F_v^\times$ and $s \in \mathbb{C}$, and $\tilde{f}_v$ is the K_v-fixed function in the space of $I(w\Lambda, w\chi)$ satisfying $\tilde{f}_v(e_v) = 1$. Let

$$r_v(w) = \prod_{\alpha>0,\ w\alpha<0} \frac{L(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee)}{L(\langle \Lambda, \alpha^\vee \rangle + 1, \chi_v \circ \alpha^\vee) \epsilon(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee, \psi_v)}.$$

We normalize the intertwining operators $A(w, \Lambda, \chi_v)$ for all v by

$$A(w, \Lambda, \chi_v) = r_v(w) R(w, \Lambda, \chi_v).$$

Let $R(w, \Lambda, \chi) = \bigotimes_v R(w, \Lambda, \chi_v)$. $R(w, \Lambda, \chi)$ satisfies the functional equation

$$R(w_1 w_2, \Lambda, \chi) = R(w_1, w_2\Lambda, w_2\chi) R(w_2, \Lambda, \chi).$$

We know, by Winarsky [32] for p-adic cases and Shahidi [26, p. 110] for real and complex cases, that

$$A(w, \Lambda, \chi_v) \prod_{\alpha>0, w\alpha<0} L(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee)^{-1} \tag{5.3}$$

is holomorphic for any v. So for any v, $R(w, \Lambda, \chi_v)$ is holomorphic for Λ with $\mathrm{Re}(\langle \Lambda, \alpha^\vee \rangle) > -1$, for all positive α with $w\alpha < 0$. For $\chi = \chi(\mu, \nu)$, $\chi \circ \alpha_1^\vee = \mu\nu^{-1}$, $\chi \circ \alpha_2^\vee = \nu$, $\chi \circ \alpha_3^\vee = \mu\nu$, $\chi \circ \alpha_4^\vee = \mu$. We list the elements of the Weyl group, together with their actions on the positive roots and on $T(F)$:

w	α_1	α_2	α_3	α_4	$w^{-1}t(a,b)w$
1	α_1	α_2	α_3	α_4	$t(a,b)$
σ	$-\alpha_1$	α_4	α_3	α_2	$t(b,a)$
τ	α_3	$-\alpha_2$	α_1	α_4	$t(a,b^{-1})$
$\sigma\tau$	α_3	$-\alpha_4$	$-\alpha_1$	α_2	$t(b^{-1},a)$
$\tau\sigma$	$-\alpha_3$	α_4	α_1	$-\alpha_2$	$t(b,a^{-1})$
$\sigma\tau\sigma$	$-\alpha_3$	α_2	$-\alpha_1$	$-\alpha_4$	$t(a^{-1},b)$
$\tau\sigma\tau$	α_1	$-\alpha_4$	$-\alpha_3$	$-\alpha_2$	$t(b^{-1},a^{-1})$
$\sigma\tau\sigma\tau$	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$t(a^{-1},b^{-1})$

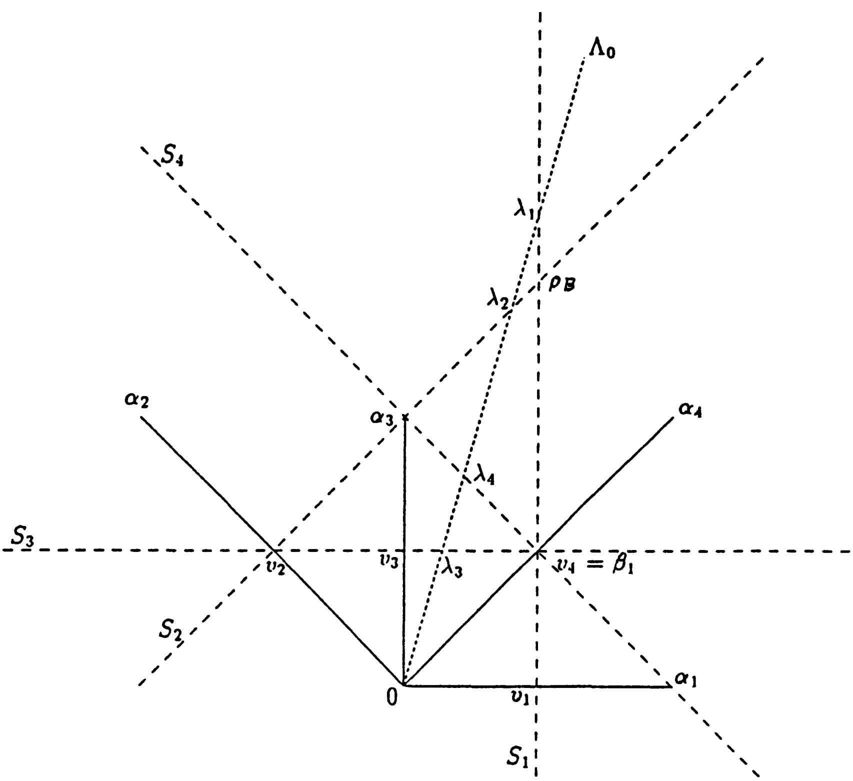
Let $S_i = \{\Lambda \in \mathfrak{a}_{\mathbb{C}}^* | \langle \Lambda, \alpha_i^\vee \rangle = 1\}$ for $i = 1,2,3,4$ and $\Lambda_{ij} = S_i \cap S_j$ for $i \neq j$. We note that $\Lambda_{12} = \rho_B$ and $\Lambda_{13} = \Lambda_{14} = \beta_1 = \frac{\alpha_4}{2}$; $\Lambda_{23} = \frac{\alpha_2}{2}$, $\Lambda_{24} = \beta_2 = \alpha_3$; $\Lambda_{34} = \beta_1$. Each S_i is rewritten as $S_i = \mathbb{C}u_i + v_i$, $1 \leqslant i \leqslant 4$, where $u_1 = \beta_2 = \alpha_3$, $u_2 = \beta_1 = \frac{\alpha_4}{2}$,

$u_3 = \alpha_1$, $u_4 = \alpha_4$ and $v_i = \frac{\alpha_i}{2}$ for $i = 1, 2, 3, 4$. Then we take a coordinate $z_i(\Lambda)$ on S_i as $\Lambda = z_i(\Lambda)u_i + v_i$ for $\Lambda \in S_i$, $1 \leqslant i \leqslant 4$.

Let $\Phi(\Lambda)$ be an entire function of Paley–Wiener type (see [33, p. 257]) on $\mathfrak{a}^*_{\mathbb{C}}$ with values in $I(\chi)$ and $\Phi(\Lambda, g)$ is the value of $\Phi(\Lambda)$ at g. Let $f(\Lambda) = \Phi(\Lambda)e^{\langle \Lambda + \rho_B, H_B()\rangle}$. Langlands' theory (see, for example, [33, p. 257]) says that $L^2(B)$ is generated by

$$\hat{\phi}(g) = \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}\,\Lambda = \Lambda_0} E(g, f(\Lambda), \Lambda)\, \mathrm{d}\Lambda$$

for all such $f(\Lambda)$, where Λ_0 satisfies $\langle \Lambda_0 - \rho_B, \alpha^\vee \rangle > 0$ for all positve roots α. In order to get discrete spectrum, we have to deform the contour $\mathrm{Re}\,\Lambda = \Lambda_0$ to $\mathrm{Re}\Lambda = 0$. Since the poles of the functions $M(w, \Lambda, \chi)$ all lie on S_i which is defined by real equations, we can represent the process of deforming the contour and the singular hyperplanes S_i as dashed lines by the following diagram in the real plane as in [18, Appendix 3].



The integral at $\mathrm{Re}\Lambda = 0$,

$$\frac{1}{(2\pi i)^2} \int_{\mathrm{Re}\Lambda = 0} E(g, f(\Lambda), \Lambda)\, \mathrm{d}\Lambda$$

gives the continuous spectrum of dimension 2. As can be seen in the diagram, if we move the contour along the dotted line indicated we may pick up residues at the points λ_1, λ_2, λ_3, λ_4:

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\,\Lambda=\lambda_i} \mathrm{Res}_{S_i} E(g, f(\Lambda), \Lambda)\,\mathrm{d}\Lambda,$$

where $\Lambda \in S_i$. Then we deform the contours $\mathrm{Re}\Lambda = \lambda_i$ to $\mathrm{Re}\Lambda = v_i$, i.e., to the point where $z_i(\Lambda) = 0$. The integrals at $\mathrm{Re}\Lambda = v_i$,

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\Lambda=v_i} \mathrm{Res}_{S_i} E(g, f(\Lambda), \Lambda)\,\mathrm{d}\Lambda$$

give the continuous spectrum of dimension 1. The square integrable residues which arise during the deformation span the discrete spectrum, i.e., $L_d^2(B)$. As we see in the diagram, we have to consider

$$\begin{aligned}
&\mathrm{Res}_{\beta_1}\mathrm{Res}_{S_1} E(g, f, \Lambda),\\
&\mathrm{Res}_{\rho_B}\mathrm{Res}_{S_1} E(g, f, \Lambda),\\
&\mathrm{Res}_{\alpha_3}\mathrm{Res}_{S_2} E(g, f, \Lambda).
\end{aligned}$$

We set

$$\begin{aligned}
M^i(w, \Lambda, \chi) &= \frac{\xi(2)}{c(F)} \mathrm{Res}_{S_i} M(w, \Lambda, \chi),\\
E^i(g, f, \Lambda) &= \frac{\xi(2)}{c(F)} \mathrm{Res}_{S_i} E(g, f, \Lambda),
\end{aligned}$$

for $\Lambda \in S_i$, $i = 1,2,3,4$. Let $W_i = \{w \in W \mid w\alpha_i < 0\}$ for $i = 1,2,3,4$. Then the constant term $E_0^i(g, f, \Lambda)$ of $E^i(g, f, \Lambda)$ along B is given by

$$E_0^i(g, f, \Lambda) = \sum_{w\in W_i} M^i(w, \Lambda, \chi) f(g).$$

We recall Langlands' square integrability criterion for autormorphic forms through their constant terms in our case ([18, p. 104] or [10, p. 187]). We write the intertwining operator $M(w, \Lambda, \chi)$ with an exponential factor as follows. Let $f = \Phi \mathrm{e}^{\langle \Lambda+\rho_B, H_B()\rangle}$, then we have

$$M(w, \Lambda, \chi) f(g) = T(w, \Lambda, \chi)\Phi(g)\,\mathrm{e}^{\langle w\Lambda+\rho_B, H_B(g)\rangle},$$

where $T(w, \Lambda, \chi)$ is a linear operator from $I(\chi)$ to $I(w\chi)$. Suppose $M^i(w, \Lambda, \chi)$ has a pole at $\Lambda = \beta$ for $w \in W_{i0} \subset W_i$. Then the residue of $E_0^i(g, f, \Lambda)$ at $\Lambda = \beta$ is

$$\begin{aligned}
\mathrm{Res}_\beta E_0^i(g, f, \Lambda) &= \sum_{w\in W_{i0}} \mathrm{Res}_\beta M^i(w, \Lambda, \chi) f(g)\\
&= \sum_{w\in W_{i0}} \mathrm{Res}_\beta T^i(w, \Lambda, \chi)\Phi(g)\mathrm{e}^{\langle w\beta+\rho_B, H_B(g)\rangle}.
\end{aligned}$$

Then we have

LEMMA (Langlands). $\mathrm{Res}_\beta E^i(g,f,\Lambda)$ *is square integrable if and only if* $\mathrm{Re}(w\beta)$ *is in* $\{-a\alpha_1 - b\alpha_2 | a,b>0\}$ *for all* $w \in W_{i0}$.

(1) $E^1(g,f,\Lambda)$. $M(w,\Lambda,\chi)$ has a pole at S_1 only when $\chi\circ\alpha_1^\vee = \mu_0$, i.e., $\mu=\nu$. Then $\chi = \chi(\mu,\mu)$. Here $W_1 = \{\sigma, \tau\sigma, \sigma\tau\sigma, \sigma\tau\sigma\tau\}$. For $\Lambda = z\alpha_3 + \frac{\alpha_1}{2}$, $\langle\Lambda,\alpha_2^\vee\rangle = z-\frac{1}{2}$, $\langle\Lambda,\alpha_3^\vee\rangle = 2z$ and $\langle\Lambda,\alpha_4^\vee\rangle = z+\frac{1}{2}$. Then

LEMMA 5.1.

$$\begin{aligned}
M^1(\sigma,\Lambda,\chi)f &= R(\sigma,\Lambda,\chi)f,\\
M^1(\tau\sigma,\Lambda,\chi)f &= \frac{\xi(z+\frac{1}{2},\mu)}{\xi(z+\frac{3}{2},\mu)}\frac{R(\tau\sigma,\Lambda,\chi)f}{\epsilon(z+\frac{1}{2},\mu)},\\
M^1(\sigma\tau\sigma,\Lambda,\chi)f &= \frac{\xi(2z,\mu^2)\xi(z+\frac{1}{2},\mu)}{\xi(2z+1,\mu^2)\xi(z+\frac{3}{2},\mu)}\frac{R(\sigma\tau\sigma,\Lambda,\chi)f}{\epsilon(2z,\mu^2)\epsilon(z+\frac{1}{2},\mu)},\\
M^1(\sigma\tau\sigma\tau,\Lambda,\chi)f &= \frac{\xi(2z,\mu^2)\xi(z-\frac{1}{2},\mu)}{\xi(2z+1,\mu^2)\xi(z+\frac{3}{2},\mu)}\frac{R(\sigma\tau\sigma\tau,\Lambda,\chi)f}{\epsilon(z-\frac{1}{2},\mu)\epsilon(z+\frac{1}{2},\mu)\epsilon(2z,\mu^2)}.
\end{aligned}$$

We note that for any v, $R(w,\Lambda,\chi_v)$ is holomorphic for $\mathrm{Re}\, z > -\frac{1}{2}$.

PROPOSITION 5.1.1. *If μ is not trivial, then only the residues at $\Lambda=\beta_1$, i.e., $z=\frac{1}{2}$, of $E^1(g,f,\Lambda)$ contribute to $L^2_d(B)$ when $\chi=\chi(\mu,\mu)$ with $\mu^2=\mu_0$.*

Proof. From Lemma 5.1, we can see that $M^1(w,\Lambda,\chi)$ has a pole only at $z=\frac{1}{2}$ when $\mu^2=\mu_0$. Then

$$\begin{aligned}
\mathrm{Res}_{\beta_1} M^1(\sigma\tau\sigma,\Lambda,\chi)f &= \frac{c(F)}{2\xi(2)}\frac{\xi(1,\mu)}{\xi(2,\mu)\epsilon(1,\mu)}R(\sigma\tau\sigma,\beta_1,\chi)f,\\
\mathrm{Res}_{\beta_1} M^1(\sigma\tau\sigma\tau,\Lambda,\chi)f &= \frac{c(F)}{2\xi(2)}\frac{\xi(0,\mu)}{\xi(2,\mu)\epsilon(1,\mu)\epsilon(0,\mu)}R(\sigma\tau\sigma\tau,\beta_1,\chi)f.
\end{aligned}$$

Here by the functional equation, $\xi(0,\mu)=\epsilon(0,\mu)\xi(1,\mu)$. Therefore

$$\begin{aligned}
\mathrm{Res}_{\beta_1} E_0^1(g,f,\Lambda) &= \frac{c(F)}{2\xi(2)}\frac{\xi(1,\mu)}{\xi(2,\mu)\epsilon(1,\mu)}\big(R(\sigma\tau\sigma,\beta_1,\chi)f + \\
&\quad + R(\sigma\tau\sigma\tau,\beta_1,\chi)f\big). \qquad (5.4)
\end{aligned}$$

By Langlands' square integrability criterion, $\mathrm{Res}_{\beta_1} E^1(g,f,\Lambda)$ is square integrable.

PROPOSITION 5.1.2. *Suppose $\mu=\mu_0$ is trivial. Then $E^1(g,f,\Lambda)$ has a pole at $\Lambda=\rho_B$ and $\Lambda=\beta_1$. Furthermore,*

(i) $\mathrm{Res}_{\rho_B} E^1(g,f,\Lambda)$ *is constant, and*

(ii) $\mathrm{Res}_{\beta_1} E^1(g,f,\Lambda)$ *is not square integrable.*

Proof. From Lemma 5.1, we see that $M^1(w,\Lambda,\chi)$ can have a pole at $z=\frac{1}{2}$, i.e., $\Lambda=\beta_1$ and at $z=\frac{3}{2}$, i.e., $\Lambda=\rho_B$.

(i) $\Lambda = \rho_B$. At $z = \frac{3}{2}$, only $M^1(\sigma\tau\sigma\tau, \Lambda, \chi)$ has a simple pole. So

$$\mathrm{Res}_{\rho_B} E_0^1(g, f, \Lambda) = \frac{c(F)}{\xi(4)} \bigotimes_v R(\sigma\tau\sigma\tau, \rho_B, \chi_v) f_v(g).$$

The map $f \mapsto \mathrm{Res}_{\rho_B} E^1(g, f, \Lambda)$ defines an intertwining map from $I(\rho_B, \chi)$ to $L_d^2(B)$. Since χ is trivial, the induced representation $I(\rho_B, \chi)$ has a K-fixed vector f_0 which is cyclic. For f_0, $\mathrm{Res}_{\rho_B} E_0^1(g, f_0, \Lambda)$ is constant. Here $\mathrm{Res}_{\rho_B} E^1(g, f_0, \Lambda)$ is orthogonal to all cusp forms. So it is constant. Since f_0 is cyclic for the whole $I(\rho_B, \chi)$, we see that $\mathrm{Res}_{\rho_B} E^1(g, f, \Lambda)$ is constant for all f.

(ii) $\Lambda = \beta_1$. At $z = \frac{1}{2}$, $M^1(\tau\sigma, \Lambda, \chi)$ has a simple pole. But since $\tau\sigma\beta_1 = -\frac{\alpha_2}{2}$, the residue from $M^1(\tau\sigma, \Lambda, \chi)$ is not square integrable. So it is enough to show the following two lemmas.

LEMMA 5.1.3. *At $z = \frac{1}{2}$, $M^1(\sigma\tau\sigma, \Lambda, \chi)$ and $M^1(\sigma\tau\sigma\tau, \Lambda, \chi)$ may have double poles. But the double poles cancel each other, i.e., the order of the pole at $z = \frac{1}{2}$ of $E^1(g, f, \Lambda)$ is at most one.*

Proof.

$$\lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right)^2 E_0^1(g, f, \Lambda) = \frac{c(F)^2}{2\xi(2)^2} \big(R(\sigma\tau\sigma, \beta_1, \chi) f - R(\sigma\tau\sigma\tau, \beta_1, \chi) f\big).$$

Here $R(\tau, \beta_1, \chi)$ is the identity since χ is the trivial character. So the above expression is zero by the cocycle relation.

LEMMA 5.1.4. *If a function f satisfies $\mathrm{Res}_{\beta_1} M^1(\tau\sigma, \Lambda, \chi) f = 0$, then $\mathrm{Res}_{\beta_1}(M^1(\sigma\tau\sigma, \Lambda, \chi) f + M^1(\tau\sigma\tau\sigma, \Lambda, \chi) f) = 0$.*

Proof. Since $\mathrm{Res}_{\beta_1} M^1(\tau\sigma, \Lambda, \chi) f = (*) R(\tau\sigma, \beta_1, \chi) f(\beta_1) = 0$, we have $R(\tau\sigma, \beta_1, \chi) f(\beta_1) = 0$. Consider $f = f(\Lambda)$ as a function of z since $\Lambda = z\alpha_3 + \frac{\alpha_1}{2}$ and we have the Taylor expansion of f at $z = \frac{1}{2}$: $f(\Lambda) = f(\beta_1) + (z - \frac{1}{2}) Df(\beta_1) + \cdots$, where $Df(\beta_1)$ is the derivative of f at $z = \frac{1}{2}$. Recall the Laurent expansion (5.1) of $\xi(z)$. We have

$$\frac{\xi(2z)}{\xi(2z+1)} = \frac{a_1}{z - \frac{1}{2}} + a_2 + \cdots,$$

$$\xi\left(z + \frac{1}{2}\right) = \frac{c(F)}{z - \frac{1}{2}} + a + \cdots, \quad \xi\left(z - \frac{1}{2}\right) = -\frac{c(F)}{z - \frac{1}{2}} + a + \cdots.$$

Let $\frac{1}{\xi(z+\frac{3}{2})} = \frac{1}{\xi(2)} + b(z - \frac{1}{2}) + \cdots$ Then we have

$$\frac{\xi(z+\frac{1}{2})}{\xi(z+\frac{3}{2})} = \frac{\frac{c(F)}{\xi(2)}}{z - \frac{1}{2}} + \left(bc(F) + \frac{a}{\xi(2)}\right) + \cdots,$$

$$\frac{\xi(z-\frac{1}{2})}{\xi(z+\frac{3}{2})} = \frac{-\frac{c(F)}{\xi(2)}}{z - \frac{1}{2}} + \left(-bc(F) + \frac{a}{\xi(2)}\right) + \cdots$$

$R(\sigma\tau\sigma, \Lambda, \chi)$ and $R(\tau, \Lambda, \chi)$ are holomorphic around $z = \frac{1}{2}$ as a function of z and we have the Taylor expansion

$$\begin{aligned} R(\sigma\tau\sigma, \Lambda, \chi) &= R(\sigma\tau\sigma, \beta_1, \chi) + \left(z - \frac{1}{2}\right) L + \cdots, \\ R(\tau, \Lambda, \chi) &= R(\tau, -\beta_1, \chi) + \left(z - \frac{1}{2}\right) N + \cdots = 1 + \left(z - \frac{1}{2}\right) N + \cdots, \end{aligned}$$

where L and N are certain intertwining operators which are derivatives at $z = \frac{1}{2}$ of $R(\sigma\tau\sigma, \Lambda, \chi)$ and $R(\tau, \Lambda, \chi)$, respectively. Here we used the fact that $R(\tau, -\beta_1, \chi)$ is the identity since χ is trivial. Then since $\sigma\tau\sigma\beta_1 = -\beta_1$,

$$\begin{aligned} R(\tau\sigma\tau\sigma, \Lambda, \chi) &= R(\tau, \sigma\tau\sigma\Lambda, \chi) R(\sigma\tau\sigma, \Lambda, \chi) \\ &= R(\sigma\tau\sigma, \beta_1, \chi) + \left(z - \frac{1}{2}\right)\big(N R(\sigma\tau\sigma, \beta_1, \chi) + L\big) + \cdots \end{aligned}$$

Then by direct computation, we can see that

$$\begin{aligned} &\mathrm{Res}_{\beta_1}\big(M^1(\sigma\tau\sigma, \Lambda, \chi) f + M^1(\tau\sigma\tau\sigma, \Lambda, \chi) f\big) \\ &= -\frac{a_1 c(F)}{\xi(2)} N R(\sigma\tau\sigma, \beta_1, \chi) f(\beta_1) + \frac{2 a a_1}{\xi(2)} R(\sigma\tau\sigma, \beta_1, \chi) f(\beta_1). \end{aligned}$$

Here $R(\sigma\tau\sigma, \beta_1, \chi) f(\beta_1) = R(\sigma, \tau\sigma\beta_1, \chi) R(\tau\sigma, \beta_1, \chi) f(\beta_1) = 0$. So the residue is zero.

(2) $E^2(g, f, \Lambda)$. $M(w, \Lambda, \chi)$ has a pole at S_2 when $\chi \circ \alpha_2^\vee = \mu_0$, i.e., $\nu = \mu_0$. Then $\chi = \chi(\mu, \mu_0)$. Here $W_2 = \{\tau, \sigma\tau, \tau\sigma\tau, \tau\sigma\tau\sigma\}$. For $\Lambda = z\beta_1 + \frac{\alpha_2}{2}$, $\langle \Lambda, \alpha_1^\vee \rangle = z - 1$, $\langle \Lambda, \alpha_3^\vee \rangle = z + 1$ and $\langle \Lambda, \alpha_4^\vee \rangle = z$. Then

LEMMA 5.2.

$$\begin{aligned} M^2(\tau, \Lambda, \chi) f &= R(\tau, \Lambda, \chi) f, \\ M^2(\sigma\tau, \Lambda, \chi) f &= \frac{\xi(z+1, \mu)}{\xi(z+2, \mu)} \frac{R(\sigma\tau, \Lambda, \chi) f}{\epsilon(z+1, \mu)}, \\ M^2(\tau\sigma\tau, \Lambda, \chi) f &= \frac{\xi(z, \mu)}{\xi(z+2, \mu)} \frac{R(\tau\sigma\tau, \Lambda, \chi) f}{\epsilon(z, \mu)\epsilon(z+1, \mu)}, \\ M^2(\sigma\tau\sigma\tau, \Lambda, \chi) f &= \frac{\xi(z-1, \mu)}{\xi(z+2, \mu)} \frac{R(\sigma\tau\sigma\tau, \Lambda, \chi) f}{\epsilon(z-1, \mu)\epsilon(z, \mu)\epsilon(z+1, \mu)}. \end{aligned}$$

We see that only when $\mu = \mu_0$, i.e., χ is the trivial character, $M^2(w, \Lambda, \chi)$ has a pole at $\Lambda = \alpha_3$, i.e., $z = 1$. We have

PROPOSITION 5.2.1. $\mathrm{Res}_{\alpha_3} E^2(g, f, \Lambda)$ *is zero.*

Proof. At $z = 1$, $M^2(\tau\sigma\tau, \Lambda, \chi)$ and $M^2(\sigma\tau\sigma\tau, \Lambda, \chi)$ both have simple poles. So

$$\mathrm{Res}_{\alpha_3} E_0^2(g, f, \Lambda) = \frac{c(F)}{\xi(3)} \big(R(\tau\sigma\tau, \alpha_3, \chi) f - R(\sigma\tau\sigma\tau, \alpha_3, \chi) f\big).$$

Here $\sigma\tau\sigma\tau = \tau\sigma\tau\sigma$ and $R(\tau\sigma\tau\sigma, \alpha_3, \chi) = R(\tau\sigma\tau, \alpha_3, \chi)R(\sigma, \alpha_3, \chi)$ and $R(\sigma, \alpha_3, \chi)$ is identity since χ is the trivial character. So the above expression is zero.

For the sake of completeness, we calculate the remaining iterated residues. We do not obtain anything new as Langlands' theory says.

PROPOSITION 5.2.2.

(i) $\mathrm{Res}_{\rho_B} E^2(g, f, \Lambda)$ *is constant.*
(ii) $\mathrm{Res}_{\frac{\alpha_2}{2}} E^2(g, f, \Lambda)$ *is not square integrable or zero.*

Proof. (i) $\Lambda = \rho_B$. From Lemma 5.2, at $z = 2$, only $M^2(\sigma\tau\sigma\tau, \Lambda, \chi)$ has a simple pole. So as in Proposition 5.1.2 (i), $\mathrm{Res}_{\rho_B} E^2(g, f, \Lambda)$ is constant.

(ii) $\Lambda = \frac{\alpha_2}{2}$. At $z = 0$, $R(\sigma\tau\sigma\tau, \Lambda, \chi_v)$ may have a pole. In that case we rewrite $M^2(\sigma\tau\sigma\tau, \Lambda, \chi)$ using (5.3) as follows:

$$M^2(\sigma\tau\sigma\tau, \Lambda, \chi)f = \frac{\xi(z-1)}{\xi_S(z, \mu_0)} \frac{\xi(z)}{\xi_S(z+1, \mu_0)} \frac{\xi(z+1)}{\xi_S(z+2, \mu_0)} \bigotimes_{v \notin S} \tilde{f}_v \otimes \bigotimes_{v \in S} \tilde{A}(\sigma\tau\sigma\tau, \Lambda, \chi_v) f_v,$$

where $\tilde{A}(\sigma\tau\sigma\tau, \Lambda, \chi_v)$ is the expression (5.3) and $\xi_S(z, \mu_0) = \prod_{v \notin S} L_v(z, \mu_0)$ is the partial Hecke L-function. $\xi_S(z, \mu_0)$ has a pole only at $z = 1$. Therefore $M^2(\sigma\tau\sigma\tau, \Lambda, \chi)$ has a simple pole at $z = 0$. But since $\sigma\tau\sigma\tau\frac{\alpha_2}{2} = -\frac{\alpha_2}{2}$, the residue is not square integrable. On the other hand, $T^2(\sigma\tau, \Lambda, \chi)$ and $T^2(\tau\sigma\tau, \Lambda, \chi)$ have simple poles at $z = 0$. But their residues cancel each other.

Remark 5.1. As the referee pointed out, the Eisenstein series $E^3(g, f, \Lambda)$ and $E^4(g, f, \Lambda)$ are related to the Eisenstein series $E^1(g, f, \Lambda)$ and $E^2(g, f, \Lambda)$ by means of the functional equations (5.2) of the Eisenstein series. Thus it is sufficient for a classification of discrete spectrum to consider contributions of $E^1(g, f, \Lambda)$ and $E^2(g, f, \Lambda)$ to $L_d^2(B)$.

(3) Conclusion

In conclusion, we have proved the following

PROPOSITION 5.3. *For χ the trivial character, the only square integrable residues of Eisenstein series are constants. Among non-trivial characters, only the quadratic characters $\chi = \chi(\mu, \mu)$, $\mu^2 = 1$, contribute to $L_d^2(B)$. In this case, the Eisenstein series have poles only at $\Lambda = \beta_1$.*

For μ a non-trivial quadratic grössencharacter, let $H(g, f, \mu) = \mathrm{Res}_{\beta_1} E^1(g, f, \Lambda)$. Then the residue map $f \mapsto H(g, f, \mu)$ defines an intertwining map

$$\mathfrak{R}_\mu \colon I(\beta_1, \chi) \mapsto L_d^2(B).$$

Let $B(\mu)$ be the image of this intertwining map. Then we have an orthogonal decomposition

$$L_d^2(B) = \bigoplus_{\mu} B(\mu) \oplus B(\mu_0),$$

where μ runs through all non-trivial quadratic grössencharacter of F and $B(\mu_0)$ is the space of constant functions which come from the trivial character.

We determine the spaces $B(\mu)$ precisely. Here a certain parity sign determined by the identities of Labesse–Langlands [16] for local intertwining operators of SL_2, plays a decisive role in determining $B(\mu)$. The components of $B(\mu)$ are exactly the elements of the global L-packet which satisfy the parity sign condition.

We look at the constant term of $H(g, f, \mu)$ along B:

$$(\text{constant})\left(\bigotimes_v R(\sigma\tau\sigma, \beta_1, \chi_v) f_v + \bigotimes_v R(\sigma\tau\sigma\tau, \beta_1, \chi_v) f_v\right).$$

Here we have the functional equation of the form

$$R(\sigma\tau\sigma\tau, \beta_1, \chi_v) = R(\sigma\tau\sigma, \tau\beta_1, \tau\chi_v) R(\tau, \beta_1, \chi_v).$$

But observe that $\tau\beta_1 = \beta_1$ and $\tau\chi_v = \chi_v$ for $\chi_v = \chi(\mu_v, \mu_v)$ with $\mu_v^2 = \mu_0$. So the constant term of $H(g, f, \mu)$ along B is given by

$$(\text{constant}) \bigotimes_v R(\sigma\tau\sigma, \beta_1, \chi_v)\left(\bigotimes_v f_v + \bigotimes_v R(\tau, \beta_1, \chi_v) f_v\right). \tag{5.5}$$

We note that $R(\tau, \beta_1, \chi_v)$ is an intertwining operator from $I(\beta_1, \chi_v)$ into itself since $\tau\beta_1 = \beta_1$ and $\tau\chi_v = \chi_v$. Here we use Shahidi's idea of inducing in stages via the non-Siegel parabolic subgroup since its Levi is $M = GL_1 \times SL_2$. We use the fact that $I(\beta_1, \chi_v) = \mathrm{Ind}_P^G(|\ |_v \mu_v \times \mathrm{Ind}_{B_0}^{SL_2}(\mu_v))$, where B_0 is the corresponding Borel subgroup of SL_2 and $|\ |_v$ is the absolute value. Suppose μ_v is not trivial. Then $\mathrm{Ind}_{B_0}^{SL_2}(\mu_v)$ is reducible. Let $\mathrm{Ind}_{B_0}^{SL_2}(\mu_v) = \pi_+(\mu_v) \oplus \pi_-(\mu_v)$, as in Labesse–Langlands [16, p. 747], i.e., with $\pi_+(\mu_v)$ generic with respect to ψ_v. Let $\epsilon(\pi_+(\mu_v)) = 1$ and $\epsilon(\pi_-(\mu_v)) = -1$. Observe that for almost all v, $\pi_+(\mu_v)$ is spherical. If μ_v is trivial, $\mathrm{Ind}_{B_0}^{SL_2}(\mu_v)$ is irreducible. In this case, we take $\pi_-(\mu_v) = 0$. Let $\pi(\mu_v) = \{\pi_+(\mu_v), \pi_-(\mu_v)\}$ and if $\pi_v \in \pi(\mu_v)$, let $\epsilon(\pi_v)$ be the corresponding sign. Then

$$I(\beta_1, \chi_v) = \mathrm{Ind}_P^G(|\ |_v \mu_v \times \pi_+(\mu_v)) \oplus \mathrm{Ind}_P^G(|\ |_v \mu_v \times \pi_-(\mu_v)).$$

Let $J_\pm(\mu_v)$ be the Langlands' quotients of $\mathrm{Ind}_P^G(|\ |_v \mu_v \times \pi_\pm(\mu_v))$, respectively. By Langlands' classification theorem, the common image of the intertwining operators $R(\sigma\tau\sigma, \beta_1, \chi_v)$ and $R(\sigma\tau\sigma\tau, \beta_1, \chi_v)$ is the direct sum of $J_\pm(\mu_v)$. Let

$$J(\mu_v) = \{J_+(\mu_v), J_-(\mu_v)\}.$$

Observe that $R(\tau, \beta_1, \chi_v)$ is the normalized intertwining operator for $\mathrm{Ind}_{B_0}^{SL_2}(\mu_v)$. By Labesse–Langlands [16, p. 747],

$$R(\tau, \beta_1, \chi_v) f_v = \begin{cases} f_v & \text{for } f_v \in \pi_+(\mu_v), \\ -f_v & \text{for } f_v \in \pi_-(\mu_v). \end{cases} \tag{5.6}$$

Then we define $J(\mu)$ to be the collection

$$J(\mu) = \left\{ \begin{aligned} \Pi &= \otimes\Pi_v \mid \Pi_v \in J(\mu_v) \text{ for all } v, \\ \Pi_v &= J_+(\mu_v) \text{ for almost all } v, \prod_v \epsilon(\pi_v) = 1 \end{aligned} \right\}.$$

We note that $\prod_v \epsilon(\pi_v)$ is well-defined and independent of the choice of ψ. Here if $\prod_v \epsilon(\pi_v) = -1$, then by (5.5) and (5.6), the constant term of $H(g, f, \mu)$ along B is zero. So it is zero.

Therefore we have proved

THEOREM 5.4. *We have an irreducible decomposition of* $L_d^2(B)$ *as follows:*

$$L_d^2(B) = \oplus_\mu B(\mu) \oplus B(\mu_0),$$

where μ *runs through all non-trivial quadratic grössencharacters of* F *and* $B(\mu_0)$ *is the space of constant functions. For each* μ*, the constant term map gives rise to an isomorphism from* $B(\mu)$ *to* $J(\mu)$*.*

Remark 5.2. In a separate paper [35], we give the Arthur parameter for the representations in $J(\mu)$ and verify Arthur's conjecture on the multiplicity formula (see [1] for Arthur's conjecture). It turns out that the parity sign $\epsilon(\pi_v)$ enters into the multiplicity formula as anticipated by Arthur [1].

Remark 5.3. Kudla–Rallis–Soudry [15] obtained also $B(\mu)$ in Theorem 5.4, for each quadratic grössencharacher μ, from Eisenstein series associated to characters of the Siegel parabolic subgroup when F is a totally real number field. But they did not show that $B(\mu)$'s exhaust $L_d^2(B)$. They described the irreducible constituents, using theta correspondence, in terms of distinguished representations which admit one family of nondegenerate Fourier coefficients (associated to a GL_2 orbit of a fixed binary form). More precisely, let χ be a grössencharacter of F. Then χ defines a character of $M = GL_2$ by $\chi(g) = \chi(\det g)$. For a function $f \in I(s, \chi) = \mathrm{Ind}_P^G \chi \otimes \exp(s, H_P(\ ))$ (recall that we identify $s \in \mathbb{C}$ with $s\beta_2 \in \mathfrak{a}_\mathbb{C}^*$ in Section 3), we define an Eisenstein series

$$E(g, s, f, P) = \sum_{\gamma \in P(F)\backslash G(F)} f(\gamma g).$$

We note that

$$\mathrm{Ind}_P^G \chi \otimes \exp\left(s, H_P(\,)\right) \subset \mathrm{Ind}_B^G \chi \otimes \exp\left(s\tilde{\alpha} + \rho_P - \rho_B, H_B(\,)\right),$$

where we consider χ as a character of B in an obvious way. Therefore, $E(g, s, f, P)$ is an automorphic form concentrated on the class of B (see [10, p. 187] for the definition). Kudla–Rallis [13] proved that the Eisenstein series $E(g, s, f, P)$ has a pole if and only if either $\chi = 1$, $s = \frac{1}{2}, \frac{3}{2}$ or $\chi^2 = 1$, $\chi \neq 1$, $s = \frac{1}{2}$.

By [30, Theorem 3.1], we know that the residues of $E(g,s,f,P)$ at $s=\frac{3}{2}$ for $\chi=1$ are constants. For $\chi=1$, the residue at $s=\frac{1}{2}$ is not square integrable. For each $\chi\neq 1$, $\chi^2=1$, by taking the residues at $s=\frac{1}{2}$, we obtain an intertwining map

$$\mathcal{R}_\chi\colon\, I\left(\frac{1}{2},\chi\right)\mapsto L^2_d(B).$$

By Lemma 5.1, for $f\in \mathrm{Ind}_B^G\chi\otimes\exp(\Lambda,H_B())$ and $\Lambda=s\beta_2+\frac{\alpha_1}{2}$,

$$E^1_0(g,f,\Lambda)=E_0\big(g,s,R(\sigma,\Lambda,\chi)f,P\big),$$

where $E_0(g,s,f,P)$ is the constant term of $E(g,s,f,P)$. Therefore we have $E^1(g,f,\Lambda)=E(g,s,R(\sigma,\Lambda,\chi)f,P)$ (see Arthur [33]). Even though $I(\frac{1}{2},\chi_v)$ might be smaller than $I(\beta_1,\chi_v)$, the image of $\mathcal{R}_\chi$ is exactly the same as $B(\chi)$ in Theorem 5.4, This follows from the work of Jantzen [36] who computed the Langlands' parameters of the irreducible subquotients of $I(\frac{1}{2},\chi_v)$, which coincide with those of $I(\beta_1,\chi_v)$ or from simply observing that the map $R(\sigma,\Lambda,\chi_v)\colon\, I(\beta_1,\chi_v)\mapsto I(\frac{1}{2},\chi_v)$ is surjective. We describe the result of Kudla–Rallis–Soudry [15] for $L^2_d(B)$ in terms of distinguished representations.

Let $V,(,)$ be an isotropic quaternary space. Write $V=V_0+V_{1,1}$, where $V_{1,1}$ is a hyperbolic plane and V_0 is a binary space and $\dim V_0$=2. Let

$$\mathcal{O}_{V_0}=\big\{\beta\in \mathrm{Sym}_2(F)\mid \exists x\in V_0(F)^2 \text{ with } (x,x)=2\beta,\ \det\beta\neq 0\big\}.$$

Then $\mathcal{O}_{V_0}$ is a $GL(2,F)$-orbit in $\mathrm{Sym}_2(F)$. Then $\mathrm{Im}(\mathcal{R}_\chi)=\oplus\Pi(V_0)$ for some binary quadratic space V_0 with $\chi_{V_0}=\chi$. Here $\Pi(V_0)$ is a certain distinguished representation attached to quadratic forms (see [15] for notations). It has the property that for $f\in\Pi(V_0)$, the non-degenerate β-th Fourier coefficient of f is zero unless $\beta\in\mathcal{O}_{V_0}$. Also $\Pi(V_0)$ is in $L^2_d(B)$ if and only if V_0 is anisotropic, i.e., $\chi_{V_0}\neq 1$. Therefore

$$L^2_d(B)=B(1)\oplus\left(\bigoplus_{V_0}\Pi(V_0)\right),$$

where V_0 runs over all binary anisotropic quadratic spaces and $B(1)$ denotes the space of constant functions. Kudla–Rallis–Soudry [15] showed that the multiplicity of $\Pi(V_0)$ is one.

Remark 5.4. It should be noted that our sign condition in Theorem 5.4 is comparable to the Hasse invariants in [15]. Therefore, our condition $\prod_v\epsilon(\pi_v)=1$ is exactly the same as their "coherent condition," i.e., the condition of existence of a global quadratic space V_0 (see [15, Proposition 2.6]).

Remark 5.5. Watanabe [31] also showed that the irreducible constituents of $L^2_d(B,K_S)$ are of multiplicity one.

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