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Some remarks on the arithmetic Hodge index conjecture

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Introduction

Let X be a regular scheme projective and flat over $\mathrm{Spec}(\mathbb{Z})$. We assume that X is of pure absolute dimension $n+1$. We will discuss the following arithmetic analogues of Grothendieck's standard conjectures which are stated by Gillet and Soulé in [7].

CONJECTURE A_p (X, ϱ). *When ϱ is an ample line bundle on X one can choose a positive hermitian metric $\|.\|$ on ϱ in such a way that if $2p \leqslant n+1$,*

i) The operator $L^{n+1-2p}_{\varrho,\|\cdot\|}\colon \widehat{CH}^p(X)_{\mathbb{R}} \longrightarrow \widehat{CH}^{n+1-p}(X)_{\mathbb{R}}$ is an isomorphism.

ii) If $x \in \widehat{CH}^p(X)_{\mathbb{R}}$, $x \neq 0$, and $L^{n+2-2p}_{\varrho,\|\cdot\|}(x) = 0$ then

$$(-1)^p\widehat{\deg}(x \,.\, L^{n+1-2p}_{\varrho,\|\cdot\|}(x)) > 0.$$

Our first result is a reduction principle. We choose a Kähler metric h on the complex manifold $X(\mathbb{C})$ associated with X. Then it is sufficient to show that A_p (X, ϱ) holds for the Arakelov Chow group of the Arakelov variety $\overline{X} = (X, h)$. This is mainly a consequence of a result from Kähler geometry. Let X be a compact Kähler manifold and denote the complex valued (p,q)-forms on X by $A^{p,q}(X)$. We prove a hard Lefschetz and a Hodge index theorem for the groups

$$A^{p,q}(X)/\mathrm{Ker}(dd^c)$$

which measure the difference between the Arakelov Chow groups and the arithmetic Chow groups.

We give two applications of the reduction principle. Namely, we discuss projective space and abelian schemes. In these cases we can split the filtration F, which in the case where X is smooth over the ring of integers in a number field, is defined as follows

$$F^kCH^p(\overline{X})_{\mathbb{R}} = \begin{cases} CH^p(\overline{X})_{\mathbb{R}} & \text{if } k=0 \\ \mathrm{Ker}(cl\colon CH^p(\overline{X})_{\mathbb{R}} \to H^{p,p}(X_{\mathbb{R}})) & \text{if } k=1 \\ \mathrm{Im}(a\colon H^{p-1,p-1}(X_{\mathbb{R}}) \to CH^p(\overline{X})_{\mathbb{R}}) & \text{if } k=2 \\ 0 & \text{if } k \geqslant 3. \end{cases}$$

As there are no homologically trivial cycles on $\mathbb{P}^n$, it is possible to prove $A_p(\mathbb{P}^n, \mathcal{O}(1))$. The situation is more difficult for abelian schemes. The statement of Lefschetz type of the conjecture above is already discussed for abelian schemes in [10].

The statement of Hodge type is related to cubical heights of homologically trivial cycles. We state a positivity conjecture about these heights which for divisors is a consequence of the positivity of the Néron-Tate height.

1. Analytic theory

Let X be a compact Kähler manifold of dimension n. By $A^{p,q}(X)$ we denote the space of complex valued differential forms on X of type (p, q). Let ω be the fundamental form of the Kähler metric and

$$L\colon A^{p,q}(X) \longrightarrow A^{p+1,q+1}(X), \quad \alpha \mapsto \alpha \wedge \omega$$

the Lefschetz operator associated with ω. For $p + q \leqslant n$, we define a hermitian inner product

$$Q\colon A^{p,q}(X) \times A^{p,q}(X) \longrightarrow \mathbb{C},$$

$$(\alpha, \beta) \mapsto \sqrt{-1}^{\,p-q}(-1)^{(p+q)(p+q-1)/2} \int_X L^{n-p-q}(\alpha) \wedge \overline{\beta}.$$

From classical Lefschetz and Hodge theory we know the following result:

THEOREM 1.1.
i) The morphism $L^{n-p-q} : A^{p,q}(X) \longrightarrow A^{n-q,n-p}(X)$ *is an isomorphism for* $p + q \leqslant n$.
ii) The form Q *is positive definite on the subspace*

$$A^{p,q}_{\text{prim}}(X) = \mathrm{Ker}(L^{n+1-p-q}|_{A^{p,q}(X)})$$

of primitive forms in $A^{p,q}(X)$.

Proof: Decompose $A^{p,q}(X)$ into eigenspaces with respect to the operation of the Laplacian and follow the proofs of [12, Cor. V.4.13] and [12, Th. V.6.1.d]. □

Recall that $d^c = (4\pi\sqrt{-1})^{-1}(\partial - \overline{\partial})$. Then, $dd^c = -(2\pi\sqrt{-1})^{-1}\partial\overline{\partial}$ is a real operator. We will prove an analogue of this theorem for the groups

$$B^{p,q}(X) = A^{p,q}(X)/\mathrm{Ker}(dd^c).$$

In the following, we denote the class in $B^{p,q}(X)$ of a form $\alpha \in A^{p,q}(X)$ by $\{\alpha\}$. The Lefschetz operator induces a map

$$L\colon B^{p,q}(X) \longrightarrow B^{p+1,q+1}(X),$$

since L commutes with d and d^c. We define a pairing

$$A^{p,q}(X) \times A^{n-1-q,n-1-p}(X) \longrightarrow \mathbb{C},$$

$$(\alpha, \beta) \mapsto \int_X \alpha \wedge dd^c(\overline{\beta}) = \int_X dd^c(\alpha) \wedge \overline{\beta}$$

for $p+q \leqslant n-1$. The last equality follows from Stoke's theorem. We obtain an induced hermitian inner product

$$\tilde{Q}\colon B^{p,q}(X) \times B^{p,q}(X) \longrightarrow \mathbb{C},$$

$$(\{\alpha\}, \{\beta\}) \mapsto \sqrt{-1}^{\,p-q}(-1)^{(p+q)(p+q-1)/2+1} \int_X L^{n-1-p-q}(\alpha) \wedge dd^c(\overline{\beta})$$

for $p+q \leqslant n-1$. The first part of the following theorem is taken from [10, 10.4].

THEOREM 1.2.
i) The morphism

$$L^{n-1-p-q}\colon B^{p,q}(X) \longrightarrow B^{n-1-q,n-1-p}(X)$$

is an isomorphism for $p+q \leqslant n-1$.
ii) The form $\tilde{Q}$ *is positive definite on the subspace*

$$B^{p,q}_{\mathrm{prim}}(X) = \mathrm{Ker}(L^{n-p-q}|_{B^{p,q}(X)})$$

of primitive forms in $B^{p,q}(X)$.

Proof: The proof of part i) is given in [10, 10.4]. I would like to thank C. Soulé for the idea for the following proof of ii). Let G be the Green's operator for the Laplacian $\Delta = dd^* + d^*d$ on X. We denote by

$$\Lambda\colon A^{p,q}(X) \longrightarrow A^{p-1,q-1}(X)$$

the adjoint of L with respect to the Hodge inner product. As in the proof of [12, Prop. VI.2.2, p. 224], we obtain

$$dd^c\alpha = -4\pi\, dd^c\, G\, \Lambda\, dd^c\alpha \tag{1}$$

for any form $\alpha \in A^{p,q}(X)$. A given class $\{\alpha\}$ in $B^{p,q}(X)$ is primitive if and only if $L^{n-p-q}(dd^c\alpha) = 0$ in $A^{n+1-q,n+1-p}(X)$. This last condition is satisfied if and only if

$$dd^c(\alpha) = \eta_0 + L\eta_1 \tag{2}$$

for primitive forms $\eta_i \in A^{p+1-i,q+1-i}(X)$.

We assume that $\{\alpha\}$ is a primitive element in $B^{p,q}(X)$ and define

$$\varepsilon_{p,q} = \sqrt{-1}^{\,p-q}(-1)^{(p+q)(p+q-1)/2}.$$

By definition and using (1), we obtain

$$\begin{aligned}\tilde{Q}(\{\alpha\},\{\alpha\}) &= 4\pi\varepsilon_{p,q}\int_X L^{n-1-p-q}(\alpha)\wedge dd^cG\Lambda dd^c(\overline{\alpha})\\ &= 4\pi\varepsilon_{p,q}\int_X L^{n-1-p-q}(dd^c\alpha)\wedge G\Lambda dd^c(\overline{\alpha}).\end{aligned}$$

We have $L^{n-1-p-q}(\eta_0)=0$ and $\Lambda(\eta_0)=0$. Hence, using (2):

$$\begin{aligned}\tilde{Q}(\{\alpha\},\{\alpha\}) &= 4\pi\varepsilon_{p,q}\int_X L^{n-1-p-q}((\eta_0+L\eta_1)\wedge G\Lambda(\overline{\eta}_0+L\overline{\eta}_1))\\ &= 4\pi\varepsilon_{p,q}\int_X L^{n-p-q}(\eta_1\wedge G\Lambda L\overline{\eta}_1).\end{aligned}$$

For $\eta_1\in A^{p,q}(X)$, we have $\Lambda L\eta_1 - L\Lambda\eta_1=(n-p-q)\eta_1$ [12, Prop. V.1.1.c] and from $\Lambda\eta_1=0$ follows

$$\tilde{Q}(\{\alpha\},\{\alpha\}) = 4\pi\varepsilon_{p,q}\,(n-p-q)\int_X L^{n-p-q}(\eta_1\wedge G\overline{\eta_1}).$$

Now, we decompose $A^{p,q}(X)$ into eigenspaces with respect to the operation of the Laplacian Δ. This decomposition is respected by L, Λ, and G. Hence all eigenvectors in the decomposition of η_1 are again primitive. Furthermore, they have strictly positive eigenvalues as (2) implies that the harmonic projection of η_1 is zero. Since the eigenspaces are orthogonal for the Hodge inner product it is sufficient to show

$$\sqrt{-1}^{\,p-q}(-1)^{(p+q)(p+q-1)/2}4\pi(n-p-q)\int_X L^{n-p-q}(\eta\wedge G\overline{\eta})>0 \qquad (3)$$

for any eigenvector $\eta\in A^{p-1,q-1}_{\text{prim}}(X)$ of Δ with positive eigenvalue λ. But in this case $G(\overline{\eta})=\lambda^{-1}\overline{\eta}$, and (3) follows from Theorem 1.1. □

2. Arithmetic intersection theory

We recall some definitions and basic facts from arithmetic intersection theory [6, 7]. Let X be a regular scheme projective and flat over $\mathrm{Spec}(\mathbb{Z})$ of absolute dimension $n+1$. Complex conjugation defines an involution $F_\infty : X(\mathbb{C})\to X(\mathbb{C})$ on the manifold $X(\mathbb{C})$. We denote by $A^{p,p}(X_\mathbb{R})$ (resp. $D^{p,p}(X_\mathbb{R})$) the space of real valued differential forms (resp. currents) α on $X(\mathbb{C})$ which are of type (p,p) and satisfy $F_\infty^*(\alpha)=(-1)^p\alpha$. Let $Z^p(X)$ be the group of codimension p cycles on X. A cycle $Z=\sum_i r_i[Z_i]\in Z^p(X)\otimes_\mathbb{Z}\mathbb{R}$ with real coefficients defines a current $\delta_Z\in D^{p,p}(X_\mathbb{R})$ by the formula

$$\delta_Z(\eta)=\sum_i r_i\int_{Z_i(\mathbb{C})}\eta.$$

The group $\hat{Z}^p(X)_{\mathbb{R}}$ of arithmetic cycles of codimension p with real coefficients is the subgroup of

$$(Z^p(X)\otimes_{\mathbb{Z}}\mathbb{R})\oplus D^{p-1,p-1}(X_{\mathbb{R}})$$

consisting of all pairs (Z,g) where g is a Green current for Z, i.e. $dd^cg+\delta_Z=\omega$ for some smooth form $\omega\in A^{p,p}(X_{\mathbb{R}})\subset D^{p,p}(X_{\mathbb{R}})$. The *real* arithmetic Chow group $\widehat{CH}^p(X)_{\mathbb{R}}$ is the quotient of $\hat{Z}^p(X)_{\mathbb{R}}$ by the real vector space generated by elements $(0,\partial u+\bar{\partial}v)$ and $\widehat{\mathrm{div}}(f)$, where W is an integral subscheme of codimension $p-1$, $f\in k(W)^*$, and $\widehat{\mathrm{div}}(f)$ is the arithmetic cycle $(\mathrm{div}(f),-[\log|f|^2])$. We set

$$\widetilde{A}^{p,p}(X_{\mathbb{R}})=\frac{A^{p,p}(X_{\mathbb{R}})}{\mathrm{Im}(\partial)+\mathrm{Im}(\bar{\partial})}\quad\text{and}\quad H^{p,p}(X_{\mathbb{R}})=\mathrm{Ker}(dd^c|_{\widetilde{A}^{p,p}(X_{\mathbb{R}})}).$$

One can show that $H^{p,p}(X_{\mathbb{R}})=\{\alpha\in H^{p,p}(X(\mathbb{C}),\mathbb{R}(p)),\quad F_\infty^*(\alpha)=(-1)^p\alpha\}$ since X is projective. We have natural maps

$$\begin{aligned}
\omega\colon \widehat{CH}^p(X)_{\mathbb{R}} &\longrightarrow A^{p,p}(X_{\mathbb{R}}), &\quad \alpha=[(Z,g)]&\mapsto\omega(\alpha)=dd^cg+\delta_Z,\\
a\colon \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}}) &\longrightarrow \widehat{CH}^p(X)_{\mathbb{R}}, &\quad [\eta]&\mapsto a(\eta)=[(0,\eta)],\\
cl\colon \widehat{CH}^p(X)_{\mathbb{R}} &\longrightarrow H^{p,p}(X_{\mathbb{R}}), &\quad \alpha&\mapsto cl(\alpha)=[\omega(\alpha)].
\end{aligned}$$

Let h be a Kähler metric on $X(\mathbb{C})$ which is invariant under F_∞. The pair $\overline{X}=(X,h)$ is called an Arakelov variety. The *real* Arakelov Chow group $CH^p(\overline{X})_{\mathbb{R}}$ of $\overline{X}$ is by definition the inverse image of the subgroup of harmonic forms in $A^{p,p}(X_{\mathbb{R}})$ under the map ω. The real Arakelov Chow group fits into an exact sequence

$$CH^{p,p-1}(X)_{\mathbb{R}}\xrightarrow{\rho}H^{p-1,p-1}(X_{\mathbb{R}})\xrightarrow{a}CH^p(\overline{X})_{\mathbb{R}}\xrightarrow{\zeta}CH^p(X)_{\mathbb{R}}\longrightarrow 0. \quad (4)$$

where $CH^{p,p-1}(X)_{\mathbb{R}}=CH^{p,p-1}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ is a K_1-analogue of the usual Chow group $CH^p(X)_{\mathbb{R}}=CH^p(X)\otimes_{\mathbb{Z}}\mathbb{R}$, ρ is a regulator map, and ζ maps the class of (Z,g) to the class of Z.

The arithmetic intersection product, constructed in [6] and denoted by '.', makes $\widehat{CH}(X)_{\mathbb{R}}=\oplus_p\widehat{CH}^p(X)_{\mathbb{R}}$ into a graded commutative ring. It induces a product on $CH(\overline{X})_{\mathbb{R}}$ if the product of harmonic forms on $X(\mathbb{C})$ is harmonic. We will make frequent use of the formula

$$a(x)\,.\,y=a(x\,.\,\omega(y)) \quad (5)$$

for $x\in\widetilde{A}^{p,p}(X_{\mathbb{R}})$ and $y\in\widehat{CH}^q(X)_{\mathbb{R}}$. The arithmetic Chow groups are contravariant for arbitrary and covariant (with a shift of the degree) for proper morphisms which are smooth in the generic fibre. If $f\colon Y\to X$ is proper and $\overline{X}=(X,h)$ is an Arakelov variety, we can define a push forward map

$$f_\#\colon\widehat{CH}(Y)_{\mathbb{R}}\longrightarrow CH(\overline{X})_{\mathbb{R}}$$

which under suitable assumptions satisfies base change, functoriality, and a projection formula [10, Sect. 4]. In particular, we obtain a natural section

$$\sigma = id_{\#}\colon \widehat{CH}^p(X)_{\mathbb{R}} \longrightarrow CH^p(\overline{X})_{\mathbb{R}} \tag{6}$$

of the inclusion map. We have a geometric and an arithmetic degree map. We define

$$\deg\colon H^{n,n}(X_{\mathbb{R}}) \longrightarrow \mathbb{R}. \tag{7}$$

by mapping $\eta \in \operatorname{Ker}(dd^c|_{\widetilde{A}^{n,n}(X_{\mathbb{R}})})$ to $\frac{1}{2}\int_{X(\mathbb{C})} \eta$. The arithmetic degree is a map

$$\widehat{\deg}\colon \widehat{CH}^{n+1}(X)_{\mathbb{R}} \longrightarrow \mathbb{R} \tag{8}$$

which satisfies $\widehat{\deg} \circ a = \deg$. The geometric degree

$$\deg\colon CH^n(X) \longrightarrow \mathbb{R}$$

is by definition the composition of the cycle class map with (7).

3. Arithmetic analogues of the standard conjectures

A line bundle ϱ on X with smooth hermitian metric $||.||$ invariant under F_∞ on the complex line bundle induced by ϱ on $X(\mathbb{C})$ is called a hermitian line bundle. We call the metric $||.||$ on ϱ positive if the first Chern form $c_1(\varrho, ||.||)$ is a positive $(1,1)$-form on $X(\mathbb{C})$. The first arithmetic Chern class $\hat{c}_1(\varrho, ||.||)$ is the class of $(\operatorname{div}(s), -[\log||s||^2])$ in $\widehat{CH}^1(X)_{\mathbb{R}}$ where s is any rational section of ϱ. We denote by

$$L_{\varrho,||\cdot||}\colon \widehat{CH}^p(X)_{\mathbb{R}} \longrightarrow \widehat{CH}^{p+1}(X)_{\mathbb{R}}, \quad \alpha \mapsto \alpha.\hat{c}_1(\varrho, ||\cdot||)$$

the Lefschetz operator associated with the hermitian line bundle $(\varrho, ||\cdot||)$.

We will discuss the following conjecture for X of pure absolute dimension $n+1$.

CONJECTURE A_p (X, ϱ) (Gillet-Soulé). *When ϱ is ample on X one can choose a positive hermitian metric $||.||$ on ϱ in such a way that if $2p \leqslant n+1$,*

i) The operator

$$L^{n+1-2p}_{\varrho,||\cdot||}\colon \widehat{CH}^p(X)_{\mathbb{R}} \longrightarrow \widehat{CH}^{n+1-p}(X)_{\mathbb{R}}$$

is an isomorphism.

ii) If $x \in \widehat{CH}^p(X)_{\mathbb{R}}$, $x \neq 0$, and $L^{n+2-2p}_{\varrho,||\cdot||}(x) = 0$ then

$$(-1)^p \widehat{\deg}(x \,.\, L^{n+1-2p}_{\varrho,||\cdot||}(x)) > 0.$$

Gillet and Soulé have shown in [7] that the conjecture holds when $p = 0$ or when X is an arithmetic surface. In the second case, the conjecture is a consequence of the Hodge index theorem of Faltings and Hriljac. Moriwaki has shown in [11] that part ii) holds in general in codimension one.

Let ϱ be an ample line bundle on X and assume that ϱ carries a positive hermitian metric $\|.\|$. There is a unique F_∞-invariant Kähler metric h on $X(\mathbb{C})$ whose fundamental form is the first Chern form $c_1(\varrho, \|.\|)$. We obtain an Arakelov variety $\overline{X} = (X, h)$. The Lefschetz operator $L_{\varrho,\|\cdot\|}$ induces a map on $CH(\overline{X})_{\mathbb{R}}$, as the product with the fundamental class respects harmonicity of forms.

PROPOSITION 3.1. *Let ϱ be an ample line bundle on X that carries a positive hermitian metric $\|.\|$. Let $\overline{X} = (X, h)$ be the associated Arakelov variety and consider the following conditions:*

i) The operator

$$L_{\varrho,\|\cdot\|}^{n+1-2p} \colon CH^p(\overline{X})_{\mathbb{R}} \longrightarrow CH^{n+1-p}(\overline{X})_{\mathbb{R}}$$

is an isomorphism.

ii) If $x \in CH^p(\overline{X})_{\mathbb{R}}$, $x \neq 0$, and $L_{\varrho,\|\cdot\|}^{n+2-2p}(x) = 0$ then

$$(-1)^p \widehat{\deg}(x \,.\, L_{\varrho,\|\cdot\|}^{n+1-2p}(x)) > 0. \tag{9}$$

Then, condition i) implies part i) of $A_p(X, \varrho)$, and ii) implies part ii) of $A_p(X, \varrho)$.

Proof: As explained in [10, 10.3], there is an exact sequence

$$0 \longrightarrow CH^p(\overline{X})_{\mathbb{R}} \longrightarrow \widehat{CH}^p(X)_{\mathbb{R}} \xrightarrow{\tau} B^{p-1,p-1}(X_{\mathbb{R}}) \longrightarrow 0 \tag{10}$$

where

$$B^{p-1,p-1}(X_{\mathbb{R}}) = A^{p-1,p-1}(X_{\mathbb{R}})/\mathrm{Ker}(dd^c).$$

Part i) of $A_p(X, \varrho)$ follows now from 1.2.i), assumption i), and the five lemma. We will show ii). The section σ from (6) defines a canonical splitting of (10), and a section $\tilde{\tau}$ of τ. Given a class $\{\alpha\}$ in $B^{p-1,p-1}(X_{\mathbb{R}})$ with $\alpha \in A^{p-1,p-1}(X_{\mathbb{R}})$, we have

$$\begin{aligned} \tilde{\tau}(\{\alpha\}) &= a(\alpha) - \sigma\, a(\alpha) && (11) \\ &= a(\alpha - H(\alpha)) \in a(\mathrm{Im}(\partial) + \mathrm{Im}(\partial^*)) && (12) \end{aligned}$$

by definition of σ, where H denotes harmonic projection.

Now take $x \in \widehat{CH}^p(X)_{\mathbb{R}}$, $x \neq 0$ with $L_{\varrho,\|\cdot\|}^{n+2-2p}(x) = 0$. We write $x = y + z$ with $y \in CH^p(\overline{X})_{\mathbb{R}}$ and $z = \tilde{\tau}(\{\beta\}) = a(\beta)$ for some $\beta \in \mathrm{Im}(\partial) + \mathrm{Im}(\partial^*)$. Let L be the Lefschetz operator associated with $c_1(\varrho, \|.\|)$. From

$$[L, \partial] = 0 \ \text{ and } \ [L, \partial^*] = \sqrt{-1}\,\bar{\partial},$$

[12, Cor. V. 4.10], we get

$$L^{n+2-2p}(\beta) \in \mathrm{Im}(\partial) + \mathrm{Im}(\overline{\partial}) + \mathrm{Im}(\partial^*).$$

The last group is contained in the orthogonal complement of the harmonic forms. Hence it equals $\mathrm{Im}(\partial) + \mathrm{Im}(\partial^*)$. This gives with (5) and (11)

$$L_{\varrho,\|\cdot\|}^{n+2-2p}(z) = a(L^{n+2-2p}(\beta)) \in \mathrm{Im}(\tilde{\tau}).$$

We have also $L_{\varrho,\|\cdot\|}^{n+2-2p}(y) \in CH^{n+2-p}(\overline{X})_{\mathbb{R}}$. As a consequence of the splitting of (10), we obtain

$$L_{\varrho,\|\cdot\|}^{n+2-2p}(y) = 0 = L_{\varrho,\|\cdot\|}^{n+2-2p}(z).$$

This implies $L^{n+2-2p}(\{\beta\}) = 0$ in $B^{n+1-p,n+1-p}(X_{\mathbb{R}})$. The computation of the left-hand side of (9) gives

$$(-1)^p\widehat{\deg}(y \,.\, L_{\varrho,\|\cdot\|}^{n+1-2p}y) + (-1)^p\, 2\,\widehat{\deg}(z \,.\, L_{\varrho,\|\cdot\|}^{n+1-2p}y)$$

$$+ (-1)^p\widehat{\deg}(z \,.\, L_{\varrho,\|\cdot\|}^{n+1-2p}z).$$

The first term is positive by assumption. The second term is zero since the Hodge decomposition

$$A^{p,p}(X) = \mathcal{H}^{p,p}(X) \oplus \mathrm{Im}(\partial) \oplus \mathrm{Im}(\partial^*)$$

is orthogonal and

$$\widehat{\deg}(a(\beta) \,.\, L_{\varrho,\|\cdot\|}^{n+1-2p}y) = \frac{1}{2}\int_{X(\mathbb{C})} \beta \wedge \omega(L_{\varrho,\|\cdot\|}^{n+1-2p}y).$$

The last term

$$(-1)^p\widehat{\deg}(a(\beta) \,.\, L_{\varrho,\|\cdot\|}^{n+1-2p}a(\beta)) = \frac{(-1)^p}{2}\int_{X(\mathbb{C})} dd^c(\beta) \wedge L^{n+1-2p}(\beta)$$

is positive by Theorem 1.2.ii). Our claim follows. □

We have seen in [10] that the Conjecture $A(X,\varrho)$ is related to the following conjecture of Beilinson. Consider the regulator map

$$\rho \colon CH^{p+1,p}(X) \longrightarrow H^{p,p}(X_{\mathbb{R}})$$

from (4) and the cycle class map

$$cl \colon CH^p(X) \longrightarrow H^{p,p}(X_{\mathbb{R}}). \tag{13}$$

Then, Beilinson's conjecture states [2, 3.7.b]:

CONJECTURE $B_p(X)$ (Beilinson).

$$H^{p,p}(X_{\mathbb{R}}) = \mathrm{Im}(\rho \otimes \mathbb{R}) \oplus \mathrm{Im}(cl \otimes \mathbb{R}).$$

Let $\overline{X} = (X, h)$ be an Arakelov variety. We assume that the scheme X is smooth over S. Under this assumption, we defined in [10, Sect. 7] a descending filtration F on $CH^p(\overline{X})_{\mathbb{R}}$ by

$$F^k CH^p(\overline{X})_{\mathbb{R}} = \begin{cases} CH^p(\overline{X})_{\mathbb{R}} & \text{if } k = 0 \\ \mathrm{Ker}\Big(cl\colon CH^p(\overline{X})_{\mathbb{R}} \to H^{p,p}(X_{\mathbb{R}})\Big) & \text{if } k = 1 \\ \mathrm{Im}\Big(a\colon H^{p-1,p-1}(X_{\mathbb{R}}) \to CH^p(\overline{X})_{\mathbb{R}}\Big) & \text{if } k = 2 \\ 0 & \text{if } k \geqslant 3. \end{cases} \tag{14}$$

It will be essential for the following proofs that we be able to split this filtration for abelian schemes and projective spaces.

4. Projective space

Let F be a number field with ring of integers $\mathcal{O}_F$ and set $\mathrm{S} = \mathrm{Spec}(\mathcal{O}_{\mathrm{F}})$. Let $\mathbb{P}^n$ be projective n-space over S. We denote the standard ample line bundle on $\mathbb{P}^n$ by $\mathcal{O}(1)$ and equip it with its canonical hermitian metric $\|.\|_F$. The goal of this section is the proof of the following theorem.

THEOREM 4.1. *Conjecture $A_p(\mathbb{P}^n, \mathcal{O}(1))$ holds for all p with the metric $\|.\|_F$.*

We collect some results to prove the theorem. First, we show that Beilinson's conjecture holds for projective space. We show this in fact more generally for Grassmannians. Let $\mathrm{G} = \mathrm{Grass_n}(\mathcal{O}_{\mathrm{S}}^{\mathrm{m+n}})$ be the Grassmannian over S representing the functor which assigns to each S-scheme T the set of locally free quotients of $\mathcal{O}_T^{m+n}$ of rank n.

LEMMA 4.2. *$B_p(\mathrm{Grass}_n(\mathcal{O}_S^{m+n}))$ holds for all p.*

Proof: Let us first compute the image of $cl \otimes \mathbb{R}$. We set $G_{\mathbb{Z}} = \mathrm{Grass}_n(\mathcal{O}_{\mathbb{Z}}^{m+n})$ and denote the canonical maps $G \to S$ and $G \to G_{\mathbb{Z}}$ by f and g. Let Σ be the set of complex embeddings of F. We have

$$H^{p,p}(G_{\mathbb{R}}) = H^{0,0}(S_{\mathbb{R}}) \otimes H^{p,p}(G_{\mathbb{Z},\mathbb{R}}) = (\oplus_{\sigma\in\Sigma} H^{p,p}(G_{\mathbb{Z},\mathbb{R}}))^{F_\infty}.$$

The cycle class map

$$cl\colon CH^0(S) = \mathbb{Z} \longrightarrow H^{0,0}(S_{\mathbb{R}}) = (\oplus_{\sigma\in\Sigma}\mathbb{R})^{F_\infty} \tag{15}$$

is the diagonal embedding and

$$cl \otimes \mathbb{R}\colon CH^p(G_{\mathbb{Z}})_{\mathbb{R}} \longrightarrow H^{p,p}(G_{\mathbb{Z},\mathbb{R}}) \tag{16}$$

is an isomorphism [8, 3.1.4]; [5, Ch. 14 and Ex. 19.1.11]. From [8, 3.1.4], we have an isomorphism

$$CH^0(S)_{\mathbb{Q}} \otimes CH^p(G_{\mathbb{Z}})_{\mathbb{Q}} \xrightarrow{\sim} CH^p(G)_{\mathbb{Q}}, \quad \alpha \otimes \beta \mapsto f^*\alpha \,.\, g^*\beta.$$

This implies that the image of

$$cl \otimes \mathbb{R}\colon CH^p(G)_{\mathbb{R}} \longrightarrow H^{p,p}(G_{\mathbb{R}}) = (\oplus_{\sigma\in\Sigma} H^{p,p}(G_{\mathbb{Z},\mathbb{R}}))^{F_\infty} \tag{17}$$

is precisely the diagonal. The map

$$\rho\colon CH^{1,0}(S) = \mathcal{O}_F^* \longrightarrow H^{0,0}(S_{\mathbb{R}}) = (\oplus_{\sigma\in\Sigma}\mathbb{R})^{F_\infty}$$

is, up to a factor of -2, the classical Dirichlet regulator map [6, 3.5.4]. Hence the image of ρ is precisely the hyperplane

$$\left\{(x_\sigma)_{\sigma\in\Sigma} \in (\oplus_{\sigma\in\Sigma}\mathbb{R})^{F_\infty}\colon \sum_{\sigma\in\Sigma} x_\sigma = 0\right\}.$$

From [8, 3.1.4], we get a canonical surjection

$$CH^{1,0}(S)_{\mathbb{Q}} \otimes CH^p(G_{\mathbb{Z}})_{\mathbb{Q}} \longrightarrow CH^{p+1,p}(G)_{\mathbb{Q}}, \quad \alpha \otimes \beta \mapsto f^*\alpha \cup g^*\beta$$

where the product comes from an identification of the different Chow groups with graded pieces of higher algebraic K-theory. Up to the factor -2, the map ρ is Beilinson's regulator map [6, 3.5]. We obtain from the multiplicativity of Beilinson's regulator the relation:

$$\rho(f^*\alpha \cup g^*\beta) = f^*\rho(\alpha) \cup g^* cl(\beta).$$

Now, our claim follows from

$$\begin{aligned}\rho(CH^{p+1,p}(G)_{\mathbb{R}}) &= f^*\rho(CH^{1,0}(S)_{\mathbb{R}}) \cup g^* cl(CH^p(G_{\mathbb{Z}})_{\mathbb{R}}) \\ &= \left\{(x_\sigma)_{\sigma\in\Sigma} \in (\oplus_{\sigma\in\Sigma} H^{p,p}(G_{\mathbb{Z},\mathbb{R}}))^{F_\infty} : \sum_{\sigma\in\Sigma} x_\sigma = 0\right\}. \qquad \square\end{aligned}$$

Let Q be the universal quotient bundle of $\mathcal{O}_S^{m+n}$ equipped with the metric $\|.\|_F$ induced from the trivial metric on $\mathcal{O}_S^{m+n}$. The first Chern form $c_1(Q, \|.\|_F)$ defines a Kähler metric invariant under F_∞ on $G(\mathbb{C})$ and $\overline{G} = (G, h)$ becomes an Arakelov variety.

COROLLARY 4.3. *For $p \geqslant 0$, there is an exact sequence*

$$0 \longrightarrow CH^{p-1}(G)_{\mathbb{R}} \xrightarrow{\tilde{a}} CH^p(\overline{G})_{\mathbb{R}} \xrightarrow{\zeta} CH^p(G)_{\mathbb{R}} \longrightarrow 0$$

where $\tilde{a}$ is the composition of the cycle class map (13) with a.

Proof: The statement is clear from (4) and the proof of the proposition. □

The line bundle $\varrho = \det(Q)$ is ample on G. Let L be the Lefschetz operator associated with $c_1(\varrho) = c_1(Q)$.

LEMMA 4.4.
i) The map $L^{mn-2p}\colon CH^p(G)_{\mathbb{R}} \longrightarrow CH^{mn-p}(G)_{\mathbb{R}}$ *is an isomorphism for* $2p \leqslant mn$.
ii) A primitive element $\alpha \in \mathrm{Ker}(L^{mn+1-2p}|_{CH^p(G)_{\mathbb{R}}})$ *satisfies*
$(-1)^p \deg(x \,.\, L^{mn-2p}(x)) > 0$.

Proof: Both statements hold for $H^{p,p}(G_{\mathbb{R}})$ as $\dim G(\mathbb{C}) = mn$. The lemma follows as $cl \otimes \mathbb{R}$ in (17) is the diagonal embedding. □

The Fubini-Study metric defines on projective space $\mathbb{P}^n$ over S in a canonical way the structure of an Arakelov variety. The fundamental form of the Kähler metric on $\mathbb{P}^n(\mathbb{C})$ is the Chern form $c_1(\mathcal{O}(1), ||.||_F)$. There is a second distinguished metric on $\mathcal{O}(1)$ that we now proceed to construct.

LEMMA 4.5. (Bost, Gillet, Soulé)

$$\widehat{\deg}(\hat{c}_1(\mathcal{O}(1), ||.||_F)^{n+1}) = [F\colon \mathbb{Q}] \cdot \sigma_n$$

where σ_n *is the Stoll number*

$$\sigma_n = \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{1}{j}.$$

Proof: [3, Lemma 3.3.1]. □

Now consider the metric $||.||_\sigma$ on $\mathcal{O}(1)$ given as $||.||_\sigma = \exp(\frac{\sigma+2\,\sigma_n}{2(n+1)}) \cdot ||.||_F$ for some $\sigma \in \mathbb{R}$. We obtain easily

$$\hat{c}_1(\mathcal{O}(1), ||.||_\sigma) = \hat{c}_1(\mathcal{O}(1), ||.||_F) - \frac{1}{(n+1)}a(\sigma + 2\sigma_n).$$

From

$$\hat{c}_1(\mathcal{O}(1), ||.||_\sigma)^{n+1} = \hat{c}_1(\mathcal{O}(1), ||.||_F)^{n+1} - (\sigma + 2\sigma_n)\,.\,a(c_1(\mathcal{O}(1), ||.||_F)^n)$$

and

$$\int_{\mathbb{P}^n(\mathbb{C})} c_1(\mathcal{O}(1), ||.||_F)^n = [F\colon \mathbb{Q}],$$

we get

$$\widehat{\deg}(\hat{c}_1(\mathcal{O}(1), \|.\|_\sigma)^{n+1}) = -[F:\mathbb{Q}] \cdot \frac{\sigma}{2}.$$

Since (8) is an isomorphism, we obtain that

$$\hat{c}_1(\mathcal{O}(1), \|.\|_\sigma)^{n+1} = 0$$

in $\widehat{CH}^{n+1}(\mathbb{P}^n)_{\mathbb{R}}$ precisely for $\sigma = 0$.

We want to prove $A_p(\mathbb{P}^n, \mathcal{O}(1))$. We are reduced to proving the prerequisites so that we may apply 3.1. In order to do this we consider a 'motivic' decomposition of the Arakelov Chow group $CH^p(\overline{\mathbb{P}^n})_{\mathbb{R}}$. Corollary 4.3 gives the exact sequence

$$0 \longrightarrow CH^{p-1}(\mathbb{P}^n)_{\mathbb{R}} \xrightarrow{\tilde{a}} CH^p(\overline{\mathbb{P}^n})_{\mathbb{R}} \xrightarrow{\zeta} CH^p(\mathbb{P}^n)_{\mathbb{R}} \longrightarrow 0. \tag{18}$$

We obtain a splitting $\tilde{\zeta}: CH^p(\mathbb{P}^n)_{\mathbb{R}} \longrightarrow CH^p(\overline{\mathbb{P}^n})_{\mathbb{R}}$ of this sequence if we take for $\tilde{\zeta}$ the unique morphism of $\mathbb{R}$-algebras mapping $c_1(\mathcal{O}(1))$ to $\hat{c}_1(\mathcal{O}(1), \|.\|_0)$. We define

$$CH^p(R^i(\overline{\mathbb{P}^n/S}))_{\mathbb{R}} = \begin{cases} \mathrm{Im}(\tilde{\zeta}) & \text{if } i = 2p, \\ \mathrm{Im}(\tilde{a}) & \text{if } i = 2p-2, \\ 0 & \text{if } i \notin \{2p, 2p-2\}. \end{cases} \tag{19}$$

This notation is inspired by the motivic considerations in [9] and [10]. The definition $CH^p(R^{2p-1}(\overline{\mathbb{P}^n/S}))_{\mathbb{R}} = 0$ reflects the fact that there are no homologically trivial cycles on $\mathbb{P}^n$. Combinig this fact with (18), we obtain a splitting of our filtration

$$F^k CH^p(\overline{\mathbb{P}^n})_{\mathbb{R}} = \bigoplus_{i \leqslant 2p-k} CH^p(R^i(\overline{\mathbb{P}^n/S}))_{\mathbb{R}}.$$

The operator

$$N: CH^p(\overline{\mathbb{P}^n})_{\mathbb{R}} \xrightarrow{\zeta} CH^p(\mathbb{P}^n)_{\mathbb{R}} \xrightarrow{\tilde{a}} CH^{p+1}(\overline{\mathbb{P}^n})_{\mathbb{R}}$$

was introduced in [10, Sect. 9]. It satisfies $N^2 = 0$ and induces an isomorphism

$$N: CH^p(R^{2p}(\overline{\mathbb{P}^n/S}))_{\mathbb{R}} \longrightarrow CH^{p+1}(R^{2p}(\overline{\mathbb{P}^n/S}))_{\mathbb{R}}.$$

We need some more notation. For $k \in \{0, 2\}$, let

$$\overline{\tau}_k: CH^p(\overline{\mathbb{P}^n})_{\mathbb{R}} \longrightarrow CH^p(R^{2p-k}(\overline{\mathbb{P}^n/S}))_{\mathbb{R}}$$

be the canonical projection, and let

$$\tau_k: CH^p(\overline{\mathbb{P}^n})_{\mathbb{R}} \longrightarrow CH^q(\mathbb{P}^n)_{\mathbb{R}}\ ,\ q = (2p-k)/2$$

be ζ for $k = 0$ and the composition of $\overline{\tau}_2$ with the inverse of $\tilde{a}$ for $k = 2$. We set $L_F = L_{\mathcal{O}(1),\|\cdot\|_F}$, $L_0 = L_{\mathcal{O}(1),\|\cdot\|_0}$, $L = L_{c_1(\mathcal{O}(1))}$, and $\widetilde{\sigma_n} = \frac{2\sigma_n}{n+1}$. The following formulas are easily verified

$$
\begin{aligned}
L_0 \circ \tilde{\zeta} &= \tilde{\zeta} \circ L, & (20)\\
L^r &= \tau_2 \circ N \circ L_0^r \circ \tilde{\zeta}, & (21)\\
L_F &= L_0 + \widetilde{\sigma_n}\, N, & (22)\\
L_F^{r+1} &= L_0^{r+1} + \widetilde{\sigma_n}(r+1)\, N L_0^r, & (23)\\
\tau_2 \circ L_F^{r+1} &= L^{r+1} \circ \tau_2 + \widetilde{\sigma_n}(r+1)\, L^r \circ \tau_0. & (24)
\end{aligned}
$$

Now we are prepared to prove $A_p(\mathbb{P}^n, \mathcal{O}(1))$. The statement 3.1.i) can be shown as in the proof of [10, Th. 12.1] or as follows. Consider the commutative diagram

$$
\begin{array}{ccccccc}
0 \longrightarrow & CH^{p-1}(\mathbb{P}^n)_{\mathbb{R}} & \xrightarrow{\tilde{a}} & CH^p(\overline{\mathbb{P}^n})_{\mathbb{R}} & \xrightarrow{\zeta} & CH^p(\mathbb{P}^n)_{\mathbb{R}} & \longrightarrow 0\\
& \downarrow L^{n+1-2p} & & \downarrow L_F^{n+1-2p} & & \downarrow L^{n+1-2p} & \\
0 \longrightarrow & CH^{n-p}(\mathbb{P}^n)_{\mathbb{R}} & \xrightarrow{\tilde{a}} & CH^{n+1-p}(\overline{\mathbb{P}^n})_{\mathbb{R}} & \xrightarrow{\zeta} & CH^{n+1-p}(\mathbb{P}^n)_{\mathbb{R}} & \longrightarrow 0
\end{array}
$$

and apply 4.4 to the outer vertical arrows. The right arrow is surjective and its kernel consists precisely of the primitive elements

$$CH^p_{\mathrm{prim}}(\mathbb{P}^n)_{\mathbb{R}} = \mathrm{Ker}(L^{n+1-2p}|_{CH^p(\mathbb{P}^n)_{\mathbb{R}}}).$$

The left arrow is injective and its cokernel is canonically isomorphic to group of coprime elements $CH^{n-p}_{\mathrm{coprim}}(\mathbb{P}^n)_{\mathbb{R}}$. The snake lemma gives us an exact sequence

$$
\begin{aligned}
&0 \longrightarrow \mathrm{Ker}(L_F^{n+1-2p}) \longrightarrow CH^p_{\mathrm{prim}}(\mathbb{P}^n)_{\mathbb{R}} \xrightarrow{\alpha}\\
&\qquad CH^{n-p}_{\mathrm{coprim}}(\mathbb{P}^n)_{\mathbb{R}} \longrightarrow \mathrm{Coker}(L_F^{n+1-2p}) \longrightarrow 0,
\end{aligned}
$$

and we have to show that α is an isomorphism. For $x \in CH^p_{\mathrm{prim}}(\mathbb{P}^n)_{\mathbb{R}}$, we see from (20) that $L_0^{n+1-2p}(\tilde{\zeta}(x))$ vanishes. This gives with (21) and (23)

$$\alpha(x) = \tau_2 \circ L_F^{n+1-2p} \circ \tilde{\zeta}(x) = \widetilde{\sigma_n}(n+1-2p)\, L^{n-2p}(x).$$

Statement 3.1.i) follows since we know from 4.4 and ordinary Lefschetz theory that

$$L^{n-2p}\colon CH^p_{\mathrm{prim}}(\mathbb{P}^n)_{\mathbb{R}} \longrightarrow CH^{n-p}_{\mathrm{coprim}}(\mathbb{P}^n)_{\mathbb{R}} \tag{25}$$

is an isomorphism.

In order to show 3.1.ii), we have to examine the condition for an element in $CH^p(\overline{\mathbb{P}^n})_{\mathbb{R}}$ to be primitive more closely. Recall that $x \in CH^p(\overline{\mathbb{P}^n})_{\mathbb{R}}$ is primitive if

$L_F^{n+2-2p}(x) = 0$ while $x \in CH^p(\mathbb{P})_{\mathbb{R}}$ is called primitive if $L^{n+1-2p}(x) = 0$. Let $x \in CH^p(\overline{\mathbb{P}^n})_{\mathbb{R}}$ be a non zero primitive element. In this case,

$$L^{n+2-2p}(\tau_0(x)) = \tau_0(L_F^{n+2-2p}(x)) = 0$$

implies

$$\tau_0(x) = \eta_0 + L\eta_1 \tag{26}$$

for primitive elements $\eta_i \in CH(\mathbb{P}^n)_{\mathbb{R}}$. We use (24) to compute

$$\tau_2(L_F^{n+2-2p}x) = L^{n+1-2p}(L\tau_2(x) + \widetilde{\sigma_n}(n+2-2p)\tau_0(x)).$$

The left-hand side of this equation vanishes. We get

$$L^{n+2-2p}(\tau_2(x) + \widetilde{\sigma_n}(n+2-2p)\eta_1) = 0,$$

if we apply (26), and observe that η_0 is primitive. This yields

$$\tau_2(x) = -\widetilde{\sigma_n}(n+2-2p)\eta_1 \tag{27}$$

as L^{n+2-2p} is an isomorphism on $CH^{p-1}(\mathbb{P}^n)_{\mathbb{R}}$. Recall that we have to show that

$$(-1)^p\widehat{\deg}(x \,.\, L_F^{n+1-2p}x) =$$
$$= (-1)^p\widehat{\deg}\Big((L_0^{n+1-2p} + \widetilde{\sigma_n}(n+1-2p)NL_0^{n-2p})(\overline{\tau}_0(x) + \overline{\tau}_2(x)^2)\Big)$$

is strictly positive. For the terms which occur, we get for $r, s \in \{0, 2\}$ that

$$\overline{\tau}_r(x) \,.\, L_0^{n+1-2p}\overline{\tau}_s(x) \in CH^{n+1}(R^{2n+2-r-s}(\overline{\mathbb{P}^n/S}))_{\mathbb{R}}$$

and

$$\overline{\tau}_r(x) \,.\, NL_0^{n-2p}\overline{\tau}_s(x) \in CH^{n+1}(R^{2n-r-s}(\overline{\mathbb{P}^n/S}))_{\mathbb{R}}$$

while $\widehat{\deg}$ defines an isomorphism

$$\widehat{\deg}\colon CH^{n+1}(R^{2n}(\overline{\mathbb{P}^n/S}))_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}.$$

This implies

$$\begin{aligned}
&(-1)^p\widehat{\deg}(x \,.\, L_F^{n+1-2p}x)\\
&\quad = (-1)^p\widehat{\deg}(2\overline{\tau}_0(x) \,.\, L_0^{n+1-2p}\overline{\tau}_2(x) + \widetilde{\sigma_n}(n+1-2p)NL_0^{n-2p}\overline{\tau}_0(x)^2)\\
&\quad = (-1)^p\deg(2\tau_0(x) \,.\, L^{n+1-2p}\tau_2(x) + \widetilde{\sigma_n}(n+1-2p)L^{n-2p}\tau_0(x)^2).
\end{aligned}$$

We replace $\tau_0(x)$ and $\tau_2(x)$ by η_0 and η_1 and obtain for the last term

$$(-1)^p\widetilde{\sigma_n}(n+1-2p)\deg(\eta_0 \,.\, L^{n-2p}\eta_0)+ \\ +(-1)^{p-1}\widetilde{\sigma_n}(n+3-2p)\deg(\eta_1 \,.\, L^{n+2-2p}\eta_1). \tag{28}$$

if we observe again that η_0 is primitive. The last expression is always non-negative by 4.4., since $\widetilde{\sigma_n} > 0$. It is zero if and only if $\eta_0 = 0 = \eta_1$ or if $2p = n+1$ and $\eta_1 = 0$. There are no primitive elements in $CH^p(\mathbb{P}^n)_{\mathbb{R}}$ for $2p = n+1$. Hence the vanishing of (28) would imply $\tau_0(x) = 0$ and from 4.4 i) and

$$L^{n+2-2p}\tau_2(x) = \tau_2\, L_F^{n+2-2p}(x) = 0,$$

we would get $\tau_2(x) = 0$ and $x = 0$. As a consequence (28) is positive for every primitive non zero element x. □

5. Abelian schemes

Let A be an abelian scheme over S of relative dimension g. The associated complex manifold $A(\mathbb{C})$ is a disjoint union of complex abelian varieties. We fix on $A(\mathbb{C})$ a F_∞-invariant Kähler metric which is translation invariant on each of these abelian varieties. In this section, we will discuss some conjectures which imply Conjecture $A_p(A, \varrho)$. Let mult(n) denote n-multiplication on A. We set

$$CH^p(R^i(\overline{A/S}))_{\mathbb{R}} = \{\alpha \in CH^p(\overline{A})_{\mathbb{R}} \mid \mathrm{mult}(n)^*(\alpha) = n^i\,\alpha\}.$$

We have seen in [10, Sect. 8] that there is a direct sum decomposition

$$CH^p(\overline{A})_{\mathbb{R}} = \bigoplus_{i=0}^{2g} CH^p(R^i(\overline{A/S}))_{\mathbb{R}}. \tag{29}$$

The following conjecture from [10, Sect. 9] relates this decomposition to the filtration F defined in (14).

CONJECTURE C_p (A). $F^kCH^p(\overline{A})_{\mathbb{R}} = \bigoplus_{i\leqslant 2p-k} CH^p(R^i(\overline{A/S}))_{\mathbb{R}}$.

Recall that a line bundle ϱ on A is called symmetric if mult$(-1)^*\varrho \cong \varrho$ in Pic(A). Let ϱ be a line bundle on A that is ample, symmetric, and rigidified along the zero section. Let $\hat{A}$ be the dual abelian scheme and $\varphi\colon A \to \hat{A}$ the polarization associated with ϱ. The morphism φ is an isogeny. We denote by $v(\varrho)$ the square root of its degree which is a positive integer.

A rigidified line bundle on an abelian scheme over S carries a canonical hermitian metric $||.||_{\mathrm{cube}}$, called the cubical metric. It is characterized by the facts that the hermitian line bundle satisfies the theorem of the cube and has a translation invariant curvature [3, 3.3.4]; [10, Sect. 7]. We use the cubical metric to define the cubical height. Let $\mathcal{L}$ be a rigidified Poincaré bundle on $A \times_S \hat{A}$ and equip $\mathcal{L}$ with the cubical metric $||.||_{\text{cube}}$. Let

$$Z \in Z^p(A \times_S \hat{A}) \otimes_{\mathbb{Z}} \mathbb{R}$$

be a cycle of codimension p. The arithmetic fundamental class $[Z]$ of Z is the class of (Z, g_Z) in $\widehat{CH}^p(\overline{A \times_S \hat{A}})_{\mathbb{R}}$ where g_Z is a normalized Green current for Z determined by the equations

$$dd^c g_Z + \delta_Z = H(\delta_Z) \tag{30}$$

and

$$H(g_Z) = 0. \tag{31}$$

The cubical height of Z is by definition [3, 3.3.4]

$$h_{\text{cube}}(Z) = \widehat{\deg}([Z] \,.\, \hat{c}_1(\mathcal{L}, ||.||_{\text{cube}})^{2g+1-p}).$$

A cycle $Z \in Z^p(A) \otimes_{\mathbb{Z}} \mathbb{R}$ is homologically trivial if the cycle class $cl([Z])$ in $H^{p,p}(A_{\mathbb{R}})$ is zero.

CONJECTURE D_p (A, ϱ). *We have*

$$(-1)^{g+p} h_{\text{cube}}((id \times_S \varphi)_*(Z \times_S Z)) > 0 \tag{32}$$

for every homologically trivial cycle $Z \in Z^p(A) \otimes_{\mathbb{Z}} \mathbb{R}$ *whose class in* $CH^p(A)_{\mathbb{R}}$ *is not zero.*

The motivation for our conjecture arises in the proof of the following theorem.

THEOREM 5.1. $B_{k-1}(A)$, $C_k(A)$, $k \in \{p, g+1-p\}$, *and* $D_p(A, \varrho)$ *imply* $A_p(A, \varrho)$.

Proof: We have already seen in [10, Sect. 12] that $A_p(A, \varrho)$ i) follows from $B_{p-1}(A)$, $B_{g-p}(A)$, $C_p(A)$, and $C_{g+1-p}(A)$ if we equip ϱ with the metric

$$||\cdot||_\varepsilon = \exp(-\varepsilon/2)\ ||\cdot||_{\text{cube}}$$

for some $\varepsilon \neq 0$. We fix some $\varepsilon > 0$. Let $2p < n + 1$. It is sufficient to show 3.1.ii). We decompose $CH^p(\overline{A})_{\mathbb{R}}$ into the direct sum of

$$CH^p(R^{2p}(\overline{A/S}))_{\mathbb{R}} \oplus CH^p(R^{2p-2}(\overline{A/S}))_{\mathbb{R}} \tag{33}$$

and

$$CH^p(R^{2p-1}(\overline{A/S}))_{\mathbb{R}}. \tag{34}$$

This decomposition is orthogonal with respect to the arithmetic intersection product as the decomposition (29) is compatible with this product and we have [10, 9.3]

$$CH^{g+1}(\overline{A})_{\mathbb{R}} = CH^{g+1}(R^{2g}(\overline{A/S}))_{\mathbb{R}}. \tag{35}$$

In addition, an element $\alpha \in CH^p(\overline{A})_{\mathbb{R}}$ satisfies $L^{n+2-2p}_{\varrho,\|\cdot\|_\varepsilon}(x) = 0$ if and only if the same is true for the projections of α in (33) and (34). Hence, it is sufficient to show 3.1.ii) for elements in (33) and (34). For elements in (33), we can proceed along the same lines as in the proof of Theorem 4.1. For α in (34), our claim will follow from $D_p(A, \varrho)$. The arithmetic Fourier transform is introduced in [1] and [10, Sect. 7]. It defines an isomorphism

$$F_{\overline{CH}}\colon CH(\overline{A})_{\mathbb{R}} \xrightarrow{\sim} CH(\overline{\hat{A}})_{\mathbb{R}}$$

which is related to the Lefschetz operator $L_{\varrho,\|\cdot\|_{\mathrm{cube}}}$ via the formula

$$\frac{1}{v(\varrho)}\varphi^* F_{\overline{CH}}(\alpha) = \frac{(-1)^g}{(g+1-2p)!}\, L^{g+1-2p}_{\varrho,\|\cdot\|_{\mathrm{cube}}}(\alpha) \tag{36}$$

for $\alpha \in \mathrm{Ker}(L^{g+2-2p}_{\varrho,\|\cdot\|_{\mathrm{cube}}}|_{CH^p(R^{2p-1}(\overline{A/S}))_{\mathbb{R}}})$. Formula (36) can be established as follows. The formula holds after an application of ζ as a consequence of [9, Th. 6.2]. Then, it follows from [10, 8.3] since both sides of (36) are contained in $CH^{g+1-p}(R^{2g+1-2p}(\overline{A/S}))_{\mathbb{R}}$.

For $\alpha \in CH^p(R^{2p-1}(\overline{A/S}))_{\mathbb{R}}$, we have

$$L_{\varrho,\|\cdot\|_\varepsilon}(\alpha) = L_{\varrho,\|\cdot\|_{\mathrm{cube}}}(\alpha) + a(\varepsilon)\,.\,\alpha = L_{\varrho,\|\cdot\|_{\mathrm{cube}}}(\alpha)$$

and we get easily from [9, Sect. 2]; [10, 8.3]

$$F_{\overline{CH}}(\alpha) = p_{2*}(p_1^*\alpha\,.\,\frac{1}{(2g+1-2p)!}\hat{c}_1(\mathcal{L}, \|\cdot\|_{\mathrm{cube}})^{2g+1-2p}).$$

These equations yield with base change and projection formula

$$\begin{aligned}
&(g+1-2p)!\varphi_*(\alpha)\,.\,F_{\overline{CH}}(\alpha)\\
&= \varphi_*(\alpha)\,.\,p_{2*}(p_1^*\alpha\,.\,\hat{c}_1(\mathcal{L}, \|\cdot\|_{\mathrm{cube}})^{2g+1-2p})\\
&= p_{2*}(p_2^*\varphi_*\alpha\,.\,p_1^*\alpha\,.\,\hat{c}_1(\mathcal{L}, \|\cdot\|_{\mathrm{cube}})^{2g+1-2p})\\
&= p_{2*}(id\times_S\varphi)_*(p_1^*\alpha\,.\,p_2^*\alpha\,.\,(id\times_S\varphi)^*\hat{c}_1(\mathcal{L}, \|\cdot\|_{\mathrm{cube}})^{2g+1-2p}).
\end{aligned}$$

For a primitive element $\alpha \in CH^p(R^{2p-1}(\overline{A/S}))_{\mathbb{R}}$, we obtain

$$(-1)^p\widehat{\deg}(\alpha\,.\,L^{g+1-2p}_{\varrho,\|\cdot\|_\varepsilon}(\alpha))$$

$$= (-1)^{g+p}\frac{(g+1-2p)!}{v(\varrho)}\widehat{\deg}(\alpha\,.\,\varphi^*F_{\overline{CH}}(\alpha))$$

$$= (-1)^{g+p}\frac{(g+1-2p)!}{v(\varrho)}\widehat{\deg}(\varphi_*(\alpha)\,.\,F_{\overline{CH}}(\alpha))$$

$$= \frac{(-1)^{g+p}}{v(\varrho)}\widehat{\deg}((id\times_S\varphi)_*(p_1^*\alpha\,.\,p_2^*\alpha)\,.\,\hat{c}_1(\mathcal{L},\|.\|_{\text{cube}})^{2g+1-2p}).$$

As a consequence of $C_p(A)$, any non trivial class in $CH^p(R^{2p-1}(\overline{A/S}))_{\mathbb{R}}$ is represented modulo $F^2CH^p(\overline{A})_{\mathbb{R}}$ by a fundamental class $[Z]$ of some homologically trivial cycle $Z\in Z^p(A)\otimes_{\mathbb{Z}}\mathbb{R}$ whose class in $CH^p(A)_{\mathbb{R}}$ is not trivial. Formula (5) implies

$$p_1^*\alpha\,.\,p_2^*\alpha = p_1^*[Z]\,.\,p_2^*[Z].$$

Therefore, our claim follows from $D_p\ (A,\varrho)$, if we show that

$$[(id\times_S\varphi)_*(Z\times_S Z)] = (id\times_S\varphi)_*(p_1^*[Z]\,.\,p_2^*[Z]).$$

This equality is easy seen to be a consequence of

$$[(Z\times_S W)] = p_1^*[Z]\,.\,p_2^*[W] \tag{37}$$

for all irreducible closed subschemes Z and W of A. We denote by g_Z and g_W normalized Green currents for Z and W. In order to prove (37), it suffices to show that the product of Green currents [6, Sect. 2]

$$(p_1^*g_Z)*(p_2^*g_W) \tag{38}$$

represents the class of a normalized Green current for $Z\times_S W$. Therefore, we have to check (30) and (31). Condition (30) holds in our situation as the product of harmonic forms is still harmonic. Condition (31) is satisfied if (38) vanishes on harmonic forms. Let α be a harmonic form of suitable degree on $(A\times_S A)(\mathbb{C})$. According to the Künneth formula, we can write $\alpha = p_1^*\alpha_1\wedge p_2^*\alpha_2$ with harmonic forms α_i on $A(\mathbb{C})$. Now our claim follows by a direct calculation from the definitions and Fubini's theorem. □

PROPOSITION 5.2. *$D_1\ (A,\varrho)$ holds for every symmetric ϱ as above and every homologically trivial divisor $Z\in Z'(A)$ whose class in $CH'(A)_{\mathbb{R}}$ is not zero.*

Proof: We see from the proof of the theorem that it is sufficient to show

$$(-1)^{g+1}\widehat{\deg}(\alpha\,.\,\varphi^*F_{\overline{CH}}(\alpha)) > 0 \tag{39}$$

for the fundamental class $\alpha=[Z]$ of any divisors Z in $Z^1(A)$ of the type considered in the conjecture. It is shown in [10, 9.3] that $Gr_F^1CH^1(\overline{A})_{\mathbb{Q}}$ is canonically isomorphic to $\mathrm{Pic}^0(A_F)_{\mathbb{Q}}$. Hence an integral multiple of the class $[Z]$ is represented modulo $F^2CH^1(\overline{A})_{\mathbb{R}}$ by an element

$$l_{\hat{a}} = (id \times_S \hat{a})^* \hat{c}_1(\mathcal{L}, \|.\|_{\text{cube}}) \in CH^1(R^1(\overline{A/S}))_{\mathbb{R}}$$

for a suitable $\hat{a} \in \hat{A}(S) = \hat{A}_F(F)$ and it becomes our task to show

$$(-1)^{g+1}\widehat{\deg}(l_{\hat{a}} \,.\, \varphi^* F_{\overline{CH}}(l_{\hat{a}})) > 0. \tag{40}$$

It is sufficient to show (40) after a finite extension of our ground field F. Hence we can assume that there exists an $a \in A(S)$ such that $\hat{a} = \varphi \circ a$. We set

$$l_a = (a \times_S id)^* \hat{c}_1(\mathcal{L}, \|.\|_{\text{cube}})$$

and get as $(id \times_S \varphi)^* \hat{c}_1(\mathcal{L}, \|.\|_{\text{cube}})$ is equal to its transpose

$$\begin{aligned} \varphi_* l_{\hat{a}} &= \varphi_*(id \times_S a)^*(id \times_S \varphi)^* \hat{c}_1(\mathcal{L}, \|.\|_{\text{cube}}) \\ &= (a \times_S id)^*(id \times_S \varphi)_*(id \times_S \varphi)^* \hat{c}_1(\mathcal{L}, \|.\|_{\text{cube}}) \\ &= v(\varrho)^2 l_a. \end{aligned}$$

From $\varphi^* \circ F_{\overline{CH}} = F_{\overline{CH}} \circ \varphi_*$ [4, Prop. 2.11]; [10, 8.3], we obtain

$$\begin{aligned} (-1)^{g+1}\widehat{\deg}(l_{\hat{a}} \,.\, \varphi^* F_{\overline{CH}}(l_{\hat{a}})) &= (-1)^{g+1}\widehat{\deg}(l_{\hat{a}} \,.\, F_{\overline{CH}}(\varphi_* l_{\hat{a}})) \\ &= (-1)^{g+1} v(\varrho)^2 \widehat{\deg}(l_{\hat{a}} \,.\, F_{\overline{CH}}(\exp(l_a))). \end{aligned}$$

The last equality follows from $F_{\overline{CH}}(l_a^q) \in CH^g(R^{2g-q}(A/S))_{\mathbb{R}}$ and (35). We have seen in the proof of [10, Prop. 8.1] that

$$F_{\overline{CH}}(a_\#[S]) = \exp(l_a)$$

where $a_\#$ is the push forward from Section 2. The Fourier inversion formula [10, 8.4] implies

$$F_{\overline{CH}}(\exp(l_a)) = (-1)^g(-a)_\#[S].$$

We obtain $\widehat{\deg} = \widehat{\deg} \circ (id_S)_\#$ as an easy consequence of the definitions. Let $\lambda\colon A \to S$ be the structure morphism. In combination with $\lambda_\# = (id_S)_\# \circ \lambda_*$, and $\lambda_\# \circ (-a)_\# = (id_S)_\#$ [10, Sect. 4], we get

$$\widehat{\deg} \circ (-a)_\# = \widehat{\deg}.$$

Now we use the projection formula for $(-a)_\#$, obtaining

$$\begin{aligned} &(-1)^{g+1}\widehat{\deg}(l_{\hat{a}} \,.\, \varphi^* F_{\overline{CH}}(l_{\hat{a}})) \\ &\quad = -v(\varrho)^2 \widehat{\deg}(l_{\hat{a}} \,.\, (-a)_\#[S]) \end{aligned}$$

$$= -v(\varrho)^2\widehat{\deg}((-a)_\#((-a)^*(id \times_S \hat{a})^*\hat{c}_1(\mathcal{L}, \|.\|_{\text{cube}}))$$

$$= v(\varrho)^2\widehat{\deg}((a \times_S \hat{a})^*\hat{c}_1(\mathcal{L}, \|.\|_{\text{cube}})).$$

The expression $\widehat{\deg}((a \times_S \hat{a})^*\hat{c}_1(\mathcal{L}, \|.\|_{\text{cube}})$ gives the Néron-Tate height of the point a [3, 3.3.4]. This height is strictly positive if a is not torsion, i.e. if the elements $l_{\hat{a}}$ and $[Z]$ are not zero in $CH^1(A)_{\mathbb{Q}}$. □

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