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A Bezout theorem for determinantal modules

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Our starting point is the classical Bezout's theorem. Given homogeneous polynomials $F_i(x_1, \ldots, x_n)$ of degree d_i for $i = 1, \ldots, n$, such that the only common zero of all of the F_i is 0. Then, Bezout's theorem counts the number of solutions (with multiplicities) to the algebraic equations $F_1 = c_1, \ldots, F_n = c_n$ (some $c_i \neq 0$). This number equals Πd_i and is independent of the particular F_i. It can be understood as the algebraic invariant $\dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]/(F_1, \ldots, F_n)$. This dimension also equals the dimension of the local algebra $\mathcal{O}_{\mathbb{C}^n,0}/(F_1, \ldots, F_n)$.

Our goal in this paper is to extend this algebraic part of the result to a general class of determinantal Cohen–Macaulay modules. The lengths of such modules appear in various geometric analogues of Bezout's theorem involving the computations of singular Milnor numbers and higher multiplicities for discriminants, linear and nonlinear arrangements, nonisolated complete intersections, generalized Zariski examples, etc. (see [DM] and [D1]).

We derive explicit formulas of these lengths in the (semi-) weighted homogeneous case. By classical results of Macaulay [Mc], these lengths are intrinsic algebraic numbers which only depend on the weights and will be called "Macaulay–Bezout numbers". First, in the homogeneous case we shall prove in Theorem 1 that these Macaulay–Bezout numbers are given by the elementary symmetric functions of the homogeneous degrees. This provides the natural extension of the formula in the Bezout theorem which is given by the nth elementary symmetric functions, i.e. the product, of the homogeneous degrees.

For the general weighted homogeneous case, we must introduce a certain universal function τ defined for all matrices. We think of the elementary symmetric functions as universal objects from which we can obtain all symmetric functions. They also can be characterized as the class of functions satisfying a generalized Pascal relation, which itself extends the classical Pascal's triangle. This function τ decouples the symmetries of the elementary symmetric functions but is universal among functions which satisfy a generalized Pascal relation. It is a type of non-skew-symmetric determinant (except it is defined for all matrices) and encodes all elementary symmetric functions.

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Theorem 2 then gives a formula for the weighted Macaulay–Bezout numbers in terms of this function τ applied to a "degree matrix". This then allows us to explicitly compute the singular Milnor numbers and higher multiplicities in [D1] leading to new results such as an extension of Terao's factorization theorem for free arrangements [T].

In Section 1, we define the Macaulay–Bezout numbers and show that classical results of Macaulay prove they are well-defined. In Section 2 we consider generalized Pascal relations and introduce τ and derive its properties. The generalized Pascal relation allows us to prove results by "Pascal induction" which generalizes the classical Pascal triangle. In Section 3, we describe how to define the degree matrix and state the general formula for the Macaulay–Bezout numbers in the weighted homogeneous case. Lastly, in Section 4, we prove the key Lemma 3.7, which establishes the generalized Pascal relation for dimensions of determinantally defined modules and rings, using results of Macaulay [Mc] and Northcott [No].

It has been pointed out to us by Terry Wall that there are related results [A], [G], and [W] which compute the dimensions of the weighted parts for various modules occurring for complete intersections. These results are complementary to the ones we obtain in the sense that there is no simple way to pass from the dimensions of each weighted part to a simple formula for length of the whole module, nor in the reverse direction.

1. Macaulay–Bezout numbers

We begin with the definition of Macaulay–Bezout numbers as the lengths of certain determinantal modules, and verify that they only depend on certain degrees (and weights).

DEFINITION 1.1. Let $F_i \in (\mathcal{O}_{\mathbb{C}^n,0})^k$ for $i = 1, \ldots, n+k-1$ be homogeneous elements of degree d_i (i.e. $F_i = (F_{i1}, \ldots, F_{ik})$ with each F_{ij} homogeneous of degree d_i). We let $\mathbf{d} = (d_1, \ldots, d_{n+k-1})$ and define the *Macaulay–Bezout number*

$$B_n(\mathbf{d}) = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_1, \ldots, F_{n+k-1}\}$$

provided this number is finite.

NOTATION. We may write the d_i in increasing order of magnitude; if the first m_1 consecutive values $d_i = b_1$, the next m_2 consecutive values $d_i = b_2$, etc. then we will write $B_n(b_1^{m_1}, \ldots, b_r^{m_r})$.

It will follow from Proposition 1.3 that this number is independent of the F_i provided that the number is finite. We first extend the definition to weighted versions of these numbers.

Suppose we have assigned weights $\mathrm{wt}(x_i) = a_i > 0$, $\mathrm{wt}(y_i) = c_i$ (where we allow weights which are nonpositive), so that the $F_i \in (\mathcal{O}_{\mathbb{C}^n,0})^k$ are weighted homogeneous with $\mathrm{wt}(F_i) = d_i$ (i.e. $F_i = (F_{i1}, \ldots, F_{ik})$ with each F_{ij} weighted homogeneous of weighted degree $d_i + c_j \geqslant 0$). We let $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{c} =$

$(c_1, \dots, c_k)$ and $\mathbf{d} = (d_1, \dots, d_{n+k-1})$ as above.

DEFINITION 1.2. We define the *weighted Macaulay–Bezout number*

$$B_n(\mathbf{d};\ \mathbf{a},\ \mathbf{c}) = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_1, \dots, F_{n+k-1}\}$$

provided this number is finite.

If we have arranged weights so that all $c_i = 0$, then we abbreviate it by $B_n(\mathbf{d};\ \mathbf{a})$.

Likewise in addition to the module dimensions we can also define the dimensions of the corresponding algebras. We also let $I_n(F_1, \dots, F_{n+k-1})$ denote the ideal generated by the $k \times k$ minors of the $k \times (n + k - 1)$ matrix (F_{ij}); and denote

$$A_n(\mathbf{d};\ \mathbf{a},\ \mathbf{c}) = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/I_n(F_1, \dots, F_{n+k-1})).$$

As earlier, we denote this in the special cases of homogeneous F_i by $A_n(\mathbf{d})$ or if all $c_i = 0$, by $A_n(\mathbf{d};\ \mathbf{a})$. That these numbers are well-defined is a consequence of the following proposition.

PROPOSITION 1.3. *Let* F_i, $F_i' \in (\mathcal{O}_{\mathbb{C}^n,0})^k$ *for* $i = 1, \dots, n + k - 1$ *be weighted homogeneous polynomial germs of weighted degrees* d_i. *Then*

- (i) *if both numbers* $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_1, \dots, F_{n+k-1}\}$ *and* $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_1', \dots, F_{n+k-1}'\}$ *are finite then they are equal*;
- (ii) *if both numbers* $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/I_n(F_1, \dots, F_{n+k-1}))$ *and* $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/ I_n(F_1', \dots, F_{n+k-1}'))$ *are finite then they are equal; and*
- (iii) $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_1, \dots, F_{n+k-1}\} < \infty$ iff $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/I_n(F_1, \dots, F_{n+k-1})) < \infty$.

Proof. The proof of (i) and (ii) are virtually identical so we give it for (i). Let N be the highest degree term appearing in any F_{ij} or F_{ij}'. First, the set of polynomial $(F_1, \dots, F_{n+k-1})$ with $F_i \in (\mathcal{O}_{\mathbb{C}^n,0})^k$ weighted homogeneous and $\mathrm{wt}(F_i) = d_i$ form a finite dimensional vector space. Using Nakayama's lemma it follows that there is a Zariski open subset of such $(n + k - 1)$-tuples for which $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_1, \dots, F_{n+k-1}\} < \infty$. If this set is nonempty, then to prove the proposition, it is sufficient to show that the dimension is locally constant.

Suppose $(F_{1,t}, \dots, F_{n+k-1,t})$ denotes a constant weight deformation of one such $(F_{1,0}, \dots, F_{n+k-1,0})$. We let

$$\mathcal{M} = (\mathcal{O}_{\mathbb{C}^{n+1},0})^k / \mathcal{O}_{\mathbb{C}^{n+1},0}\{F_{1,t}, \dots, F_{n+k-1,t}\}$$

defines a module on $\mathbb{C}^{n+1}$. The support of this module is defined by the vanishing of the $k \times k$ minors of the $k \times (n + k - 1)$ matrix $(F_{i,j,t})$. Since by assumption, on $\mathbb{C}^n \times \{0\}$ the support is at the origin, $\mathrm{supp}(\mathcal{M})$ is 1-dimensional (and hence 0-dimensional in any $\mathbb{C}^n \times \{t\}$). Thus, as in Proposition 5.2 of [DM], the push-forward

$\pi_*\mathcal{M}$ for the projection $\pi\colon \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$ is Cohen–Macaulay of dim $= 1$ and thus free. Thus, $\dim_{\mathbb{C}}(\pi_*\mathcal{M}/\underline{m}_{t-t_0}\pi_*\mathcal{M})$ is constant. Since the deformation has constant weight, the minors of $F_{1,t}, \dots, F_{n+k-1,t}$ will only simultaneously vanish at 0 for t small (the set where they vanish will be a union of $\mathbb{C}^*$-orbits, and for t constant will be discrete using that $a_i > 0$). Hence, $\operatorname{supp}(\mathcal{M}) = \{0\} \times \mathbb{C}$. Thus

$$\dim_{\mathbb{C}}(\pi_*\mathcal{M}/\underline{m}_{t-t_0}\pi_*\mathcal{M}) = \dim_{\mathbb{C}}((\mathcal{O}_{\mathbb{C}^n,0})^k/\mathcal{O}_{\mathbb{C}^n,0}\{F_{1,t_0}, \dots, F_{n+k-1,t_0}\})$$

is locally constant. This completes (i).

For (iii), we observe by Cramer's rule that

$$I_n(F_1, \dots, F_{n+k-1}) \cdot (\mathcal{O}_{\mathbb{C}^n,0})^k \subseteq \mathcal{O}_{\mathbb{C}^n,0}\{F_1, \dots, F_{n+k-1}\}$$

implying "$\Leftarrow$". For the converse, $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k/\mathcal{O}_{\mathbb{C}^n,0}\{F_1, \dots, F_{n+k-1}\} < \infty$ implies $\operatorname{supp}((\mathcal{O}_{\mathbb{C}^n,0})^k/\mathcal{O}_{\mathbb{C}^n,0}\{F_1, \dots, F_{n+k-1}\}) = \{0\}$. Then, by Macaulay [Mc] or Northcott [No], this implies that $\mathcal{O}_{\mathbb{C}^n,0}/I_n(F_1, \dots, F_{n+k-1})$ is Cohen–Macaulay of dimension 0, i.e. $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/I_n(F_1, \dots, F_{n+k-1})) < \infty$. □

The relation between the lengths of the modules and algebras is given in the homogeneous case by the following theorem.

THEOREM 1. *In the homogeneous case, the Macaulay–Bezout numbers satisfy*

$$B_n(\mathbf{d}) = A_n(\mathbf{d}) = \sigma_n(\mathbf{d}),$$

where $\sigma_n(\mathbf{d})$ denotes the nth elementary symmetric function in $(d_1, \dots, d_{n+k-1})$.

This theorem actually follows from the more general result Theorem 2 to be given in Section 3; however, in this homogeneous form we see, for example, the equality

$$B_n(d_1, \dots, d_n) = \sigma_n(\mathbf{d}) = \prod d_i$$

which is just a restatement for Bezout's theorem. Also, $B_j(1^p) = \binom{p}{j}$ (in the notation following 1.1) which will suggest another way to view Pascal's triangle in Section 2.

SEMI-WEIGHTED HOMOGENEOUS CASE

In [DM] or [D1], computing singular Milnor fibers in terms of Macaulay–Bezout numbers, we often find it necessary to go beyond the weighted homogeneous case. Even if $f_0\colon \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is homogeneous and $V \subset \mathbb{C}^p$ is homogeneous defined by a homogeneous H, it may not be that there are generators of $T\mathcal{K}_{H,e} \cdot f_0$ which are weighted homogeneous. However, they are often semi-weighted homogeneous in an appropriate sense which we describe next.

Let $F_i \in (\mathcal{O}_{\mathbb{C}^n,0})^k$ for $i = 1, \ldots, n+k-1$ be germs, with $F_i = (F_{i1}, \ldots, F_{ik})$. Suppose as above we have assigned weights $\mathrm{wt}(x_i) = a_i > 0$, $\mathrm{wt}(y_i) = c_i$ (where we allow weights which are nonpositive) and have chosen weighted degrees d_i, so that each F_{ij} consists of terms of weighted degree $\geqslant d_i + c_j \geqslant 0$. We let F_{0ij} denote the terms of F_{ij} of weight $d_i + c_j$; and let $F_{0i} = (F_{0i1}, \ldots, F_{0ik})$. We will write $F_{0i} = \mathrm{in}(F_i)$, the *initial part* of F_i relative to the given weights. As earlier we let $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{c} = (c_1, \ldots, c_k)$ and $\mathbf{d} = (d_1, \ldots, d_{n+k-1})$.

DEFINITION 1.4. The $\{F_1, \ldots, F_{n+k-1}\}$ are *semi-weighted homogeneous* (with respect to the weights $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{d}$) if

$$\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_{01}, \ldots, F_{0n+k-1}\} < \infty.$$

We also let $I_n(F_1, \ldots, F_{n+k-1})$ denote the ideal generated by the $k \times k$ minors of the $k \times (n+k-1)$ matrix (F_{ij}).

COROLLARY 1.5. *If the* $\{F_1, \ldots, F_{n+k-1}\}$ *are semi-weighted homogeneous (with respect to the weights* $\mathbf{a}$, $\mathbf{c}$ *and* $\mathbf{d}$ *then*

(i) $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0} / I_n(F_1, \ldots, F_{n+k-1}) < \infty$,
(ii) $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_1, \ldots, F_{n+k-1}\} = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0} / I_n(F_1, \ldots, F_{n+k-1})$
and both $= B_n(\mathbf{d}; \mathbf{a}, \mathbf{c})$.

Proof. We shall show that if $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_{01}, \ldots, F_{0n+k-1}\} < \infty$, then $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_1, \ldots, F_{n+k-1}\} = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_{01}, \ldots, F_{0n+k-1}\}$, and similarly for $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0} / I_n(F_1, \ldots, F_{n+k-1}))$. The result will then follow by Proposition 1.3.

We consider the deformation $F_{it} = (1-t)F_{0i} + tF_i$. We note that $\mathrm{in}(F_{it}) = F_{0i}$ for all $0 \leqslant t \leqslant 1$. Then, by Proposition 1.3

$$\begin{aligned}&\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_{01}, \ldots, F_{0n+k-1}\} < \infty \text{ iff}\\ &\quad \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0} / I_n(F_{01}, \ldots, F_{0n+k-1}) < \infty.\end{aligned}$$

We denote the determinantal generators of $I_n(F_{01}, \ldots, F_{0n+k-1})$ by $\{G_{01}, \ldots, G_{0\ell}\}$, with say $\mathrm{wt}(G_{0j}) = m_j$. Then, if the corresponding generators of $I_n(F_{1t}, \ldots, F_{n+k-1t})$ are $\{G_{1t}, \ldots, G_{\ell t}\}$, then G_{jt} consists of terms of weight $\geqslant m_j$ with G_{0j} giving the terms of weight m_j. If we let $G_t = (G_{1t}, \ldots, G_{\ell t})\colon \mathbb{C}^n, 0 \to \mathbb{C}^\ell, 0$, then $G_0^{-1}(0) = \{0\}$. Now applying Lemma 1.13 [D2], we conclude $G_{t_0}^{-1}(0) = \{0\}$ for any t_0.

Now we can repeat the argument of the proof of Proposition 1.3 to the deformation $(F_{1t}, \ldots, F_{n+k-1t})$ to conclude that $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_{1t}, \ldots, F_{n+k-1t}\}$ is locally constant and hence constant. A similar argument works for $\mathcal{O}_{\mathbb{C}^n,0} / I_n(F_1, \ldots, F_{n+k-1})$. $\square$

2. A universal decoupling of symmetries

We introduce the function τ which simultaneously extends all of the elementary symmetric functions by being universal for a "generalized Pascal relation" which characterizes them. In doing so, it decouples in a universal fashion their symmetries. This function will be a type of non-skew-symmetric determinant except that it is defined for all size matrices. The elementary symmetric functions are defined for m-tuples, allowing varying m, and hence can be thought of as being defined for certain size matrices; the function τ when restricted to various matrices will yield all of the elementary symmetric functions.

EXAMPLE 2.1. *Generalized Pascal's Triangle.* Given a sequence of numbers d_i, we construct a generalized Pascal's triangle as follows. In Fig. 1.2, at level ℓ, we let $b_k^{(\ell)} = b_k^{(\ell-1)} + d_\ell \cdot b_{k-1}^{(\ell-1)}$, where by definition $b_{\ell+1}^{(\ell)} = 0$ and $b_0^{(\ell)} = 1$. Thus, as in Pascal's triangle, a number is obtained from the two directly above it except the left hand number is multiplied by the level factor d_ℓ.

$$
\begin{array}{lccccccccccccl}
\text{Level} & & & & & & 1 & & & & & & & \\
1 & & & & & 1 & & b_1^{(1)} & & & & & & d_1 \\
2 & & & & 1 & & b_1^{(2)} & & b_2^{(2)} & & & & & d_2 \\
3 & & & 1 & & b_1^{(3)} & & b_2^{(3)} & & b_3^{(3)} & & & & d_3 \\
4 & & 1 & & b_1^{(4)} & & b_2^{(4)} & & b_3^{(4)} & & b_4^{(4)} & & & d_4 \\
5 & 1 & & b_1^{(5)} & & b_2^{(5)} & & b_3^{(5)} & & b_4^{(5)} & & b_5^{(5)} & & d_5 \\
1 & & . & & . & & . & & . & & . & & . &
\end{array}
$$

Fig. 2.2.

The entries of this generalized Pascal triangle can be given very simply as follows.

LEMMA 2.3. *In the situation of 2.2, $b_k^{(\ell)} = \sigma_k(d_1, \ldots, d_\ell)$, where σ_k denotes the kth elementary symmetric function.*

Proof. We observe that the elementary symmetric functions satisfy the following generalized Pascal relation

$$\sigma_k(d_1, \ldots, d_\ell) = \sigma_k(d_1, \ldots, d_{\ell-1}) + d_\ell \cdot \sigma_{k-1}(d_1, \ldots, d_{\ell-1}). \tag{2.3}$$

Then we have two Pascal triangles with the same multipliers at each level. Then by "Pascal-type induction" on the level ℓ we conclude $b_k^{(\ell)} = \sigma_k(d_1, \ldots, d_\ell)$. It is trivially true at level 1; and if it is true though level $\ell - 1$, by (2.3) together with the inductive definition of $b_k^{(\ell)}$, it is true at level ℓ. $\square$

Next, we strengthen this by introducing a function which satisfies a more general form of the Pascal relation.

We consider, for any $n,\ k > 0$, an $n \times k$ matrix $\mathbf{D}$ given in Fig. 2.4.

$$\mathbf{D} = \begin{pmatrix} d_{11} & d_{12} & d_{13} & \cdots & d_{1k} \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{n1} & d_{n2} & d_{n3} & \cdots & d_{nk} \end{pmatrix}.$$

Fig. 2.4.

Let Mat($\mathbf{R}$) $= \cup M_{n,k}(\mathbf{R})(n,\ k > 0)$ where $M_{n,k}(\mathbf{R})$ denotes the set of $n \times k$ matrices over the commutative ring $\mathbf{R}$

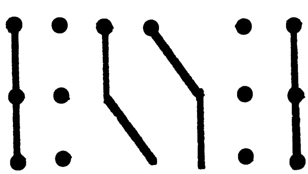
DEFINITION 2.5. We define $\tau\colon$ Mat($\mathbf{R}$) $\to \mathbf{R}$ by

$$\tau(D) = \sum_{j_1 \leqslant j_2 \leqslant \cdots \leqslant j_n} d_{1j_1} d_{2j_2} \cdots d_{nj_n}.$$

Each term of this sum is a product of n factors, one from each row, such that any factor lies beneath or to the right of the other factors coming from the rows above it.

EXAMPLE 2.6. For $\mathbf{D}$ given below, there will be four terms in the sum

$$\mathbf{D} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \end{pmatrix}$$



$$\tau(\mathbf{D}) = d_{11}d_{21}d_{31} + d_{11}d_{21}d_{32} + d_{11}d_{22}d_{32} + d_{12}d_{22}d_{32}.$$

Fig. 2.7.

If

$$\mathbf{D} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 4 \end{pmatrix} \quad \text{then,} \quad \tau(\mathbf{D}) = 1 + 4 + 8 + 16 = 29.$$

In [D1] it is shown that $\mathbf{D}$ is "degree matrix" obtained from the weight vectors for the classical Zariski example $(x^2 + y^2)^3 + (y^3 + z^3)^2$ and $\tau(\mathbf{D})$ computes the number of singular vanishing cycles, from which we can obtain its vanishing Euler characteristic (see [D1]).

EXAMPLE 2.8. Given integers $n,\ k > 0$ and an $n+k-1$ tuple $\mathbf{d} = (d_1, \dots, d_{n+k-1})$, we define the Pascal matrix

$$P_n(\mathbf{d}) = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 & \cdots & d_k \\ d_2 & d_3 & d_4 & \cdots & & d_{k+1} \\ d_3 & d_4 & \cdots & & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ d_n & d_{n+1} & d_{n+2} & & \cdots & d_{n+k-1} \end{pmatrix}.$$

Fig. 2.9.

LEMMA 2.10. $\tau(P_n(\mathbf{d})) = \sigma_n(\mathbf{d})$ *where again* $\sigma_n(\mathbf{d})$ *denotes the nth elementary symmetric function in* $(d_1, \dots, d_{n+k-1})$.

The proof will follow from the "General Pascal Relation" which is one of the key properties of τ which we consider next. To give it we introduce several elementary operations on matrices. Given an $n \times k$ matrix $\mathbf{D}$, we let $\mathbf{D}^r$ denote the $(n-1) \times k$ matrix obtained from $\mathbf{D}$ by deleting the top row. Similarly we let $\mathbf{D}^c$ denote the $n \times (k-1)$ matrix obtained from $\mathbf{D}$ by deleting the first column. More generally we let $\mathbf{D}^{(i,j)}$ denote the $i \times j$ matrix obtained from $\mathbf{D}$ by deleting the first $n-i$ rows and $k-j$ columns; and $\mathbf{D}'^{(i,j)}$ denote the $i \times j$ matrix obtained from $\mathbf{D}$ by deleting the last $n-i$ rows and $k-j$ columns. We also let $\mathbf{D}^{r'}$ denote the matrix obtained by removing the bottom row.

EXAMPLE 2.11. For the Pascal matrix $P_n(\mathbf{d})$ defined for $\mathbf{d} = (d_1, \dots, d_{n+k-1})$, we observe

$$P_n(\mathbf{d})^r = P_{n-1}(\mathbf{d}') \quad \text{and} \quad P_n(\mathbf{d})^c = P_n(\mathbf{d}'),$$

where $\mathbf{d}' = (d_2, \dots, d_{n+k-1})$; also, more generally,

$$P_n(\mathbf{d})'^{(i,j)} = P_i(\mathbf{d}_{i+j-1}) \quad \text{and} \quad P_n(\mathbf{d})^{(i,j)} = P_i(\mathbf{d}'_{i+j-1}),$$

where $\mathbf{d}_{i+j-1} = (d_1, \dots, d_{i+j-1})$ and $\mathbf{d}'_{i+j-1} = (d_{n+k-i-j+1}, \dots, d_{n+k-1})$. These properties correspond via Lemma 2.10 and Proposition 2.12 to the properties of elementary symmetric functions. Now we can state the properties of τ.

PROPOSITION 2.12. τ *satisfies the following properties* (1)–(7).

(1) *Normalization:* $\tau(a) = a$, *for a* 1×1 *matrix a identified with an element of* $\mathbf{R}$;

(2) *General Pascal Relation:*

$$\tau(\mathbf{D}) = d_{11} \cdot \tau(\mathbf{D}^r) + \tau(\mathbf{D}^c)$$

(for the degenerate cases we define $\tau(\mathbf{D}) = 1$ for $\mathbf{D}$ a $0 \times k$ matrix and $\tau(\mathbf{D}) = 0$ for $\mathbf{D}$ an $n \times 0$ matrix with both $n,\ k > 0$);

Determinantal-type Properties

(3) *Multilinearity: $\tau(\mathbf{D})$ is multilinear in the rows of $\mathbf{D}$;*

(4) *Expansion along a row: for any i, with $1 \leqslant i \leqslant n$,*

$$\tau(\mathbf{D}) = \sum_{j=1}^{k} d_{ij}\tau(\mathbf{D}'^{(i-1,j)})\tau(\mathbf{D}^{(n-i,k-j+1)});$$

(5) *Expansion along a column: for any j, with $1 \leqslant j \leqslant k$*

$$\tau(\mathbf{D}) = \tau(\hat{\mathbf{D}}_j) + \sum_{i=1}^{n} d_{ij}\tau(\mathbf{D}'^{(i-1,j-1)})\tau(\mathbf{D}^{(n-i,k-j+1)}),$$

where $\hat{\mathbf{D}}_j$ = the matrix obtained from $\mathbf{D}$ by removing the jth column.

(6) *Product formula: For $n \times k$ and $n \times \ell$ matrices $\mathbf{A}$ and $\mathbf{B}$, let $(\mathbf{A} \mid \mathbf{B})$ denote the $n \times (k + \ell)$ matrix obtained by adjoining the columns of $\mathbf{B}$ to those of $\mathbf{A}$. Then,*

$$\begin{aligned}\tau(\mathbf{A} \mid \mathbf{B}) = {} & \tau(\mathbf{A}) + \tau(\mathbf{A}^{r'})\tau(\mathbf{B}^{r^{n-1}}) + \tau(\mathbf{A}^{r'^2})\tau(\mathbf{B}^{r^{n-2}}) \cdots \\ & + \tau(\mathbf{A}^{r'^{n-1}})\tau(\mathbf{B}^{r}) + \tau(\mathbf{B})\end{aligned}$$

(here e.g. $\mathbf{B}^{r^k}$ denotes iterating $(\ )^r$ k times, i.e. removing k rows)

(7) *Multiplicative property: For $\mathbf{D}$ of the form*

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{pmatrix},$$

$$\tau(\mathbf{D}) = \tau(\mathbf{D}_1) \cdot \tau(\mathbf{D}_2).$$

REMARK. The multilinearity (1) is definitely restricted to rows and is false for columns; also, the form of expansion along rows (4) is different from (5) the expansion along columns. In addition, very much like determinants, there does not seem to be a simple formula for $\tau(\mathbf{A}+\mathbf{B})$ for matrices $\mathbf{A}$ and $\mathbf{B}$. In Proposition 2.19 we shall give such a formula for two important classes of matrices $\mathbf{A}$ and $\mathbf{B}$.

Proof. First (1) is immediate from the definition. For (2) we can break up the terms in τ as the terms containing d_{11} and the other terms

$$\begin{aligned}\tau(\mathbf{D}) &= d_{11} \cdot \sum_{j_2 \leqslant j_3 \leqslant \cdots \leqslant j_n} d_{2j_2} \cdots d_{nj_n} + \sum_{1 < j_1 \leqslant \cdots \leqslant j_n} d_{1j_1} d_{2j_2} \cdots d_{nj_n} \\ &= d_{11} \cdot \tau(\mathbf{D}^r) + \tau(\mathbf{D}^c).\end{aligned}$$

For (3) we just observe that in each term of the sum for τ, there is only one element from say the jth row. Hence, if we fix all other rows, it is linear in the jth; thus so is the sum. Then, (4) follows by an argument similar to (2) collecting together the terms involving each d_{ij}. For (5) there may be more than one term from the jth column; hence, we collect together the terms for which d_{ij} is the first term of the column appearing, and the terms in which no term appears giving $\hat{\mathbf{D}}_j$.

For (6), we collect together those terms for which the first factor from $\mathbf{B}$ occurs in the kth row. The sum of these terms is exactly $\tau(\mathbf{A}^{r'^k})\tau(\mathbf{B}^{r^{n-k}})$. Summing over $0 \leqslant k \leqslant n$ gives the decomposition for $\tau(\mathbf{A} \mid \mathbf{B})$. Lastly, (7) follows by applying (6) to $\tau(\mathbf{D})$, for there is only one nonzero term $\tau(\mathbf{D}_1)\tau(\mathbf{D}_2)$. $\square$

We can uniquely characterize τ as follows.

LEMMA 2.13. *τ is uniquely characterized by the General Pascal property* (2) *and normalization* (1) *in the following strong sense:*
if $\mathcal{B} \subset$ Mat($\mathbf{R}$) is a subset of matrices closed under the operations $(\ )^r$ and $(\ )^c$ and $\tau'\colon \mathcal{B} \to \mathbf{R}$ satisfies General Pascal property (2) *and normalization* (1) *then $\tau' = \tau \mid \mathcal{B}$.*

Proof. We use a Pascal–type induction on the level $\ell = n + k - 1$. We note that the levels for both $\mathbf{D}^r$ and $\mathbf{D}^c$ are one less than that of $\mathbf{D}$. Thus, if τ' is another function satisfying (1) and (2) then τ and τ' agree at level 1 by normalization and the induction step follows by the Pascal property so they always agree. $\square$

As a corollary we also immediately obtain a proof of Lemma 2.10. In fact, by the General Pascal relation and Example 2.11,

$$\tau(P_n(\mathbf{d})) = d_1 \cdot \tau(P_{n-1}(\mathbf{d}')) + \tau(P_n(\mathbf{d}')).$$

Hence, both $\tau(P_n(\mathbf{d}))$ and $\sigma_n(\mathbf{d})$ form generalized Pascal triangles with the same multipliers (placed in the reverse order); hence they agree. $\square$

For $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_\ell)$, we apply the product formula to $(P_n(\mathbf{a}, 0^p) \mid P_n(0^q, \mathbf{b})) = P_n(\mathbf{a}, 0^{p+q-n}, \mathbf{b})$ $(p, q \geqslant n)$ and using $\sigma_n(\mathbf{d}, 0^k) = \sigma_n(\mathbf{d})$ we obtain the standard product formula $\sigma_n(\mathbf{a}, \mathbf{b}) = \Sigma_{i=0}^n \sigma_i(\mathbf{a}) \cdot \sigma_{n-i}(\mathbf{b})$.

A SECOND IMPORTANT CLASS OF FUNCTIONS

For integers n, $k > 0$ and a k-tuple $\mathbf{a} = (a_1, \ldots, a_k)$, we let $\mathbf{a}^n$ denote the $n \times k$ matrix all of whose rows are given by $\mathbf{a}$. Then, we make several remarks concerning $\tau(\mathbf{a}^n)$. First, by its definition, it is exactly the sum of all monomials of degree n in $a_1, \ldots, a_k$.

However, $\tau(\mathbf{a}^n)$ is an intrinsic function much as the elementary symmetric functions $\sigma_n(\mathbf{a})$ are. For this reason we use the special notation

$$s_n(\mathbf{a}) \stackrel{\text{def}}{=} \tau(\mathbf{a}^n).$$

The General Pascal relation implies

$$\tau(\mathbf{a}^n) = a_1 \cdot \tau(\mathbf{a}^{n-1}) + \tau(\mathbf{a}'^n) \quad \text{where } \mathbf{a}' = (a_2, \dots, a_k),$$

or we obtain *general Pascal relation for the functions* $s_n(\mathbf{a})$

$$s_n(\mathbf{a}) = a_1 \cdot s_{n-1}(\mathbf{a}) + s_n(\mathbf{a}'). \tag{2.14}$$

If $n = 0$, then by our convention, $s_0(\mathbf{a}) = 1$; this fits in well with the formulas we will obtain. If $k = 1$, $s_n(a) = a^n$, while for $k = 2$, we have the closed form

$$s_n(a, b) = \begin{cases} (a^{n+1} - b^{n+1})/(a-b) & \text{if } a \neq b, \\ (n+1)a^n & \text{if } a = b. \end{cases} \tag{2.15}$$

Applying the product formula to $(\mathbf{a}^n \mid \mathbf{b}^n)$ yields $s_n(\mathbf{a}, \mathbf{b}) = \Sigma_{i=0}^n s_i(\mathbf{a}) \cdot s_{n-i}(\mathbf{b})$. Then, either applying a special case of this product formula or expanding along the last column yields a recursive formula

$$s_n(\mathbf{a}) = s_n(\mathbf{a}'') + \sum_{i=1}^{n} s_{n-i}(\mathbf{a}'') \cdot a_k^i \tag{2.16}$$

where $\mathbf{a}'' = (a_1, \dots, a_{k-1})$.

In the special case where all $a_i = a$ then $s_n(\mathbf{a})$ equals (number of monomials of degree n in k-variables) $\cdot\, a^n$, i.e. $s_n(\mathbf{a}) = \binom{n+k-1}{n} \cdot a^n$.

EXAMPLE. *Stirling Numbers*. These numbers appear in combinatorics in a number of different settings, see e.g. [Ri]. There are two types of such numbers. Stirling numbers of the first kind $s(n, k)$ are the coefficients of x^k in $x(x-1)(x-2)\cdots(x-(n-1))$; hence,

$$s(n, k) = \sigma_{n-k}(0, -1, \dots, -(n-1)) = (-1)^{n-1}\sigma_{n-k}(1, \dots, n-1).$$

Those of the second kind $S(n, k)$ are defined $= 0$ if $n < k$, $= 1$ if $n = k$ and then for $n > k$ they are defined inductively by

$$S(n, k) = k \cdot S(n-1, k) + S(n-1, k-1). \tag{2.17}$$

However, if we write $n = k + m$ then we see from (2.17) that $S(m+k, k)$ satisfies the normalization condition and general Pascal relation for $s_m(k, k-1, \dots, 1)$. Thus,

$$S(m+k, k) = s_m(k, k-1, \dots, 1) \quad \text{for all } m, k \geqslant 0.$$

For example, various identities involving these numbers follow from the more general properties we obtain for σ_k and s_k.

Part of our interest in these functions s_k is their use together with σ_k for the computations of τ for certain classes of matrices which are important for the computations of singular Milnor numbers. These calculations will demonstrate the importance of the general Pascal relation.

These calculations are centered around a formula for $\tau(\mathbf{A} + \mathbf{B})$ for important matrices $\mathbf{A}$ and $\mathbf{B}$. For an $n + k - 1$ tuple $\mathbf{d} = (d_1, \ldots, d_{n+k-1})$ and a k-tuple $\mathbf{a} = (a_1, \ldots, a_k)$ we let $\mathbf{D}(\mathbf{d}, \mathbf{a}) = P_n(\mathbf{d}) + \mathbf{a}^n$. Specifically

$$\mathbf{D}(\mathbf{d}, \mathbf{a}) =$$

$$\begin{pmatrix} a_1 + d_1 & a_2 + d_2 & a_3 + d_3 & \cdots & a_{k-1} + d_{k-1} & a_k + d_k \\ a_1 + d_2 & a_2 + d_3 & a_3 + d_4 & \cdots & a_{k-1} + d_k & a_k + d_{k+1} \\ a_1 + d_3 & a_2 + d_4 & a_3 + d_5 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1 + d_{n-1} & \cdots & \cdots & \cdots & a_{k-1} + d_{n+k-3} & a_k + d_{n+k-2} \\ a_1 + d_n & a_2 + d_{n+1} & \cdots & \cdots & a_{k-1} + d_{n+k-2} & a_k + d_{n+k-1} \end{pmatrix}. \tag{2.18}$$

There is the following formula for $\tau(\mathbf{D}(\mathbf{d}, \mathbf{a}))$.

PROPOSITION 2.19. For $\mathbf{D}(\mathbf{d}, \mathbf{a}) = P_n(\mathbf{d}) + \mathbf{a}^n$ as in (2.18)

$$\tau(\mathbf{D}(\mathbf{d}, \mathbf{a})) = \sigma_n(\mathbf{d}) + \sigma_{n-1}(\mathbf{d}) \cdot s_1(\mathbf{a}) + \cdots + \sigma_1(\mathbf{d}) \cdot s_{n-1}(\mathbf{a}) + s_n(\mathbf{a}).$$

Proof. Let $\mathbf{R} = \mathbb{C}$ and let $\mathcal{B}$ = set of matrices of the form (2.18) with $a_i, d_j \in \mathbb{C}$. We observe $\mathcal{B} \subset \mathrm{Mat}(R)$ is closed under $(\ )^r$ and $(\ )^c$. For such matrices, we define $\tau' \colon \mathcal{B} \to \mathbf{R}$ by

$$\tau'(\mathbf{D}(\mathbf{d}, \mathbf{a})) = \text{RHS of (2.19)}.$$

Then we wish to show that τ' satisfies the normalization condition and general Pascal relation.

However, there is a subtle point which we must mention. Strictly speaking, as we have defined τ', it is not immediately clear that τ' is well-defined since $(\mathbf{d}, \mathbf{a})$ is not uniquely determined by the matrix $\mathbf{D}(\mathbf{d}, \mathbf{a})$. To avoid having to prove that τ' is well-defined, and in fact to deduce it as a consequence of the characterization of τ, we can proceed by "making the $\mathbf{a}$ formal variables". We consider instead matrices obtained by replacing $\mathbf{a}$ by all sequences $(x_r, \ldots, x_{r+k-1})$ of variables and work in $\mathbf{R} = \mathbb{C}[x_\infty]$, the polynomial ring on an infinite number of variables. Then, the new $\mathcal{B} \subset \mathrm{Mat}(\mathbf{R})$ is still closed under $(\ )^r$ and $(\ )^c$, and τ' is well-defined. Once we conclude $\tau' = \tau \mid \mathcal{B}$, it will follow that τ' is independent of the representation of $\mathbf{D}(\mathbf{d}, \mathbf{a})$ and this yields various identities involving σ_k and s_k. With this said, we still proceed to verify the conditions using **a** with the understanding that we can replace it as need be with any sequence $(x_r, \ldots, x_{r+k-1})$.

First, for $n = k = 1$, the definition gives

$$\tau'(a_1 + d_1) = \sigma_1(d_1) + s_1(a_1) = d_1 + a_1.$$

Second, we establish the general Pascal relation for τ'. Then,

$$(a_1 + d_1) \cdot \tau'(\mathbf{D}(\mathbf{d},\, \mathbf{a})^r) = (a_1 + d_1)(\sigma_{n-1}(\mathbf{d}') + \sigma_{n-2}(\mathbf{d}') \cdot s_1(\mathbf{a}) + \cdots \\ + s_{n-1}(\mathbf{a})),$$

and

$$\tau'(\mathbf{D}(\mathbf{d},\, \mathbf{a})^c) = \sigma_n(\mathbf{d}') + \sigma_{n-1}(\mathbf{d}') \cdot s_1(\mathbf{a}') + \cdots \\ + \sigma_1(\mathbf{d}') \cdot s_{n-1}(\mathbf{a}') + s_n(\mathbf{a}')$$

(where $\mathbf{d}' = (d_2, \ldots, d_{n+k-1})$ and $\mathbf{a}' = (a_2, \ldots, a_k)$).

Adding these two equations, we break up the sum of terms into two types. First, the terms only involving **d**, respectively **a**, are:

$$d_1 \cdot \sigma_{n-1}(\mathbf{d}') + \sigma_n(\mathbf{d}') \quad \text{and} \quad a_1 \cdot s_{n-1}(\mathbf{a}) + s_n(\mathbf{a}').$$

However, by the general Pascal relation for σ_n and s_n, these are $\sigma_n(\mathbf{d})$ and $s_n(\mathbf{a})$. The remaining terms can be grouped into collections of 3 terms of the form

$$a_1 \cdot \sigma_k(\mathbf{d}') \cdot s_{n-k-1}(\mathbf{a}) + \sigma_k(\mathbf{d}') \cdot s_{n-k}(\mathbf{a}') \\ + \; d_1 \cdot \sigma_{k-1}(\mathbf{d}') \cdot s_{n-k}(\mathbf{a}), \quad \text{for } k = 1, \ldots, n-1. \tag{2.20}$$

Then, in (2.20), by the general Pascal relation for s_n, the sum of the first two terms equals $\sigma_k(\mathbf{d}') \cdot s_{n-k}(\mathbf{a})$. When this is added to the third term of (2.20), we can again apply the general Pascal relation for σ_k to obtain for (2.20) $\sigma_k(\mathbf{d}) \cdot s_{n-k}(\mathbf{a})$. Thus, the sum of all of the terms of the two types gives exactly $\tau'(\mathbf{D}(\mathbf{d},\, \mathbf{a}))$; hence this establishes the general Pascal relation for τ'. By the characterization of τ (Lemma 2.13), $\tau' = \tau \mid \mathcal{B}$. □

We deduce consequences for two special cases which are important for computations of singular Milnor numbers.

We first consider for $\mathbf{d} = (d_1, \ldots, d_{n+k-1})$ the matrix

$$P_n(\mathbf{d},\, x) = P_n(\mathbf{d}) + (0, \ldots, 0,\, x)^n$$

which is the matrix obtained from $P_n(\mathbf{d})$ by adding x to each entry of the last column. Since $s_k(0, \ldots, 0,\, x) = s_k(x) = x^k$, we obtain a first Corollary of 2.19.

COROLLARY 2.21. *Let* $\mathcal{P}_n(\mathbf{d}) = x^n + \sigma_1(\mathbf{d}) \cdot x^{n-1} + \cdots + \sigma_{n-1}(\mathbf{d}) \cdot x + \sigma_n(\mathbf{d})$ *then*

$$\tau(P_n(\mathbf{d}, x)) = \mathcal{P}_n(\mathbf{d}).$$

REMARK. We note that in the case $k = 1$, $\mathcal{P}_n(\mathbf{d})$ has a natural interpretation as the polynomial with roots $-d_i$; however, if $k > 1$ there is no such simple interpretation. Corollary 2.21 gives a common origin for all of these polynomials for all k. The expression $\mathcal{P}_n(\mathbf{d})(e)$ gives the number of relative vanishing cycles for an arrangement with exponents $\mathbf{d}$ on an isolated hypersurface singularity of degree $e+1$ see [OT] and [D1].

Second, we consider for an integer k with $0 \leqslant \ell \leqslant n+k-1$, and a k-tuple $\mathbf{a} = (a_1, \ldots, a_k)$, then $n \times k$ matrix

$$\mathbf{D}_n(\ell, \mathbf{a}) = P_n(0^{n+k-1-\ell}, x^{\ell}) + \mathbf{a}^n \tag{2.22}$$

(where we use the notation for repeated entires so x^{ℓ} denotes $(x, \ldots, x)$ with ℓ entries of x). This matrix is obtained from $\mathbf{a}^n$ by adding x to each entry which can be reached from the lower right hand position by making a combination of $\leqslant \ell - 1$ vertical or horizontal moves. This time

$$\sigma_k(0^{n+k-1-\ell}, x^{\ell})(= \sigma_k(0, \ldots, 0, x, \ldots, x) = \sigma_k(x, \ldots, x)\ell \text{ entries of } x)$$
$$= \binom{l}{k} \cdot x^k,$$

where we make the convention $\binom{\ell}{k} = 0$ if $k > \ell$. Again applying Proposition 2.19, there is the following general formula for $\tau(\mathbf{D}_n(\ell, \mathbf{a}))$.

COROLLARY 2.23.

$$\tau(\mathbf{D}_n(\ell, \mathbf{a})) = s_n(\mathbf{a}) + \binom{\ell}{1} \cdot s_{n-1}(\mathbf{a}) \cdot x + \cdots$$
$$+ \binom{\ell}{n-1} \cdot s_1(\mathbf{a}) \cdot x^{n-1} + \binom{\ell}{n} \cdot x^n,$$

where $\binom{\ell}{k} = 0$ *if* $k > \ell$.

For example, the $n \times 2$ matrix (2.24) is of the form $\mathbf{D}_n(1, (a, b))|_{x=1}$;

$$\mathbf{D} = \begin{pmatrix} a & b \\ a & b \\ \cdots & \cdots \\ a & b \\ a & b+1 \end{pmatrix}. \tag{2.24}$$

By Corollary 2.23

$$\tau(\mathbf{D}) = s_n(a, b) + s_{n-1}(a, b)$$
$$= \begin{cases} (a^{n+1} - b^{n+1} + a^n - b^n)/(a-b) & \text{if } a \neq b, \\ (n+1)a^n + na^{n-1} & \text{if } a = b. \end{cases} \tag{2.25}$$

In the special case when $a = b$, the matrix $\mathbf{D}$ in (2.24) has Pascal form so also $\tau(\mathbf{D}) = \sigma_n(a^n, a+1)$ ($= \sigma_n(a, \dots, a, a+1)$ with n entries of a).

EXAMPLE 2.26. For $\mathbf{a} = (a_1, \dots, a_k)$ we let $\mathbf{a} + x = (a_1 + x, \dots, a_k + x)$. Then, $s_n(a + x)$ is the sum of all monomials of degree n in $a_1 + x, \dots, a_k + x$. We can write this as a degree n polynomial in x by $\tau(\mathbf{D}_n(\ell, \mathbf{a}))$ with $\ell = n + k - 1$.

$$s_n(\mathbf{a} + x) = s_n(\mathbf{a}) + \binom{n+k-1}{1} \cdot s_{n-1}(\mathbf{a}) \cdot x + \cdots$$
$$+ \binom{n+k-1}{n-1} \cdot s_1(\mathbf{a}) \cdot x^{n-1} + \binom{n+k-1}{n} \cdot x^n.$$

REMARK. The formulas in (2.19)–(2.25) occur in the calculation of the singular Milnor numbers for a variety of circumstances including nonlinear arrangements of hypersurfaces and nonisolated complete intersection singularities. It also yields an alternate method to obtain Milnor numbers of isolated complete intersection singularities. For example, it follows from [D1] that the isolated complete intersection singularity in $\mathbb{C}^n$ formed from hypersurfaces of degrees $a + 1$ and $b + 1$ has Milnor number given by $\tau(\mathbf{D}) - a^n - b^n$, where $\mathbf{D}$ is given by (2.24). By using the formulas for τ, we recover in [D1] the general formulas obtained by Giusti and Greuel–Hamm.

3. The weighted homogeneous case

Now we turn to the main objective of this paper, namely to compute the weighted Macaulay–Bezout numbers.

DEFINITION 3.1. Consider a triple of weights $(\mathbf{d};\ \mathbf{a},\ \mathbf{c})$ with $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{c} = (c_1, \dots, c_k)$ and $\mathbf{d} = (d_1, \dots, d_{n+k-1})$ used to define a weighted Macaulay–Bezout number. We assume (after a possible permutation) that $\mathbf{c}$ and $\mathbf{d}$ are placed in nondecreasing order, $c_1 \leqslant \cdots \leqslant c_k$ and $d_1 \leqslant \cdots \leqslant d_{n+k-1}$. We can define the *degree matrix* for the weights as the $n \times k$ matrix of non-negative integers $\mathbf{D} = (d_{ij})$ where

$$d_{ij} = d_{i+j-1} + c_j.$$

Since $d_i + c_j$ is a nondecreasing function of j for fixed $i - j$, $\mathbf{D}$ is nondecreasing along a row as j increases.

EXAMPLE 3.2. For the weights $\mathbf{a} = (1, 2)$, $\mathbf{d} = (1, 2, 4)$, and $\mathbf{c} = (0, 0)$, the matrix of weight vectors and the degree matrix $\mathbf{D}$ are given by (3.3).

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 4 & 4 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \tag{3.3}$$

$$Q(\mathbf{D}) = \begin{pmatrix} x_1^{d_{1,1}} & 0 & 0 & 0 & \cdots & 0 \\ x_2^{d_{2,1}} & x_1^{d_{1,2}} & 0 & 0 & \cdots & 0 \\ x_3^{d_{3,1}} & x_2^{d_{2,2}} & x_1^{d1,3} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & & 0 \\ x_n^{d_{n,1}} & x_{n-1}^{d_{n-1,2}} & \cdots & \cdots & \cdots & x_1^{d_{1,k}} \\ 0 & x_n^{d_{n,2}} & \cdots & \cdots & \cdots & x_2^{d_{2,k}} \\ 0 & 0 & x_n^{d_{n,3}} & \cdots & \cdots & x_3^{d_{3,k}} \\ 0 & 0 & 0 & \cdots & \cdots & \\ 0 & 0 & 0 & 0 & 0 & x_n^{d_{n,k}} \end{pmatrix}.$$

Fig. 3.4.

To understand this definition and the associated degree matrix $\mathbf{D}$, we consider the monomial matrix $Q(\mathbf{D})$ in (3.4) defined for a matrix $\mathbf{D} = (d_{ij})$ of non-negative integers nondecreasing along rows, with $d_{ij} = d_{i+j-1} + c_j$ for nondecreasing sequences $c_1 \leqslant \cdots \leqslant c_k$ and $d_1 \leqslant \cdots \leqslant d_{n+k-1}$. If we assign weights $\mathrm{wt}(x_i) = 1$ for all i, then the entries of $Q(\mathbf{D})$ are weighted homogeneous for these weights. Then, (3.1) would associate the degree matrix $\mathbf{D}$ to $Q(\mathbf{D})$.

We also note that for a set of weights $(\mathbf{d};\ \mathbf{a},\ \mathbf{c})$, the corresponding degree matrix is exactly the matrix $\mathbf{D}(\mathbf{d},\ \mathbf{c})$ using the notation of (2.18).

Then we can give a formula for the weighted Macaulay–Bezout numbers.

THEOREM 2. *Let* $(\mathbf{d};\ \mathbf{a},\ \mathbf{c})$ *be a set weights and* $\mathbf{D}$ *the associated degree matrix defined by* (3.1). *Then,*

$$A_n(\mathbf{d};\ \mathbf{a},\ \mathbf{c}) = B_n(\mathbf{d};\ \mathbf{a},\ \mathbf{c}) = (1/a) \cdot \tau(\mathbf{D}),$$

where $a = \Pi_{i=1}^{n} a_i$.

Since the degree matrix is given by $\mathbf{D}(\mathbf{d},\ \mathbf{c})$, we can combine the theorem with Proposition 2.19 to give a closed formula for the weighted Macaulay–Bezout numbers.

COROLLARY 3. *The weighted Macaulay–Bezout number corresponding to a set weights* $(\mathbf{d};\ \mathbf{a},\ \mathbf{c})$ *is given by*

$$(1/a)\cdot(\sigma_n(\mathbf{d})+\sigma_{n-1}(\mathbf{d})\cdot s_1(\mathbf{c})+\cdots+\sigma_1(\mathbf{d})\cdot s_{n-1}(\mathbf{c})+s_n(\mathbf{c})),$$

where $a=\Pi_{i=1}^n a_i$.

REMARK. The theorem includes Theorem 1 as a special case since in the homogeneous case, the matrix $\mathbf{D}$ obtained from the weights $(\mathbf{d};\ 1^n,\ 0^k)$ is noneother than the Pascal matrix $P_n(\mathbf{d})$. Hence, Theorem 1 follows from Theorem 2 and Lemma 2.10. We also see this in Corollary 3 where we can choose $\mathbf{c}=0$ in the homogeneous case. □

Although Corollary 3 gives a closed formula, Theorem 2 is often more useful in some cases where it is better to work directly from the properties of τ.

Proof. To prove the theorem we reduce to a simpler case.

We let $\{F_1,\ldots,F_{n+k-1}\}$ denote the rows of $Q(\mathbf{D})$, and $I(\mathbf{D})$ denote the ideal generated by the $k\times k$ minors of $Q(\mathbf{D})$. It is easily seen that the $k\times k$ minors of $Q(\mathbf{D})$ only vanish at $\{0\}$. Then, we define the module $M(\mathbf{D})$ and ring $A(\mathbf{D})$ by

$$\begin{aligned} M(\mathbf{D}) &= (\mathcal{O}_{\mathbb{C}^n,0})^k/\mathcal{O}_{\mathbb{C}^n,0}\{F_1,\ldots,F_{n+k-1}\},\\ A(\mathbf{D}) &= \mathcal{O}_{\mathbb{C}^n,0}/I(\mathbf{D}). \end{aligned} \tag{3.5}$$

We reduce the computations to those for $M(\mathbf{D})$ and $A(\mathbf{D})$ by the following lemma.

LEMMA 3.6. *For* $\mathbf{D}$ *the degree matrix obtained from a set of weights* $(\mathbf{d};\ \mathbf{a},\ \mathbf{c})$ ·

$$B_n(\mathbf{d};\ \mathbf{a},\ \mathbf{c})=(1/a)\dim_{\mathbb{C}}M(\mathbf{D})\quad \textit{and}$$

$$A_n(\mathbf{d};\ \mathbf{a},\ \mathbf{c})=(1/a)\dim_{\mathbb{C}}A(\mathbf{D}),$$

where $a=\Pi_{i=1}^n a_i$.

We prove this lemma after indicating its role in the proof of Theorem 2. By the lemma, it is sufficient to compute $\dim_{\mathbb{C}}M(\mathbf{D})$ and $\dim_{\mathbb{C}}A(\mathbf{D})$. We do this even in the case $\mathbf{D}$ is not the degree matrix for a set of weights.

LEMMA 3.7. *For* $\mathbf{D}$ *a matrix of non-negative integers nondecreasing along rows*

(i) $\dim_{\mathbb{C}}M(\mathbf{D})=d_{11}\cdot\dim_{\mathbb{C}}A(\mathbf{D}^r)+\dim_{\mathbb{C}}M(\mathbf{D}^c)$,

(ii) $\dim_{\mathbb{C}}A(\mathbf{D})=d_{11}\cdot\dim_{\mathbb{C}}A(\mathbf{D}^r)+\dim_{\mathbb{C}}A(\mathbf{D}^c)$.

The proof of this lemma will be given in Section 4. Lastly, using the two lemmas we are able to prove Theorem 2.

By Lemma 3.6, it is sufficient to prove $\tau(\mathbf{D}) = \dim_{\mathbb{C}} M(\mathbf{D}) = \dim_{\mathbb{C}} A(\mathbf{D})$ for matrices $\mathbf{D}$ of non-negative integers which are nondecreasing along rows. The set of such matrices is closed under the operations $(\ )^r$ and $(\ )^c$. Thus, it is sufficient to prove that $\dim_{\mathbb{C}} A(\mathbf{D}) = \dim_{\mathbb{C}} M(\mathbf{D})$ and that either satisfies the General Pascal relation and the normalization condition on such matrices and then apply Lemma 2.13. However, first (ii) of Lemma 3.7 establishes the General Pascal Relation for $\dim_{\mathbb{C}} A(\mathbf{D})$ and the normalization is immediate.

For $\dim_{\mathbb{C}} M(\mathbf{D})$, (i) and (ii) of Lemma 3.7 together allow us to use induction on the level $\ell = n + k - 1$ to conclude $\dim_{\mathbb{C}} M(\mathbf{D}) = \dim_{\mathbb{C}} A(\mathbf{D})$. □

Proof of Lemma 3.6. The proofs for $A_n(\mathbf{d};\ \mathbf{a},\ \mathbf{c})$ and $B_n(\mathbf{d};\ \mathbf{a},\ \mathbf{c})$ are similar so we give it for $M(\mathbf{D})$. The proof uses an algebraic version of an argument due to Milnor–Orlik [MO] for a weighted version of the classical Bezout's theorem.

Let $\{F_1, \dots, F_{n+k-1}\}$ be weighted homogeneous for the triple of weights $(\mathbf{d};\ \mathbf{a},\ \mathbf{c})$ so that $B_n(\mathbf{d};\ \mathbf{a},\ \mathbf{c}) = \dim_{\mathbb{C}} M$ where

$$M = (\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_1, \dots, F_{n+k-1}\}.$$

We define $f_0\colon \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ by $f_0(x_1, \dots, x_n) = (x_1^{a_1}, \dots, x_n^{a_n})$. We denote the ring homorphism $f_0^*\colon \mathcal{O}_{\mathbb{C}^n,0} \to \mathcal{O}_{\mathbb{C}^n,0}$ by $f_0^*\colon R \to S$ to distinguish between the two copies of $\mathcal{O}_{\mathbb{C}^n,0}$. Then, S is a free R-module of rank $a = \Pi a_i$.

If $\varphi\colon (\mathcal{O}_{\mathbb{C}^n,0})^{n+k-1} \to (\mathcal{O}_{\mathbb{C}^n,0})^k$ is defined by $\varphi(e_i) = F_i$, then M has the presentation

$$R^{n+k-1} \xrightarrow{\varphi} R^k \to M \to 0. \tag{3.8}$$

If we tensor (3.8) with $\otimes_R S$, we obtain

$$S^{n+k-1} \xrightarrow{\varphi'} S^k \to M \otimes_R S \to 0, \tag{3.9}$$

where now $\varphi'(e_i) = F_i \circ f_0$. Thus

$$M \otimes_R S \stackrel{\text{def}}{=} M' = (\mathcal{O}_{\mathbb{C}^n,0})^k / \mathcal{O}_{\mathbb{C}^n,0}\{F_1 \circ f_0, \dots, F_{n+k-1} \circ f_0\};$$

and the weights for M' are now $(\mathbf{d};\ 1^n,\ \mathbf{c})$. By Proposition 1.3

$$\dim_{\mathbb{C}} M' = \dim_{\mathbb{C}} M(\mathbf{D}).$$

Finally, as S is a free R-module of rank a, (3.9) is a direct sum of a copies of (3.8); hence, as vector spaces $\operatorname{coker}(\varphi')$ is a direct sum of a copies of $\operatorname{coker}(\varphi)$. Thus

$$\dim_{\mathbb{C}} M(\mathbf{D}) = \dim_{\mathbb{C}} M' = a \cdot \dim_{\mathbb{C}} M = a \cdot B_n(\mathbf{d};\ \mathbf{a},\ \mathbf{c}).$$ □

4. Completing the proofs of theorems 1 and 2

It only remains to prove Lemma 3.7,

Proof of Lemma 3.7.
We let

$$b(\mathbf{D}) = \dim_{\mathbb{C}} M(\mathbf{D}) \quad \text{and} \quad a(\mathbf{D}) = \dim_{\mathbb{C}} A(\mathbf{D}).$$

Then we wish to prove

$$b(\mathbf{D}) = b(\mathbf{D}^c) + d_{11} \cdot a(\mathbf{D}^r) \tag{4.1}$$

and

$$a(\mathbf{D}) = a(\mathbf{D}^c) + d_{11} \cdot a(\mathbf{D}^r). \tag{4.2}$$

We first prove the "degenerate cases" where either $n = 1$ or $k = 1$. For $n = 1$, $Q(\mathbf{D})$ is a $k \times k$ diagonal matrix with diagonal entries $x_1^{d_{1i}}$; hence,

$$\begin{aligned} b(\mathbf{D}) &= \dim_{\mathbb{C}}(\oplus(\mathcal{O}_{\mathbb{C},0}/(x_1^{d_{1i}}))) \quad \text{where the sum is over } i = 1, \dots, k \\ &= \textstyle\sum d_{1i} = b(\mathbf{D}^c) + d_{11} \cdot 1. \end{aligned}$$

Also, as the determinant equals x_1^d where $d = \Sigma d_{1i}$,

$$a(\mathbf{D}) = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C},0}/(x_1^d)) = d = a(\mathbf{D}^c) + d_{11} \cdot 1.$$

Secondly, we consider the case $k = 1$. Then, $\mathbf{D}$ is a $n \times 1$ column matrix with $Q(\mathbf{D})$ having entries $x_j^{d_{j1}}$. Then both

$$a(\mathbf{D}) = b(\mathbf{D}) = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C},0}/(x_1^{d_{11}}, \dots, x_n^{d_{n1}})) = \prod d_{j1}$$

$(= \sigma_n(d_{11}, \dots, d_{n1})$ which is the case in Bezout's theorem) and so

$$a(\mathbf{D}) = d_{11} \cdot a(\mathbf{D}^r) + 0 \quad \text{and} \quad b(\mathbf{D}) = d_{11} \cdot a(\mathbf{D}^r) + 0.$$

To establish (4.1) and (4.2) in general, we use the notation of Section 3. First for (4.1), we let L be the submodule of $(\mathcal{O}_{\mathbb{C}^n,0})^k$ generated by $\varepsilon_1 = (1, 0, \dots, 0)$. Also, we let

$$N(\mathbf{D}) = \mathcal{O}_{\mathbb{C}^n,0}\{F_1, \dots, F_{n+k-1}\},$$

where $\{F_1, \dots, F_{n+k-1}\}$ denote the rows of $Q(\mathbf{D})$. Then

$$\begin{aligned} b(\mathbf{D}) &= \dim_{\mathbb{C}}(M(\mathbf{D})) = \dim_{\mathbb{C}}((\mathcal{O}_{\mathbb{C}^n,0})^k/N(\mathbf{D})) \\ &= \dim_{\mathbb{C}}((\mathcal{O}_{\mathbb{C}^n,0})^k/(N(\mathbf{D}) + L)) + \dim_{\mathbb{C}}((N(\mathbf{D}) + L)/N(\mathbf{D})). \end{aligned} \tag{4.3}$$

For the first dimension in (4.3), we have by projecting off the first factor

$$(\mathcal{O}_{\mathbb{C}^n,0})^k/(N(\mathbf{D})+L) \simeq (\mathcal{O}_{\mathbb{C}^n,0})^{k-1}/N(\mathbf{D}^c).$$

Thus,

$$\dim_{\mathbb{C}}((\mathcal{O}_{\mathbb{C}^n,0})^k/(N(\mathbf{D})+L)) = b(\mathbf{D}^c).$$

For the second term, we have

$$(N(\mathbf{D})+L)/N(\mathbf{D}) \simeq L/(L\cap N(\mathbf{D})).$$

We claim

$$L/(L\cap N(\mathbf{D})) \simeq \mathcal{O}_{\mathbb{C}^n,0}/(I(\mathbf{D}^r)+(x_1^{d_{11}})), \tag{4.4}$$

where we remark that here $I(\mathbf{D}^r)$ is defined using the variables $(x_2,\dots,x_n)$. Since

$$\mathcal{O}_{\mathbb{C}^n,0}/(I(\mathbf{D}^r)+(x_1^{d_{11}})) \simeq \mathcal{O}_{\mathbb{C}^{n-1},0}/I(\mathbf{D}^r)\otimes_{\mathbb{C}}\mathcal{O}_{\mathbb{C},0}/(x_1^{d_{11}}),$$

this yields

$$\dim_{\mathbb{C}}((N(\mathbf{D})+L)/N(\mathbf{D})) = d_{11}\cdot a(\mathbf{D}^r).$$

To establish (4.4), we use results of Macaulay–Northcott [Mc, Sect. 53] [No, Theorems 10, 11]. We consider the matrix obtained from $Q(\mathbf{D})$ by deleting the first row and column. This new matrix has the same form as $Q(\mathbf{D})$, in fact, it is just $Q(\mathbf{D}^c)$. We denote its rows by $\{\tilde{F}_2,\dots,\tilde{F}_{n+k-1}\}$; these generate the submodule $N(\mathbf{D}^c)$. It is easily seen that the $(k-1)\times(k-1)$ minors of this new matrix only vanish at 0, so $\operatorname{supp}((\mathcal{O}_{\mathbb{C}^n,0})^{k-1}/N(\mathbf{D}^c)) = \{0\}$ has codimension $= n$. We claim

$$\sum_{i=2}^{n+k-1}\varphi_i\tilde{F}_i = 0 \tag{4.5}$$

implies that $(\varphi_2,\dots,\varphi_{n+k-1})$ belongs to the module of determinantal relations on $(\tilde{F}_2,\dots,\tilde{F}_{n+k-1})$.

Recall that this module is generated by the determinantal relations obtained as follows. Since the $\tilde{F}_i$ have $k-1$ entries, if we choose any k of the $\tilde{F}_i$, say $\{\tilde{F}_{i_1},\dots,\tilde{F}_{i_k}\}$, then, a determinantal relation is obtained by letting $\varphi_{i_j}=$ the $(k-1)\times(k-1)$ minor of $(\tilde{F}_{i_j\ell})$ obtained by deleting the jth row, i.e. $\tilde{F}_{i_j}$, and letting the other $\varphi_i = 0$.

Then (4.5) follows from the following lemma (which, in turn, also follows from the results of Macaulay–Northcott). Let J denote the ideal generated by the $k\times k$

minors of the matrix $Q(\mathbf{D})^r$ which is the $(n-1)\times k$ matrix obtained from $Q(\mathbf{D})$ by deleting the first row.

LEMMA 4.6. *$R = \mathcal{O}_{\mathbb{C}^n,0}/J$ is a Cohen–Macaulay ring of dimension 1 and*

$$J \cap (x_1^{d_{11}}) = (x_1^{d_{11}}) \cdot J.$$

First, we indicate how (4.5) allows us complete (4.4), then we deduce (4.5) from Lemma 4.6; the proof of the lemma will be delayed until we have finished the proof of the main lemma. By (4.5)

$$\sum_{i=2}^{n+k-1} \varphi_i F_{i1} \in I(Q(\mathbf{D})^r),$$

where $I(Q(\mathbf{D})^r)$ denotes the ideal generated by the $k\times k$ minors. Conversely, by Cramer's rule, any $k\times k$ minor generating $I(Q(\mathbf{D})^r)$ is of this form. In fact, since the first row of $Q(\mathbf{D})$ is $(x_1^{d_{11}}, 0, \ldots, 0)$ and $d_{11} \leqslant d_{1j}$ for all j

$$I(Q(\mathbf{D})^r) + (x_1^{d_{11}}) = I(\mathbf{D}) + (x_1^{d_{11}}) = I(\mathbf{D}^r) + (x_1^{d_{11}}).$$

Thus,

$$L/(L \cap N(\mathbf{D})) \simeq \mathcal{O}_{\mathbb{C}^n,0}/(I(\mathbf{D}^r) + (x_1^{d_{11}})), \tag{4.7}$$

so (4.4) follows from (4.7).

To establish (4.5) from Lemma 4.6, we note that the $\tilde{F}_i$ are the rows of $Q(\mathbf{D}^c)$ and so it is sufficient to prove it for the rows F_i of $Q(\mathbf{D})$ for $n\times k$ matrices $\mathbf{D}$ by induction on k. For $k = 1$, the result follows for the entries for the single column form a regular sequence. Suppose the result holds for $k' < k$ and that $\mathbf{D}$ is $n \times k$. As the last $k-1$ entries of (4.5) (for $\mathbf{D}$ and F_i) are 0, by induction the $(\varphi_2, \ldots, \varphi_{n+k-1})$ belongs to the module of relations for $I(\mathbf{D}^c)$. Also, by the results of Macaulay–Northcott, the relations between the determinantal generators of $I(\mathbf{D})$ are determinantal. Hence, the first entry in (4.5) is a sum of terms from the ideals $I(Q(\mathbf{D}^c))$ and $(x_1^{d_{11}})$ (since the first term is only nonzero in its first entry). Since the sum is zero, we can apply Lemma 4.6 to conclude that the term in $(x_1^{d_{11}})$ is, in fact, in $(x_1^{d_{11}})\cdot J$. Since $J \subset I(\mathbf{D}^c)$, it is a term in $I(\mathbf{D})$. Thus, the first entry of (4.5) is a sum of elements from $I(\mathbf{D})$ which equals 0, and hence is a determinantal relation for $I(\mathbf{D})$.

This completes the proof of (4.1). For (4.2) we perform a similar analysis.

$$\begin{aligned} a(\mathbf{D}) &= \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/I(\mathbf{D})) \\ &= \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/(I(\mathbf{D}) + (x_1^{d_{11}}))) + \dim_{\mathbb{C}}((I(\mathbf{D}) \end{aligned}$$

$$+ (x_1^{d_{11}}))/I(\mathbf{D})). \tag{4.8}$$

For the first dimension in (4.8), again since $d_{11} \leqslant d_{1j}$ for all j, we have

$$\mathcal{O}_{\mathbb{C}^n,0}/(I(\mathbf{D}) + (x_1^{d_{11}})) \simeq \mathcal{O}_{\mathbb{C}^n,0}/(I(\mathbf{D}^r) + (x_1^{d_{11}})),$$

which by (4.4) has dimension

$$\dim_{\mathbb{C}}((\mathcal{O}_{\mathbb{C}^n,0})^k/(I(\mathbf{D}^r) + (x_1^{d_{11}}))) = d_{11} \cdot a(\mathbf{D}^r).$$

Again

$$(I(\mathbf{D}) + (x_1^{d_{11}}))/I(\mathbf{D}) \simeq (x_1^{d_{11}})/(I(\mathbf{D}) \cap (x_1^{d_{11}})).$$

We claim

$$I(\mathbf{D}) \cap (x_1^{d_{11}}) = (x_1^{d_{11}})I(\mathbf{D}^c). \tag{4.9}$$

We may break up the generators of $I(\mathbf{D})$ into those that involve the first row and those that do not. If a $k \times k$ minor involves the first row then it is of the form $x_1^{d_{11}} \cdot S$ where S is a $(k-1) \times (k-1)$ minor of the matrix obtained by deleting the first row and column. Hence, these generate $(x_1^{d_{11}})I(\mathbf{D}^c)$. Thus, the inclusion "$\supseteq$" for (4.9) follows.

For the reverse inclusion, as in Lemma 4.6, we let J denote the ideal generated by the $k \times k$ minors of the matrix $Q(\mathbf{D})^r$. Such minors do not involve the first row. Then, $I(\mathbf{D}) = J + (x_1^{d_{11}})I(\mathbf{D}^c)$; thus, (4.9) follows from Lemma 4.6. This is sufficient because each $k \times k$ minor of $Q(\mathbf{D})^r$ expands along the first column to give a sum of terms in $I(\mathbf{D}^c)$ so $(x_1^{d_{11}}) \cdot J \subseteq (x_1^{d_{11}})I(\mathbf{D}^c)$. Hence, by (4.9)

$$\begin{aligned}\dim_{\mathbb{C}}((I(\mathbf{D}) + (x_1^{d_{11}}))/I(\mathbf{D})) &= \dim_{\mathbb{C}}((x_1^{d_{11}})/(x_1^{d_{11}})I(\mathbf{D}^c)) \\ &= \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/I(\mathbf{D}^c)) = a(\mathbf{D}^c).\end{aligned}$$

Here the second equality follows because multiplication by $x_1^{d_{11}}$ induces an isomorphism of $\mathcal{O}_{\mathbb{C}^n,0}$-modules

$$\mathcal{O}_{\mathbb{C}^n,0}/I(\mathbf{D}^c) \xrightarrow{\sim} (x_1^{d_{11}})/(x_1^{d_{11}})I(\mathbf{D}^c).$$

This completes (4.2) and the proof of the main lemma (and the theorems). □

Proof of Lemma 4.6. We first note that if $x_1 = 0$ then $J = I(\mathbf{D}^r)$, again defined using the variables $(x_2, \dots, x_n)$. Thus, when $x_1 = 0$, J only vanishes at 0. Hence, $R = \mathcal{O}_{\mathbb{C}^n,0}/J$ has dimension $\leqslant 1$ as a ring. However, J is defined by the $k \times k$ minors of an $(n+k-2) \times k$ matrix and so by Macaulay–Northcott it has dimension $\geqslant 1$. Thus, its dimension is exactly one (and again by Macaulay–Northcott) it is a Cohen–Macaulay ring. However, $\operatorname{supp}(R/(x_1^{d_{11}})) = \{0\}$. Thus, $R/(x_1^{d_{11}})$ is 0-dimensional. Then, a theorem in Zariski–Samuel [ZS vol. II, Appendix 6, Theorem 2] implies that $x_1^{d_{11}}$ is not a zero divisor in R. Hence, if $g \cdot x_1^{d_{11}} \in J$, then $g \in J$, implying the lemma.

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