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Arithmetic models for Hilbert modular varieties

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Notations

F	$=$	totally real field of degree d over the field of rational numbers $\mathbf{Q}$
$\mathcal{O}$	$=$	the ring of integers of F
$\mathcal{A}$	$=$	a square free ideal of $\mathcal{O}$ (the level ideal)
$\mathcal{L}$	$=$	a fractional ideal of F (the polarization ideal)
n	$=$	a positive integer relatively prime to $\mathcal{A}$
$\mathcal{D}$	$=$	the different of $\mathcal{O}$ over the integers $\mathbf{Z}$
Δ	$=$	the discriminant of F over $\mathbf{Q}$
p	$=$	a prime number
$\mathbf{Z}_{(p)}$	$=$	the localization of $\mathbf{Z}$ at the prime ideal (p)
$\mathbf{Z}_p$	$=$	the completion of $\mathbf{Z}_{(p)}$ at the prime ideal (p).

1. Introduction

In this paper, we consider the problem of describing integral models for certain Hilbert–Blumenthal moduli varieties.

Let F be a totally real field of degree d over $\mathbf{Q}$, with ring of integers $\mathcal{O}$ and let $\mathcal{A}$ be a square-free ideal of $\mathcal{O}$ (i.e. a product of distinct prime ideals of $\mathcal{O}$). Denote by Δ the discriminant of F over $\mathbf{Q}$. The group $\mathrm{SL}(2, \mathcal{O})$ acts on the product of d copies of the complex upper half plane $\mathcal{H}$ via the d embeddings of $\mathrm{SL}(2, \mathcal{O})$ into $\mathrm{SL}(2, \mathbf{R})$ induced by the embeddings of F into $\mathbf{R}$. Let n be an integer relatively prime to $\mathcal{A}$. We denote by $\Gamma(n)$ the kernel of the reduction

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homomorphism $\mathrm{SL}(2, \mathcal{O}) \to \mathrm{SL}(2, \mathcal{O}/n\mathcal{O})$. For this introduction, we assume that $n \geqslant 3$. We set

$$\Gamma_0(\mathcal{A}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathcal{O}) \mid c \in \mathcal{A} \right\},$$

$$\Gamma_{00}(\mathcal{A}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathcal{O}) \mid c,\ a-1,\ d-1 \in \mathcal{A} \right\}.$$

The Hilbert–Blumenthal varieties we are considering are $H_\Gamma = \mathcal{H}^d/\Gamma$ where Γ is either $\Gamma_0(\mathcal{A}, n) = \Gamma_0(\mathcal{A}) \cap \Gamma(n)$, or $\Gamma_{00}(\mathcal{A}, n) = \Gamma_{00}(\mathcal{A}) \cap \Gamma(n)$. These are defined over $\mathbf{Q}(\zeta_n) \subset \mathbf{C}$, where $\zeta_n = \mathrm{e}^{2\pi i/n}$. We will study models for H_Γ over $\operatorname{Spec} \mathbf{Z}[\zeta_n, 1/n]$. In particular, we are interested in the local structure of the reduction at the rational primes p with $(p, n) = 1$ and $p|\mathrm{Norm}(\mathcal{A})$.

These Hilbert–Blumenthal modular varieties are moduli spaces of abelian varieties with $\mathcal{O}$-multiplication, taken with an appropriate polarization isomorphism and $\mathcal{A}$-level structure. The theory of integral models was first considered by Rapoport in [9]. Using his approach, one can describe well-behaved moduli spaces over $\mathbf{Z}[\zeta_n, 1/(n \cdot \Delta \cdot \mathrm{Norm}(\mathcal{A}))]$. They provide us with models for H_Γ, which are actually smooth over $\mathbf{Z}[\zeta_n, 1/(n \cdot \Delta \cdot \mathrm{Norm}(\mathcal{A}))]$. In [4], we have shown how the moduli spaces of [9] can be extended to well-behaved moduli spaces over the primes dividing Δ but not $n \cdot \mathrm{Norm}(\mathcal{A})$. The corresponding models, are no longer smooth. They are relative local complete intersections, and the fibers over primes which divide the discriminant Δ have singularities in codimension 2.

However, when we ask for models over $\mathbf{Z}[\zeta_n, 1/n]$, there are added complications: In general, when the level of the subgroup is not invertible in the base scheme, it is not always clear what the moduli problem should be. The problem is, as we shall see, that it is not easy to decide what to take as a definition of level structure. There is a "naive" notion of level structure, but its existence forces the abelian variety to be ordinary (this notion was used by Deligne and Ribet, Katz etc.) The corresponding moduli spaces are non-proper over the moduli space without level structure, since the whole non-ordinary locus is missing from the image. Of course, we can brutally compactify them by taking normalizations, but then there is no control of the singularities that will appear in these normalizations over the non-ordinary locus.

In our approach, we try to understand proper models by considering a well behaved notion of $\mathcal{A}$-level structure for all abelian varieties over the primes p, $p|\mathrm{Norm}(\mathcal{A})$. For $\Gamma = \Gamma_0(\mathcal{A}, n)$ there is a natural choice ($\mathcal{O}$-linear $\mathcal{A}$-isogenies) which as it turns out works well. The corresponding moduli space is singular but well behaved. We devote some effort, to the study of the combinatorics of the singularities. The main result is that the moduli spaces are normal and relative local complete intersections (see Theorem 2.2.2). The singularities are the same for Shimura varieties associated to certain forms of the reductive group. This way, we obtain a generalization of results of Langlands and Zink ([11]) on the bad reduction of certain Shimura varieties associated to totally indefinite division quaternion algebras over F. The calculation of the local structure employs the method of handling the crystalline deformation theory which was introduced in [4] (see also [2]). See 4.5 for some explicit calculations of the singularities in low dimensions.

The situation for $\Gamma = \Gamma_{00}(\mathcal{A},\ n)$ is more complicated. The case $d = 1$ (this is the case of modular curves) was first treated by Deligne and Rapoport ([5]). Katz and Mazur ([8]) later showed that (for any ideal $\mathcal{A}$) a definition of level structure which was first used by Drinfeld gives regular models. They also generalized Drinfeld's idea so that it applies to the higher dimensional setting. In this paper, we calculate the local structure of the model for $\Gamma = \Gamma_{00}(\mathcal{A},\ n)$ given by the DKM-level structure. The calculation is made possible by the fact that the kernel of the universal isogeny is a group scheme of type $(p, \ldots, p)$. Raynaud obtained explicit universal descriptions for such group schemes. We use them to also calculate explicitly their subschemes of generators in the sense of DKM. Unfortunately, the resulting moduli space is not well behaved. The problem, first noticed by Chai and Norman in [1], is that DKM-level structure is not rigid enough in higher dimensions. It has been suggested that the needed rigidification, can be provided by adding the choice of a filtration of the kernel of the isogeny by group schemes. We show that in the case of Hilbert modular surfaces, and for square-free $\mathcal{A}$, this indeed works. There is an explicit description of the singularities, which once more are local complete intersections. In fact, the resulting moduli spaces are either regular or they can be easily desingularized. We use this to obtain a construction of regular scheme models for certain Hilbert–Blumenthal surfaces of the form $H_{\Gamma_{00}(\mathcal{A},1)}$ over the full Spec $\mathbf{Z}$ with no primes inverted (see Theorem 2.4.3).

The cases of non-square free level (even for Γ_0) seem to require new insight, except when $\mathcal{A}$ is prime to the discriminant and all the prime ideals dividing $\mathcal{A}$ have as residue field the prime field. In this last case, it seems that the DKM definition is the only logical choice, however unless different primes of $\mathcal{A}$ lie over different rational primes, there are still problems to getting good models. We will not deal with this case here.

This paper improves on some of the results in the author's thesis at Columbia University. The improvement was made possible by the use of techniques which were introduced in [4]. I would like to thank P. Deligne for his generosity. I would also like to thank my advisor, T. Chinburg for his support, C.-L. Chai for sharing his ideas and F. Oort for a helpful conversation.

2. Moduli problems

In this paragraph, we introduce the moduli problems that we will study and state the main results on their structure.

Throughout the paper we use the language of algebraic stacks. Our main reference is [3] Section 4 (see also [7] Ch. 1 Sect. 4). By a stack, we will mean a stack fibered in groupoids over some category of schemes with the étale topology. The reader who is unfamiliar with this language can assume that the integer n which gives the auxiliary full n-level structure of 2.1 is always sufficiently large ($n \geqslant 3$ will be enough). All the results will then refer to algebraic spaces (or even schemes).

Suppose $A \to S$ is an abelian scheme. We will denote by $A^* \to S$ the dual abelian scheme. It exists by [7] Section 3. There is a natural isomomorphism $A \simeq (A^*)^*$. If $\phi\colon A \to B$ is a homomorphism of abelian schemes, we denote its dual $B^* \to A^*$ by ϕ^*. An abelian scheme with real multiplication by $\mathcal{O}$ is, by definition, an abelian scheme $\pi : A \to S$ of relative dimension d, together with a ring homomorphism $i\colon \mathcal{O} \to \mathrm{End}_S(A)$. The dual of an abelian scheme with real multiplication acquires real multiplication by $a \to i(a)^*$. An abelian scheme homomorphism $\psi\colon A \to A^*$ is called symmetric, when the composition $A \simeq (A^*)^* \xrightarrow{\psi^*} A^*$ is equal to ψ. If $A \to S$ is an abelian scheme with real multiplication, we will denote by $\mathrm{Hom}_{\mathcal{O}}(A, A^*)^{\mathrm{sym}}$ the $\mathcal{O}$-module of symmetric $\mathcal{O}$-linear homomorphisms $A \to A^*$. We refer to [4], Section 3 for the definition of the cone of polarizations in $\mathrm{Hom}_{\mathcal{O}}(A, A^*)^{\mathrm{sym}}$.

2.1. Suppose that S is a scheme over Spec $\mathbf{Z}[1/n]$. We consider objects consisting of

1. An abelian scheme with real multiplication by $\mathcal{O}$, $A \to S$.
2. An $\mathcal{O}$-linear homomorphism from $\mathcal{L}$ to the module of $\mathcal{O}$-linear symmetric homomorphisms from A to A^*, $\lambda \to \psi(\lambda)$, such that:

   α. The set of totally positive elements of $\mathcal{L}$ maps into the positive cone defined by polarizations, and

   β. The induced morphism of sheaves on the large étale site of S, $A \otimes_{\mathcal{O}} \mathcal{L} \to A^*$ is an isomorphism.

3. An $\mathcal{O}$-linear isomorphism $(\frac{1}{n}\mathcal{O}/\mathcal{O})_S \xrightarrow{\gamma} A[n]$, between the constant group scheme defined by $\frac{1}{n}\mathcal{O}/\mathcal{O}$ and the kernel of multiplication by n on A.

The following moduli stack has been introduced in [4]:

DEFINITION 2.1.1. The moduli stack $\mathcal{H}_n^{\mathcal{L}}$ of abelian schemes with $\mathcal{O}$-multiplication, $\mathcal{L}$ polarization and full n level structure, is the stack over $\mathrm{Sch}/\mathbf{Z}[1/n]$ whose objects over the scheme S are given as before.

As it can be seen by standard methods, $\mathcal{H}_n^{\mathcal{L}}$ is a separated algebraic stack of finite type over Spec $\mathbf{Z}[1/n]$. When $n \geqslant 3$, the objects of $\mathcal{H}_n^{\mathcal{L}}$ do not have automorphisms and Artin's method shows that $\mathcal{H}_n^{\mathcal{L}}$ is an algebraic space. Using the techniques of Mumford's geometric invariant theory we can see that when $n \geqslant 3$, this algebraic space is in fact a quasi-projective scheme. We recall the following theorem of [4].

THEOREM 2.1.2 ([4]). *The algebraic stack $\mathcal{H}_n^{\mathcal{L}}$ is smooth over* Spec $\mathbf{Z}[1/n\Delta]$; *it is flat and a relative local complete intersection over* Spec $\mathbf{Z}[1/n]$. *If p is a rational prime dividing Δ then the fiber of $\mathcal{H}_n^{\mathcal{L}}$ over p is smooth outside a closed subset of codimension* 2.

2.1.3. A simple calculation using the local charts of 4.3 [4], shows that $\mathcal{H}_n^{\mathcal{L}}$ is regular when $d = 2$.

2.2. Suppose that S is a scheme over Spec $\mathbf{Z}[1/n]$, and consider $\mathcal{O}$-linear isogenies $\phi\colon A_1 \to A_2$ of degree Norm$(\mathcal{A})$, where A_1 and A_2 correspond to objects of $\mathcal{H}_n^{\mathcal{L}}$ and $\mathcal{H}_n^{\mathcal{AL}}$ over S, and which satisfy:

1. The kernel of ϕ is annihilated by $\mathcal{A}$.
2. For every $\lambda \in \mathcal{AL} \subset \mathcal{L}$, we have

$$\phi^* \cdot \psi_2(\lambda) \cdot \phi = \psi_1(\lambda).$$

3. The n level structures γ_1, γ_2 are compatible: $\gamma_2 = \phi_{|A_1[n]} \cdot \gamma_1$.

DEFINITION 2.2.1. The moduli stack of $\mathcal{O}$-linear isogenies $\mathcal{H}_0(\mathcal{A})_n^{\mathcal{L}}$ is the stack over Sch$/\mathbf{Z}[1/n]$ whose objects over the scheme S are given by $\mathcal{O}$-linear isogenies $\phi\colon A_1 \to A_2$ as before.

There exists a forgetful morphism

$$\mathcal{H}_0(\mathcal{A})_n^{\mathcal{L}} \to \mathcal{H}_n^{\mathcal{L}}.$$

As it is seen by standard methods, this morphism is representable and proper. The stack $\mathcal{H}_0(\mathcal{A})_n^{\mathcal{L}}$ is a separated algebraic stack of finite type over Spec $\mathbf{Z}[1/n]$. When $n \geqslant 3$, it is an algebraic space and in fact a scheme.

From here and on, we shall surpress all indexing pertaining to the polarization ideal $\mathcal{L}$.

In Section 3, using the methods of crystalline deformation theory as in [4], we give étale local charts for $\mathcal{H}_0(\mathcal{A})_n$. These have a linear algebra description which is analyzed in Section 4. We show:

THEOREM 2.2.2. *The morphism $\mathcal{H}_0(\mathcal{A})_n \to$* Spec $\mathbf{Z}[1/n]$ *is a flat local complete intersection of relative dimension d. If p is a rational prime dividing* Norm$(\mathcal{A})$ *then the fiber of $\mathcal{H}_0(\mathcal{A})_n$ over p is smooth outside a closed subset of codimension* 1.

COROLLARY 2.2.3. *$\mathcal{H}_0(\mathcal{A})_n$ is normal and Cohen–Macaulay.*

In 3.4, we show that the same étale local charts describe the bad reduction of certain moduli stacks which are associated to compact Shimura varieties obtained from totally indefinite division quaternion algebras over F.

2.3. We now introduce the Drinfeld–Katz–Mazur (DKM) definition for $\Gamma_{00}(\mathcal{A})$-level structure.

Suppose that G is a group scheme which is finite and locally free of rank $\mathrm{Norm}(\mathcal{A})$ over the scheme S, and has an action of the ring $\mathcal{O}/\mathcal{A}$. The constant group scheme $(\mathcal{O}/\mathcal{A})_S$ associated to $\mathcal{O}/\mathcal{A}$, provides us with an example of such a group scheme. If $\phi: A_1 \to A_2$ is the isogeny corresponding to an object of $\mathcal{H}_0(\mathcal{A})_n$ over S, then the kernel $\ker(\phi)$ is also a group scheme which satisfies these conditions.

DEFINITION 2.3.1. An $\mathcal{O}/\mathcal{A}$-generator of G over S is a point $P \in G(S)$ such that the $\mathcal{O}/\mathcal{A}$-linear homomorphism of group schemes over S

$$(\mathcal{O}/\mathcal{A})_S \to G$$

which is induced by $1 \to P$, is an $\mathcal{O}/\mathcal{A}$-structure on G in the sense of [8] 1.10 (see 5.1.1).

DEFINITION 2.3.2. The moduli stack $\mathcal{H}_{00}(\mathcal{A})_n$ of DKM $\Gamma_{00}(\mathcal{A})$-level structure over $\mathrm{Sch}/\mathbf{Z}[1/n]$, is the stack whose objects over S are pairs, consisting of an object of $\mathcal{H}_0(\mathcal{A})_n$ over S together with an $\mathcal{O}/\mathcal{A}$-generator of the kernel of the corresponding isogeny $\phi\colon A_1 \to A_2$.

There is a forgetful morphism

$$\pi_0\colon \mathcal{H}_{00}(\mathcal{A})_n \to \mathcal{H}_0(\mathcal{A})_n$$

which is representable and finite by [8] 1.10.13. Therefore, $\mathcal{H}_{00}(\mathcal{A})_n$ is also a separated algebraic stack of finite type over $\mathbf{Z}[1/n]$. We will show that the morphism π_0 is flat by using results of Raynaud on $(p,\ldots,p)$-group schemes. We can then deduce from 2.2.2 and 2.2.3,

THEOREM 2.3.3. *$\mathcal{H}_{00}(\mathcal{A})_n$ is Cohen–Macaulay and flat over* $\mathrm{Spec}\,\mathbf{Z}[1/n]$.

In fact, we explicitly determine $\mathcal{H}_{00}(\mathcal{A})_n$ as a cover of $\mathcal{H}_0(\mathcal{A})_n$. However, $\mathcal{H}_{00}(\mathcal{A})_n$ is rarely normal; if $d = 2$, we will see that $\mathcal{H}_{00}(\mathcal{A})_n$ is normal if and only if different prime factors of $\mathcal{A}$ lie over different rational primes and the residue fields of all the primes factors of $\mathcal{A}$ are the prime field.

2.4. In what follows, we concentrate on the case of Hilbert modular surfaces $(d = 2)$. We shall see from the calculation of the étale local charts of $\mathcal{H}_0(\mathcal{A})_n$, that either $\mathcal{H}_0(\mathcal{A})_n$ is regular, or it can be desingularized by a simple blow-up. Regarding $\mathcal{H}_{00}(\mathcal{A})_n$, we study in Section 6, the moduli spaces which are obtained by the following modification of the notion of $\mathcal{O}/\mathcal{A}$-generator:

Suppose first that $\mathcal{A} = (p)$, where (p) is a rational prime inert in F. There is a filtration

$$\mathrm{Fil}\,:\, 0 \subset \mathbf{Z}/p\mathbf{Z} \subset \mathcal{O}/(p).$$

By definition, an $(\mathcal{O}/\mathcal{A},\,\mathrm{Fil})$-structure on $G := \ker(\phi)$, is a pair $(H,\,P)$ of a locally free subgroup scheme of H of rank p of G, together with a point $P \in H(S) \subset G(S)$

such that P is a $\mathbf{Z}/p\mathbf{Z}$-generator of H and an $\mathcal{O}/(p)$-generator of G (both in the sense of 2.3.1).

In general, write $\mathcal{A}$ as a product of distinct prime ideals of $\mathcal{O}$, $\mathcal{A} = \mathcal{P}_1 \cdots \mathcal{P}_r$ and correspondingly $\ker(\phi) = G_1 \times \cdots \times G_r$, where G_i is the part of $\ker(\phi)$ annihilated by $\mathcal{P}_i$. An $(\mathcal{O}/\mathcal{A},\ \mathrm{Fil})$-structure on $\ker(\phi)$ is, by definition, a collection of $\mathcal{O}/\mathcal{P}_i$-generators of G_i for the $\mathcal{P}_i$ with $\mathcal{O}/\mathcal{P}_i = \mathbf{Z}/p\mathbf{Z}$, together with $(\mathcal{O}/\mathcal{P}_i,\ \mathrm{Fil})$-structures for $\mathcal{P}_i = (p_i)$ with p_i rational primes.

DEFINITION 2.4.1 ($d = 2$). The moduli stack $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_n$ of filtered DKM $\Gamma_{00}(\mathcal{A})$-level structure over $\mathrm{Sch}/\mathbf{Z}[1/n]$, is the stack whose objects over S are pairs consisting of an object of $\mathcal{H}_0(\mathcal{A})_n$ over S together with an $(\mathcal{O}/\mathcal{A},\ \mathrm{Fil})$-structure on the kernel of the corresponding isogeny $\phi\colon A_1 \to A_2$.

There is a forgetful morphism

$$\pi_{00}\colon \mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_n \to \mathcal{H}_{00}(\mathcal{A})_n.$$

We can see (cf. Sect. 6), that the morphism π_{00} is representable and proper. It is an isomorphism on the complement of the fibers over primes dividing $\mathrm{Norm}(\mathcal{A})$.

THEOREM 2.4.2 ($d = 2$). *The algebraic stack $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_n$ is a flat relative local complete intersection over* $\mathrm{Spec}\,\mathbf{Z}[1/n]$. *If different prime factors of $\mathcal{A}$ lie over different rational primes, it is regular outside a closed subset of codimension 2. If in fact, in addition all prime factors of $\mathcal{A}$ have residue fields the prime field, then $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_n \simeq \mathcal{H}_{00}(\mathcal{A})_n$ is regular.*

The singularities are again explicitly described. In 6.2, we show that when different prime factors of $\mathcal{A}$ lie over different rational primes, we arrive at a regular model by repeated blow-up of $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_n$ along the singular locus. The fibers of the regular model over primes which divide $\mathrm{Norm}(\mathcal{A})$ but are prime to the discriminant Δ are divisors with non-reduced normal crossings.

Consider now the case $n = 1$. The standard arguments show that there is an integer N, such that when $\mathrm{Norm}(\mathcal{A}) \geqslant N$, the objects of $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})$ over $\mathbf{Z}[1/\mathrm{Norm}(\mathcal{A})]$ do not have any non-trivial automorphisms. Therefore, if $\mathcal{A} = \mathcal{A}' \cdot \mathcal{A}''$ for two ideals with norms which are relatively prime and both $\geqslant N$, then the same is true for objects of $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})$ over $\mathrm{Spec}\,\mathbf{Z}$. Mumford's geometric invariant theory shows that under these conditions $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})$ is represented by a scheme.

As a corollary of our analysis we then obtain:

THEOREM 2.4.3 ($d = 2$). *Suppose that $\mathcal{A} = \mathcal{A}' \cdot \mathcal{A}''$ is as before. Then there exists a scheme M over* $\mathrm{Spec}\,\mathbf{Z}$ *which is regular, flat over* $\mathrm{Spec}\,\mathbf{Z}$, *and such that*

(α) $M \times_{\mathbf{Z}} \mathbf{Z}[1/\mathrm{Norm}(\mathcal{A})] \simeq \mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A}) \times_{\mathbf{Z}} \mathbf{Z}[1/\mathrm{Norm}(\mathcal{A})] \simeq$
$\simeq \mathcal{H}_{00}(\mathcal{A}) \times_{\mathbf{Z}} \mathbf{Z}[1/\mathrm{Norm}(\mathcal{A})]$.

(β) *There exists a proper representable morphism $M \to \mathcal{H}$ which extends the forgetful morphism $\mathcal{H}_{00}(\mathcal{A}) \times_{\mathbf{Z}} \mathbf{Z}[1/\mathrm{Norm}(\mathcal{A})] \to \mathcal{H} \times_{\mathbf{Z}} \mathbf{Z}[1/\mathrm{Norm}(\mathcal{A})]$.*

2.4.4. Suppose that $A \to S$ gives an object of $\mathcal{H}_n$ over S. The polarization isomorphism of 2.1(2) induces a non-degenerate alternating $\mathcal{O}/n\mathcal{O}$-bilinear form

$$A_n \times A_n \to \mathcal{L}^{-1}\mathcal{D}^{-1} \otimes \mu_n := \mathcal{L}^{-1}\mathcal{D}^{-1}(1),$$

which gives an isomorphism

$$\bigwedge^2_{\mathcal{O}/n\mathcal{O}} A_n \simeq \mathcal{L}^{-1}\mathcal{D}^{-1}(1).$$

Composing this with $\mathcal{O}/n\mathcal{O} = \wedge^2_{\mathcal{O}/n\mathcal{O}}\mathcal{O}/n\mathcal{O} \simeq \wedge^2_{\mathcal{O}/n\mathcal{O}}A_n$ induced by the n-level structure of 2.1(3), we obtain an $\mathcal{O}$-linear isomorphism

$$\beta_A\colon \mathcal{O}/n\mathcal{O} \xrightarrow{\simeq} \mathcal{L}^{-1}\mathcal{D}^{-1}(1).$$

Suppose now that $\mathcal{L} = \mathcal{D}^{-1}$ and $n \geqslant 3$. Consider the closed and open subscheme of the moduli schemes $\mathcal{H}_n \times_{\mathbf{Z}[1/n]} \mathbf{Q}(\zeta_n)$ (resp. $\mathcal{H}_0(\mathcal{A})_n \times_{\mathbf{Z}[1/n]} \mathbf{Q}(\zeta_n)$) on which $\beta_A(1) = \zeta_n$ (resp. $\beta_{A_1}(1) = \zeta_n$). This scheme is the Hilbert–Blumenthal variety $H_{\Gamma(n)}$ (resp. $H_{\Gamma_0(\mathcal{A},n)}$) of the introduction.

However, the situation about $H_{\Gamma_{00}(\mathcal{A},n)}$ is more subtle. Suppose again that $\mathcal{L} = \mathcal{D}^{-1}$ and that $\mathcal{A}$, n are such that $\mathcal{H}_{00}(\mathcal{A})_n$, $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_n$ are representable. Then the previous construction, applied to $\mathcal{H}_{00}(\mathcal{A})_n \times_{\mathbf{Z}[1/n]} \mathbf{Q}(\zeta_n) = \mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_n \times_{\mathbf{Z}[1/n]} \mathbf{Q}(\zeta_n)$ does not yield the "canonical" model of $H_{\Gamma_{00}(\mathcal{A},n)}$ but a twist of it. The two varieties become isomorphic after a base extension to the field obtained from $\mathbf{Q}(\zeta_n)$ by adjoining the pth roots of unity for all primes in the support of Norm$(\mathcal{A})$. To obtain moduli stacks which provide integral models for the canonical model, we have to modify our definitions and consider, instead of $\mathcal{O}/\mathcal{A}$-structures on the kernel of the modular isogeny, $\mathcal{D}^{-1}/\mathcal{A}\mathcal{D}^{-1}$-structures on its Cartier dual. Let us explain this in the case of $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_n$.

Define the notion of a $(\mathcal{D}^{-1}/\mathcal{A}\mathcal{D}^{-1}$, Fil$)$-structure similarly to 2.4, by observing that, when p is a prime number inert in F, there is a natural $\mathcal{O}$-linear isomorphism $\mathcal{D}^{-1}/(p)\mathcal{D}^{-1} \simeq \mathcal{O}/(p)$. Then, consider the moduli stack $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})'$ whose objects over S are pairs consisting of an object $\phi\colon A_1 \to A_2$ of $\mathcal{H}_0(\mathcal{A})$, together with a $(\mathcal{D}^{-1}/\mathcal{A}\mathcal{D}^{-1}$, Fil$)$-structure on the Cartier dual of ker(ϕ). Over $\mathbf{Z}[1/\mathrm{Norm}(\mathcal{A})]$, a choice of a $(\mathcal{D}^{-1}/\mathcal{A}\mathcal{D}^{-1}$, Fil$)$-structure on the Cartier dual of ker(ϕ) is equivalent to a choice of an $\mathcal{O}$-linear homomorphism

$$\mathcal{O}/\mathcal{A}(1) \xrightarrow{\simeq} \ker(\phi),$$

where (1) again denotes the Tate twist. This is the notion of $\Gamma_{00}(\mathcal{A})$-level structure used in [6]. The calculations of Sections 5 and 6, will make obvious the fact that the moduli stack $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})'$ has the same local structure with $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})$. Therefore, the Theorem 2.4.3 will remain true if we replace $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})$ by $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})'$.

The usual machinery of toroidal compactification applies to the normal algebraic stacks $\mathcal{H}_0(\mathcal{A})$ and (for $d = 2$ when $\mathcal{A}$ has prime factors over different rational primes) $\mathcal{H}_{00}^{\rm Fil}(\mathcal{A})$. The singular locus of $\mathcal{H}_{00}^{\rm Fil}(\mathcal{A})$ avoids the boundary, so we also obtain toroidal compactifications of the regular models obtained by blow-up. One can verify that these compactifications are regular at the boundary. We will leave the details for another occasion.

3. Local model for the moduli of isogenies

3.1. Suppose that $n'|n$. Then, there is a forgetful morphism $\mathcal{H}_0(\mathcal{A})_n \to \mathcal{H}_0(\mathcal{A})_{n'}$ which is étale. Therefore, the local structure of $\mathcal{H}_0(\mathcal{A})_n$ for the étale topology does not depend on n, and for the rest of this paragraph we will assume that $n = 1$ and drop the index from the notation.

Let us fix a rational prime p. Write the ideal $\mathcal{A}$ of $\mathcal{O}$ as $\mathcal{A} = \mathcal{A}' \cdot \mathcal{B}$, where $\mathcal{B}$ is relatively prime to p and the prime divisors of $\mathcal{A}'$ divide p. There is a forgetful morphism $\mathcal{H}_0(\mathcal{A})_{\mathbf{Z}_{(p)}} \to \mathcal{H}_0(\mathcal{A}')_{\mathbf{Z}_{(p)}}$ which is finite étale and so for the study of the local structure over p, we can assume that $\mathcal{A}$ is supported over p, i.e. that $\mathcal{A} = \mathcal{A}'$.

DEFINITION 3.1.1. An étale local model for the algebraic stack $\mathcal{M}$ is a scheme $\mathcal{S}$ such that there is a scheme U, an étale surjective morphism $U \to \mathcal{M}$ and an étale morphism $U \to \mathcal{S}$.

3.2. We will describe an étale local model for $\mathcal{H}_0(\mathcal{A})_{\mathbf{Z}_{(p)}}$. Recall, we assume that $\mathcal{A}$ is supported over p. Let us fix an integer N relatively prime to p, such that $\mathcal{L}[1/N]$, $\mathcal{D}[1/N]$ and $\mathcal{A}[1/N]$ are free $\mathcal{O}[1/N]$-modules. We fix a generator l of $\mathcal{L}[1/N]$, and $a \in \mathcal{O}[1/N]$ such that $\mathcal{A}[1/N] = a \cdot \mathcal{O}[1/N]$. From 2.1(2), we have $\mathcal{O}$-linear homomorphisms (we abuse notation slightly)

$$\psi_1 \colon \mathcal{L}[1/N] \to \mathrm{Hom}_{\mathcal{O}}(A_1, A_1^*)^{\rm sym} \otimes_{\mathcal{O}} \mathcal{O}[1/N],$$

$$\psi_2 \colon \mathcal{A}\mathcal{L}[1/N] \to \mathrm{Hom}_{\mathcal{O}}(A_2, A_2^*)^{\rm sym} \otimes_{\mathcal{O}} \mathcal{O}[1/N].$$

Using 2.β we see that $\psi_1(l)$ and $\psi_2(al)$ are invertible (cf. 2.6 [4]). Now 2.2(2) applied to $a \cdot l$ gives

$$\phi^* \cdot \psi_2(al) \cdot \phi = \psi_1(al) = a \cdot \psi_1(l).$$

Setting

$$\phi^t = \psi_1(l)^{-1} \cdot \phi^* \cdot \psi_2(al)$$

we thus obtain $\phi^t \cdot \phi = a$. Then we also have $\phi \cdot \phi^t = a$.

3.3. If S is any scheme over $\mathbf{Z}[1/N]$, we consider the coherent $\mathcal{O}_S$-module $(\mathcal{O}_S \otimes \mathcal{O})^2$, together with the $\mathcal{O}_S \otimes \mathcal{O}$-linear alternating form Ψ_0, given by

$\Psi_0((x, 0), (0, y)) = x \cdot y$. We denote by $u_0 = \mathrm{diag}(1, a)$ the $\mathcal{O}_S \otimes \mathcal{O}$-module homomorphism $(\mathcal{O}_S \otimes \mathcal{O})^2 \to (\mathcal{O}_S \otimes \mathcal{O})^2$ given by $u_0((1, 0)) = 1$, $u_0((0, 1)) = a$.

We now consider the functor H_0 on schemes over $\mathbf{Z}[1/N]$ which to such a scheme S, associates the set of pairs (F, F') of $\mathcal{O}_S \otimes \mathcal{O}$-submodules of $(\mathcal{O}_S \otimes \mathcal{O})^2$, which are equal to their orthogonal with respect to Ψ_0, are both locally on S direct summands as $\mathcal{O}_S$-modules and satisfy the relation $u(F) \subset F'$. The functor H_0 is represented by a scheme $\mathcal{N}_0(a)$ over Spec $\mathbf{Z}[1/N]$ which is a closed subscheme of a fiber product of two Grassmanians.

The main theorem of this paragraph is the following. The proof closely resembles the proof of similar results in [4].

THEOREM 3.3.1. *The scheme $\mathcal{N}_0(a)_{\mathbf{Z}_{(p)}}$ is an étale local model for $\mathcal{H}_0(\mathcal{A})_{\mathbf{Z}_{(p)}}$.*

Proof. Suppose that x is a closed point of $\mathcal{H}_0(\mathcal{A})_{\mathbf{Z}_{(p)}}$. Since $\mathcal{H}_0(\mathcal{A})_{\mathbf{Z}_{(p)}}$ is of finite type over $\mathbf{Z}_{(p)}$, we can find a scheme U of finite type over Spec $\mathbf{Z}_{(p)}$ together with an étale morphism $U \to \mathcal{H}_0(\mathcal{A})_{\mathbf{Z}_{(p)}}$ covering a neighborhood of x in $\mathcal{H}_0(\mathcal{A})_{\mathbf{Z}_{(p)}}$.

Let us denote by $\phi\colon A_1 \to A_2$ the $\mathcal{O}$-linear isogeny over U, which corresponds to $U \to \mathcal{H}_0(\mathcal{A})_{\mathbf{Z}_{(p)}}$. Denote by f_1, f_2 the structure morphisms $A_1 \to U$, $A_2 \to U$.

We consider the DeRham cohomology sheaves

$$H^1_{\mathrm{dR}}(A_1) = R^1 f_{1*}(\Omega^\bullet_{A_1/U})$$

and their $\mathcal{O}_U$-duals ("DeRham homology") $H_i := H_1^{\mathrm{dR}}(A_i/U)$. The H_i's are extensions

$$0 \to \mathrm{Lie}(A_i^*/U)^\vee \to H_i \to \mathrm{Lie}(A_i/U) \to 0,$$

where $\mathrm{Lie}(A_i^*/U)^\vee$ is the $\mathcal{O}_U$-dual of the $\mathcal{O}_U$-locally free Lie algebra $\mathrm{Lie}(A_i^*/U)$. The sheaf $\mathrm{Lie}(A_i^*/U)^\vee$ is an $\mathcal{O}_U \otimes \mathcal{O}$-subsheaf of H_i which is Zariski locally on U a direct summand as an $\mathcal{O}_U$-sheaf.

By [9] Lemma 1.3, we see that after shrinking U to an open subscheme, we can assume that the H_i are free $\mathcal{O}_U \otimes \mathcal{O}$-modules. The isogenies $\phi\colon A_1 \to A_2$ and ϕ^t induce $\mathcal{O}_U \otimes \mathcal{O}$-linear homomorphisms

$$u\colon H_1 \to H_2, \quad {}^t u\colon H_2 \to H_1$$

and we have $u(\mathrm{Lie}(A_1^*/U)^\vee) \subset \mathrm{Lie}(A_2^*/U)^\vee$. By [4] 2.12, we see that the polarization isomorphisms of 2.1.2 induce non-degenerate alternating $\mathcal{O}_U \otimes \mathcal{O}$-bilinear pairings

$$H_1 \times H_1 \to \mathcal{O}_U \otimes \mathcal{L}^{-1}\mathcal{D}^{-1},$$

$$H_2 \times H_2 \to \mathcal{O}_U \otimes \mathcal{A}^{-1}\mathcal{L}^{-1}\mathcal{D}^{-1}.$$

Using the trivializations introduced in 3.2, we obtain non degenerate alternating $\mathcal{O}_U \times \mathcal{O}$-bilinear pairings

$$\Psi_i \colon H_i \times H_i \to \mathcal{O}_U \otimes \mathcal{O}$$

which satisfy

$$\Psi_2(ux,\ y) = \Psi_1(x,\ {}^t uy), \quad \text{for } x \in H_1,\ y \in H_2.$$

By [4] 2.12, the $\mathcal{O}_U \otimes \mathcal{O}$-sheaves $\mathrm{Lie}(A_i^*/U)^\vee$ are equal to their own orthogonal for Ψ_i, and so also for every other non-degenerate $\mathcal{O}_U \otimes \mathcal{O}$-bilinear alternating form on H_i.

The point now is the following "rigidity" lemma (cf. [4] 5.5).

LEMMA 3.3.2. *The modular triple $(H_1,\ H_2,\ u)$ is locally for the Zariski topology on U, isomorphic to the "constant" triple $((\mathcal{O}_U \otimes \mathcal{O})^2,\ (\mathcal{O}_U \otimes \mathcal{O})^2,\ u_0)$.*

Proof. The lemma is interesting only for a neighborhood of a point s of U with residue characteristic p. Indeed, u is an isomorphism at a neighborhood of a point s with residue characteristic 0, and then it is enough to choose an isomorphism $(\mathcal{O}_U \otimes \mathcal{O})^2 \simeq H_1$. Look at the fibers of H_1 and H_2 over s; we have morphisms

$$u(s) := u \otimes k(s) \colon H_{1_s} \to H_{2_s}, \quad {}^t u(s) := {}^t u \otimes k(s) \colon H_{2_s} \to H_{1_s}.$$

We will show that $\mathrm{coker}({}^t u(s)) \simeq \mathrm{coker}(u(s)) \simeq k(s) \otimes (\mathcal{O}/a\mathcal{O})$ as a $k(s) \otimes \mathcal{O}/a\mathcal{O}$-modules. For this, using descent, we can reduce to the case $k(s)$ is perfect. Then the $k(s) \otimes \mathcal{O}$-modules H_{is} can be viewed as the reduction mod p of the $W(k(s))$-duals $H_i^{\mathrm{cr}} := H^1_{\mathrm{cr}}(A_{is})^\vee$ of the crystalline cohomology of the fibers A_{is}. The modules H_i^{cr} are free of rank 2 over $W(k(s)) \otimes \mathcal{O}$ by [9] 1.3. The restriction of the isogenies ϕ and ${}^t\phi$ on the fibers induce $W(k(s)) \otimes \mathcal{O}$-linear morphisms u_s and ${}^t u_s$ between H_1^{cr} and H_2^{cr} and we have $u_s^t u_s = a,\ {}^t u_s u_s = a$. We conclude that the cokernels of u_s and ${}^t u_s$ are annihilated by a. Since a is a local generator of a square free ideal, they are $k(s)$-vector spaces and they coincide with the cokernels of ${}^t u(s)$ and $u(s)$. The conditions 2.2 show that their dimension over $k(s)$ is equal to $\mathrm{Norm}(a) = \mathcal{O}/a\mathcal{O}$ and so $\mathrm{coker}({}^t u(s)) \simeq \mathrm{coker}(u(s)) \simeq k(s) \otimes (\mathcal{O}/a\mathcal{O})$.

The statement about the cokernel of $u(s)$ allows us to choose elements v_1 and v_2 in H_{1_s} and H_{2_s} respectively, such that $(u(s)(v_1),\ v_2)$ form an $k(s) \otimes \mathcal{O}/a\mathcal{O}$-basis of H_{2_s}. Then $(v_1,\ {}^t u(s)(v_2))$ is also a $k(s) \otimes \mathcal{O}$-basis of H_{1_s}. Lift v_1 and v_2 to elements V_1 and V_2 of H_1 and H_2. Then the $\mathcal{O}_U \otimes \mathcal{O}$-module homomorphisms

$$(\mathcal{O}_U \otimes \mathcal{O}) \oplus (\mathcal{O}_U \otimes \mathcal{O}) \xrightarrow{V_1 \oplus {}^t u(V_2)} H_1,$$

$$(\mathcal{O}_U \otimes \mathcal{O}) \oplus (\mathcal{O}_U \otimes \mathcal{O}) \xrightarrow{u(V_1) \oplus V_2} H_2,$$

give an isomorphism in a neighborhood of s in U as required. □

We now complete the proof of 3.3.1, by arguing as in [4] Section 3. Any choice of an isomorphism as in Lemma 1.2 provides us with an U-valued point of $\mathcal{N}_0(a)_{\mathbf{Z}_{(p)}}$, i.e. a morphism

$$U \to \mathcal{N}_0(a)_{\mathbf{Z}_{(p)}}.$$

Consider a point x' of U which lies above x. The residue field of x' is finite and therefore perfect. Let U_1 be the first characteristic p infinitesimal neighborhood of x' in U. The triple $(H_1,\ H_2,\ u)$ is trivialized on U_1 by the Gauss–Manin connection. The proof of 3.3.2, shows that we can choose the isomorphism of 3.3.2, so that it extends the natural isomorphism on U_1 which is provided by the Gauss–Manin connection (cf [4] Sect. 3). We now see exactly like in loc. cit (3.4, 3.5 also 5.6) that crystalline deformation theory implies that the corresponding morphism $U \to \mathcal{N}_0(a)_{\mathbf{Z}_{(p)}}$ is étale at x'. This concludes the proof of 3.3.1. □

3.4. In this paragraph, we show that the schemes $\mathcal{N}_0(a)$ also give local models for the non-smooth reduction of certain moduli stacks which are associated to compact Shimura varieties, obtained from totally indefinite quaternion division algebras over F.

Suppose that B is a totally indefinite quaternion algebra over F, and that $\mathcal{A}$ divides the discriminant of B. For simplicity of exposition, we will assume that $\mathcal{A} = \mathcal{P}_1 \dots \mathcal{P}_r$ is supported over a single rational prime p. We choose a totally real splitting field E of B. This is a quadratic extension of F, and we can take it such that the prime divisors of p in F remain prime in E. Denote by σ the non-trivial automorphism of E over F. Using the Skolem–Noether theorem, we can choose an element $c \in B$, such that $ce = e^\sigma c$ for $e \in E$. We have $c^2 \in F$, and in fact we can suppose that $-c^2 = \delta$ is a totally positive element of $\mathcal{O}$ with $\mathrm{ord}_{\mathcal{P}_i}(\delta) = 1$ at every prime divisor of $\mathcal{A}$ and $\mathrm{ord}_{\mathcal{P}_i}(\delta) = 0$ at all other primes of F over p. The order $\mathcal{O}_B := \mathcal{O}_E[c]$ is maximal at all primes of F over p. Denote by $b \to \bar{b}$ the canonical involution of B. The involution $'$ of B, defined by $b \to b' = c\bar{b}c^{-1}$, is a positive involution which preserves $\mathcal{O}_B$.

We now consider the moduli stack $\mathcal{B}$ over Spec $\mathbf{Z}_p$, ("pseudo Hilbert–Blumenthal isogenies") whose objects over the $\mathbf{Z}_p$-scheme S are:

(1) Abelian schemes $A \to S$ of relative dimension $2d$ together with a ring homomorphism $i\colon\ \mathcal{O}_B \to \mathrm{End}_S(A)$,
(2) A polarization $\phi\colon\ A \to A^*$ which is principal at p, and $\mathcal{O}_B$-linear with the $\mathcal{O}_B$ structure on A^* given by $b \to i(b')^*$.

We can see that $\mathcal{B}$ is a separated algebraic stack of finite type over $\mathbf{Z}_p$. A variant of this moduli stack, has been considered by R. Langlands in the case $d = 2$ and by T. Zink ([11]) in general, when p is inert in F and $\mathcal{A} = (p)$.

We view δ as an element of the completion $\mathcal{O}_p = \mathcal{O}_{(p)} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_p$. It is the generator of a square-free ideal of $\mathcal{O}_p$. We can consider the scheme $\mathcal{N}_0(\delta)_{\mathbf{Z}_p}$, defined as in 3.3.

THEOREM 3.4.1. *$\mathcal{N}_0(\delta)_{\mathbf{Z}_p}$ is an étale local model for $\mathcal{B}$.*

Proof. The idea is the following (cf. [4] 5.10–5.12): After an étale base change the quaternion algebra splits and then the linear Hodge–DeRham data that determine the deformation theory of the moduli stack $\mathcal{B}$, have the same shape with the data that determine the deformation theory of the moduli of $\mathcal{O}$-linear isogenies.

We obtain an embedding $\mathcal{O}_B \subset M_2(\mathcal{O}_E)$ by sending $e \to \left(\begin{smallmatrix} e & 0 \\ 0 & e^\sigma \end{smallmatrix}\right)$ and $c \to \left(\begin{smallmatrix} 0 & 1 \\ -\delta & 0 \end{smallmatrix}\right)$. We have

$$\mathcal{O}_B \simeq \left\{ \left(\begin{array}{cc} x_1 & x_2 \\ -\delta x_2^\sigma & x_1^\sigma \end{array} \right) \middle| x_1,\, x_2 \in \mathcal{O}_E \right\} \subset M_2(\mathcal{O}_E).$$

Let W be a DVR finite and étale over $\mathbf{Z}_p$ such that $\mathcal{O} \otimes W$ contains the product of the completions of $\mathcal{O}_E$ at all primes of E over p. Then if $F(W)$ is the fraction field of W, $B \otimes_{\mathbf{Q}} F(W) = B \otimes_F (F \otimes_{\mathbf{Q}} F(W))$ decomposes as a product of matrix algebras. We have

$$\mathcal{O}_B \otimes W = \mathcal{O}_B \otimes_{\mathcal{O}} (\mathcal{O} \otimes W) \simeq \left(\begin{array}{cc} * & * \\ \delta * & * \end{array} \right) \subset M_2(\mathcal{O} \otimes W). \tag{3.1}$$

Under this isomorphism, the involution $b \to b'$ of $\mathcal{O}_B$ becomes

$$\left(\begin{array}{cc} x & y \\ \delta z & t \end{array} \right) \to \left(\begin{array}{cc} x & z \\ \delta y & t \end{array} \right). \tag{3.2}$$

Suppose now that we have an U-point of $\mathcal{B}$, where U is a scheme over Spec W. Then $H^1_{\mathrm{dR}}(A/U)$ is a module over $\mathcal{O}_U \otimes \mathcal{O}_B = \mathcal{O}_U \otimes_W (\mathcal{O}_B \otimes W)$.

In view of (3.1), we can consider

$$e_{11}\mathrm{Lie}(A^*/U)^\vee \subset e_{11} H^1_{\mathrm{dR}}(A/U), \tag{3.3}$$

and

$$e_{22}\mathrm{Lie}(A^*/U)^\vee \subset e_{22} H^1_{\mathrm{dR}}(A/U), \tag{3.4}$$

where e_{11} and e_{22} are the diagonal idempotents of $M_2(\mathcal{O} \otimes W)$. By [9] Lemma 1.3, $H^1_{\mathrm{dR}}(A/U)$ is locally on U free of rank 2 over $\mathcal{O}_U \otimes \mathcal{O}_E$. Therefore, $e_{11} H^1_{\mathrm{dR}}(A/U)$ and $e_{22} H^1_{\mathrm{dR}}(A/U)$ are locally on U, free of rank 2 over $\mathcal{O}_U \otimes_W (\mathcal{O} \otimes W)$. In view of (3.2), we see, exactly like in [4] 2.12 (cf. proof of 3.3.1), that the polarization ϕ of 2) induces nondegenerate $\mathcal{O}_U \otimes_W (\mathcal{O} \otimes W)$-bilinear alternating forms on $e_{11} H^1_{\mathrm{dR}}(A/U)$ and $e_{22} H^1_{\mathrm{dR}}(A/U)$ for which $e_{11}\mathrm{Lie}(A^*/U)^\vee$ and $e_{11}\mathrm{Lie}(A^*/U)^\vee$ are equal to their own orthogonal.

Now consider the $\mathcal{O} \otimes W$-linear morphisms

$$u\colon e_{11}H^1_{\mathrm{dR}}(A) \to e_{22}H^1_{\mathrm{dR}}(A), \quad \text{and}$$

$$^t u\colon e_{22}H^1_{\mathrm{dR}}(A) \to e_{11}H^1_{\mathrm{dR}}(A)$$

induced by multiplication by $\left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ respectively. We can see that we can take $e_{11}H^1_{\mathrm{dR}}(A/U)$ and $e_{22}H^1_{\mathrm{dR}}(A/U)$ to play the role that $H^1_{\mathrm{dR}}(A_1/U) = H_1$ and $H^1_{\mathrm{dR}}(A_2/U) = H_2$ play in the proof of Theorem 3.3.1 (cf. [4] 5.10–5.12). In particular, after choosing trivializations, (3.3) and (3.4) provide us with morphisms of U into $\mathcal{N}_0(\delta)_W$. The rest now falls exactly along the lines of the proof of 3.3.1. □

Using the results of the next paragraph on the scheme $\mathcal{N}_0(\delta)$ we shall see that

THEOREM 3.4.2. *The algebraic stack $\mathcal{B}$ is a flat relative local complete intersection of relative dimension d over* Spec $\mathbf{Z}_p$. *It is smooth outside a subset of codimension* 1 *in the special fiber.*

4. Singularities for $\Gamma_0(\mathcal{A})$-structure

In this paragraph, we study the functor H_0 of Section 3. Combining the study of H_0 with Theorems 3.3.1 and 3.4.1, we will show 2.2.2 and 3.4.2.

4.1. We start by introducing a family of functors, of which H_0 is a special member: Let $S = \operatorname{Spec} A$ be an affine Noetherian scheme, R an A-finite locally free A-algebra R and take $a \in R$.

We consider the functor $H(S, R)$ (resp. $H_0(S, R, a)$) which to a scheme $f\colon T \to S$ associates the set of all $R_T := R \otimes_{\mathcal{O}_S} \mathcal{O}_T$-submodules of F (resp. pairs (F, F') of submodules) of R^2_T which are locally direct summands as $\mathcal{O}_T$-modules, are equal to their orthogonal with respect to the standard R_T-bilinear alternating form on R^2_T (resp. and satisfy $u(F) \subset F'$ where $u = \mathrm{diag}(1, a)$). If (F, F') is in $H_0(S, R, a)(T)$ and $^t u = \mathrm{diag}(a, 1)$, then we also have $^t u(F') \subset F$.

The functor H_0 of Section 3 is $H_0 = H_0(\operatorname{Spec} \mathbf{Z}[1/N], \mathcal{O}[1/N], a)$ where as we recall $a \cdot \mathcal{O}[1/n] = \mathcal{A}[1/N]$.

The functor $H(S, R)$ (resp. $H_0(S, R, a)$) is represented by a scheme $\mathcal{N}(S, R)$ (resp. $\mathcal{N}_0(S, \mathcal{R}, a)$) which is proper over S. Indeed, the natural morphism

$$\mathcal{N}_0(S, R, a) \to \mathcal{N}(S, R) \times_S \mathcal{N}(S, R)$$

identifies $\mathcal{N}_0(S, R, a)$ with a closed subscheme of the product. Since $\mathcal{N}(S, R)$ is a subscheme of a Grassmanian, the schemes $\mathcal{N}(S, R)$ and $\mathcal{N}_0(S, R, a)$ are proper over S.

The schemes $\mathcal{N}_0(S, R, T)$ satisfy the following functoriality properties (similarly for $\mathcal{N}(S, R)$, see Sect. 4 of [4]).

4.1.1. For $f: T \to S$, $\mathcal{N}_0(T, R_T, f^*a)$ is obtained from $\mathcal{N}_0(S, R, a)$ by base change.

4.1.2. For a finite étale $f: T \to S$, $\mathcal{N}_0(S, f_*R', a')$ is obtained from $\mathcal{N}_0(T, R', a')$ by Weil restriction of scalars:

$$\mathcal{N}_0(S, f_*R', a') = \prod_{T/S} \mathcal{N}_0(T, R', a').$$

To every T-valued point of $\mathcal{N}_0(S, R, a)$ we associate the diagram of morphisms:

$$\bigwedge F \overset{\wedge u}{\to} \bigwedge F' \overset{\wedge^t u}{\to} \bigwedge F,$$

where $\wedge$ signifies the highest exterior power of $\mathcal{O}_T$-modules. Then:

LEMMA 4.1.3. *Assume that the A-dual of R, which we denote by $R^\vee$, is a free R-module of rank* 1. *Then the composition of the two maps $\wedge u$ and $\wedge^t u$ is multiplication by* $\mathrm{Norm}_{R/A}(a)$.

Proof. As before, we will use the notation R_T for $R \otimes_A \mathcal{O}_T$. Also, we will denote by $M^\vee$ the $\mathcal{O}_T$-dual of an $\mathcal{O}_T$-module M. Since F is equal to its orthogonal with respect to the standard alternating pairing on R_T^2, we have an isomorphism

$$F \to \mathrm{Hom}_{R_T}(R_T^2/F, R_T).$$

Under our assumption, we have $R_T \simeq \mathrm{Hom}_{\mathcal{O}_T}(R_T, \mathcal{O}_T)$, and so we obtain an isomorphism

$$F \to \mathrm{Hom}_{R_T}(R_T^2/F, R_T) \simeq \mathrm{Hom}_{R_T}(R_T^2/F, \mathrm{Hom}_{\mathcal{O}_T}(R_T, \mathcal{O}_T))$$
$$= \mathrm{Hom}_{\mathcal{O}_T}(R_T^2/F, \mathcal{O}_T)$$

(similarly for F'). Since R_T^2/F and R_T^2/F' are also locally free $\mathcal{O}_T$-modules, we obtain perfect $\mathcal{O}_T$-pairings

$$F \times R_T^2/F \to \mathcal{O}_T, \quad F' \times R_T^2/F' \to \mathcal{O}_T,$$

and isomorphisms

$$F^\vee \simeq R_T^2/F, \quad F'^\vee \simeq R_T^2/F',$$

under which the $({}^t u)^\vee : F^\vee \to F'^\vee$ becomes the morphism induced by u, $\bar{u} : R_T^2/F \to R_T^2/F'$. We have $\wedge R_T^2 \simeq \wedge F \otimes \wedge(R_T^2/F) \simeq \wedge F' \otimes \wedge(R_T^2/F')$ and the proof follows now from this and the fact that $\wedge u \colon \wedge R_T^2 \to \wedge R_T^2$ is multiplication by $\mathrm{Norm}_{R_T/\mathcal{O}_T}(a) = \mathrm{Norm}_{R/A}(a)$. □

Assume again that $R^\vee \simeq R$. If T is sufficiently small, we obtain by choosing a trivialization of $\wedge F \otimes_{\mathcal{O}_T} (\wedge F')^{-1}$ a T-valued point of the scheme Spec $A[X, Y]/(XY - \mathrm{Norm}_{R/A}(a))$.

4.2. Suppose now that $A = k$ is a field, $R = k[T]/(T^N)$ with N a non-negative integer and $a = T$.

For simplicity of notation denote $\mathcal{N}_0(\mathrm{Spec}\, k,\ k[T]/(T^N),\ T)$ by $\mathcal{N}_0(N)$.

For every pair of non-negative integers $(b,\ b')$, let us denote by $\mathcal{N}_0(b,\ b')$ the closed subscheme of $\mathcal{N}_0(N)$ which classifies pairs $(F,\ F')$ such that $T^{N-b}R^2 \subset F \subset T^b R^2$, and $T^{N-b'}R^2 \subset F' \subset T^{b'} R^2_T$.

This subscheme is empty when b or b' is $> N/2$. We have

$$\mathcal{N}_0(b,\ b') \subset \mathcal{N}_0(b_1,\ b_1')$$

when $b_1 \leqslant b$ and $b_1' \leqslant b'$.

4.2.1. There is an isomorphism

$$\mathcal{N}_0(b,\ b) \simeq \mathcal{N}_0(N - 2b).$$

The Lemma 4.1.3 implies that if U is a sufficiently small open subset of $\mathcal{N}_0(N)$, then there is a (non-canonical) morphism

$$h\colon\ U \to \mathrm{Spec}\, k[X,\ Y]/(XY) := \mathcal{W} \subset \mathcal{N}_0(1).$$

Denote by s_0 the singular point of $\mathcal{W}$, $s_0 = (X,\ Y)$.

The following proposition, together with 4.2.1, describes a stratification of $\mathcal{N}_0(N)$. In particular, they imply that $\mathcal{N}_0(N)$ is purely of dimension N and it is nonsingular in codimension 0.

PROPOSITION 4.2.2. *The morphism h is smooth of relative dimension $N - 1$ when restricted to $\mathcal{N}_0(N) - \mathcal{N}_0(1,\ 1)$. When $N > 1$*

$$h^{-1}(s_0) = (\mathcal{N}_0(0,\ 1) \cup \mathcal{N}_0(1,\ 0)) \cap U - \mathcal{N}_0(1,\ 1).$$

We will later prove 4.2.2 by an explicit calculation. First we show how to describe affine charts for the schemes $\mathcal{N}_0(S,\ R,\ a)$.

4.3. In this section, we will always assume that $R = A[T]/(P(T))$ where $P(T)$ is a monic polynomial of degree N (A is not necessarily a field) and $a = T$. We will describe affine charts for the schemes $\mathcal{N}_0(S,\ R,\ T)$.

Suppose that $(F,\ F')$ corresponds to a U-point of $\mathcal{N}_0(S,\ R,\ a)$ where U is a Noetherian scheme and that $x \in U$ is such that the non-leading coefficients of $P(T)$ vanish in the residue field $k(x)$.

LEMMA 4.3.1. *Assume $x \in U$ is as before. Then, locally on a Zariski neighborhood of x in U there are elements $f \in F$* (resp. $f' \in F'$) *such that F* (resp. F') *are generated as $R \otimes_A \mathcal{O}_U$-modules by f, ${}^t u(f')$* (resp. f', $u(f)$).

Proof. $F'/u(F)$ and $F/{}^tu(F')$ are each generated locally on U by a single element. Indeed, for $y \in U$ with residue field $k(y)$ we have

$$\dim_{k(y)} \ker(u \otimes_{\mathcal{O}_U} k(y)|_{F\otimes_{\mathcal{O}_U} k(y)}) \leqslant \dim_{k(y)} \ker(u \otimes_{\mathcal{O}_U} k(y)) \leqslant 1,$$

so $\dim_{k(y)} u(F\otimes_{\mathcal{O}_U} k(y)) \geqslant N-1$ and therefore $\dim_{k(y)}(F'/u(F)) \otimes_{\mathcal{O}_U} k(y) \leqslant 1$. Similarly for $F/{}^tu(F')$. Now locally on a neighborhood of $x \in U$, choose $f \in F$ and $f' \in F'$ generating $F/{}^tu(F')$ and $F'/u(F)$ respectively. It is easy to see that F (resp. F') is generated by f, ${}^tu(f')$ (resp. f', $u(f)$) modulo TF (resp. TF') and therefore also modulo $T^N F$ (resp. $T^N F'$). This is enough by Nakayama's lemma. □

LEMMA 4.3.2. *Suppose that $A = k$ is a field. If (F, F') is a point of $\mathcal{N}_0(\operatorname{Spec} k, k[T]/(T^N), T)$ over k, then there are bases (e_1, e_2) and (e_1', e_2') of $R^2 = (k[T]/(T^N))^2$, and non-negative integers i, j, i', j' with $i + j = N$, $i' + j' = N$ such that*

(a) $u(e_1) = e_1'$, $u(e_2) = Te_2'$ *and* ${}^tu(e_1') = Te_1$, ${}^tu(e_2') = e_2$.
(b) *F and F' are generated by $T^i e_1$, $T^j e_2$ and $T^{i'} e_1'$, $T^{j'} e_2'$ respectively.*
(c) *We either have $i = i'$ and $j = j'$, or $i' = i - 1$ and $j' = j + 1$.*

Proof. Apply 4.3.1 to find $f \in F$, $f' \in F'$ such that F (resp. F') is generated as an R-module by $(f, {}^tu(f'))$ (resp. $(u(f), f')$). We have $u(f)\wedge f' = f\wedge {}^tu(f') = 0$. Write $f = T^a e_1$, $f' = T^{a'} e_2'$, with $e_1, e_2' \notin TR$. If either F or F' is free over $k[T]/(T^N)$, i.e. either a or a' is zero, the proof is simpler and is left to the reader. Assume otherwise, i.e. that both a and a' are positive, and consider the pairs $(e_1, {}^tu(e_2'))$ and $(u(e_1), e_2')$ of R^2. These pairs are both linearly independent over k. If they form bases of R^2 the lemma is proven; we have $a + a' = N$ because F and F' are equal to their own orthogonal. Otherwise, $u(e_1)$ and ${}^tu(e_2')$ are in TR^2, and we can write $u(e_1) = Te_1'$, ${}^tu(e_2') = Te_2$ with $e_1', e_2 \notin TR^2$. Then $({}^tu(e_1'), e_2)$ and $(e_1', u(e_2))$ are bases which have the desired property. Indeed, take $g' = T^{a'} u(e_2)$, $g = T^a\, {}^tu(e_1')$. Then $g' \equiv f' \bmod T^{N-1}R^2$, $g \equiv f \bmod T^{N-1}R^2$. Under the assumption that neither F nor F' is free, $T^{N-1}R^2 \subset F, F'$, and so g, ${}^tu(g')$ and g', $u(g)$ generate F and F' respectively. Since F and F' are equal to their own orthogonal, we have $a + a' + 1 = N$. □

4.3.3. Suppose s is a point of $S = \operatorname{Spec} A$, such that all non-leading coefficients of $P(T)$ vanish in $k(s)$. Take bases (e_1, e_2) and (e_1', e_2') of R^2 such that $u(e_1) = e_1'$, $u(e_2) = Te_2'$ and ${}^tu(e_1') = Te_1$, ${}^tu(e_2') = e_2$. Given positive integers i, j, i', j' which satisfy $i + j = i' + j' = N$ and $i = i'$, $j = j'$, or $i' = i - 1$, $j' = j + 1$ we can consider the point x on the fiber of $\mathcal{N}_0(S, R, T)$ over s given by $F = (T^i \bar{e}_1, T^j \bar{e}_2)$, $F' = (T^{i'} \bar{e}_1, T^{j'} \bar{e}_2)$ where $\bar{e}_i$, $\bar{e}_i'$ denote the images of e_i, e_i' in $R \otimes_A k(s)$.

We will describe an affine chart for $\mathcal{N}_0(S, R, T)$ at the point x.

In the case $i' = i,\ j' = j$, write

$$f = T^i e_1 + ae_1 + be_2, \quad f' = T^j e_2' + ce_1' + de_2', \tag{4.1}$$

where

$$a = a_0 + \cdots + a_{i-1}T^{i-1}, \quad c = c_0 + \cdots + c_{i-1}T^{i-1},$$

$$b = b_0 + \cdots + b_{j-1}T^{j-1}, \quad d = d_0 + \cdots + d_{j-1}T^{j-1}.$$

In the case $i' = i - 1,\ j' = j + 1$, write

$$f = T^j e_2 + ce_1 + de_2, \quad f' = T^{i'} e_1' + ae_1' + be_2', \tag{4.2}$$

where

$$a = a_0 + \cdots + a_{i'-1}T^{i'-1}, \quad c = c_0 + \cdots + c_{i-1}T^{i-1},$$

$$b = b_0 + \cdots + b_{j'-1}T^{j'-1}, \quad d = d_0 + \cdots + d_{j-1}T^{j-1}.$$

We think of f and f' as being defined over $\mathbf{A}_A^{2N}$, the $2N$-dimensional affine space over A with coordinates $a_k,\ b_l,\ c_m,\ d_n$. Consider the subscheme Y of $\mathbf{A}_A^{2N}$ defined by the N equations given by

$$u(f) \wedge f' = 0 \tag{4.3}$$

(or equivalently $f \wedge^t u(f') = 0$).

The following lemma will show that Y gives an affine chart for $\mathcal{N}_0(S,\ R,\ T)$ at the point x.

LEMMA 4.3.4. *The R_Y-modules F and F' generated by f, ${}^t u(f')$ and $u(f)$, f' respectively, define a point of $\mathcal{N}_0(S,\ R,\ T)$ with values in Y. The corresponding morphism $Y \to \mathcal{N}_0(S,\ R,\ T)$ is an isomorphism in a neighborhood of x.*

Proof. Let $(y,\ y')$ be the image of $x \in \mathcal{N}_0(S,\ R,\ T)$ under the closed immersion

$$\mathcal{N}_0(S,\ R,\ T) \to \mathcal{N}(S,\ R) \times_S \mathcal{N}(S,\ R)$$

Affine charts for $\mathcal{N}(S,\ R)$ at points like y and y' have been described in [4] 4.3. We refer to 4.3 for their description which is used in the proof of this lemma. We obtain affine charts for the product $\mathcal{N}(S,\ R) \times_S \mathcal{N}(S,\ R)$ which in turn give affine charts for $\mathcal{N}_0(S,\ R,\ T)$ at x. These are given by considering the open subscheme Y' of $\mathcal{N}_0(S,\ R,\ T)$ where the submodules F and F' are graphs of linear functions of a certain type (see [4], 4.3) which satisfy $u(F) \subset F'$. We will show that Y identifies with an open neighborhood of Y'.

It is easy to see that the F, F' defined as before are graphs of $\mathcal{O}_Y$-linear functions of this type and therefore, there is a morphism $Y \to \mathcal{N}_0(S, R, a)$ which in fact factors through $\alpha: Y \to Y'$. The description of the affine charts of *loc. cit.* 4.3 and an argument like in the proof of Lemma 4.3.2 shows that there is an open neighborhood U of x in Y' where we can take the modules F and F' to be generated by $u(f)$, f' and f, ${}^tu(f')$ respectively, where f and f' are given by (4.1) or (4.2). This shows that there is a section $\beta: U \to \alpha^{-1}(U)$. We also have $\beta \cdot \alpha = \mathrm{id}$, and this proves the lemma. □

We will now use the affine charts provided by 4.3.4 to prove Proposition 4.2.2.

Proof of 4.2.2. We will first show that h is smooth. By descent, we can assume that the field k is algebraically closed. Let x be a closed point of $\mathcal{N}_0(N) - \mathcal{N}_0(1, 1)$. We will show that h is smooth at x. By 4.3.2, we can use the affine charts of 4.3.3–4.3.4 for the calculation.

In the notations of 4.3.2, we can take $i = i' = 0$ (or $j = j' = 0$) when x is in $\mathcal{N}_0(N) - (\mathcal{N}_0(0, 1) \cup \mathcal{N}_0(1, 0))$, and $i' = 0$ and $i = 1$ (resp. $j' = 1$, $j = 0$) when x is in $\mathcal{N}_0(0, 1) - \mathcal{N}_0(1, 1)$ (resp. $\mathcal{N}_0(1, 0) - \mathcal{N}_0(1, 1)$). Consider $x \in \mathcal{N}_0(0, 1) - \mathcal{N}_0(1, 1)$, in which case $i' = 0$ and $i = 1$ (then we have $j = N - 1$ and $j' = N$).

The submodules F and F' are given by taking as in (4.2)

$$f = T^{N-1}e_2 + c_0 e_1 + (d_0 + d_1 T + \cdots + d_{N-2}T^{N-2})e_2,$$

$$f' = e_1' + (b_0 + b_1 T + \cdots + b_{N-1}T^{N-1})e_2',$$

where (by (4.3))

$$b_0 c_0 = 0, \qquad d_k = b_{k+1}c_0, \quad 0 \leqslant k \leqslant N-2. \tag{4.4}$$

A basis of F (resp. F') as an $\mathcal{O}_Y$-module is formed by f, $T^l\,{}^tu(f')$, $0 \leqslant l \leqslant N-2$ (resp. by $T^k f'$), $0 \leqslant k \leqslant N-1$. Using (4.4) we find $u(f) = c_0 f'$, ${}^tu(T^{N-1}f') = d_0 f$ and therefore

$$(\wedge u)\,(f \wedge {}^tu(f') \wedge \cdots \wedge T^{N-2}\,{}^tu(f'))$$
$$= c_0(f' \wedge Tf' \wedge \cdots \wedge T^{N-1}f'),$$

and

$$(\wedge\,{}^tu)(f' \wedge Tf' \wedge \cdots \wedge T^{N-1}f')$$
$$= (-1)^{N-1}d_0(f \wedge {}^tu(f') \wedge \cdots \wedge T^{N-2}\,{}^tu(f')).$$

Together with (4.4) this shows that h is smooth at $x \in \mathcal{N}_0(0, 1) - \mathcal{N}_0(1, 1)$. The case $x \in \mathcal{N}_0(1, 0) - \mathcal{N}_0(1, 1)$ is similar. In both cases $h(x) = s_0$. When x is in $\mathcal{N}_0(N) - (\mathcal{N}_0(0, 1) \cup \mathcal{N}_0(1, 0))$, both F and F' are free over R and we have either $u(F) = F'$ or ${}^t u(F') = F$. A similar calculation shows that h is again smooth at x and that $h(x) \neq s_0$. We will leave the details to the reader. □

Proof of 2.2.2 and 3.4.2. By 2.1.2 and the observations in the beginning of 3.1, we see that for the proof of 2.2.2 it is enough to show the statement for $\mathcal{H}_0(\mathcal{A})_{\mathbf{Z}_{(p)}}$ where $\mathcal{A}$ is supported over p. By 3.3.1 we see that it is enough to show the same statements for the scheme $\mathcal{N}_0(a)_{\mathbf{Z}_{(p)}}$ (recall that a is a local generator for the ideal $\mathcal{A}$). Of course, by 3.4.1, this will suffice for the proof of 3.4.2 also.

As in [4] 4.5, using 4.1.1 and 4.1.2, we see that it is enough to prove our claims for $\mathcal{N}_0(\operatorname{Spec} W,\ W[T]/(P(T)),\ T)$ where $W = W(\bar{\mathbf{F}}_p)$ is the ring of Witt vectors and $R = W[T]/(P(T))$ with $P(T)$ a monic polynomial with non-leading coefficients vanishing in the residue field. By 4.1.1, 4.2.2 and 4.2.1, $\mathcal{N}_0(\operatorname{Spec} W,\ W[T]/(P(T)),\ T)$ has closed fiber of dimension N which is smooth outside a subscheme of codimension 1. The description of the affine charts given by 4.3.3–4.3.4, shows that every point in the closed fiber of $\mathcal{N}_0(\operatorname{Spec} W,\ W[T]/(P(T)),\ T)$ has a neighborhood which is a flat relative local complete intersection over $\operatorname{Spec} W$ of relative dimension N. Since $\mathcal{N}_0(\operatorname{Spec} W,\ W[T]/(P(T)),\ T)$ is proper over $\operatorname{Spec} W$, this implies that the morphism $\mathcal{N}_0(\operatorname{Spec} W,\ W[T]/(P(T)),\ T) \to \operatorname{Spec} W$ is a flat relative local complete intersection of relative dimension N. □

4.4. We now study the kernel of the universal isogeny. The main observation is that this kernel is a product of $(p, \ldots, p)$-group schemes of the type studied by Raynaud in [10]. We will see that the explicit description of such group schemes given in [10], provides us with Cartier divisors on the moduli stack. We show in Proposition 4.4.4 that these agree with divisors which are given using the étale local charts of 4.1–4.3.

Let us write $\mathcal{A}$ as a product of distinct prime ideals of $\mathcal{O}$: $\mathcal{A} = \mathcal{P}_1 \cdots \mathcal{P}_r$ (we are always assuming that these divide the fixed rational prime p). Write $\mathcal{P}_1, \ldots, \mathcal{P}_k$, $k \geqslant r$ for the ideals of F which lie above p. Denote by $W(\mathcal{P}_i)$ the maximal unramified extension of $\mathbf{Z}_p$ contained in the completion $\mathcal{O}_{\mathcal{P}_i}$. Let $W = W(\bar{\mathbf{F}}_p)$ be the ring of Witt vectors and fix embeddings $W(\mathcal{P}_i) \to W$. Set $W_i = \mathcal{O}_{\mathcal{P}_i} \otimes_{W(\mathcal{P}_i)} W$. W_i is a DVR which is totally ramified over W. Let π_i be a uniformizer of W_i.

If S is a scheme over $\operatorname{Spec} W$ and ϕ the isogeny corresponding to an S-valued point of $\mathcal{H}_0(\mathcal{A})$, consider its kernel $G := \ker(\phi)$. It decomposes as a product of finite locally free group schemes:

$$G = G_1 \times \cdots \times G_r,$$

where G_i is the part of G annihilated by $\mathcal{P}_i$. Our conditions on ϕ imply that the

G_i have rank Norm$(\mathcal{P}_i)$. They are $(p, \ldots, p)$ group schemes, in fact $\mathcal{O}/\mathcal{P}_i$-vector space schemes.

4.4.1. Let $q = p^f$ and fix the Teichmuller character $\chi_0 : (\mathbf{F}_q)^* \to W^*$. In [10] 1.2, Raynaud associates to any $\mathbf{F}_q$-vector space scheme H which is finite and locally free over the W-scheme S, a collection of coherent locally free sheaves $\mathcal{L}_j$, $j \in \mathbf{Z}/f\mathbf{Z}$ over S together with $\mathcal{O}_S$-linear morphisms $\Delta_j : \mathcal{L}_j^{\otimes p} \to \mathcal{L}_{j+1}$, $\Gamma_j : \mathcal{L}_{j+1} \to \mathcal{L}_j^{\otimes p}$. There is a uniformizer π of W (in [10], it would be denoted by w) such that $\Gamma_j \cdot \Delta_j$, and $\Delta_j \cdot \Gamma_j$ are for all j, multiplication by π. The sheaf $\mathcal{L}_j$ is the direct summand of the augmentation ideal $\mathcal{J}_G$ of $\mathcal{O}_G$, on which $b \in \mathbf{F}_q^*$ acts via multiplication by $\chi_j(b) := \chi_0^{p^j}(b)$. The morphisms Δ_j and Γ_j are induced by multiplication and comultiplication respectively.

We will say that H is a Raynaud $\mathbf{F}_q$-vector space scheme, if the $\mathcal{L}_j$ are all invertible sheaves ([10] (**)). Raynaud proves (*loc. cit.* 1.4.1) that in this case H is realized as the closed subscheme of Spec(Symm$_{\mathcal{O}_S}(\oplus_{j\in\mathbf{Z}/f\mathbf{Z}}\mathcal{L}_j)$) defined by $(\Delta_j - 1)\mathcal{L}_j^{\otimes p}$, $j \in \mathbf{Z}/f\mathbf{Z}$. If H is Raynaud, then the same is true for its Cartier dual H^*. The line bundles for H^* are $\mathcal{L}_j^* = \mathcal{L}_j^{\otimes -1}$, the morphisms Γ_j^*, Δ_j^* are respectively the duals of Δ_j, Γ_j.

LEMMA 4.4.2. *The components G_i of the kernel of ϕ are Raynaud $\mathcal{O}/\mathcal{P}_i$-vector space schemes.*

Proof. It follows from [10] 1.2.2, since by 2.2.2 and 2.2.3 $\mathcal{H}_0(\mathcal{A})$ is normal and flat over $\mathbf{Z}$. □

4.4.3. Let us set Norm$(\mathcal{P}_i) = p^{f_i}$, $B_i = \operatorname{Spec} W[X_j, Y_j]/(X_jY_j - \pi)_{j\in\mathbf{Z}/f_i\mathbf{Z}}$. Assume that $S = U = \operatorname{Spec} C$ is a local neighborhood of the closed point $x \in \mathcal{H}_0(\mathcal{A})_W$. Then by 4.4.2 and [10] 1.4.1, we obtain, after choosing trivializations for the invertible sheaves of 4.4.1, morphisms:

$$g_i \colon U \to B_i,$$

such that over $U = \operatorname{Spec} C$, G_i and its Cartier dual G_i^* are given by

$$G_i \simeq \operatorname{Spec} C[T_j]/(T_j^p - g_i^*(X_j)T_{j+1})_{j\in\mathbf{Z}/f_i\mathbf{Z}}, \tag{4.5}$$

$$G_i^* \simeq \operatorname{Spec} C[T_j]/(T_j^p - g_i^*(Y_j)T_{j+1})_{j\in\mathbf{Z}/f_i\mathbf{Z}}. \tag{4.6}$$

Denote by $\alpha \colon U \to \mathcal{N}_0(\operatorname{Spec}\mathbf{Z}[1/N], \mathcal{O}[1/N], a)_W$ the étale morphism defined in Section 3 (it also depends on the choice of a trivialization, this time of the deRham cohomology). By 4.1.1 and 4.1.2 we deduce that $\mathcal{N}_0(\operatorname{Spec}\mathbf{Z}[1/N], \mathcal{O}[1/N], a)_W$ is isomorphic to

$$\prod_{i=1}^{r} \prod_{j\in\mathbf{Z}/f_i\mathbf{Z}} \mathcal{N}_0(\operatorname{Spec} W, W_i, \pi_i) \times \prod_{i=r+1}^{k} \prod_{j\in\mathbf{Z}/f_i\mathbf{Z}} \mathcal{N}(\operatorname{Spec} W, W_i).$$

By 4.1.3, there are morphisms from every local neighborhood of $\mathcal{N}_0(\operatorname{Spec} W,\ W_i,\ \pi_i)$ to $W[X,\ Y]/(XY - \operatorname{Norm}(\pi_i))$ (here the norm is taken from W_i down to W). Since W_i is totally ramified over W, $\operatorname{Norm}(\pi_i)$ is a uniformizer of W and therefore, we obtain morphisms

$$h_i\colon \mathcal{N}_0(\operatorname{Spec}\mathbf{Z}[1/N],\ \mathcal{O}[1/N],\ a)_W \cap \alpha(U) \to B_i.$$

Denote by $g_i^*((X_j))$, $g_i^*((Y_j))$ (resp. $(h_i \cdot \alpha)^*((X_j))$, $(h_i \cdot \alpha)^*((Y_j))$) the pull-back on U by g_i (resp. $h_i \cdot \alpha$) of the Cartier divisors of B_i defined by X_j and Y_j. These Cartier divisors are of course independent of the choice of the various trivializations.

LEMMA 4.4.4. *For every* $i = 1, \ldots, r$, *and* $j \in \mathbf{Z}/f_i\mathbf{Z}$ *we have* $(h_i \cdot \alpha)^*((X_j)) = g_i^*((X_j))$ *and* $(h_i \cdot \alpha)^*((Y_j)) = g_i^*((Y_j))$.

Proof. Since $\mathcal{H}_0(\mathcal{A})_W$ is normal and flat over W (2.2.2–2.2.3), and $X_jY_j = \pi$, it is enough to check the equalities of the proposition for Y_j only, after localizing on points of codimension 1 of $U = \operatorname{Spec} C$. The local rings of U at such points are discrete valuation rings. By (4.6), the module of invariant differentials of G_i^* is

$$\omega_i = \oplus_{j\in\mathbf{Z}/f_i\mathbf{Z}} C/g_i^*(Y_j)C.$$

On the other hand, if $\phi\colon\ A_1 \to A_2$ is the modular isogeny over U, we have an exact sequence

$$\operatorname{Lie}(A_1^*/U)^\vee \to \operatorname{Lie}(A_2^*/U)^\vee \to \bigoplus_{i=1}^{r} \omega_i \to 0.$$

The statement now follows from the definitions of the morphisms α, g_i and h_i and the theorem on the structure of finitely generated modules over discrete valuation rings. □

REMARK 4.4.5. For all the above constructions, we only need to assume that W is a DVR, finite and étale over $\mathbf{Z}_p$, which contains $W(\mathcal{P}_i)$, $i = 1, \ldots, r$.

4.5. EXAMPLES. The first part of (a) and (b) will also be used in Section 6.

(a) Suppose that $[F:\mathbf{Q}] = 2$ and p is a rational prime ramified in F, $p = \mathcal{P}^2 = \mathcal{P}_1^2$. Write an Eisenstein equation $\pi_1^2 + \lambda\pi_1 + \mu = 0$, $\lambda, \mu \in \mathbf{Z}_p$ for the uniformizer π_1 of $\mathcal{O}_\mathcal{P}$.

An easy calculation shows that the affine chart of 4.3.3 which covers the point on the special fiber of $\mathcal{N}_0(\operatorname{Spec}\mathbf{Z}_p,\ \mathcal{O}_\mathcal{P},\ \pi_1)$ with the worst singularity $(i = i' = j = j' = 1)$ is $\operatorname{Spec}\mathbf{Z}_p[U,\ V,\ T]/(U(VT - U + \lambda) + \mu)$. In fact, we can see that all the other affine charts of 4.3.3 are covered by open sets isomorphic to open sets of $\operatorname{Spec}\mathbf{Z}_p[U,\ V,\ T]/(U(VT - U + \lambda) + \mu)$. This shows that $\operatorname{Spec}\mathbf{Z}_p[U,\ V,\ T]/(U(VT - U + \lambda) + \mu)$ is an étale local model for $\mathcal{H}_0(\mathcal{P})_{\mathbf{Z}_p}$. We can see that the Cartier divisors on $\mathcal{H}_0(\mathcal{P})_{\mathbf{Z}_p}$ which are given locally by

$g_1^*(X_0)$ and $g_1^*(Y_0)$ correspond to (U) and $(VT-U+\lambda)$ (cf. Remark 4.4.5). Since $\mu \in (p)-(p^2)$, we notice that the scheme $\operatorname{Spec} \mathbf{Z}_p[U, V, T]/(U(VT-U+\lambda)+\mu)$ is regular. Therefore the same is true for $\mathcal{H}_0(\mathcal{P})_{\mathbf{Z}_p}$. Together with 2.1.3, this shows that $\mathcal{H}_0(\mathcal{P})$ is regular.

Let $\mathcal{H}_F$ (resp. $\mathcal{H}_V$) be the closed subset of the points of the fiber over p of $\mathcal{H}_0(\mathcal{P})$ where the kernel of the modular isogeny is annihilated by the Frobenius (resp. the Verschiebung). The stack $\mathcal{H}_F$ (resp. $\mathcal{H}_V$) is given étale locally by the equation $g_1^*(X_0)=0$ (resp. $g_1^*(Y_0)=0$) and therefore is smooth.

With a little more work one can derive the following description for the fiber of $\mathcal{H}_0(\mathcal{P})$ over p: The closed subsets $\mathcal{H}_F$ and $\mathcal{H}_V$ are smooth and irreducible and they form the two irreducible components of the fiber of $\mathcal{H}_0(\mathcal{P})$ over p. They intersect in $\mathcal{H}_0(\mathcal{P})_{\mathbf{F}_p}$ along a closed subset of dimension 1 which is smooth except on a finite set of points where it has two transverse branches. The intersection of $\mathcal{H}_F$ and $\mathcal{H}_V$ is transversal outside those isolated points. The local equations for $\mathcal{H}_0(\mathcal{P})_{\mathbf{Z}_p}$ there, are $XY=-\mu$, with variables X, Y, Z. The local equations of $\mathcal{H}_0(\mathcal{P})_{\mathbf{Z}_p}$ on the isolated points where the intersection of $\mathcal{H}_F$ and $\mathcal{H}_V$ is not transversal are

$$U(VT-U+\lambda)=-\mu.$$

(b) Suppose again $d=2$. Assume that $\mathcal{A}=(p)$ with p a rational prime inert in F. We can take $W=W(\mathbf{F}_{p^2})$ (remark 4.4.5). In this case we have

$$\begin{aligned}&\mathcal{N}_0(\operatorname{Spec}\mathbf{Z}[1/N],\ \mathcal{O}[1/N],\ a)_W\\ &\quad=\mathcal{N}_0(\operatorname{Spec}W,\ W,\ p)\times_W\mathcal{N}_0(\operatorname{Spec}W,\ W,\ p).\end{aligned}$$

We can see that

$$\mathcal{N}_0(\operatorname{Spec}W,\ W,\ p)=\widetilde{\mathbf{P}^1_W},$$

where $\widetilde{\mathbf{P}^1_W}$ is the blow-up of $\mathbf{P}^1_W$ at the 0-point of the special fiber. The scheme $\widetilde{\mathbf{P}^1_W}\times_W\widetilde{\mathbf{P}^1_W}$ is covered by open sets isomorphic to open sets of

$$\mathcal{U}=\operatorname{Spec}W[X_0,\ X_1,\ Y_0,\ Y_1]/(X_0Y_0-p,\ X_1Y_1-p).$$

Therefore, $\mathcal{U}$ is an étale local model for $\mathcal{H}_0(p)_W$: For every closed point x of $\mathcal{H}_0(p)_W$, there is an étale local neighborhood U of $\mathcal{H}_0(p)_W$ and an étale morphism $\alpha: U \to \mathcal{U}$. In fact, in this case, $\mathcal{U}\simeq B_1$ and by 4.4.4 we can choose this isomorphism so that α is identified with g_1.

The scheme $\mathcal{U}\simeq B_1$ is not regular, but we shall see in 6.1.1 that it is desingularized by a blow-up along the ideal (X_0, X_1). Therefore, the blow-up of $\mathcal{H}_0(p)_W$ along the ideal given locally by $(g_1^*(X_0),\ g_1^*(X_1))$ is also regular (cf. 6.1.1).

(c) Suppose now that $[F:\mathbf{Q}]=3$ and that p is a rational prime totally ramified in F, $p=\mathcal{P}^3$. For simplicity, assume that $p\neq 3$. Then we can find a uniformizer

π_1 of $\mathcal{O}_{\mathcal{P}}$ such that $\pi_1^3 = \mu$, μ a uniformizer of $\mathbf{Z}_p$. A calculation using the charts of 4.3.3, shows that the scheme

$$\operatorname{Spec} \mathbf{Z}_p[X, Y, Z, T, W]/(XY - \mu, W^2 - ZTW - TX - ZY)$$

is an étale local model for $\mathcal{H}_0(\mathcal{P})_{\mathbf{Z}_p}$. This scheme is not regular. Its fiber over p, has two irreducible components, defined by $X = 0$, $Y = 0$. Their intersection is $\operatorname{Spec} \mathbf{F}_p[Z, T, W]/(W^2 - ZTW)$ which is the special fiber of the local model for the Hilbert–Blumenthal surface of (a). This phenomenon is partially explained by 4.2.1 and 4.2.2.

5. Level structures on group-schemes of type $(p, \ldots, p)$

In this paragraph, we study DKM (Drinfeld–Katz–Mazur)-level structures on Raynaud $(p, \ldots, p)$-group schemes. The main result, given by Proposition 5.1.5, is an explicit description of the subscheme of DKM-generators of such a group scheme. The result applies to the kernel of the universal isogeny over $\mathcal{H}_0(\mathcal{A})$ which by 4.4.2 splits into a product of Raynaud group schemes. We use this to obtain a proof of Theorem 2.3.3. In 5.2, we prove two lemmas on subgroups of Raynaud group schemes which are an essential ingredient of the local structure calculations of Section 6.

5.1. Suppose $G \xrightarrow{\pi} S$ is a finite flat commutative group scheme of finite presentation over the scheme S. Equivalently, $\pi\colon G \to S$ is locally free over S, of finite rank say r. Let A be a finite abelian group of order r. If $(A)_S$ denotes the constant group scheme over S given by A, then a group scheme homomorphism $\epsilon\colon (A)_S \to G$ provides G with sections $\epsilon^*(a)\colon \mathcal{O}_G \to \mathcal{O}_S$ for each $a \in A$.

DEFINITION 5.1.1 (cf. [8] 1.10). *The group scheme homomorphism $\epsilon\colon (A)_S \to G$ is an A-structure on G, if for every affine scheme T, $T \to S$, and every $h \in \mathcal{O}_{G\times_S T}$ we have:*

$$\operatorname{Norm}_{G\times_S T/T}(h) = \prod_{a\in A} (\epsilon^*(a) \otimes_{\mathcal{O}_S} \mathrm{id})(h). \tag{5.1}$$

REMARK 5.1.1.a. When G is étale over S, then $\epsilon\colon (A)_S \to G$ is an A-structure on G, if and only if ϵ is an isomorphism ([8], 1.10.12).

We now continue with the notations of the previous paragraphs.

5.1.2. Suppose that S is a scheme over $\operatorname{Spec} W$ and let $G \to S$ be a Raynaud $\mathbf{F}_{p^f}$-vector space scheme. If $(\mathbf{F}_{p^f})_S$ denotes the constant Raynaud $\mathbf{F}_{p^f}$-vector space scheme over S, then by definition, a point $P \in G(S)$ is an $\mathbf{F}_{p^f}$-generator of G, if the $\mathbf{F}_{p^f}$-linear morphism $(\mathbf{F}_{p^f})_S \to G$ determined by $1 \to P$ is an $\mathbf{F}_{p^f}$-structure on G (cf. 2.3.1).

5.1.3. The notion of an $\mathbf{F}_{p^f}$-generator generalizes immediately to all finite locally free schemes $G \to S$ of rank p^f, which support an action of the group $\mathbf{F}^*_{p^f}$ and have a distinguished "zero" section.

5.1.4. If $\mathcal{L}_j$, $j \in \mathbf{Z}/f\mathbf{Z}$ are the line bundles associated to G as in 4.4.1, we can consider the closed subscheme G' of $\mathrm{Spec}(\mathrm{Symm}_{\mathcal{O}_S}(\oplus_{j\in\mathbf{Z}/f\mathbf{Z}}\mathcal{L}_j))$ defined by

$$(\Delta_j - 1)\mathcal{L}_j^{\otimes p}, \quad j \in \mathbf{Z}/f\mathbf{Z}, \quad (\otimes_{j\in\mathbf{Z}/f\mathbf{Z}}\Delta_j - 1)(\otimes_{j\in\mathbf{Z}/f\mathbf{Z}}\mathcal{L}_j^{\otimes(p-1)}),$$

where we think of $\otimes_j\Delta_j\colon \otimes_j\mathcal{L}_j^{\otimes p} \to \otimes_j\mathcal{L}_j$ as a morphism $\otimes_j\mathcal{L}_j^{\otimes(p-1)} \to \mathcal{O}_S$.

By [10] 1.4.1, the first equations define the group scheme G, and so G' is a closed subscheme of G of finite presentation over S. It is easy to see that G' is flat over S. Indeed, we can reduce to the case of the universal base $S = \mathrm{Spec}\, W[X_j, Y_j]/(X_jY_j - \pi)_{j\in\mathbf{Z}/f\mathbf{Z}}$ where the $\mathcal{L}_j$ are trivial and X_j (resp. Y_j) gives the morphisms Δ_j (resp. Γ_j). Then S is the spectrum of a Noetherian integral domain. For every point $s \in S$ we have $\dim_{k(s)}(\mathcal{O}_{G'}\otimes_{\mathcal{O}_S}k(s)) = p^f - 1$ and so G' is flat over S.

PROPOSITION 5.1.5. *The subscheme G' represents the functor which to $S' \to S$ associates the set of $\mathbf{F}_{p^f}$-generators of $G\times_S S'$.*

Proof. By [8] 1.10.13 the functor of $\mathbf{F}_{p^f}$-generators of G is represented by a closed subscheme of G, which is of finite presentation over S. We will denote it by $G^\dagger$. We can reduce the proof of $G^\dagger = G'$ to the case that the base is the spectrum of a local Noetherian ring C. Then write

$$G = \mathrm{Spec}\, C[U_j]/(U_j^p - \delta_jU_{j+1})_{j\in\mathbf{Z}/f\mathbf{Z}},$$

$$G' = \mathrm{Spec}\, C[V_j]/(V_j^p - \delta_jV_{j+1}, (V_0\cdots V_{f-1})^{p-1} - \delta_0\cdots\delta_{f-1}).$$

The action of $a \in \mathbf{F}^*_{p^f}$ on G is given by $U_j \to \chi_j(a)U_j$. Set $A = C[U_j]/(U_j^p - \delta_jU_{j+1})$, $A' = C[V_j]/(V_j^p - \delta_jV_{j+1}, (V_0\cdots V_{f-1})^{p-1} - \delta_0\cdots\delta_{f-1})$. We will abuse notation by denoting also by U_j and V_j the images of U_j and V_j in A and A' respectively. The morphism Φ defined by $U_j \to V_j$, realizes G' as a closed subscheme of G. Consider the universal point of G over G':

$$G' \overset{P_0}{\to} G\times_C G' \quad \text{where } P_0 = \Phi\times_C \mathrm{Id}.$$

The following lemma establishes that G' is a closed subscheme of $G^\dagger$.

LEMMA 5.1.6. *P_0 is a $\mathbf{F}_{p^f}$-generator of $G\times_C G'$.*

Proof. It will be enough to treat the universal case $C = W[X_j, Y_j]/(X_jY_j - \pi)_{j\in\mathbf{Z}/f\mathbf{Z}}$ and $G = \mathrm{Spec}\, C[U_j]/(U_j^p - X_jU_{j+1})$. (Here we drop the assumption that C is local). It is now easy to see that A' does not have zero divisors. Indeed, since A' is flat over C, it is enough to check this over the generic point η of C;

we have $(A')_\eta \simeq C_\eta[V]/(V^{p^f-1} - X_0^{p^{f-1}} \cdots X_{f-1})$, which has no zero divisors. By [8] 1.9.1 there is a closed subscheme T of $G' = \operatorname{Spec} A'$ over which the point P_0 is an $\mathbf{F}_{p^f}$-generator. It will be enough to show that T contains the generic point of G'. Denote by K the fraction field of A'; $K = (A')_\eta \simeq C_\eta[V]/(V^{p^f-1} - X_0^{p^{f-1}} \cdots X_{f-1})$. The group scheme $G \times_C \operatorname{Spec} K = (G \times_C G') \times_{G'} \operatorname{Spec} K$ is étale, and we can see that

$$\operatorname{Spec} K \xrightarrow{(P_0)_K} G \times_C \operatorname{Spec} K,$$

is a section which is not the 0-section. We can now see, using Remark 5.1.1a, that $(P_0)_K$ is a generator of $G \times_C \operatorname{Spec} K$ and the proof follows. □

Now let us continue with the proof of Proposition 5.1.5. We will prove that the surjective ring homomorphism $\mathcal{O}_{G^\dagger} \to A'$, which was obtained by Lemma 5.1.6, is an isomorphism. Since A' is flat over C, it will be enough to prove this modulo the maximal ideal $\mathcal{M}$ of C. We distinguish two cases:

(a) all δ_j are non zero modulo $\mathcal{M}$. Then G is étale and $G' = G^\dagger$ by reasons of rank (Remark 5.1.1.a).

(b) at least one of the δ_j is zero modulo $\mathcal{M}$. Then $p \in \mathcal{M}$. Suppose $U_j \to T_j$ defines a point of $G^\dagger$ with values in a ring $\mathcal{D}$. Apply (5.1) to $h = 1 - U_0 \cdots U_{f-1}$. Since multiplication by $U_0U_1 \ldots U_{f-1}$ is nilpotent modulo $\mathcal{M}$, we have $\operatorname{Norm}_{\mathcal{D}\otimes_{\mathcal{O}} A/\mathcal{D}}(h) = 1 \bmod \mathcal{M}\mathcal{D}$. Therefore, we can see that (5.1) implies:

$$1 = (1 - (T_0 \cdots T_{f-1})^{p-1})^{\frac{p^f-1}{p-1}} \bmod \mathcal{M}\mathcal{D}. \tag{5.2}$$

From this and from $(T_0 \cdots T_{f-1})^p = \delta_0 T_{f-1} \cdots \delta_{f-1} T_0 = 0 \bmod \mathcal{M}\mathcal{D}$, we deduce that $(T_0 \cdots T_{f-1})^{p-1} = 0 = \delta_0 \ldots \delta_{f-1} \bmod \mathcal{M}\mathcal{D}$. This concludes the proof of Proposition 5.1.5. □

Proof of 2.3.3. The forgetful morphism $\mathcal{H}_{00}(\mathcal{A})_n \to \mathcal{H}_{00}(\mathcal{A})$ is étale and so it is enough to treat the case $n = 1$. By 4.4.2, we can write the kernel of the universal isogeny $\phi\colon A_1 \to A_2$, as a product of Raynaud vector space schemes: $\ker(\phi) = G_1 \times \cdots \times G_r$. By [8] 1.10.14, the functor of $\mathcal{O}/\mathcal{A}$-generators of $\ker(\phi)$ is the product of the functors of $\mathcal{O}/\mathcal{P}_i$-generators of the group schemes G_i for $i = 1, \ldots, r$. By applying 5.1.5 to G_i, $i = 1, \ldots, r$ we conclude that the forgetful morphism $\pi_0\colon \mathcal{H}_{00}(\mathcal{A}) \to \mathcal{H}_0(\mathcal{A})$ is finite and flat. By 2.2.2 and 2.2.3, $\mathcal{H}_0(\mathcal{A})$ is Cohen–Macaulay and flat over $\mathbf{Z}$, and the Theorem 2.3.3 now follows from EGA IV 6.4.1. □

5.2. In this section, we study p-rank subgroups of Raynaud group schemes. For simplicity, we will restrict to the case of (p, p)-group schemes ($f = 2$). The results are used in Section 6.

Let $f = 2$ and suppose G is a Raynaud $\mathbf{F}_{p^2}$-vector space scheme over the W-scheme S. We consider the functor $\mathcal{F}$ which to $T \to S$ associates the set of all finite locally free subgroup schemes of rank p of $G \otimes_S T$. The group $K := \mathbf{F}_{p^2}^*/\mathbf{F}_p^*$ acts on $\mathcal{F}$. An element $k \in \mathbf{F}_{p^2}^*/\mathbf{F}_p^*$ acts by taking the subgroup H to $k \cdot H$. Recall the notations of 4.4.1; we have invertible sheaves $\mathcal{L}_j$, $j \in \mathbf{Z}/2\mathbf{Z}$ over S, and morphisms $\Delta_j : \mathcal{L}_j^{\otimes p} \to \mathcal{L}_{j+1}$ and $\Gamma_j : \mathcal{L}_{j+1} \to \mathcal{L}_j^{\otimes p}$ induced respectively by multiplication and comultiplication.

The following proposition shows that the functor $\mathcal{F}$ is represented by a subscheme of the projective line bundle $\mathbf{P}(\mathcal{L}_0 \oplus \mathcal{L}_1)$ over S.

PROPOSITION 5.2.1. *The functor $\mathcal{F}$ is naturally isomorphic to the functor $\mathcal{G}$ which to $T \to S$ associates the set of locally free quotients of rank* 1, $(\mathcal{L}_0 \oplus \mathcal{L}_1) \otimes_{\mathcal{O}_S} \mathcal{O}_T \xrightarrow{\alpha} \mathcal{M}$ *over T, such that if we write $\alpha = \alpha_0 + \alpha_1$ with $\alpha_0 : \mathcal{L}_0 \otimes_{\mathcal{O}_S} \mathcal{O}_T \to \mathcal{M}$ and $\alpha_1 : \mathcal{L}_1 \otimes_{\mathcal{O}_S} \mathcal{O}_T \to \mathcal{M}$ we have:*

$$(\Delta_0 \oplus \Delta_1)(\ker(\alpha_0^{\otimes p} + \alpha_1^{\otimes p})) \subset \ker(\alpha), \tag{5.3a}$$

$$(\Gamma_0 \oplus \Gamma_1)(\ker(\alpha_1 + \alpha_0)) \subset \ker(\alpha_0^{\otimes p} + \alpha_1^{\otimes p}). \tag{5.3b}$$

Before giving the proof, let us write down the explicit shape of (5.3a–b) when $T = \operatorname{Spec} R$, with R a local ring. Choose X_0, X_1 and X generators of the line bundles $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{M}$, and elements γ_j, δ_j, $j \in \mathbf{Z}/2\mathbf{Z}$ of R such that

$$\Delta_j(X_j^{\otimes p}) = \delta_j X_{j+1}, \quad \Gamma_j(X_{j+1}) = \gamma_j X_j^{\otimes p}.$$

Suppose that α is given by $X_0 \to \xi_0 X$, $X_1 \to \xi_1 X$. Then our two conditions become:

$$\delta_1 \xi_0^{p+1} = \delta_0 \xi_1^{p+1}, \quad \gamma_0 \xi_0^{p+1} = \gamma_1 \xi_1^{p+1}. \tag{5.3a′–b′}$$

Proof. Let H be a p-rank subgroup scheme of G, over T. H can be thought of as a $\mathbf{F}_p$-vector space scheme. It satisfies the condition (**) of [10] 1.2 (this case of rank p was actually first treated by Oort and Tate). Let $\mathcal{M} = \mathcal{M}_0$ be the component of the augmentation ideal $\mathcal{J}_H$ where $\mathbf{F}_p^*$ acts through multiplication by $\chi := \chi_j|_{\mathbf{F}_p^*}$. $\mathcal{M}$ is an invertible sheaf and the epimorphism $\mathcal{O}_G \to \mathcal{O}_H$ restricts to an epimorphism $\mathcal{L}_0 \oplus \mathcal{L}_1 \to \mathcal{M}$. Iterated multiplication induces the morphism $\Delta : \mathcal{M}^{\otimes p} \to \mathcal{M}$, such that the diagrams for $j \in \mathbf{Z}/2\mathbf{Z}$ are commutative:

$$\begin{array}{ccc} \mathcal{L}_j^{\otimes p} & \xrightarrow{\Delta_j} & \mathcal{L}_{j+1} \\ \Big\downarrow{\scriptstyle \alpha_j^{\otimes p}} & & \Big\downarrow{\scriptstyle \alpha_{j+1}} \\ \mathcal{M}^{\otimes p} & \xrightarrow{\Delta} & \mathcal{M} \end{array} .$$

This shows that (5.3a) is verified. For (5.3b), we can restrict to the case that T is the spectrum of a local ring R. Then (5.3b′) follows readily from the explicit description of the comultiplication on H and G_T ([10], 1.5.1). Indeed, we just have to trace the images of generators for $\mathcal{L}_0$ and $\mathcal{L}_1$ around the commutative diagram of comultiplications:

$$\begin{array}{ccc} \mathcal{O}_{G_T} & \xrightarrow{c_{G_T}} & \mathcal{O}_{G_T} \otimes_R \mathcal{O}_{G_T} \\ \downarrow & & \downarrow \\ \mathcal{O}_H & \xrightarrow{c_H} & \mathcal{O}_H \otimes_R \mathcal{O}_H \end{array} .$$

Conversely, given a surjection $\alpha\colon (\mathcal{L}_0 \oplus \mathcal{L}_1) \otimes_{\mathcal{O}_S} \mathcal{O}_T \to \mathcal{M}$, we consider the subscheme H' of G_T defined by the ideal generated by the kernel of α. To check that it is a subgroup scheme of rank p, we can again restrict to the case that T is the spectrum of a local ring R. Denote by X a generator of $\mathcal{M}$. Then, (5.3a′) implies that we can write $\mathcal{O}_{H'} = R[X]/(X^p - \delta X)$ where $\delta = \delta_0 \cdot \xi_1/\xi_0^p = \delta_1 \cdot \xi_0/\xi_1^p$ (at least one of the ξ_0, ξ_1 is a unit in R.)

The condition (5.3b) allows us to provide H' with a comultiplication compatible with the comultiplication of G_T. Indeed, it is enough to pick the constant of comultiplication (see *loc. cit* 1.5.1) to be $\gamma = \gamma_0 \cdot \xi_0^p/\xi_1 \cdot \epsilon = \gamma_1 \cdot \xi_1^p/\xi_0 \cdot \epsilon$ ($\epsilon = \pi'/\pi$ is the ratio of the uniformizers of Raynaud for $\mathbf{F}_p$ and $\mathbf{F}_{p^2}$-vector space schemes). The formula of [10] 1.5.1, shows that with this comultiplication H' is a subgroup of G_T. □

We end this paragraph with a lemma on generators of p-rank subgroups of Raynaud group schemes. We continue with the same assumptions and notations. Let H be a locally free subgroup scheme of G of rank p and P a point in $H(S) \subset G(S)$.

LEMMA 5.2.2. *If P is an $\mathbf{F}_p$-generator of H, it is also an $\mathbf{F}_{p^2}$-generator of G.*

Proof. Note that the statement is obviously true in characteristic zero, in view of Remark 5.1.1a. We will use the notations of 5.2.1 and its proof. It is enough to consider the case $S = \operatorname{Spec} R$ with R a local ring. By 5.2.1 and its proof, we can find ξ_0, ξ_1 such that $\mathcal{O}_H = \operatorname{Spec} R[X]/(X^p - \delta X)$ where $\delta = \delta_0 \cdot \xi_1/\xi_0^p = \delta_1 \cdot \xi_0/\xi_1^p$. By 5.1.5, the scheme of $\mathbf{F}_p$-generators of H is defined by the ideal $(X^{p-1} - \delta)$. Suppose that $P \in H(S)$ is an $\mathbf{F}_p$-generator of H, given by

$$R[X]/(X^p - \delta X) \to R[X]/(X^{p-1} - \delta) \to R.$$

We obtain by composing with $X_0 \to \xi_0 X$, $X_1 \to \xi_1 X$ the corresponding point of G:

$$R[X_0, X_1]/(X_0^p - \delta_0 X_1, X_1^p - \delta_1 X_0) \to R.$$

In view of (5.3a′–b′), it is easy to see that this morphism factors through the quotient defined by the ideal

$$(X_0^p - \delta_0 X_1,\ X_1^p - \delta_1 X_0,\ X_0^{p-1} X_1^{p-1} - \delta_0\delta_1)$$

By 5.1.5 this defines the subscheme of F_{p^2}-generators of G. □

6. Filtered $\Gamma_{00}(\mathcal{A})$-level structure

In this paragraph, we restrict to Hilbert–Blumenthal surfaces ($d = 2$) and prove 2.4.2 and 2.4.3. We continue with the notations of the previous paragraphs. In particular, W is the ring of Witt vectors $W(\bar{\mathbf{F}}_p)$.

For the study of the local structure of $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})$ over p, we can and will eventually assume that $\mathcal{A}$ is supported over the rational prime p. We start by considering the most interesting case:

6.1. We assume that p is inert in F and $\mathcal{A} = (p)$. In this case, by Example 4.5(b)

$$B_1 \simeq \mathcal{U} = \mathrm{Spec}\ W[X_0,\ X_1,\ Y_0,\ Y_1]/(X_0Y_0 - p,\ X_1Y_1 - p)$$

is an étale local model for $\mathcal{H}_0(p)_W$.

Set $R = W[X_0,\ X_1,\ Y_0,\ Y_1]/(X_0Y_0 - p,\ X_1Y_1 - p)$.

Denote by $\rho\colon \widetilde{\mathcal{H}_0(p)}_W \to \mathcal{H}_0(p)_W$ the blow-up of $\mathcal{H}_0(p)_W$ along the sheaf of ideals $\mathcal{I}$ defined (in the notations of 4.4.3) by $g_1^*(X_0),\ g_1^*(X_1)$. The blow-up for algebraic stacks can be constructed using a variant of [3] 4.20.

LEMMA 6.1.1. *$\widetilde{\mathcal{H}_0(p)}_W$ is regular with special fiber a divisor with normal crossings.*

Proof. By 4.5(b), we see that it is enough to show the same statement for the blow-up of $\mathcal{U}$ along the subscheme defined by $(X_0,\ X_1)$. This blow-up is the subscheme of $\mathbf{P}^1_R = \mathrm{Proj}(R[x_0,\ x_1])$ defined by $X_1x_0 = X_0x_1,\ Y_0x_0 = Y_1x_1$. It is covered by two open affine subsets

$$\mathrm{Spec}\ W[X_0,\ Y_1,\ (x_0/x_1)]/(X_0(x_0/x_1)Y_1 - p),$$
$$\mathrm{Spec}\ W[X_1,\ Y_0,\ (x_1/x_0)]/(X_1(x_1/x_0)Y_0 - p),$$

which are regular. In fact, we can see that the special fiber is a divisor with simple normal crossings. □

Let us now define the moduli stack of an "intermediate" level structure:

DEFINITION 6.1.2. Let $\mathcal{H}_0^{\mathrm{Fil}}(p)$ be the algebraic stack over Spec $\mathbf{Z}$ whose objects over the scheme S, are pairs consisting of an object of $\mathcal{H}_0(p)$ over S together with a locally free rank p subgroup scheme of the kernel of the corresponding isogeny.

There is a natural action of $K = \mathbf{F}^*_{p^2}/\mathbf{F}^*_p$ on $\mathcal{H}_0^{\mathrm{Fil}}(p)$. The element $k \in K$ takes the intermediate subgroup H to $k \cdot H$. Also, there are forgetful morphisms

$$\mathcal{H}_{00}^{\mathrm{Fil}}(p) \xrightarrow{\sigma} \mathcal{H}_0^{\mathrm{Fil}}(p) \xrightarrow{\tau} \mathcal{H}_0(p)$$

with composition $\pi_{00} = \tau \cdot \sigma$.

PROPOSITION 6.1.3.

(a) *The morphism τ is representable and proper.*
(b) *The morphism σ is representable, finite and flat. In fact, for every morphism $S \to \mathcal{H}_0^{\mathrm{Fil}}(p)$ where S is a scheme, $\mathcal{H}_{00}^{\mathrm{Fil}}(p) \times_{\mathcal{H}_0^{\mathrm{Fil}}(p)} S$ is the subscheme of $\mathbf{F}_p$-generators of the corresponding p-rank group scheme over S.*

Proof. (a) follows immediately from 5.2.1. (b) follows 5.2.2 and 5.2.1. □

6.1.4. We will now apply 5.2.1 to the kernel of the modular isogeny over a local étale neighborhood $U = \operatorname{Spec} C$ of the closed point $x \in \mathcal{H}_0(p)_W$. Choose trivializations for $\mathcal{L}_0$ and $\mathcal{L}_1$. Then, Γ_j and Δ_j provide us with the elements $\gamma_j = g_1^*(Y_j)$ and $\delta_j = g_1^*(X_j)$ of C. The remark after 5.2.1, shows that $\mathcal{H}_0^{\mathrm{Fil}}(p) \times_{\mathcal{H}_0(p)} U$ is represented by the subscheme $\mathcal{Z}$ of $\mathbf{P}(\mathcal{L}_0 \oplus \mathcal{L}_1) \simeq \operatorname{Proj}(C[\xi_0, \xi_1]) = \mathbf{P}^1_C$ which is defined by:

$$\delta_0 \cdot \xi_1^{p+1} = \delta_1 \cdot \xi_0^{p+1}, \quad \text{and} \quad \gamma_0 \cdot \xi_0^{p+1} = \gamma_1 \cdot \xi_1^{p+1}. \tag{6.1}$$

The action of $K = \mathbf{F}^*_{p^2}/\mathbf{F}^*_p$ is given by $k \cdot (\xi_0, \xi_1) = (\chi_0(k)\xi_0, \chi_1(k)\xi_1)$.

Denote by H the universal subgroup scheme of rank p over $\mathcal{H}_0^{\mathrm{Fil}}(p) \times_{\mathcal{H}_0(p)} U$ and let V_j be the open affine subset of $\mathcal{H}_0^{\mathrm{Fil}}(p) \times_{\mathcal{H}_0(p)} U$ which is the restriction of $\operatorname{Spec} C[\xi_{j+1}/\xi_j] \subset \mathbf{P}^1_C$. The proof of 5.2.1 shows that we have:

$$H_{V_j} \simeq \operatorname{Spec} \mathcal{O}_{V_j}[T]/(T^p - \delta_{j+1} \cdot (\xi_{j+1}/\xi_j^p)T),$$

or (by renaming T)

$$H_{V_j} \simeq \operatorname{Spec} \mathcal{O}_{V_j}[T]/(T^p - \delta_{j+1} \cdot (\xi_{j+1}/\xi_j)T).$$

By applying Proposition 5.1.5 to H_{V_j}, we see that the subscheme of $\mathbf{F}_p$-generators of H over V_j is

$$H^{\dagger}_{V_j} \simeq \operatorname{Spec} \mathcal{O}_{V_j}[T]/(T^{p-1} - \delta_{j+1} \cdot \xi_{j+1}/\xi_j). \tag{6.2}$$

By 6.1.3, (6.2) gives an open affine subset of $\mathcal{H}_0^{\mathrm{Fil}}(p) \times_{\mathcal{H}_0(p)} U$.

PROPOSITION 6.1.5.

(a) *The morphism τ_W factors through the blow-up:*

$$\mathcal{H}_0^{\mathrm{Fil}}(p)_W \xrightarrow{\tilde{\tau}} \widetilde{\mathcal{H}_0(p)}_W \xrightarrow{\rho} \mathcal{H}_0(p)_W,$$

where $\rho \cdot \tilde{\tau} = \tau_W$ and $\tilde{\tau}$ is representable finite and flat of rank $p + 1$.

(b) *The algebraic stack $\mathcal{H}_0^{\mathrm{Fil}}(p)$ is regular. It is flat over* Spec $\mathbf{Z}$ *and its fiber over p is a divisor with (non-reduced) normal crossings.*

Proof. All statements of (a) are of local nature on the base $\mathcal{H}_0(p)_W$. For simplicity of notation we will drop the subscript W. We show first that τ factors through the blow-up ρ. If U = Spec C is an étale local neighborhood of $\mathcal{H}_0(p)$, then by (6.1) the pull back of the ideal $(g_1^*(X_0), g_1^*(X_1)) = (\delta_0, \delta_1) \subset C$ on $\mathcal{H}_0^{\mathrm{Fil}}(p) \times_{\mathcal{H}_0(p)} U$ is locally generated by a single element. This is enough by the universal property of the blow-up.

To show the other statements, we recall (Example 4.5(b), that in this case $g_1 \colon U \to B_1 \simeq \mathcal{U}$ is étale.

By the proof of the Lemma 6.1.1 the blow-up $\tilde{U}$ of U along $(g_1^*(X_0), g_1^*(X_1))$ is covered by two open affine subsets which are isomorphic to étale open subsets of

$$\operatorname{Spec} W[X_j, Y_{j+1}, (x_j/x_{j+1})]/(X_j(x_j/x_{j+1})Y_{j+1} - p), \quad j \in \mathbf{Z}/2\mathbf{Z}.$$

The equations (6.1) now show that $\mathcal{H}_0^{\mathrm{Fil}}(p) \times_{\mathcal{H}_0(p)} U$ is covered by two open affine subsets which are isomorphic to étale open subsets of

$$\operatorname{Spec} W[X_j, Y_{j+1}, (\xi_j/\xi_{j+1})]/(X_j(\xi_j/\xi_{j+1})^{p+1}Y_{j+1} - p).$$

The morphism $\tilde{\tau}_U \colon \mathcal{H}_0^{\mathrm{Fil}}(p) \times_{\mathcal{H}_0(p)} U \to \tilde{U}$ is given by restricting the morphism given by

$$x_j/x_{j+1} \to \xi_j/\xi_{j+1}.$$

The proof of (a) now follows. The proof of (b) follows from 2.1.2 and the above calculation. □

6.1.6. Using 6.1.3, (6.2) and the proof of 6.1.5, we see that

$$\operatorname{Spec} W[T, X, Y, Z]/(T^{p-1} - XY, XY^{p+1}Z - p) \tag{6.3}$$

is an étale local model for $\mathcal{H}_{00}^{\mathrm{Fil}}(p)_W$. This scheme is obviously a relative complete intersection. It is easy to check that it is a regular outside the codimension 2 reduced subscheme defined by (X, Y, T). By EGA IV 5.8.6, it is normal.

COROLLARY 6.1.7. *Suppose $\mathcal{A} = (p)$ with p inert in F. Then $\mathcal{H}_{00}^{\mathrm{Fil}}(p)_W$ is the normalization (see [3] p. 104) of the blow-up $\widetilde{\mathcal{H}_0(p)}_W$ in $\mathcal{H}_{00}(p)_{W[1/p]}$.*

Proof. By 6.1.3 and 6.1.5 $\tilde{\tau}\cdot\sigma\colon \mathcal{H}_{00}^{\mathrm{Fil}}(p)_W \to \widetilde{\mathcal{H}_0(p)}_W$ is a finite morphism. Therefore, the corollary follows from 6.1.6 and the fact that the forgetful morphism $\pi_{00}\colon \mathcal{H}_{00}^{\mathrm{Fil}}(p)_W \to \mathcal{H}_{00}(p)_W$ is an isomorphism over $W[\frac{1}{p}]$. □

Proof of Theorem 2.4.2. First of all, we can assume that $n = 1$, since the forgetful morphism $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_n \to \mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_{n'}$ is étale when $n'|n$. Fix a prime p and suppose that $\mathcal{A} = \mathcal{A}'\cdot\mathcal{B}$ with $\mathcal{B}$ relatively prime to p. The forgetful morphisms $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A}) \to \mathcal{H}$ and $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_{\mathbf{Z}_{(p)}} \to \mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A}')_{\mathbf{Z}_{(p)}}$ are finite étale over $\mathbf{Z}[1/\mathrm{Norm}(\mathcal{A})]$ and $\mathbf{Z}_{(p)}$ respectively. Using Theorems 2.1.2 and 2.1.3, we see that we can restrict to the case in which $\mathcal{A}$ is supported over a single rational prime p, and prove the statement of the theorem for $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})_W$. The case $\mathcal{A} = (p)$ with p inert in F, is taken care by 6.1.6 above. In all the other cases we have $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A}) = \mathcal{H}_{00}(\mathcal{A})$.

Suppose p ramifies; $p = \mathcal{P}^2$ and $\mathcal{A} = \mathcal{P}$. By Example 4.5a, every closed point x of $\mathcal{H}_0(\mathcal{P})_W$ has an étale local neighborhood $U = \mathrm{Spec}\, C$ isomorphic to an étale local neighborhood of

$$W[V, Y, Z]/(V(YZ - V + \lambda) + \mu),$$

and $\delta_0 = g_1^*(X_0) = V\cdot x$, with x a unit in C. By 5.1.5,

$$\mathcal{H}_{00}(\mathcal{P}) \times_{\mathcal{H}_0(\mathcal{P})} U \simeq \mathrm{Spec}\, C[T]/(T^{p-1} - V\cdot x).$$

This scheme is regular.

The case of a split $p = \mathcal{P}\cdot\mathcal{P}'$ is treated similarly. An étale local model for $\mathcal{H}_{00}(\mathcal{A})$ is

$$\mathrm{Spec}\, W[T, U, U']/(T^{p-1}U - p),$$

where $\mathcal{A} = \mathcal{P}$, and

$$\mathrm{Spec}\, W[T, U]/(T^{p-1}U - p) \times_W \mathrm{Spec}\, W[T', U']/(T'^{p-1}U' - p),$$

when $\mathcal{A} = \mathcal{P}\cdot\mathcal{P}'$. Both schemes are complete intersections and flat over Spec W. The first scheme is regular. This concludes the proof of Theorem 2.4.2. □

6.2. We will now prove 2.4.3. Let us discuss the desingularization of $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})$. The usual argument, shows that we can suppose that $\mathcal{A}$ is supported over a single rational prime p. By 2.1.3 and the fact that the forgetful morphism $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A}) \to \mathcal{H}$ is then étale over $\mathbf{Z}[1/p]$, $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})$ is regular away from p. In fact, by 2.1.3 and the proof of the Theorem 2.4.2 above, $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})$ is regular when $p = \mathcal{P}^2$ and $\mathcal{A} = \mathcal{P}$, or when $p = \mathcal{P}\cdot\mathcal{P}'$ and $\mathcal{A} = \mathcal{P}$. In the case $p = \mathcal{P}\cdot\mathcal{P}' = \mathcal{A}$, a blow-up along the radical of the ideal defined in the notations of 4.4.3 by $g_1^*(X_1)$, $g_2^*(X_1)$, desingularizes $\mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{P}\cdot\mathcal{P}')$. This ideal is defined over $\mathbf{Z}$, and it corresponds to (T, T') in the description of the local model that appears in the proof of 2.4.2

above. The resulting algebraic stack is flat over Spec $\mathbf{Z}$. We will leave the details of this case to the reader and concentrate on the remaining case $\mathcal{A} = (p)$ with p inert in F.

If $\mathcal{M}$ is a separated algebraic stack of finite type over $\mathbf{Z}$, denote by $\Sigma(\mathcal{M})$ the reduced closed subset of singular points of $\mathcal{M}$ ([3] p. 102). Set $\mathcal{M}_0 = \mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A}) = \mathcal{H}_{00}^{\mathrm{Fil}}(p)$ and inductively, for $i \geqslant 1$, define $\mathcal{M}_i$ as the blow-up of $\mathcal{M}_{i-1}$ along $\Sigma(\mathcal{M}_{i-1})$.

PROPOSITION 6.2.1. *If $p = 2$, then $\mathcal{M}_0 = \mathcal{H}_{00}^{\mathrm{Fil}}(p)$ is regular and flat over* Spec $\mathbf{Z}$. *Otherwise, $\mathcal{M}_{(p-1/2)}$ is regular and flat over* Spec $\mathbf{Z}$, *with fiber over p a divisor with (non-reduced) normal crossings.*

Proof. Set $A = W[T, X, Y, Z]/(T^{p-1} - XY, XY^{p+1}Z - p)$. Notice that Spec A is a toroidal embedding.

Using 2.1.3 and 6.1.6, we see that it is enough to show that the same procedure desingularizes $\mathcal{V}_0 :=$ Spec A. If $p = 2$, $\mathcal{V}_0$ is already regular. Assume that $p \neq 2$. The reduced subscheme of singular points is defined by (T, X, Y), and the first blow-up is certainly a closed subscheme of

$$\begin{aligned}\mathcal{V}_1 = {} & \operatorname{Proj} A[T_0, T_1, T_2]/(XT_1 - YT_0, XT_2 - TT_0, YT_2 - TT_1, \\ & T^{p-3}T_2^2 - T_0T_1).\end{aligned}$$

This scheme is covered by three open affine subsets:

$$\begin{aligned}\mathcal{V}_1^1 = {} & \operatorname{Spec} A[T_1/T_0, T_2/T_0]/(Y - XT_1/T_0, T - XT_2/T_0, \\ & T^{p-3}(T_2/T_0)^2 - T_1/T_0)\end{aligned}$$

which is: Spec $W[Z, X_0, T_2/T_0]/(ZX^{2p+3}(T_2/T_0)^{p^2-1} - p)$.

$$\begin{aligned}\mathcal{V}_1^2 = {} & \operatorname{Spec} A[T_0/T_1, T_2/T_1]/(X - YT_0/T_1, T - YT_2/T_1, \\ & T^{p-3}(T_2/T_1)^2 - (T_0/T_1))\end{aligned}$$

which is: Spec $W[Z, Y, T_2/T_1]/(ZY^{2p-1}(T_2/T_1)^{p-1} - p)$.

$$\begin{aligned}\mathcal{V}_2^3 = {} & \operatorname{Spec} A[T_0/T_2, T_1/T_2]/(X - TT_0/T_2, Y - TT_1/T_2, \\ & T^{p-3} - (T_0/T_2)(T_1/T_2))\end{aligned}$$

which is: Spec $W[Z, T, T_0/T_2, T_1/T_2]/(ZT^{2p-1}(T_1/T_2)^p - p, T^{p-3} - (T_0/T_2)(T_1/T_2))$.

The affine subschemes $\mathcal{V}_1^1$ and $\mathcal{V}_1^2$ are regular and flat over Spec W. The scheme $\mathcal{V}_1^3$ has a singularity similar to the one we started with, but with lower exponent of T. In particular, $\mathcal{V}_1^3$ is a normal relative complete intersection which is flat over Spec W. Therefore, $\mathcal{V}_1$ is normal and flat over Spec W and we can conclude that $\mathcal{V}_1$

is indeed the blow-up of $\mathcal{V}_0$ along (T, X, Y). The reduced locus of non-regularity of $\mathcal{V}_1^3$ is defined by the ideal $(T, T_0/T_2, T_1/T_2)$. The second blow-up again reduces the exponent of T by 2. We can see that after repeating $(p-1)/2$ times we obtain a regular scheme, flat over Spec W, whose special fiber is a divisor with non-reduced normal crossings. □

The conclusion of the above discussion can be phrased as follows ($\mathcal{A}$ is any square-free ideal):

PROPOSITION 6.2.2. *There is an algebraic stack M such that*

(a) *M is regular and flat over* Spec $\mathbf{Z}$.
(b) *There is a proper representable morphism $M \to \mathcal{H}_{00}^{\mathrm{Fil}}(\mathcal{A})$ which is an isomorphism outside a closed subset supported on the fibers over primes dividing* Norm$(\mathcal{A})$.

In view of the discussion before 2.4.3, the proof of 2.4.3 now follows immediately. □

Note added in proof. Rapoport and Zink in the preprint "Period spaces for p-divisible groups" provide a construction of étale local models for many moduli problems of abelian varieties with level structure of parahoric type, which generalizes theorems 3.3.1 and 3.4.1. The local structure of $\mathcal{H}_0(\mathcal{A})$ when $\mathcal{A}$ is a rational prime inert in F was also treated by H. Stamm (doctoral thesis, Bergische Universität). Theorems 2.1.2 and 2.2.2 verify special cases of a conjecture of Rapoport and Zink on the flatness of local models.

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