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The real Seifert form and the spectral pairs of isolated hypersurface singularities

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Abstract. The mixed-Hodge theoretical data of an isolated hypersurface singularity f are codified by the set of spectral pairs $\mathrm{Spp}(f) \in \mathbf{N}[\mathbf{Q} \times \mathbf{N}]$. We prove that its projection in $\mathbf{N}[\mathbf{Q}/2\mathbf{Z} \times \mathbf{N}]$ is equivalent to the real Seifert form of f. In order to prove and illustrate this, we discuss and classify the sesqui-linear forms from the viewpoint of the Hodge theory.

1. Introduction

The final goal of this paper is the description of the connection between the complex (real) Seifert form and the collection of spectral pairs $\mathrm{Spp}(f)$ of an isolated hypersurface singularity f. The Seifert form is defined topologically and coordinates the intersection form and the monodromy. On the other hand, the powerful discret invariant $\mathrm{Spp}(f)$, defined in the free abelian group generated by $\mathbf{Q} \times \mathbf{N}$, is equivalent to the collection of Hodge numbers $\{h_\lambda^{p,q}\}$ [11, 12, 13].

In this paper we prove the following

THEOREM. *Consider the image* $\mathrm{Spp}_{\text{mod-}2}(f)$ *of* $\mathrm{Spp}(f)$ *under the projection induced by* $\mathbf{Q} \times \mathbf{N} \to (\mathbf{Q}/2\mathbf{Z}) \times \mathbf{N}$. *Then the information contained in* $\mathrm{Spp}_{\text{mod-}2}(f)$ *is equivalent to the information contained in the real Seifert form of the singularity. Moreover, this correspondence is very explicit (see 6.1 and 6.5).*

Therefore, we can distinguish three levels of invariants. The first one is determined by the monodromy. The collection of the eigenvalues and the weight filtration (measuring the dimension of the blocks) can be codified in $\mathbf{N}[(\mathbf{Q}/\mathbf{Z}) \times \mathbf{N}]$. The second one is the level of the real Seifert form codified in $\mathbf{N}[(\mathbf{Q}/2\mathbf{Z}) \times \mathbf{N}]$. The additional $\mathbf{Z}_2$-invariants can be identified with some signatures. The analytic invariant $\mathrm{Spp}(f)$ is codified in $\mathbf{N}[\mathbf{Q} \times \mathbf{N}]$. The relation between them is the corresponding factorizations. So, the Seifert form contains complete information about the weight filtration and a $\mathbf{Z}_2$-(Hodge) decomposition. The latter can be understood in the following way. We collapse the mixed Hodge structure (in fact the Hodge filtration) of f (or the spectral pairs) corresponding to the signs given by the polarization.

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This result is similar to the Hodge signature theorem in the case of smooth projective varieties. In that classical case, the signature is given by the collapsed $\mathbf{Z}_2$-("even–odd")-Hodge decomposition, where the collapse is induced by the polarization. In our case, the real Seifert form of an isolated singularity is equivalent to the collapsed mixed Hodge structure associated with the singularity.

This correspondence motivates an intensive study of sesqui-linear forms: we develop the "theory of mixed Hodge structures" at this level. In fact, it is convenient to study the Seifert form together with the hermitian intersection form and the monodromy operator. This triplet forms a "variation structure". Since we did not find a convenient presentation form our point of view in the literature, we start in Section 2 with the classification of the variation structures. In Section 3 we introduce their spectral invariants and in the next section we relate them to the signature-type invariants. In Section 5 we recall some properties of the mixed Hodge structure associated with an isolated hypersurface singularity. Section 6 contains the proofs of the main theorems and some examples.

As an application, we find new obstructions for the *algebraic* Seifert forms, compute the complex Seifert forms of quasi-homogeneous isolated singularities, establish different connections between our invariants (for example, we compute the equivariant signatures corresponding to the eigenvalues $\neq 1$ in terms of mod-2-spectral *numbers*).

2. ε-hermitian variation structures

If U is a finite dimensional vector space then U^* is its dual $\mathrm{Hom}_{\mathbf{C}}(U, \mathbf{C})$. We have the natural isomorphism $\theta\colon U \to U^{**}$ given by $\theta(u)(\varphi) = \varphi(u)$. We denote by $\bar{\cdot}$ the complex conjugation. If $\varphi \in \mathrm{Hom}_{\mathbf{C}}(U, U')$, then $\bar{\varphi} \in \mathrm{Hom}_{\mathbf{C}}(U, U')$ is defined by $\bar{\varphi}(x) := \overline{\varphi(\bar{x})}$. The dual $\varphi^*\colon U'^* \to U^*$ of φ is defined by $\varphi^*(\psi) = \psi \circ \varphi$.

It is convenient to write $\varepsilon = \pm 1$ in the form $\varepsilon = (-1)^n$.

2.1. DEFINITION. An ε-hermitian variation structure (abbreviated as HVS) over $\mathbf{C}$ is a system $(U; b, h, V)$, where

(a) U is a finite dimensional $\mathbf{C}$-vector space,
(b) $b : U \to U^*$ is a $\mathbf{C}$-linear endomorphism with $\overline{b^* \circ \theta} = \varepsilon b$; (i.e. it is ε-hermitian).
(c) h is b-orthogonal automorphism of U, i.e. $\overline{h^*} \circ b \circ h = b$.
(d) $V\colon U^* \to U$ is a $\mathbf{C}$-linear endomorphism, with
   (i) $\overline{\theta^{-1} \circ V^*} = -\varepsilon V \circ \bar{h}^*$, (i.e. V is "ε-h-hermitian")
   (ii) $V \circ b = h - I$ (the "Picard-Lefschetz relation").

2.2. The endomorphism b defines an ε-symmetric hermitian form $B : U \otimes U \to \mathbf{C}$ by $B(u, v) = b(u)(\bar{v})$. Indeed, $\overline{B(v, u)} = \overline{b(v)(\bar{u})} = \varepsilon\overline{b^* \circ \theta(v)(\bar{u})} = \varepsilon b^*\theta(\bar{v})(u) = \varepsilon\theta(\bar{v})(b(u)) = \varepsilon b(u)(\bar{v}) = \varepsilon B(u, v)$. Condition (c) is equivalent to $B(hx, hy) = B(x, y)$ for any x and y.

We have two immediate properties:

2.3. LEMMA. $b \circ V = (\bar{h}^*)^{-1} - I$, *and* $h \circ V \circ \bar{h}^* = V$.

Proof. The first identity follows from $\overline{b^* \circ \theta \circ \theta^{-1} \circ V^*} = \bar{h}^* - 1$. For the second one, one has: $h \circ V \circ \bar{h}^* = (h - I)V\bar{h}^* + V\bar{h}^* = VbV\bar{h}^* + V\bar{h}^* = V(\bar{h}^{*,-1} - I)\bar{h}^* + V\bar{h}^* = V$. □

2.4. DEFINITION. The HVS $(U;\, b,\, h,\, V)$ is called *non-degenerate* (resp. *simple*) if b (resp. V) is an isomorphism.

2.5. DEFINITION. Two ε-hermitian variation structures $(U;\, b,\, h,\, V)$ and $(U';\, b',\, h',\, V')$ are isomorphic (denoted by $\approx$) if there exists a (**C**-linear) isomorphism $\varphi\colon U \to U'$ such that $b = \bar{\varphi}^* b' \varphi$, $h = \varphi^{-1} h' \varphi$, and $V = \varphi^{-1} V' (\bar{\varphi}^*)^{-1}$.

2.6. REMARKS.

(a) If b is an isomorphism then $V = (h - I)b^{-1}$ and the HVS $(U;\, b,\, h,\, V)$ is completely determined by the *isometric structure* $(U;\, b,\, h)$ (i.e. triplets with axioms a–b–c and with non-degenerate form b). The classification (up to isomorphism) of the isometric structures is equivalent to the classification of the conjugate classes in the orthogonal group $O(b)$. For this classification, see, for example, the papers of Milnor [4] and Neumann [7].

(b) If V is an isomorphism, then $h = -\varepsilon V(\overline{\theta^{-1} \circ V^*})^{-1}$ and $b = -V^{-1} - \varepsilon(\overline{\theta^{-1} \circ V^*})^{-1}$. In particular, the classification of *simple* HVS-s is equivalent to the classification of **C**-linear isomorphisms $V:\, U^* \to U$ or to the classification of sesqui-linear forms on finite dimensional vector spaces. Here $V\colon U^* \to U$ and $V'\colon U'^* \to U'$ are isomorphic if $V' = \varphi V \bar{\varphi}^*$ for an isomorphism $\varphi\colon U \to U'$.

If we would like to emphasize ε then the ε-HVS determined by V is denoted by ${}_\varepsilon\mathcal{V}$.

(c) Any base $\{e_i\}_i$ of U defines a dual base $\{e_i^*\}_i$ of U^* by $e_j^*(e_i) = 1$ if $j = i$ and $= 0$ else. In all our matrix notations we will use the matrix representation in a convenient base and its dual base. (Notice that θ corresponds to the identity matrix. If the endomorphism $\varphi\colon U \to U'$, in a given base, has matrix representation A, then φ^* in the dual base is represented by the transposed matrix A^*.)

2.7. EXAMPLES.

1. Define the *trivial* structure $\mathcal{T}$ by $U = \mathbf{C}$, $b = 0$, $h = 1_{\mathbf{C}}$, $V = 0$.

2. Let S^1 be the unit circle. Any $\xi \in S^1 - \{\varepsilon\}$ defines an ε-hermitian 1-dimensional simple structure $\mathcal{V}(\xi)$ by

$$\mathcal{V}(\xi) = \left(\mathbf{C};\, \frac{1-\xi}{1-\varepsilon\xi},\, \xi,\, \varepsilon\xi - 1\right).$$

3. If $\mathcal{V}_i = (U_i;\, b_i,\, h_i,\, V_i)$ $(i = 1, 2)$ are variation structures, then $\mathcal{V}_1 \oplus \mathcal{V}_2 = (U_1 \oplus U_2;\, b_1 \oplus b_2,\, h_1 \oplus h_2,\, V_1 \oplus V_2)$ is their direct sum in this category. $n\mathcal{V}$

denotes the direct sum of n copies of $\mathcal{V}$. If $\mathcal{V} = (U;\, b,\, h,\, V)$ then $-\mathcal{V}$ denotes $(U;\, -b,\, h,\, -V)$ with the same ε.

If $\mathcal{V}_i$, $(i = 1, 2)$ are *simple* ε_i-hermitian variation structures, then the tensor product $V_1 \otimes V_2$ defines a new simple ε-structure. The corresponding automorphisms are related by ${}^{\otimes}h = -\varepsilon\varepsilon_1\varepsilon_2 h_1 \otimes h_2$. If we want to emphasize the sign of ε in the tensor product, we write $\mathcal{V}_1 \otimes_\varepsilon \mathcal{V}_2$. (In this paper always $\varepsilon\varepsilon_1\varepsilon_2 = -1$, i.e. $h = h_1 \otimes h_2$.)

The conjugate of $\mathcal{V} = (U;\, b,\, h,\, V)$ is $\bar{\mathcal{V}} = (U;\, \bar{b},\, \bar{h},\, \bar{V})$.

4. In the next examples J_k denotes the $k \times k$-Jordan block:

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}.$$

Consider $\lambda \in \mathbf{C}^* - S^1$. The ε-HVS $\mathcal{V}^{2k}(\lambda)$ is defined by:

$$\mathcal{V}_\lambda^{2k} = \left(\mathbf{C}^{2k};\, \begin{pmatrix} 0 & I \\ \varepsilon I & 0 \end{pmatrix},\, \begin{pmatrix} \lambda J_k & 0 \\ 0 & \frac{1}{\bar\lambda} J_k^{*,-1} \end{pmatrix},\, \begin{pmatrix} 0 & \varepsilon(\lambda J_k - I) \\ \frac{1}{\bar\lambda} J_k^{*,-1} - I & 0 \end{pmatrix}\right).$$

Note that $\mathcal{V}_\lambda^{2k} \approx \mathcal{V}_{1/\bar\lambda}^{2k} \approx -\mathcal{V}_\lambda^{2k}$.

5. We are looking for *non-degenerate* $(k \times k)$-matrix b such that $\bar{b}^* = \varepsilon b$ and $J_k^* b J_k = b$. It is immediate that $b_{ij} = 0$ if $i+j \leqslant k$ and $b_{k+1-i,i} = (-1)^{i+1} b_{k,1}$. By [4] the isomorphism class of $(b,\, J_k)$ is determined by $b_{k,1}$. Since b is non-degenerate $b_{k,1} \neq 0$. Since for any $t \in (0, \infty)$ one has $(U;\, b,\, J_k,\, V) \approx (U;\, t^2 b,\, J_k,\, t^{-2} V)$, we can assume that $b_{k,1} = \omega \in S^1$. By the hermitian property of b one has $\bar\omega = \varepsilon(-1)^{k-1}\omega$. This equation has two solutions. In conclusion, there are exactly two non-degenerate forms $b = b_\pm^k$ (up to isomorphism) with $\bar{b}^* = \varepsilon b$ and $J_k^* b J_k = b$. Their representatives are chosen so that $(b_\pm^k)_{k,1} = \pm i^{-n^2-k+1}$; (this strange choice has a Hodge-theoretical motivation and it will simplify the description of the results in the next sections). Note that $b_{k,1} = B(e_k,\, e_1) = B(e_k,\, (J_k - I)^{k-1} e_k) = B(e_k,\, (\log J_k)^{k-1} e_k)$. (Here $\{e_l\}_l$ denotes the standard base of $\mathbf{C}^k$.)

Let $\lambda \in S^1$. If $h = \lambda J_k$, then by the above argument, there are exactly two *non-degenerate* ε-HVS-s (up to isomorphism):

$$\mathcal{V}_\lambda^k(\pm 1) = (\mathbf{C}^k;\, b_\pm^k,\, \lambda J_k,\, (\lambda J_k - I)(b_\pm^k)^{-1})$$

where $\omega = (b_\pm^k)_{k,1} = \pm i^{-n^2-k+1}$.

If $\lambda \neq 1$, then by (2.1.d-ii) any HVS with $h = \lambda J_k$ is non-degenerate. If $h = J_k$, then there are some degenerate structures, too.

6. Suppose that $k \geqslant 2$ and $h = J_k$ but b is degenerate. Since $\ker b \subset \ker(h - I)$ (by 2.1.d-ii), and $\dim \ker(J_k - I) = 1$, one has $\ker b = \ker(h - I)$. Similarly as above, any degenerated form b with $\ker b = \ker(J_k - I)$ and $\bar{b}^* = \varepsilon b$ and $\bar{h}^* b h = b$ has the properties $b_{i,j} = 0$ if $i + j \leqslant k + 1$, and $b_{k+2-i,i} = (-1)^i b_{k,2}$. Therefore $b_{k,2} \neq 0$ and in the isomorphism class of the structure there is a representative with $b_{k,2} = \omega \in S^1$. By symmetry, $\bar{\omega} = (-1)^{n+k}\omega$ and b is completely determined by $b_{k,2}$ modulo an isomorphism. So, we have exactly two solutions $\tilde{b}^k_{\pm}$ (up to isomorphism) with $(\tilde{b}^k_{\pm})_{k,2} = \pm(-1)^{n+1} i^{-(n+1)^2-k+1}$ (notice the shift $n \mapsto n+1$ in the exponent of i). Moreover, V is completely determined by h and b (up to isomorphism). In particular, there are exactly two degenerate structures with $h = J_k$ and $k \geqslant 2$:

$$\tilde{\mathcal{V}}^k_1(\pm 1) = (\mathbf{C}^k; \tilde{b}^k_{\pm}, J_k, \tilde{V}^k_{\pm}),$$

where $(\tilde{b}^k_{\pm})_{k,2} = B^k_{\pm}(e_k, (\log J_k)^{k-2} e_k) = \pm(-1)^{n+1} i^{-(n+1)^2-k+1} = \pm i^{-n^2-k+2}$. In fact:

$$b = \tilde{b}^k_{\pm} = \begin{pmatrix} 0 & 0 \\ 0 & b^{k-1}_{\pm} \end{pmatrix}.$$

Note that the structure can also be recognized from $((\tilde{V}^k_{\pm})^{-1})_{k,1} = \pm i^{-n^2-k+2}$; (use the identity $Vb = h - I$).

By computation we get that $\tilde{V}^k_{\pm}$ is an isomorphism. In particular, the variation structures $\mathcal{V}^k_{\lambda}(\pm 1)$, where $\lambda \in S^1 - \{1\}$ resp. $k \geqslant 1$, and $\tilde{\mathcal{V}}^k_1(\pm 1)$ where $k \geqslant 2$, are simple. They are determined by the corresponding isometric structures $(\mathbf{C}^k; b, h)$.

7. Suppose that $U = \mathbf{C}$ and $h = 1_{\mathbf{C}}$. Then there are exactly five HVS-s (up to isomorphism):

$$\mathcal{V}^1_1(\pm 1) = (\mathbf{C}; \pm i^{-n^2}, 1_{\mathbf{C}}, 0);$$

$$\tilde{\mathcal{V}}^1_1(\pm 1) = (\mathbf{C}; 0, 1_{\mathbf{C}}, \pm i^{n^2-1});$$

and

$$\mathcal{T} = (\mathbf{C}; 0, 1_{\mathbf{C}}, 0).$$

Note that in $\tilde{\mathcal{V}}^1_1(\pm 1)$ the variation structure is *not* determined by its underlying isometric structure.

8. In order to unify the notations of the *simple* structures, we introduce: $\mathcal{W}^k_{\lambda}(\pm 1) = \mathcal{V}^k_{\lambda}(\pm 1)$ if $\lambda \in S^1 - \{1\}$, and $= \tilde{\mathcal{V}}^k_1(\pm 1)$ if $\lambda = 1$. Set $s = 1$ if $\lambda = 1$ and $= 0$ otherwise. Then: $\bar{\mathcal{W}}^k_{\lambda}(\pm 1) = \mathcal{W}^k_{\bar{\lambda}}(\pm(-1)^{-n^2-k+1+s})$.

9. Consider the following matrices:

$$\mathbf{b} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

They define an indecomposable $(+1)$-HVS, but the automorphism h has two Jordan blocks. Note that even the associated (degenerate) isometric structure (b, h) is indecomposable (cf. 2.9.b).

2.8. For the completeness of the discussion, we recall (the complex version of) Milnor's result [4] (see also [7]):

Any isometric structure $(U;\, b,\, h)$ is a sum of indecomposable ones. The indecomposable ones are the corresponding isometric structures of $\mathcal{V}_\lambda^k(\pm 1)$, where $\lambda \in S^1$; and of $\mathcal{V}_\lambda^{2k}$, where $\lambda \in \mathbf{C}^* - S^1$.

The main result of this section is:

2.9. THEOREM.

(a) *An ε-hermitian variation structure is uniquely expressible as a direct sum $\mathcal{V}' \oplus \mathcal{V}''$ so that $h' - I$ is an isomorphism (in particular, $\mathcal{V}'$ is simple and non-degenerate), and $h'' - I$ is nilpotent.*

(b) *A simple ε-hermitian variation structure is uniquely expressible as a sum of indecomposable ones up to order of summands and isomorphism. The indecomposable structures are:*

$\mathcal{W}_\lambda^k(\pm 1)$ *where* $k \geqslant 1$; $\lambda \in S^1$; *and*

$\mathcal{V}_\lambda^{2k}$ *where* $k \geqslant 1$; $0 < |\lambda| < 1$.

2.10. REMARK. Part (b) of this theorem gives a classification of complex sesqui-linear forms (with respect to complex conjugation) over finite dimensional $\mathbf{C}$-vector spaces (cf. 2.6.b).

If two real non-degenerate bilinear forms are isomorphic as sesqui-linear forms over $\mathbf{C}$, then they are isomorphic as real bilinear forms. In particular, the study of *real* simple variation structures is equivalent to the study of the complex ones. This follows from the comparation of (2.9) and the corresponding real classification result [4].

2.11. PROOF of 2.9. We start with the following

Fact. Suppose that $(U;\, b,\, h,\, V) = (U' \oplus U'';\, b' \oplus b'',\, h' \oplus h'',\, V)$. If b' or b'' is non-degenerate, then $V = V' \oplus V''$, in particular $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$ (with the obvious notations).

The proof is a direct verification.

Set $U_\lambda = \{u \in U \,:\, (h - \lambda I)^N u = 0$ for N sufficiently large$\}$. Then $U_\lambda \perp U_\mu$ (B-orthogonal) if $\lambda\bar{\mu} \neq 1$ and $U = \oplus_\lambda U_\lambda$. Define $U_{\neq 1} = \oplus_{\lambda \neq 1} U_\lambda$; $U_1^* = \{\varphi \in$

$U^* : \bar{\varphi}|U_{\neq 1} = 0\}$ resp. $U^*_{\neq 1} = \{\varphi \in U^* : \bar{\varphi}|U_1 = 0\}$. The subspaces U_1 and $U_{\neq 1}$ are h-invariants, U_1^* and $U^*_{\neq 1}$ can be considered as the duals of U_1 resp. $U_{\neq 1}$. Moreover, $b(U_1) \subset U_1^*$ and $b(U_{\neq 1}) \subset U^*_{\neq 1}$. Since $b_{\neq 1}$ is non-degenerated, by the above Fact: $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_{\neq 1}$. This gives the first part.

Now, $\mathcal{V}_{\neq 1}$ is non-degenerate, hence it is completely determined by the isometric structure $(U_{\neq 1}; b_{\neq 1}, h_{\neq 1})$. Thus the result follows from the result of Milnor [loc. cit.].

The decomposition of $\mathcal{V}_1 = (U_1; b_1, h_1, V_1)$ follows from the decomposition of $\mathcal{V}_1 \otimes \mathcal{V}(\xi)$ (where $\xi \in S^1 - \{\pm 1\}$) by the same argument as above. □

2.12. EXAMPLE. If $V^{-1} = -\Gamma_m$, where

$$\Gamma_m = \begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix} \quad (m \times m \text{ block})$$

and $\mathcal{V}$ is the simple ε-HVS defined by V, then $\mathcal{V} = \oplus \mathcal{V}(\xi)$, where the sum is over the roots of $\xi^{m+1} = \varepsilon^{m+1}$ with $\xi \neq \varepsilon$.

In the sequel, the ε-sign of $\mathcal{V}(\xi)$ is always $\varepsilon(\mathcal{V}(\xi)) = +1$; i.e. $\mathcal{V}(\xi) = (\mathbf{C}; 1, \xi, \xi - 1) \approx \mathcal{V}_\xi^1(+1)$.

2.13. PROPOSITION. The ξ-suspension property.
Set $\xi \neq 1$. Then

$$(-1)^{n+1}[{}_{(-1)^n}\tilde{\mathcal{V}}_1^k(\pm 1)] \otimes_{(-1)^{n+1}} [\mathcal{V}(\xi)] = {}_{(-1)^{n+1}}\mathcal{V}_\xi^k(\pm 1).$$

Proof. If $k \geqslant 2$, then the bilinear form of the tensor product is ${}^{\otimes}b = (\tilde{V}_\pm^k)^{-1} + (\tilde{b}_\pm^k)\frac{\xi}{\xi - 1}$. Therefore $({}^{\otimes}b)_{k,1} = ((\tilde{V}_\pm^k)^{-1})_{k,1}$. Using the identity $\tilde{b}_\pm^k = (\tilde{V}_\pm^k)^{-1}(J_k - I)$, we get ${}^{\otimes}b_{k,1} = (\tilde{b}_\pm^k)_{k,2} = \pm(-1)^{n+1} i^{-(n+1)^2 - k + 1} = (-1)^{n+1}({}_{(-1)^{n+1}}b_\pm^k)_{k,1}$.

If $k = 1$, then ${}^{\otimes}b = [\pm(-1)^{n+1} i^{(n+1)^2}]^{-1} = (-1)^{n+1}({}_{(-1)^{n+1}}b_\pm^1)$. □

2.14. If we do not want to relate this presentation and classification to the singularity and Hodge theory, then the sign-convention can be simplified. The sign of $(b_\pm^k)_{k,1}$ is motivated by the polarization formula (see 5.6.ii). The fact, that the structures with $\lambda = 1$ and with weight filtration centered at n have the same behaviour as the structures with $\lambda \neq 1$ and with weight filtration centered at $n + 1$, is central in the mixed Hodge theory of singularities. This motivates the shift in the definition of $(\tilde{b}_\pm^k)_{k,2}$. The additional sign $(-1)^{n+1}$ comes from Deligne's (or Sakamoto's) theorem about the Seifert form of singularities with separable variables.

2.15. REMARK. For the multiplicative properties of the hermitian variation structures, see [6].

3. Hodge numbers and spectral pairs associated with variation structures

Fix an integer n and set $\varepsilon = (-1)^n$.

Let $\mathcal{V} = (U;\, b,\, h,\, V)$ be a ε-symmetric hermitian *simple* variation structure.

In the sequel, we assume that the eigenvalues of the automorphism h are on the unit circle S^1. Recall that $s = s(\lambda) = 0$ if $\lambda \neq 1$ and $= 1$ otherwise.

We want to construct a weight filtration W on U_λ^* and a $\mathbf{Z}_2$-decomposition on $Gr_*^W U_\lambda^*$. By the decomposition Theorem (2.9), it is enough to define them on the indecomposable elements $\mathcal{W}_\lambda^{r+1}(u)$, where $r \geqslant 0$, $\lambda \in S^1$, and $u = \pm 1$. The weight filtration is given by $h^{*,-1}$, the center of the filtration is, by definition, $n + s$. In fact, it is the unique filtration with center $n + s$ and with the properties: $\dim Gr_l^W(\mathcal{W}_\lambda^{r+1}(u)) = 1$ if $l = n + s + r - 2t$, where $t = 0, 1, \ldots, r$, and $\log \bar{J}_{r+1}^{*,-1}(W_l) \subset W_{l-2}$.

The $\mathbf{Z}_2$-decomposition

$$Gr_l^W \mathcal{W}_\lambda^{r+1}(u) = F_{+1} Gr_l^W \mathcal{W}_\lambda^{r+1}(u) \oplus F_{-1} Gr_l^W \mathcal{W}_\lambda^{r+1}(u)$$

is given by:

$$\dim F_v Gr_{n+s+r-2t}^W \mathcal{W}_\lambda^{r+1}(u) = \begin{cases} 1 & \text{if } uv(-1)^t = 1 \\ 0 & \text{otherwise} \end{cases}.$$

In other words: $Gr_{n+s+r-2t}^W \mathcal{W}_\lambda^{r+1}(u) = F_{(-1)^t u} Gr_{n+s+r-2t}^W \mathcal{W}_\lambda^{r+1}(u)$.

If $\mathcal{V}_\lambda = \Sigma_{u=\pm 1, r\geqslant 0} p_\lambda^{r+1}(u) \mathcal{W}_\lambda^{r+1}(u)$ then we redefine $\mathbf{p}_\lambda^{n+s+r,u} = p_\lambda^{r+1}(u)$, $u = \pm 1$, $r \geqslant 0$; ($n + s + r$ is the weight of the "primitive element" of $\mathcal{W}_\lambda^{r+1}(u)$); and we define $\mathbf{h}_\lambda^{w,u} = \dim F_u Gr_w^W U_\lambda^*$. By these notations, we have the following relations:

$$\mathbf{p}_\lambda^{w,u} = \mathbf{h}_\lambda^{w,-u} - \mathbf{h}_\lambda^{w+2,u}, \quad w \geqslant n + s;$$

$$\mathbf{h}_\lambda^{w,u} = \sum_{l \geqslant 0} \mathbf{p}_\lambda^{w+2l,(-1)^l u}, \quad w \geqslant n + s; \tag{3.1}$$

$$\mathbf{h}_\lambda^{n+s-k,u} = \mathbf{h}_\lambda^{n+s+k,(-1)^k u}.$$

In particular, $\mathcal{V} = \Sigma p_\lambda^{r+1}(u) \mathcal{W}_\lambda^{r+1}(u)$ is completely determined by the numbers $\{\mathbf{p}_\lambda^{w,u};\, w \geqslant n + s\}$ or by the set $\{\mathbf{h}_\lambda^{w,u}\}$.

There is a dual weight filtration (with center $n - s(\lambda)$ for $\mathcal{W}_\lambda^{r+1}(u)$) induced by h, and a dual $\mathbf{Z}_2$-decomposition on U_λ. We prefer the theory on U^* because it can be easily compared to the mixed Hodge theory of singularities defined on the cohomology groups.

The (mod-2)-spectral pairs associated with a variation structure lies in

$$\mathbf{N}[(\mathbf{R}/2\mathbf{Z})\times\mathbf{N}] = \left\{ \sum_{(\alpha,w)} (\alpha,\, w),\ \alpha\in\mathbf{R}/2\mathbf{Z},\ w\in\mathbf{N}\right\}.$$

The system of equations:

$$\begin{cases} e^{-2\pi i\alpha} = \lambda \\ (-1)^{w-n-[-\alpha]} = u \end{cases}$$

has exactly one solution $\alpha = \alpha_{\lambda,w,u} \in \mathbf{R}/2\mathbf{Z}$. We associate with the space $F_u Gr_w^W U_\lambda^*$ the spectral pairs $(\alpha,\, w-s(\lambda))$ with multiplicity $\mathbf{h}_\lambda^{w,u} = \dim F_u Gr_w^W U_\lambda^*$. The collection of the spectral pairs of $\mathcal{V}$ is:

$$\mathrm{Spp}(\mathcal{V}) = \sum_{\lambda,w,u} \mathbf{h}_\lambda^{w,u}(\alpha_{\lambda,w,u},\, w - s(\lambda)).$$

It is clear that passing to the spectral pairs we do not lose any information: we can recuperate $\mathcal{V}$ from its spectral invariants. Moreover, $\mathrm{Spp}(\mathcal{V}_1\oplus\mathcal{V}_2) = \mathrm{Spp}(\mathcal{V}_1) + \mathrm{Spp}(\mathcal{V}_2)$.

The symmetry of the weight filtration gives the invariance of $\mathrm{Spp}(f)$ with respect to the transformation $(\alpha,\, n+k) \leftrightarrow (\alpha - k,\, n-k)$. If the structure comes from a real one, the stability with respect to the complex conjugation gives an addition invariance with respect to the transformation $(\alpha,\, n+k) \leftrightarrow (n-1-\alpha,\, n-k)$.

4. Other invariants of the variation structures

4.1. Let $\mathcal{V}$ be a HVS with eigenvalues on the unit circle. The null-space of a bilinear form b is denoted by μ_0. If b is $(+1)$-symmetric, then $\mu_+(b)$ resp. $\mu_-(b)$ denote the maximal dimension of a positive resp. negative definite subspace of b. If b is (-1)-symmetric then $\mu_\pm(b)$ is defined as $\mu_\pm(i\cdot b)$. The signature of b is $\sigma = \mu_+ - \mu_-$. (By this notation, since our form b is $(-1)^n$-symmetric, $\mu_\pm(b)$, by definition, is $\mu_\pm(i^{n^2}b)$.)

Now, consider the homotopy $t\mapsto b_\pm^k(t)$; $b_\pm^k(t)_{ij} = t^{c(i,j)}(b_\pm^k)_{ij}$, where $c(i,j) = 0$ if $i+j = k+1$ and $=1$ otherwise. Since the determinant is non-zero for $t\in[0,1]$: $\mu_\pm(b_\pm^k) = \mu_\pm(b_\pm^k(0))$. Using this, we obtain that the μ-invariants of the *simple* indecomposable variation structures are:

	$\mu_\pm$	$\mu_\mp$	μ_0	σ
$\mathcal{V}_\lambda^k(\pm 1)$	$[\frac{k+1}{2}]$	$[\frac{k}{2}]$	0	$\pm\frac{1+(-1)^{k+1}}{2}$
$\tilde{\mathcal{V}}_1^k(\pm 1)$	$[\frac{k}{2}]$	$[\frac{k-1}{2}]$	1	$\pm\frac{1+(-1)^k}{2}$

4.2. In the sequel, we describe the relation between the spectral pairs and the μ-invariants of a variation structure $\mathcal{V}$.

The first entry of a spectral pair is called spectral number. Their collection defines two numbers corresponding to $v = \pm 1$:

$$\mathrm{Sp}_\lambda(v)(\mathcal{V}) = \#\{\alpha \mid \alpha \text{ is a spectral number with } e^{-2\pi i\alpha} = \lambda \text{ and } (-1)^{[\alpha]} = v\}.$$

PROPOSITION. *Let* $\mathcal{V} = \oplus_\lambda \oplus p^l_\lambda(\pm 1)\mathcal{W}^l_\lambda(\pm 1)$. *Then:*

(a) *For any* λ *and* $v = \pm 1$ *one has:*

$$\mathrm{Sp}_\lambda(v)(\mathcal{V}) = \sum_{\substack{r \text{ even} \\ u=\pm 1}} \frac{r+1-uv}{2} \cdot p^{r+1}_\lambda(u) + \sum_{\substack{r \text{ odd} \\ u=\pm 1}} \frac{r+1}{2} \cdot p^{r+1}_\lambda(u).$$

In particular:

$$\mathrm{Sp}_\lambda(-1)(\mathcal{V}) - \mathrm{Sp}_\lambda(+1)(\mathcal{V}) = \sum_{\substack{r \text{ even} \\ u=\pm 1}} u \cdot p^{r+1}_\lambda(u).$$

(b) *If* $\lambda \neq 1$, *then:* $\mathrm{Sp}_\lambda(\pm 1)(\mathcal{V}) = \mu_{\mp 1}(\mathcal{V}_\lambda)$, *in particular:*

$$\sigma(\mathcal{V}_\lambda) = \mathrm{Sp}_\lambda(-1)(\mathcal{V}) - \mathrm{Sp}_\lambda(+1)(\mathcal{V}) = \sum_{\substack{r \text{ even} \\ u=\pm 1}} u \cdot p^{r+1}_\lambda(u).$$

(Compare with the classical Hodge signature formula.)

(c)

$$\sigma(\mathcal{V}_1) = \sum_{\substack{r \text{ odd} \\ u=\pm 1}} u \cdot p^{r+1}_1(u).$$

Proof. Use (4.1) and the definition of the spectral pairs. □

4.3. By the above proposition, the equivariant signatures corresponding to the eigenvalues $\lambda \neq 1$, are determined by the (mod-2) spectral *numbers*. On the other hand, $\sigma(\mathcal{V}_1)$ cannot be computed from the spectral numbers alone. It can be recovered from the spectral pairs in many ways, for example:

$$\sigma(\mathcal{V}_1) = 2 \sum_{\substack{(\alpha,w)\in \mathrm{Spp}(\mathcal{V}_1) \\ \alpha \geqslant n+2}} (-1)^\alpha + \sum_{\substack{(\alpha,w)\in \mathrm{Spp}(\mathcal{V}_1) \\ \alpha = n+1}} (-1)^\alpha.$$

4.4. Consider the filtration $0 \subset U^{(1)}_\lambda \subset \cdots \subset U^{(m)}_\lambda = U_\lambda$, where $U^{(k)}_\lambda = \ker((h - \lambda I)^k; U_\lambda)$ with dimension $n^{(k)}_\lambda = \dim_{\mathbf{C}} U^{(k)}_\lambda$. On $U^{(k)}_\lambda / U^{(k-1)}_\lambda$ we can define a (± 1)-hermitian form by $B^{(k)}_\lambda(x, y) = B(x, \lambda^{1-k}(h - \lambda I)^{k-1} y)$. Let $\sigma^{(k)}_\lambda$ be its signature.

4.5. PROPOSITION. *The number of the indecomposable components of the direct sum* $\oplus p^l_\lambda(\pm 1)\mathcal{W}^l_\lambda(\pm 1)$ *is determined by the collection of numbers* $n^{(k)}_\lambda$ *and* $\sigma^{(k)}_\lambda$.

Proof. Since $n^{(m)}_\lambda = \#\{\text{all } \lambda\text{-blocks}\}$ and $n^{(k)}_\lambda - n^{(k-1)}_\lambda = \Sigma_{l\geqslant k}\#\{l-\lambda\text{-blocks}\}$, $(k \leqslant m)$, the numbers $n^{(k)}_\lambda$ determine the number of $l-\lambda$-blocks. Since we have only two types of $l-\lambda$-blocks, they can be separated by $\sigma^{(k)}_\lambda$. In fact $\mu_0(B^{(k)}_\lambda) = \Sigma_{l>k}\#\{l-\lambda\text{-blocks}\}$ and $\mu_\pm(B^{(k)}_\lambda) = \#\mathcal{W}^k_\lambda(\pm 1)$. □

5. Isolated hypersurfaces singularities

We review some topological and Hodge-theoretical definitions and results as a preparation for the next section. The basic references are [11, 12, 13, 10, 9] and the first chapter of [1].

5.1. Consider an isolated hypersurface singularity $f\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$. We recall the definitions of the main invariants.

For ε sufficiently small and $0 < \delta \ll \varepsilon$ define $S^1_\delta = \{w : |w| = \delta\} \subset \mathbf{C}$ and $E := f^{-1}(S^1_\delta) \cap \{z : |z| \leqslant \varepsilon\} \subset \mathbf{C}^{n+1}$. Then the induced map $f\colon (E, \partial E) \to S^1_\delta$ is a locally trivial fibration with fiber $(F, \partial F)$, such that $f|\partial E$ is trivial. The (Milnor) fiber F is homotopically equivalent to a bouquet $\bigvee S^n$ therefore its reduced (real) homology (cohomology) is concentrated in $U_\mathbf{R} = \tilde{H}_n(F, \mathbf{R})$ ($U^*_\mathbf{R} = \tilde{H}^n(F, \mathbf{R})$). The characteristic map of the above fibration at (co)homological level defines the algebraic monodromies $h_\mathbf{R} : U_\mathbf{R} \to U_\mathbf{R}$ and $T_\mathbf{R} = h^{*,-1}_\mathbf{R} : U^*_\mathbf{R} \to U^*_\mathbf{R}$. The natural, real intersection form is denoted by $b_\mathbf{R} : U_\mathbf{R} \to U^*_\mathbf{R}$. Fixing a trivialization of $f|\partial E$ one defines a variation map $\mathrm{Var}\colon U^*_\mathbf{R} \to U_\mathbf{R}$.

These invariants satisfy the relations: $\mathrm{Var} \circ b_\mathbf{R} = h_\mathbf{R} - I$; $h^*_\mathbf{R} \circ b_\mathbf{R} \circ h_\mathbf{R} = b_\mathbf{R}$; $b^*_\mathbf{R} \circ \theta = \varepsilon b_\mathbf{R}$; and $\mathrm{Var}^* = -\varepsilon\, \mathrm{Var} \circ h^*_\mathbf{R}$, where $\varepsilon = (-1)^n$.

In particular, the complex maps $b = b_\mathbf{R} \otimes 1_\mathbf{C}$, $h = h_\mathbf{R} \otimes 1_\mathbf{C}$, and $V = \mathrm{Var} \otimes 1_\mathbf{C}$ define a $(-1)^n$-HVS on $U = U_\mathbf{R} \otimes \mathbf{C}$. It is denoted by $\mathcal{V}(f)$. (We recall that b is a hermitian form rather that a bilinear form.)

It is well-known that V is an isomorphism (see, for example, [1, p. 41]), therefore our variation structure is simple. The real Seifert form L can be defined as follows. If $\langle\, ,\, \rangle$ denotes the pairing between $H_n(F, \partial F, \mathbf{R})$ and $H_n(F, \mathbf{R})$, then for $a, b \in H_n(F, \mathbf{R})$ one has $L(a, b) := \langle \mathrm{Var}^{-1}(a), b\rangle$. By our notation, $\langle\, ,\, \rangle$ identifies $H_n(F, \partial F, \mathbf{R})$ with U^*, therefore Var can be identified with the inverse of the Seifert form (cf. [3] or [1, p. 41]).

5.2. Consider the Jordan decompositions $h = h_s h_u$ and $T = T_s T_u$ into semisimple and unipotent part; and the generalized eigenspaces $U_\lambda = \ker(h_s - \lambda I)$ resp. $U^*_\lambda = \ker(T_s - \lambda I)$; and the corresponding decomposition $\log h_u = {}_cN = \oplus_c N_\lambda$ resp. $\log T_u = N = \oplus N_\lambda$. Let $s = 0$ if $\lambda \neq 1$, and $= 1$ if $\lambda = 1$.

5.3. The space U_λ^* carries a mixed Hodge structure with weight filtration centered at $n+s$. For $r \geqslant 0$, the space

$$P_{r,\lambda} = \ker(N_\lambda^{r+1}\colon Gr^W_{n+s+r}U_\lambda^* \to Gr^W_{n+s-r-2}U_\lambda^*)$$

carries an induced Hodge structure of weight $n+s+r$:

$$P_{r,\lambda} = \bigoplus_{a+b=n+s+r} P_\lambda^{a,b}.$$

By the monodromy theorem and [9]: $a+b=n+s+r \leqslant 2n$.

The discrete invariants of the Hodge and of the weight filtrations are collected in the Hodge numbers: $h_\lambda^{p,q} = \dim Gr_F^p Gr^W_{p+q} U_\lambda^*$ (and $h^{p,q} = \Sigma_\lambda h_\lambda^{p,q}$). We will use the dimensions (of the primitive spaces) $p_\lambda^{a,b} = \dim P_\lambda^{a,b}$ ($r = a+b-n-s \geqslant 0$), too. Since N_λ is a morphism of Hodge structures of type $(-1,-1)$, one has:

$$p_\lambda^{a,b} = h_\lambda^{a,b} - h_\lambda^{a+1,b+1}, \quad (r = a+b-n-s \geqslant 0). \tag{5.4}$$

This system of linear equation can be solved in the Hodge numbers, thus

$$h_\lambda^{a,b} = \sum_{l \geqslant 0} p_\lambda^{a+l,b+l}, \quad (a+b \geqslant n+s). \tag{5.5}$$

In fact, this can be also regarded as a consequence of the direct sum decomposition:

$$Gr_F^a Gr^W_{a+b} U_\lambda^* = \bigoplus_{l \geqslant 0} N_\lambda^l P_\lambda^{a+l,b+l}, \quad (a+b \geqslant n+s).$$

Moreover, since $h_\lambda^{p,q} = h_{\bar\lambda}^{n+s-p,n+s-q}$, the system of numbers $\{h_\lambda^{p,q}\}_{p,q}$ is equivalent to the system of numbers $\{p_\lambda^{a,b}\}_{a+b \geqslant n+s}$.

5.6. We want to relate $\mathcal{V}(f)$ to the mixed Hodge structure of the singularity. Since the former object (more precisely b) is defined on U and the latter on U^*, we will consider the dual of this mixed Hodge structure, too.

We identify U with $\tilde{H}_c^n(F, \mathbf{C})$. By [13]:

(i) The space U_λ carries a mixed Hodge structure with weight filtration centered at $n-s$. For $r \geqslant 0$, the space

$${}_cP_{r,\lambda} = \ker({}_cN_\lambda^{r+1}\colon Gr^W_{n-s+r}U_\lambda \to Gr^W_{n-s-r-2}U_\lambda)$$

carries an induced Hodge structure of weight $n-s+r$:

$${}_cP_{r,\lambda} = \bigoplus_{a+b=n-s+r} {}_cP_\lambda^{a,b}.$$

(ii) If $\lambda \neq 1$ then the form $Q_{r,\lambda}\colon {}_cP_{r,\lambda} \otimes {}_cP_{r,\lambda} \to \mathbf{C}$ given by $Q_{r,\lambda}(x, \bar{y}) = B(x, {}_cN_\lambda^r y)$ has the (polarization) properties:

(a) $({}_cP_\lambda^{a,b}) \perp_{Q_r} ({}_cP_\lambda^{c,d})$ if $(a, b) \neq (c, d)$; and

(b) $(-1)^{n(n-1)/2} i^{a-b} Q_{r,\lambda}(x, \bar{x}) > 0$ if $x \in {}_cP_\lambda^{a,b}$.

From the duality, dim ${}_cP_\lambda^{a,b} = {}_cp_\lambda^{a,b}$ is equal to $p_\lambda^{a,b}$, provided that $\lambda \neq 1$. (For more precise description of the duality, see [13].)

5.7. An excellent codification of the Hodge numbers is the collection of the spectral pairs considered in the free abelian group $\mathbf{Z}[\mathbf{Q} \times \mathbf{N}]$, generated by $\mathbf{Q} \times \mathbf{N}$:

$$\mathrm{Spp}(f) = \sum_{(\alpha,w)} h_{\exp(-2\pi i\alpha)}^{n+[-\alpha],w+s-n-[-\alpha]}(\alpha, w).$$

Since $h_\lambda^{p,q}$ is the coefficient of $(\alpha, p+q-s)$, where α is the unique solution of $\lambda = \exp(-2\pi i\alpha)$ with $n + [-\alpha] = p$, the collection of the Hodge numbers is equivalent to $\mathrm{Spp}(f)$. The symmetry of the Hodge numbers gives the invariance of the spectral pairs under $(\alpha, w) \mapsto (n-1-\alpha, 2n-w)$.

If we forget the weight filtration, then the information of the equivariant Hodge filtration is codified in the spectrum:

$$\mathrm{Sp}(f) = \sum \alpha \in \mathbf{Z}[\mathbf{Q}] \text{ (the sum over the spectral pairs } (\alpha, w)).$$

Any spectrum number α is in the interval $(-1, n)$.

5.8. EXAMPLE. If $f(x) = x^{m+1}$, then $U = \mathbf{C}^m$, $n = 0$, $\varepsilon = +1$, and $V = -\Gamma_m^{-1}$. By (2.12), $\mathcal{V}(f) = \oplus_{l=1}^m \mathcal{V}(\exp(\frac{2\pi il}{m+1}))$. $U = Gr_0^W U$ is pure of weight $w = 0$ and

$$\mathrm{Spp}(f) = \sum_{l=1}^m h_{\exp(\frac{2\pi il}{m+1})}^{0,0}\left(-\frac{l}{m+1}, 0\right) = \sum_{l=1}^m \left(-\frac{l}{m+1}, 0\right).$$

5.9. In Section 6, we compare the spectrum pairs and the μ-type and signature-type invariants of f. These are defined as follows: $\mu_\pm(\lambda)(f) = \mu_\pm(b; U_\lambda)$, $\sigma_\lambda(f) = \mu_+(\lambda)(f) - \mu_-(\lambda)(f)$ for $\lambda \in S^1$; and $\mu_0(f) = \mu_0(b)$, $\mu_\pm(f) = \Sigma_\lambda \mu_\pm(\lambda)(f)$, $\sigma(f) = \Sigma_\lambda \sigma_\lambda(f)$.

6. Topology and Hodge structure

Let $f\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be an isolated hypersurface singularity. The connection between the topological invariant $\mathcal{V}(f)$ and the Hodge theoretical invariants $p_\lambda^{a,b}(f)$ is given in the following

6.1. THEOREM.

$$\mathcal{V}(f) = \bigoplus_{\lambda} \bigoplus_{2n \geqslant a+b \geqslant n+s} p_\lambda^{a,b}(f) \mathcal{W}_\lambda^{r+1}((-1)^b),$$

where $s = 0$ if $\lambda \neq 1$, $s = 1$ if $\lambda = 1$ and $r = a + b - n - s \geqslant 0$. In particular, the Hodge numbers determine the real Seifert form.

Proof. Let $\lambda \neq 1$. Then by (5.6.ii) one has

$$\bigoplus_{l=0}^{r} {}_cN_\lambda^l({}_cP_\lambda^{a,b}) = {}_cp_\lambda^{a,b}\mathcal{V}_\lambda^{r+1}((-1)^b).$$

Now, since

$$U_\lambda = \bigoplus_{2n \geqslant a+b \geqslant n} \bigoplus_{l=0}^{n} {}_cN_\lambda^l({}_cP_\lambda^{a,b}),$$

the result follows for these λ-components from the polarization properties.

Suppose $\lambda = 1$. Consider the germ $(z, z_{n+1}) \mapsto f(z) + z_{n+1}^{m+1} : (\mathbf{C}^{n+2}, 0) \to (\mathbf{C}, 0)$. Then, by a result of Deligne (see [2]) (which solves the Sebastiani–Thom problem at variation-map level) (or equivalently, by a result of Sakamoto [8] which solves the Sebastiani–Thom problem at Seifert matrix level):

$$\mathcal{V}(f + z_{n+1}^{m+1}) = (-1)^{n+1}\mathcal{V}(f) \otimes \mathcal{V},$$

where $\mathcal{V}$ is associated with $-\Gamma_m^{-1}$. The HVS associated with $(z \mapsto z^{m+1})$ is $(+1)$-symmetric, hence by (2.12) $\mathcal{V} = \oplus_{\xi^{m+1}=1, \xi \neq 1}\mathcal{V}(\xi)$. Consider m so that the monodromies h_f and $h_{(z \mapsto z^{m+1})}$ have no common eigenvalues. Then:

$$\mathcal{V}_\xi(f + z_{n+1}^{m+1}) = (-1)^{n+1}\mathcal{V}_1(f) \otimes \mathcal{V}(\xi). \qquad (*)$$

Moreover, by the solution of the Sebastiani–Thom problem at the mixed Hodge structure level [9]:

$$h_\xi^{p,q}(f + z_{n+1}^{m+1}) = h_1^{p,q}(f)h_\xi^{0,0} = h_1^{p,q}(f) \qquad (**)$$

(since $h_\xi^{0,0} = 1$ by (5.8)). Now, $\mathcal{V}_1(f) \otimes \mathcal{V}(\xi) =$

$$= (-1)^{n+1}\mathcal{V}_\xi(f + z_{n+1}^{m+1}) \qquad \text{by } (*)$$

$$= (-1)^{n+1}\Sigma_{a+b \geqslant n+1} p_\xi^{a,b}(f + z_{n+1}^{m+1}) \cdot {}_{(-1)^{n+1}}\mathcal{V}_\xi^{r+1}((-1)^b) \qquad \text{by the case } \lambda \neq 1$$

$$= \Sigma_{a+b \geqslant n+1} p_1^{a,b}(f) \cdot (-1)^{n+1} \cdot {}_{(-1)^{n+1}}\mathcal{V}_\xi^{r+1}((-1)^b) \qquad \text{by } (**) \text{ and } (5.4)$$

$$= \Sigma_{a+b \geqslant n+1} p_1^{a,b}(f)\tilde{\mathcal{V}}_1^k((-1)^b) \otimes \mathcal{V}(\xi) \qquad \text{by } (2.12).$$

Since $\cdot \otimes \mathcal{V}(\xi)$ is one-to-one, the result follows. □

6.2. EXAMPLE (The case of quasi-homogeneous polynomials). Let $f : \mathbf{C}^{n+1} \to \mathbf{C}$ be a quasi-homogeneous polynomial of type $(w_0, \ldots, w_n)$ with isolated singularity at the origin. Let $\{z^\alpha \,|\, \alpha \in \mathcal{I} \subset \mathbf{N}^{n+1}\}$ be a set of monomials in $\mathbf{C}[z]$ whose residue classes form a bases for the Milnor algebra $\mathbf{C}[[z]]/(\partial f)$. For $\alpha \in \mathcal{I}$ let $l(\alpha) = \Sigma_{i=0}^{n}(\alpha_i + 1)w_i$. By [12]:

$$\sum_{\substack{a+b=n+s \\ (-1)^b=\pm 1}} h_\lambda^{a,b} = \#\{\alpha \in \mathcal{I},\ \lambda = \exp(2\pi i l(\alpha)),\ (-1)^{[l(\alpha)]} = \pm 1\}.$$

Therefore, our result becomes:

6.3. THEOREM. *If f is as in (6.2), then:*

$$\mathcal{V}(f) = \bigoplus_{\alpha \in \mathcal{I}} \mathcal{W}^1_{\exp(2\pi i l(\alpha))}((-1)^{[l(\alpha)]})$$

where $[\cdot]$ denotes integral part.

This, in particular, determines the dimension of the null-space $\mu_0 = \#\{\alpha \in \mathcal{I},\ l(\alpha) \in \mathbf{Z}\}$, too, as proved in [12]. But our result gives a supplementary connection between the topological invariants of the singularity and the combinatorics of the lattice points; (i.e. it gives significance to the parity of $l(\alpha)$ when $l(\alpha) \in \mathbf{Z}$).

6.4. Theorem 6.1 can be formulated in terms of spectral pairs as follows.

The projection $\mathbf{Q} \times \mathbf{N} \to (\mathbf{Q}/2\mathbf{Z}) \times \mathbf{N}$ induces a natural map

$$pr_{\text{mod-2}}:\ \mathbf{Z}[\mathbf{Q} \times \mathbf{N}] \to \mathbf{Z}[(\mathbf{Q}/2\mathbf{Z}) \times \mathbf{N}]$$

at the level of the free abelian groups generated by $\mathbf{Q} \times \mathbf{N}$ resp. $(\mathbf{Q}/2\mathbf{Z}) \times \mathbf{N}$.

DEFINITION. The mod-2-*spectral* pairs of f are defined as the images of the spectral pairs by $pr_{\text{mod-2}}$. The element $pr_{\text{mod-2}}(\mathrm{Spp}(f))$ is denoted by $\mathrm{Spp}_{\text{mod-2}}(f)$.

6.5. THEOREM. *Giving the real Seifert form of an isolated hypersurface singularity is equivalent to giving the* mod-2-*spectral pairs of f.*

In fact, $\mathrm{Spp}_{\text{mod-2}}(f) = \mathrm{Spp}(\mathcal{V}(f))$, *where $\mathcal{V}(f)$ is the variation structure given by Seifert form $V(f)$ of f. Moreover:*

$$\mathbf{h}_\lambda^{w,u} = \sum_{\substack{a+b=w \\ (-1)^b=u}} h_\lambda^{a,b}.$$

Proof. Use theorem 6.1 and the corresponding definitions. □

6.6. COROLLARY. *For any $\lambda \in S^1$:*

$$\mu_\pm(\lambda)(f) = \sum_{2n \geqslant a+b \geqslant n+s} p_\lambda^{a,b}\left[\frac{2(a+b-n)+3\pm(-1)^b}{4} - s\right],$$

$$\sigma_\lambda(f) = \sum_{2n\geqslant a+b\geqslant n+s} p_\lambda^{a,b}(-1)^b \cdot \frac{1+(-1)^{a+b-n}}{2}, \quad \sigma_{\bar\lambda}(f) = (-1)^n\sigma_\lambda(f),$$

and

$$\mu_0(f) = \sum_{2n\geqslant a+b\geqslant n+1} p_1^{a,b}.$$

($[\cdot]$ *denotes integral part.*)

Proof. Use (4.2): $\mu_u(\mathcal{W}_\lambda^k(v)) = [\frac{2k+1-2s+uv}{4}]$, where u, $v = \pm 1$. □

6.7. EXAMPLES. The case $n = 1$ is: $\mu_+(\lambda) = h_\lambda^{1,1} + h_\lambda^{1,0}$, $\mu_-(\lambda) = h_\lambda^{1,1} + h_\lambda^{0,1}$, $\sigma_\lambda = h_\lambda^{1,0} - h_\lambda^{0,1}$ for $\lambda \neq 1$; and $\mu_0 = h_1^{1,1}$. In this case $\mu_\pm(1) = 0$. But for $n \geqslant 2$, the invariants $\mu_\pm(1)$ might be non-trivial.

If $n = 2$, then by computation:

$\mu_+(\lambda) = 2h_\lambda^{2,2} + h_\lambda^{2,1} + h_\lambda^{1,2} + h_\lambda^{0,2} + h_\lambda^{2,0}$, $(\lambda \neq 1)$;
$\mu_-(\lambda) = h_\lambda^{2,1} + h_\lambda^{1,2} + h_\lambda^{1,1}$, $(\lambda \neq 1)$;
$\sigma_\lambda = 2h_\lambda^{2,2} + h_\lambda^{0,2} + h_\lambda^{2,0} - h_\lambda^{1,1}$, $(\lambda \neq 1)$;
$\mu_+(1) = h_1^{2,2}$; $\mu_-(1) = 0$ and $\mu_0 = h_1^{2,2} + h_1^{2,1} + h_1^{1,2}$.

Using these and the relations $h_\lambda^{p,q} = h_{\bar\lambda}^{q,p}$, we get:

(a) $\mu_0 + \mu_+ = 2(h^{2,2} + h^{2,1} + h^{2,0})$
(b) $\sigma = 2h^{2,2} + h^{0,2} + h^{2,0} - h^{1,1}$.

(These relations for $n = 2$ were proved for globalisable smoothings of normal singularities by J. Wahl; for complete intersection by A. Durfee; and for smoothings of isolated singularities by J. Steenbrink.)

The sum $\Sigma_k h^{2,k}$ in the right hand side of (a) is $\dim Gr_F^2 U^*$, and it can be identified with the geometric genus p_g of the singularity f [13].

6.8. EXAMPLE. Theorem 6.5 for $n = 1$ and $n = 2$ can be formulated as follows.

If $n = 1$ then $\alpha \in (-1, 1)$ for any spectral pair (α, w). In conclusion, the real Seifert form is equivalent to the spectral pairs. This result is proved in [10] by the complete computation of the spectral pairs of curve singularities in terms of the resolution graph and using a characterization of the Seifert form of curve singularities, given by Neumann (cf. 6.15).

Assume $n = 2$. Let $\mathrm{Spp}_I(f) = \{\Sigma(\alpha, w);\ (\alpha, w)$ is a spectral pair with $\alpha \in I\}$ and $\#\mathrm{Spp}_I(f)$ its cardinality. Then, by Theorem 6.5, $\mathrm{Spp}_{[0,1]}(f)$ is completely determined by the real Seifert form. In particular its cardinality must be a remarkable invariant of the Seifert form. Indeed, suppose that $\alpha \in (0, 1)$. Take $\lambda = \exp(-2\pi i\alpha)$. Since $p = n + [-\alpha] = 1$, by (6.7): $\#\{\alpha|\alpha$ is spectrum number with $\lambda = \exp(-2\pi i\alpha)$ and $\alpha \in (0, 1)\} = h_\lambda^{1,0} + h_\lambda^{1,1} + h_\lambda^{1,2} = h_\lambda^{2,1} + h_\lambda^{1,1} + h_\lambda^{1,2} = \mu_-(\lambda)$. Since $\mu_-(1) = 0$ (again by 6.7), we get that:

$$\#\mathrm{Spp}_{(0,1)}(f) = \mu_-, \quad \text{and}$$

$$\#\mathrm{Spp}_{(-1,0]}(f) = \tfrac{1}{2}(\mu - \mu_-) = \tfrac{1}{2}(\mu_0 + \mu_+) = p_g.$$

Moreover, $\#\mathrm{Spp}_{\{0\}}(f) = \#\mathrm{Spp}_{\{1\}}(f) = \frac{1}{2} \cdot \dim U_1$.

6.9. In fact, for any n, 4.2 implies the following relation between spectrum *numbers* and μ-invariants. Denote: $\mathrm{Sp}_\lambda(\pm 1)(f) = \#\{\alpha | \alpha$ is a spectral number of f with $e^{-2\pi i\alpha} = \lambda$, and $(-1)^{[\alpha]} = \pm 1\}$. Then by (4.2), one has:

PROPOSITION. *Suppose* $\lambda \neq 1$. *Then* $\mathrm{Sp}_\lambda(\pm 1)(f) = \mu_\mp(\lambda)(f)$, *in particular,* $\sigma_\lambda(f) = \mathrm{Sp}_\lambda(-1)(f) - \mathrm{Sp}_\lambda(+1)(f)$.

6.10. EXAMPLE. Let $f_I = f_{-1,-1;1,1}(x, y) + z^2$ and $f_{II} = f_{-1,1;-1,1} + z^2$, where $f_{k,l;m,n} = ((y - x^2)^2 - x^{5+k})((y + x^2)^2 - x^{5+l})((x - y^2)^2 - y^{5+m})((x + y^2)^2 - y^{5+n})$.

Then f_I and f_{II} have the same spectral numbers [10], in particular the same equivariant signatures for $\lambda \neq 1$. But their spectral pairs differ [10], the nonequal pairs are: $(0, 3)$ and $(1, 1)$ for the first germ, $(0, 2)$ and $(1, 2)$ for the second one. In particular, their signatures differ: $\sigma_I = \sigma_{II} + 1$.

6.11. In the end of this section we discuss some properties of variation structures which are satisfied by the Seifert form of the isolated singularities.

Let $\#\mathcal{V}$ be the number of $\mathcal{V}$-components in $\mathcal{V}(f)$.

There are several obstructions of the decomposition of $\mathcal{V}(f)$. The first is the stability of $\mathcal{V}(f)$ with respect to the complex conjugation. Using either (2.2.8), or the symmetry of the Hodge numbers, we get:

$$\begin{aligned} \#\mathcal{W}_\lambda^{r+1}(\pm 1) &= \sum_{(-1)^b = \pm 1} p_\lambda^{a,b} = \sum_{(-1)^a = \pm(-1)^{n+r+s}} p_{\bar\lambda}^{b,a} \\ &= \#\mathcal{W}_{\bar\lambda}^{r+1}(\pm(-1)^{n+r+s}), \end{aligned}$$

(where $a + b - n - s = r$).

6.12. Now, since $a \leqslant n$ and $b \leqslant n$, $\mathcal{V}(f)$ determines the numbers $p_\lambda^{a,b}$ where $(a, b) = (n, n)$, $(n, n-1)$, $(n-1, n)$ and $(n-1, n-1)$ (with $a + b \geqslant n + s$.) For these pairs:

$$p_\lambda^{a,b} = \#\mathcal{W}_\lambda^{a+b-n-s+1}((-1)^b). \tag{6.13}$$

In particular, for $n = 1$, the system of Hodge numbers is completely determined by $\mathcal{V}(f)$. For $n = 2$, only $\{p_1^{a,b}\}_{a,b}$; $p_\lambda^{2,2}$; $p_\lambda^{2,1}$; $p_\lambda^{1,2}$; $p_\lambda^{1,1}$ and the sum $p_\lambda^{0,2} + p_\lambda^{2,0} = \#\mathcal{V}_\lambda^1(+1)$ are determined by $\mathcal{V}(f)$.

Our main obstruction, as a consequence of (6.13), is:

6.14. PROPOSITION. *The structures* $\mathcal{W}_\lambda^{n+1-s}((-1)^{n+1})$ *do not appear in the decomposition of* $\mathcal{V}(f)$ *for any isolated singularity* $f\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$.

This obstruction is nontrivial even for $n = 1$: $\mathcal{V}_\lambda^2(+1)$, $\lambda \neq 1$ and $\tilde{\mathcal{V}}_1^1(+1)$ cannot be components of an algebraic Seifert form. Both cases ($n = 1$; $s = 0$ and $s = 1$) were proved by Neumann [7] using the splice geometry of curve singularities.

6.15. EXAMPLE. Let us describe the possible decomposition when $n = 1$:

$$\mathcal{V}_1(f) = h_1^{1,1}\tilde{\mathcal{V}}_1^1(-1) \text{ and for } \lambda \neq 1:$$

$$\mathcal{V}_\lambda(f) = h_\lambda^{1,1}\mathcal{V}_\lambda^2(-1) + h_\lambda^{1,0}\mathcal{V}_\lambda^1(+1) + h_\lambda^{0,1}\mathcal{V}_\lambda^1(-1).$$

By the notation of (4.4): $h_1^{11} = n_1^{(1)}$, $h_\lambda^{1,1} = n_\lambda^{(2)} - n_\lambda^{(1)}$, $h_\lambda^{1,1} + h_\lambda^{1,0} + h_\lambda^{0,1} = n_\lambda^{(1)}$, $\sigma_\lambda^{(1)} = h_\lambda^{1,0} - h_\lambda^{0,1}$. Therefore, $n_\lambda^{(1)}$, $n_\lambda^{(2)}$, $\sigma_\lambda^{(1)}$ determine $\mathcal{V}(f)$. This is Neumann's characterization of the real Seifert form in terms of the characteristic polynomials Δ and Δ^1 and the equivariant signatures σ_λ^-.

If $n = 2$ then the Hodge numbers $h_\lambda^{p,q}$ $(p + q \geqslant 3)$ and $h_\lambda^{1,1}$ are determined by the real Seifert form, in particular, by the numbers $\{n_\lambda^{(1)}, n_\lambda^{(2)}, n_\lambda^{(3)}, \sigma_\lambda^{(1)}, \sigma_\lambda^{(2)}\}$. (But the number $h_\lambda^{0,2}$ is not.)

In general, the obstruction (6.14) implies $\sigma_\lambda^{(n+1)} = (-1)^n n_\lambda^{(n+1)}$, therefore the real Seifert form is determined by $n_\lambda^{(k)}$ $(1 \leqslant k \leqslant n+1)$ and $\sigma_\lambda^{(k)}$ $(1 \leqslant k \leqslant n)$. (cf. 4.5)

References

1. Arnold, V. I., Gusein-Zade, S. M. and Varchenko, A. N.: *Singularities of Differentiable Mappings*, Vol. 2, Birkhauser, Boston, 1988.
2. Demazure, M.: Classification des germes a point critique isolé et a nombres de modules 0 ou 1 (d'apres Arnold). *Sém. Bourbaki* 26^e année 1973/74, 443 (1974).
3. Durfee, A. and Kaufman, L.: Periodicity of branched cyclic covers. *Math. Ann.* 218 (1975) 157–174.
4. Milnor, J.: On Isometries of Inner Product Spaces. *Inventiones Math.* 8 (1969) 83–97.
5. Némethi, A.: The equivariant signature of hypersurface singularities, will appear in *Topology*.
6. Némethi, A.: The semi-ring structure and the spectral pairs of sesqui-linear forms, *Algebra Colloq.* 1:1 (1994) 85–95.
7. Neumann, W. D.: Invariants of plane curve singularities. *Monographie N° 31 de L'Enseignement Math.* (1983).
8. Sakamoto, K.: Milnor fiberings and their Characteristic Maps. *Proc. Intern. Conf. on Manifolds and Related topics in Topology.* Tokyo (1973).
9. Scherk, J. and Steenbrink, J. H. M.: On the Mixed Hodge Structure on the Cohomology of the Milnor Fibre. *Math. Ann.* 271 (1985) 641–665.
10. Schrauwen, R. and Steenbrink, J. and Stevens, J.: Spectral Pairs and the Topology of Curve Singularities. *Proc. of Symposia in Pure Math.* 53 (1991) 305–327.

11. Steenbrink, J. H. M.: Mixed Hodge Structures on the Vanishing Cohomology. *Nordic Summer School/NAVF, Symposium in Math.* Oslo (1976).
12. Steenbrink, J. H. M.: Intersection form for quasi-homogeneous singularities. *Comp. Math.* 34 (1977) 211–233.
13. Steenbrink, J. H. M.: Mixed Hodge structures associated with isolated singularities. *Proc. of Symp. in Pure Math.* 40 Part 2 (1983).