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A. VARCHENKO

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Critical points of the product of powers of linear functions and families of bases of singular vectors

A. VARCHENKO*
Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599, USA
E-mail address: varchenko@math.unc.edu

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Abstract. The quasiclassical asymptotics of the Knizhnik–Zamolodchikov equation with values in the tensor product of sl_2 representations are considered. The first term of asymptotics is an eigenvector of a system of commuting operators. We show that the norm of this vector with respect to the Shapovalov form is equal to the determinant of the matrix of second derivatives of a suitable function. This formula is an analog of the Gaudin and Korepin formulae for the norm of the Bethe vectors. We show that the eigenvectors form a basis under certain conditions.

Introduction

Consider the Lie algebra sl_2 with standard generators e, f, h such that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Let L^n be the $(n+1)$-dimensional irreducible sl_2 module. The module is generated by its singular element v_n such that $ev_n = 0$ and $hv_n = nv_n$. The elements $v_n, fv_n, \dots, f^n v_n$ form a basis of L^n. The Shapovalov form on L^n is the bilinear form B^n such that

$$B^n(f^k v_n, f^k v_n) = k!n!/(n-k)!, \qquad B^n(f^k v_n, f^l v_n) = 0 \quad \text{for } k \neq l.$$

The tensor product of irreducible representations is direct sum of irreducible representations: $L^n \otimes L^m = L^{m-n} \oplus L^{m-n+2} \oplus \cdots \oplus L^{m+n}$ for $m \geqslant n$, and a singular vector of L^{m+n-2k} has the form

$$\omega_{m+n-2k} = \sum_{p=0}^{k} (-1)^p \binom{k}{p} \frac{\prod_{j=0}^{k-1}(m+n-2k+j+2)}{\prod_{j=0}^{p-1}(n-j) \cdot \prod_{j=0}^{k-p-1}(m-j)} \cdot f^p v_n \otimes f^{k-p} v_m.$$

Let $B = B^n \otimes B^m$ be the Shapovalov form on the tensor product.

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Consider the rational function

$$\Phi(t) = \prod_{j=1}^{k} t_j^{-n}(t_j - 1)^{-m} \prod_{1 \leqslant j < i \leqslant k} (t_i - t_j)^2.$$

Let $t^0 = (t_1, \ldots, t_k)$ be a critical point of Φ such that $t_i \neq t_j$ for $i \neq j$.

THEOREM.

$$B(\omega_{n+m-2k}, \omega_{n+m-2k}) = \det\left(\frac{\partial^2}{\partial t_i\, \partial t_j} \ln \Phi(t^0)\right).$$

The formula is an analog of the Gaudin and Korepin formulae for the norm of the Bethe vectors in the theory of quantum integrable models [Ga, K]. We prove the formula and its generalizations.

The main results of this work are Theorem (1.2.1), Corollary (2.4.6), and Theorem (2.5.1).

This work is inspired by [RV] in which connections between the quasiclassical asymptotics of solutions of the Knizhnik–Zamolodchikov equation and the Bethe ansatz vectors for the Gaudin model are explained.

The author thanks R. Askey, I. Cherednik, N. Reshetikhin, and V. Tarasov for stimulating and very useful discussions.

1. Critical points

(1.1) Conjecture

Let $f_j: \mathbb{C}^k \to \mathbb{C}$, $j = 1, \ldots, N$, be pairwise different polynomials of degree 1. For every i denote by H_i the hyperplane in $\mathbb{C}^k$ defined by $f_j = 0$. Let $\mathcal{C} = \{H_j\}_{j=1}^N$ be the configuration of the hyperplanes,

$$T = \mathbb{C}^k - \bigcup_{j=1}^{N} H_j$$

the complement to the union of hyperplanes. Let $\Lambda = \{\lambda_j\}_{j=1}^N$ be a collection of complex numbers. Consider a function

$$\Phi_\Lambda = \prod_{j=1}^{N} f_j^{\lambda_j}. \tag{1.1.1}$$

Φ_Λ is a multivalued holomorphic function on T. A point $t \in T$ is critical for Φ_Λ if its first derivatives vanish at t. First derivatives vanish at t for all branches of Φ_Λ simultaneously, since the ratio of every two branches is constant.

Assume that the configuration has a vertex.

CONJECTURE. *For generic Λ all critical points of Φ_Λ are nondegenerate and the number of critical points is equal to the absolute value of the Euler characteristic of T.*

The conjecture is proved below for the case in which all polynomials $\{f_j\}$ have real coefficients.

REMARKS.

(a) An edge of a configuration is a nonempty intersection of some of its hyperplanes. A vertex is a zero dimensional edge.

(b) Generic Λ means that there exists an algebraic subset $\Sigma \subset \mathbb{C}^N$ such that the conjecture is true if $\Lambda \in \mathbb{C}^N - \Sigma$.

(c) The defining equations for critical points of Φ_Λ have the form:

$$\sum_{j=1}^{N} \lambda_j \, \frac{\partial f_j}{\partial t_l} \Big/ f_j = 0, \quad l = 1, \ldots, k.$$

(d) According to [OS], the Euler characteristic $\chi(T)$ is defined combinatorially in terms of the lattice of edges of the configuration:

$$|\chi(T)| = \left| \sum_{E \subset \mathcal{C}} \mu(E) \right|,$$

where the sum is over all edges of $\mathcal{C}$, $\mu(E)$ is the value of the Mobius function of $\mathcal{C}$ at E.

(e) There are theorems on Newton polygons in which one considers a polynomial system of equations depending on parameters. Under certain conditions on the system, the number of solutions of the system for generic values of parameters is defined combinatorially in terms of Newton polygons of equations, see [BKK]. It would be interesting to find a connection between those theorems and the statement of the conjecture.

(f) A multidimensional hypergeometric integral is an integral of the form

$$\int_\gamma \Phi_\Lambda R \, dt_1 \wedge \cdots \wedge dt_k,$$

where $R: T \to \mathbb{C}$ is a rational function and $\gamma \subset T$ is a suitable cycle. If the polynomials $\{f_j\}$ depend on additional parameters, such an integral becomes a function of additional parameters called a multidimensional hypergeometric function, see [A, G, V]. Multidimensional hypergeometric functions satisfy remarkable differential equations. For example, the Knizhnik–Zamolodchikov equation in conformal field theory is solved in hypergeometric functions [SV]. In the application to the KZ equation the exponents $\{\lambda_j\}$ have the form $\{\lambda_j = \alpha_j / \kappa\}$ where κ is a parameter of the equation. Studying asymptotics of solutions of the KZ equation as κ tends to 0 leads to studying critical points of the function Φ_Λ. This problem motivated the conjecture.

(g) If the configuration has no vertices, then there exist linear coordinates $u_1, \dots, u_k$ in $\mathbb{C}^k$ such that all polynomials $\{f_j\}$ do not depend on $u_1, \dots, u_r$ for some r and the configuration, cut by $\mathcal{C}$ in $u_1 = \dots = u_r = 0$, has a vertex.

(1.2) Real Configuration

Assume that all polynomials $\{f_j\}$ have real coefficients:

$$f_j = a_j^0 + a_j^1 t_1 + \cdots a_j^k t_k, \quad j = 1, \dots, N,$$

and all numbers $\{a_j^m\}$ are real.

Let $T_{\mathbb{R}} = T \cap \mathbb{R}^k$. Let $T_{\mathbb{R}} = \cup_\alpha D_\alpha$ be the decomposition into the union of connected components. Each component is a convex polytope.

By [BBR], the number of bounded components is given by the formula

$$\# = \left| \sum_{E \subset \mathcal{C}} \mu(E) \right|.$$

(1.2.1) THEOREM. *Let all numbers $\{\lambda_j\}$ be positive. Then the union of all critical points of Φ_Λ is contained in the union of all bounded components of $T_{\mathbb{R}}$. Each bounded component contains exactly one critical point. All critical points are non-degenerate.*

The theorem implies the conjecture for the case in which all polynomials $\{f_j\}$ are real.

Proof. First we formulate a trivial but useful lemma.

Let t be the complex coordinate on $\mathbb{C}$, and r, ϕ the polar coordinates on $\mathbb{C}$, $t = r\,\exp(i\phi)$. Let $v = a\partial/\partial r + b\partial/\partial\varphi$ be a tangent vector at a point $t = t^0$ in $\mathbb{C} - 0$. Denote by $L_v \ln t$ the derivative of $\ln t$ along v.

(1.2.2) LEMMA. *If $a > 0$, then* $\mathrm{Re}(L_v \ln t) > 0$.

The theorem is implied by the following three lemmas.

(1.2.3) LEMMA. *Let D be an unbounded component of $T_{\mathbb{R}}$. Then there are no critical points in D.*

Proof. Let $p \in D$. By [BBR] there exists a vector $v \in \mathbb{R}^k$ such that the ray $p(s) = p + sv$, $s \in \mathbb{R}_{\geqslant 0}$, has no intersection with the union of hyperplanes of $\mathcal{C}$. By Lemma (1.2.2) we have

$$\mathrm{Re}(L_v(\ln \Phi_\Lambda)(p)) > 0. \tag{1.2.4}$$

Hence p is not critical.

(1.2.5) LEMMA. *Let $p \in T - \mathbb{R}^k$. Then p is not critical.*

Proof. Let $p = w + iv$ where $w,\ v \in \mathbb{R}^k$. The ray $p(s) = p + isv,\ s \in \mathbb{R}_{\geqslant 0}$, has no intersection with the union of hyperplanes of $\mathcal{C}$. By Lemma (1.2.2) we have inequality (1.2.4). Hence p is not critical.

(1.2.6) LEMMA. *Let D be a bounded component of $T_{\mathbb{R}}$. Then D contains exactly one critical point, and this critical point is non-degenerate.*

Proof. We have an equality in D:

$$\ln \Phi = S + \text{const}.$$

where $S = \Sigma \lambda_j \ln|f_j|$. Hence Φ_Λ and S have the same critical points in D. We have $S(p) \to -\infty$ as $p \to \partial D$. So S has a critical point in D. The critical point is unique since S is convex. The critical point is non-degenerate because the matrix of second derivatives of S is positive definite.

(1.3) Example

Let

$$\Phi(t) = \Phi(t;\ \alpha,\ \beta,\ \gamma) = \prod_{j=1}^{k} t_j^{\alpha}(1 - t_j)^{\beta} \prod_{1 \leqslant j < i \leqslant k} (t_i - t_j)^{2\gamma},$$

where $\alpha,\ \beta,\ \gamma$ are complex parameters. We describe critical points of Φ.

The critical set of Φ is invariant with respect to the group of permutations of coordinates. By Theorem (1.2.1) the number of critical points is not greater than $k!$.

Let $\lambda_1 = t_1 + \cdots + t_k,\ \lambda_2 = \Sigma t_i t_j, \ldots, \lambda_k = t_1 \cdot \cdots \cdot t_k$ be the standard symmetric functions. Let $\mu_1 = (1 - t_1) + \cdots + (1 - t_k),\ \mu_2 = \Sigma(1 - t_i)(1 - t_j), \ldots, \mu_k = (1 - t_1) \cdot \cdots \cdot (1 - t_k)$, $\delta = \prod_{1 \leqslant i < j \leqslant k}(t_i - t_j)^2$.

(1.3.1) THEOREM. *If $(t_1, \ldots, t_k)$ is a critical point of Φ, then*

$$\lambda_l = \binom{k}{l} \prod_{j=1}^{l} \frac{\alpha + (k-j)\gamma}{\alpha + \beta + (2k - j - 1)\gamma}, \tag{1.3.2}$$

$$\mu_l = \binom{k}{l} \prod_{j=1}^{l} \frac{\beta + (k-j)\gamma}{\alpha + \beta + (2k - j - 1)\gamma},$$

for all l.

Proof. The defining equations for critical points are

$$\frac{\alpha}{t_i} + \frac{\beta}{t_i - 1} + \sum_{j \neq i} \frac{2\gamma}{t_i - t_j} = 0, \quad i = 1, \ldots, k.$$

Multiplying the ith equation by $t_i/t_1 \cdot \cdots \cdot t_k$ and taking the sum of the equations we get

$$\frac{k(\alpha+(k-1)\gamma)}{t_1\cdot \cdots \cdot t_k}+\beta\left(\frac{1}{t_1\cdot \cdots \cdot t_{k-1}(t_k-1)}+\cdots+\frac{1}{(t_1-1)t_2\cdot \cdots \cdot t_k}\right)=0.$$

Similarly, for every $p=0,\dots,k-1$, we get

$$(k-p)(\alpha+(k-p-1)\gamma)\sum_{1\leqslant i_1<\cdots<i_p\leqslant k}\frac{1}{(t_{i_1}-1)}\cdots\frac{1}{(t_{i_p}-1)}\prod_{j\notin(i_1,\dots,i_p)}\frac{1}{t_j}$$

$$+(p+1)(\beta+p\gamma)\sum_{1\leqslant i_1<\cdots<i_{p+1}\leqslant k}\frac{1}{(t_{i_1}-1)}\cdots\frac{1}{(t_{i_{p+1}}-1)}\prod_{j\notin(i_1,\dots,i_{p+1})}\frac{1}{t_j}=0.$$

This system of equations implies

$$(k-p)(\alpha+(k-p-1)\gamma)\sum_{j=0}^{k}(-1)^j\binom{k-j}{p}\lambda_{k-j}$$

$$+(p+1)(\beta+p\gamma)\sum_{j=0}^{k}(-1)^j\binom{k-j}{p+1}\lambda_{k-j}=0, \qquad (1.3.3)$$

for $p=0,\dots,k-1$. Here $\lambda_0=1$.

(1.3.4) LEMMA. *System (1.3.3) is equivalent to the system:*

$$(p+1)(\alpha+p\gamma)\lambda_{k-p-1}=(k-p)(\alpha+\beta+(k+p-1)\gamma)\lambda_{k-p},$$

for $p=0,\dots,k-1$.

Lemma (1.3.4) is proved by induction on p. Lemma (1.3.4) proves Theorem (1.3.1).

REMARK. After this work was written Prof. R. Askey informed me that Theorem (1.3.1) could be deduced from T.J. Stieltjes' Theorem [Sz, Th. 6.7.1] proved more than hundred years ago: if $(t_1,\dots,t_k)$ is a critical point of Φ, then $t_1,\dots,t_k$ are zeros of the Jacobi polynomial $P_k^{(a,b)}(2t-1)$, where $a=\alpha/\gamma-1$, $b=\beta/\gamma-1$. This Stieltjes' Theorem was rediscovered many times, in particular, by I. Schur, C.L. Siegel, F. Calogero, ...

Set

$$A_{k-p,p}(t;\,\alpha,\,\beta,\,\gamma)=\sum_{1\leqslant i_1<\cdots<i_p\leqslant k}\frac{1}{(t_{i_1}-1)}\cdots\frac{1}{(t_{i_p}-1)}\prod_{j\notin(i_1,\dots,i_p)}\frac{1}{t_j},$$

for $p=0,\dots,k$.

(1.3.5) LEMMA. *If* $(t_1^0, \dots, t_k^0)$ *is a critical point of the function* $\Phi(t;\ \alpha,\ \beta,\ \gamma)$, *then*

$$A_{k-p,p}(t^0;\ \alpha,\ \beta,\ \gamma) = (-1)^p \binom{k}{p} \frac{\prod_{j=1}^{k}(\alpha+\beta+(2k-j-1)\gamma)}{\prod_{j=0}^{k-p-1}(\alpha+j\gamma)\cdot\prod_{j=0}^{p-1}(\beta+j\gamma)}.$$

The vector $A_k = (A_{k,0}, \dots, A_{0,k})$ has the following interpretation.

Consider the Lie algebra sl_2 with the standard generators $e,\ f,\ h$. For $\alpha \in \mathbb{C}$ let V_α be an sl_2 module with highest weight α, that is, the module V_α is generated by a vector v_α such that $ev_\alpha = 0$ and $hv_\alpha = \alpha v_\alpha$. For $\alpha,\ \beta \in \mathbb{C}$ consider the vector

$$F_k(\alpha,\ \beta) = A_{k,0}(t^0;\ \alpha,\ \beta,\ -1) f^k v_\alpha \otimes v_\beta + \cdots + A_{0,k}(t^0;\ \alpha,\ \beta,\ -1) v_\alpha \otimes f^k v_\beta$$

of the tensor product $V_\alpha \otimes V_\beta$. We have $hF_k = (\alpha+\beta-2k)F_k$.

COROLLARY OF LEMMA (1.3.5). *The vector* $F_k(\alpha,\ \beta)$ *is a singular vector:* $eF_k = 0$.

Explanations of this fact see in [RV] and in Section 2.

The Shapovalov form on V_α is the unique symmetric bilinear form B_α defined by the conditions:

$$B_\alpha(v_\alpha,\ v_\alpha) = 1, \quad B_\alpha(fx,\ y) = B_\alpha(x,\ ey),$$

for all $x,\ y \in V_\alpha$. Consider the bilinear form $B = B_\alpha \otimes B_\beta$ on $V_\alpha \otimes V_\beta$.

(1.3.6) LEMMA. *We have*

$$B(F(\alpha,\ \beta),\ F(\alpha,\ \beta)) = k! \prod_{l=0}^{k-1} \frac{(\alpha+\beta-2k+l+2)^3}{(\alpha-l)(\beta-l)}.$$

The proof easily follows from the formula

$$\prod_{l=0}^{k-1}(\alpha+\beta-2k+l+2) = \sum_{p=0}^{k}\binom{k}{p}\prod_{l=0}^{p-1}(\alpha-k+l+1) \prod_{l=0}^{k-p-1}(\beta-k+l+1),$$

and the formula can be proved by induction.

Denote the number $B(F(\alpha,\ \beta),\ F(\alpha,\ \beta))$ by $b(\alpha,\ \beta;\ k)$.

(1.4) Asymptotics of Selberg Integral

The Selberg formula

$$k! \int_\Delta \prod_{j=1}^{k} t_j^\alpha (1-t_j)^\beta \prod_{1\leqslant i<j\leqslant k} (t_i - t_j)^{2\gamma}\, dt_1 \wedge \cdots \wedge dt_k$$

$$= \prod_{l=0}^{k-1} \frac{\Gamma(\alpha + l\gamma + 1)\Gamma(\beta + l\gamma + 1)\Gamma((l+1)\gamma + 1)}{\Gamma(\alpha + \beta + (2k - l - 2)\gamma + 2)\Gamma(\gamma + 1)} \tag{1.4.1}$$

where $\Delta = \{t \in \mathbb{R}^k \,|\, 0 < t_1 < \cdots < t_k < 1\}$ (see [A, As, M, S]), has beautiful applications, in particular, in conformal field theory [DF].

Assume that $\alpha = a/\kappa$, $\beta = b/\kappa$, $\gamma = c/\kappa$ where a, b, c, κ are positive numbers and κ tends to zero. We compute asymptotics of both sides of formula (1.4.1).

The method of steepest descent gives the following asymptotics for the left hand side of (1.4.1):

$$lhs \sim k! \cdot (2\pi\kappa)^{k/2} \cdot \Phi(t^0;\, a/\kappa,\, b/\kappa,\, c/\kappa) \cdot \mathrm{Hess}(-S(t^0;\, a,\, b,\, c))^{-1/2},$$

where

$$S(t;\, a,\, b,\, c)) = \kappa \ln \Phi(t;\, a/\kappa,\, b/\kappa,\, c/\kappa)$$

$$= \sum_{j=1}^{k} (a \ln t_j + b \ln(1 - t_j)) + \sum_{1\leqslant i<j\leqslant k} 2c \ln(t_i - t_j),$$

t^0 is the critical point of S in Δ

$$\mathrm{Hess}(-S) = \det\left(-\frac{\partial^2 S}{\partial t_i\, \partial t_j}\right),$$

$$-\frac{\partial^2 S}{\partial t_i^2} = a\frac{1}{t_i^2} + b\frac{1}{(t_i - 1)^2} + 2c \sum_{j\neq i} \frac{1}{(t_i - t_j)^2},$$

$$-\frac{\partial^2 S}{\partial t_i\, \partial t_j} = -2c\frac{1}{(t_i - t_j)^2}.$$

The symmetric functions of coordinates $t_1^0, \ldots, t_k^0$ are given by formula (1.3.2) in which α, β, γ must be replaced by a, b, c, respectively.

Let $\lambda_k = t_1^0 \cdot \cdots \cdot t_k^0$, $\mu_k = (1 - t_1^0) \cdot \cdots \cdot (1 - t_k^0)$, $\delta = \prod_{i<j} (t_i^0 - t_j^0)^2$. Then

$$lhs \sim k! \cdot (2\pi\kappa)^{k/2} \cdot \lambda_k^{a/\kappa} \cdot \mu_k^{b/\kappa} \cdot \delta^{c/\kappa}\, \mathrm{Hess}(-S(t^0;\, a,\, b,\, c))^{-1/2}.$$

Asymptotics of the right hand side of (1.4.1) can be computed by the Stirling formula. Comparing both asymptotics we get the following formulae:

$$\delta = \prod_{l=0}^{k-1} \frac{(l+1)^{l+1} c^l (a+lc)^l (b+lc)^l}{(a+b+(2k-l-2)c)^{2k-l-2}}, \tag{1.4.2}$$

$$\mathrm{Hess}(-S(t^0;\, a,\, b,\, c)) = k! \prod_{l=0}^{k-1} \frac{(a+b+(2k-l-2)c)^3}{(a+lc)(b+lc)}. \tag{1.4.3}$$

Formula (1.4.3) is an example of the series of rather surprising formulae in which the determinant of a bilinear form is the product of very simple factors [V1-3, L, LS, SV, BV]. The list of such examples includes formula (1.4.1), the Legandre equation, the Vandermonde determinant, and many others.

I was informed by Prof. R. Askey that Formula (1.4.2) is due to Stieltjes, Hilbert, Schur, see [Sz, Th. 6.71], the formula gives the discriminant of the Jacobi polynomial.

Comparing formula (1.4.3) and Lemma (1.3.6) we get

(1.4.4) THEOREM. *Let $F(\alpha, \beta) \in V_\alpha \otimes V_\beta$ be the vector defined in Section (1.3), B the Shapovalov form on $V_\alpha \otimes V_\beta$. Then*

$$B(F(\alpha, \beta), F(\alpha, \beta)) = \mathrm{Hess}(-S(t^0, \alpha, \beta, -1)).$$

This theorem implies the theorem formulated in the introduction.

2. Families of bases of singular vectors

(2.1) KZ Equation

Consider the Lie algebra $\mathfrak{g} = sl_2$ with the generators e, f, h. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ generated by h, let $\alpha \in \mathfrak{h}^*$ be the simple root, $(\,,)$ the bilinear form on $\mathfrak{h}^*$ such that $(\alpha, \alpha) = 2$.

Denote by Ω the element $h \otimes h/2 + e \otimes f + f \otimes e \in \mathfrak{g} \otimes \mathfrak{g}$ corresponding to the Killing form.

Let $V_1, \dots, V_n$ be $\mathfrak{g}$ modules, $V = V_1 \otimes \cdots \otimes V_n$. For $i < j$ let $\Omega_{i,j}$ be the linear operator on V acting as Ω on $V_i \otimes V_j$ and as the identity operator on other factors.

The *Knizhnik–Zamolodchikov equation (KZ)* on an V-valued function $\Psi(z_1, \dots, z_n)$ is the system of equations

$$\kappa \frac{\partial \Psi}{\partial z_i} = H_i \Psi, \quad i = 1, \dots, n,$$

where κ is a parameter of the equation and

$$H_i = \sum_{j \neq i} \frac{\Omega_{i,j}}{z_i - z_j}.$$

For $\Lambda_1, \dots, \Lambda_n \in \mathfrak{h}^*$ let $V_1, \dots, V_n$ be highest weight $\mathfrak{g}$ modules with highest weights $\Lambda_1, \dots, \Lambda_n$ and highest weight vectors $v_1, \dots, v_n$, respectively.

For a nonnegative integer k let

$$(V)_k = \left\{ v \in (V)_k \mid hv = \left(\sum_{i=1}^{n} \Lambda_i - k\alpha,\, \alpha \right) v \right\}$$

be the weight space and

$$\mathrm{Sing}(V)_k = \{ v \in (V)_k \mid ev = 0 \}$$

the subspace of singular vectors of weight k.

The KZ equation preserves the subspace of singular vectors.

Let B_i be the Shapovalov form on V_i. Denote by B the bilinear form $B_1 \otimes \dots \otimes B_n$ on V. If $V_1, \dots, V_n$ are irreducible, then B is nondegenerate. It is easy to see that the operators $H_1, \dots, H_n$ are symmetric:

$$B(H_i x,\, y) = B(x,\, H_i y)$$

for all i and all $x,\, y \in V$.

(2.2) Integral Representations

There is an integral representation for solutions of the KZ equation with values in $\mathrm{Sing}(V)_k$ [SV].

Set

$$\Phi(t,\, z) = \prod_{1 \leqslant i < j \leqslant n} (z_i - z_j)^{(\Lambda_i, \Lambda_j)/\kappa} \prod_{1 \leqslant j < i \leqslant k} (t_i - t_j)^{2/\kappa} \prod_{i,j} (t_i - z_j)^{-(\Lambda_j, \alpha)/\kappa}.$$

A *monomial* of weight k is an element of $(V)_k$ of the form

$$f_I = f^{i_1} v_1 \otimes \dots \otimes f^{i_n} v_n,$$

here $I = (i_1, \dots, i_n)$, $i_1 + \dots + i_n = k$.

For a monomial f_I, define a differential k-form in t and z:

$$\eta(f_I) \quad = A_I(t,\, z)\, \mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_k,$$

$$A_I(t,\, z) \;=\; \sum_{\sigma \in S(k, i_1, \dots, i_n)} \prod_{i=1}^{k} \frac{1}{(t_i - z_{\sigma(i)})}.$$

The sum is over the set $S(k,\, i_1, \dots, i_n)$ of maps σ from $\{1, \dots, k\}$ to $\{1, \dots, n\}$, such that for all m the cardinality of $\sigma^{-1}(m)$ is i_m.

Let $\mathcal{C} = \mathcal{C}(z_1, \dots, z_n)$ be the configuration of hyperplanes

$$t_i = t_j,\ 1 \leqslant i < j \leqslant k, \quad t_i = z_j,\ i = 1, \dots, k,\ j = 1, \dots, n,$$

in $\mathbb{C}^k$. Let $T(z_1, \dots, z_n)$ be the complement to the union of hyperplanes of $\mathcal{C}$ in $\mathbb{C}^k$.

Assume that $z_1, \dots, z_n$ are pairwise different real numbers. Let $D(z_1, \dots, z_n)$ be a bounded component of $T(z_1, \dots, z_n) \cap \mathbb{R}^k$ continuously depending on z.

(2.2.1) THEOREM [SV]. *The function*

$$\Psi_D(z) = \sum_{f_I \in V_k} \int_{D(z)} \Phi(t, z) \cdot \eta(f_I) \cdot f_I$$

takes values in $\mathrm{Sing}(V)_k$ *and satisfies the KZ equation.*

REMARK. If the integrals diverge, then their value must be taken in the sense of analytic continuation with respect to parameters $\Lambda_1, \dots, \Lambda_n$, κ. If $z_1, \dots, z_n$ are not real, then $D(z)$ must be replaced by a k-cycle in $T(z)$ with coefficients in a suitable local system, see [SV, V5].

(2.3) Basis of Solutions

For $i = 1, \dots, n$, let V_i be the Verma module with highest weight Λ_i. This means that V_i is an infinite dimensional module generated by a vector v_i such that $hv_i = (\Lambda_i, \alpha)v_i$ and $ev_i = 0$.

Consider the set of solutions $\{\Psi_D\}$ where D ranges over all bounded components of $T - \mathbb{R}^k$. According to [SV], $\{\Psi_D\}$ generate all solutions of the KZ equation with values in $\mathrm{Sing}(V)_k$ for generic $\Lambda_1, \dots, \Lambda_n$, κ. Below we will give a formula for a suitable determinant which will make the above statement more explicit.

Assume that $z_1 < z_2 < \cdots < z_n$. We say that a bounded component $D(z) \in T(z) - \mathbb{R}^k$ is *admissible* if it lies in the cone $z_1 < t_1 < t_2 < \cdots < t_k$. Admissible components have the form

$$D(z) = \{t \in \mathbb{R}^k \mid z_1 < t_1 < \cdots < t_{k_2} < z_2 < \\ \cdots < z_{n-1} < t_{k_{n-1}+1} < \cdots < t_k < z_n\},$$

where $0 \leqslant k_2 \leqslant \cdots \leqslant k_n = k$.

A monomial $f_I \in (V)_k$ for $I = (0, i_2, \dots, i_n)$, $i_2 = \cdots + i_n = k$, will be called *admissible*.

The number of admissible components is equal to the number of admissible monomials. Denote this number by N.

Set

$$\bar{\Phi}(t, z) = \prod_{1 \leqslant j < i \leqslant k} (t_i - t_j)^{2/\kappa} \prod_{i=1}^{k} \prod_{j=1}^{n} (t_i - z_j)^{-(\Lambda_j, \alpha)/\kappa}.$$

Consider the determinant $\det(\int_{D(z)} \bar{\Phi}\cdot\eta(f_I))$ where $D(z)$ ranges over all admissible components and f_I ranges over all admissible monomials. We will give a formula for this determinant.

For any admissible domain D and for any function g of the form

$$(t_i - t_j)^{2/\kappa}, \quad 1 \leqslant i < j \leqslant k,$$

$$(t_i - z_j)^{-(\Lambda_j,\alpha)/\kappa}, \quad i = 1, \dots, k, j = 1, \dots, n, \tag{2.3.1}$$

fix a branch of g over D. This choice determines a branch of $\bar{\Phi}$ and, hence, a branch of the determinant.

For any such g and for any admissible D, let t be a point of the closure of D which is the most remote point from the hyperplane of singularities of g. The number $g(t)$ will be called the *extreme value* of g on D and will be denoted by $c(D, g)$.

For every g and D, the number $c(D, g)$ is equal to $(z_l - z_m)^{-(\Lambda_l,\alpha)/\kappa}$, or $(z_l - z_m)^{-(\Lambda_m,\alpha)/\kappa}$, or $(z_l - z_m)^{2/\kappa}$, for suitable l and m.

For any admissible monomial f_I, set

$$b(f_I) = \prod_{l=2}^{n} i_l! \kappa^{-i_l} \prod_{j=0}^{i_l-1} (-(\Lambda_l, \alpha) + j).$$

(2.3.2) THEOREM [V3]. *We have*

$$\det\left(\int_{D(z)} \bar{\Phi} \cdot \eta(f_I)\right) = \pm \prod_{D(z),g} c(D(z), g) \cdot \prod_{f_I} b(f_I)^{-1} \cdot$$

$$\cdot \prod_{i=0}^{k-1} \left(\frac{\Gamma\left(\frac{i+1}{\kappa} + 1\right)^{n-1} \prod_{l=1}^{n} \Gamma\left(\frac{-(\Lambda_l,\alpha)+i}{\kappa} + 1\right)}{\Gamma\left(\frac{1}{\kappa} + 1\right)^{n-1} \Gamma(\sum_{l=1}^{n} \frac{-(\Lambda_l,\alpha)}{\kappa} + \frac{2k-i-2}{\kappa} + 1)} \right)^{p_i}, \tag{2.3.3}$$

where $p_i = \binom{n+k-i-3}{k-i-1}$, *the first product is over all admissible domains and over all functions described in (2.3.1), the second product is over all admissible monomials.*

The sign $\pm$ in the formula depends on the choice of the orientation of domains, see [V1, V2].

It is easy to see that

$$\dim \operatorname{Sing}(V)_k = N$$

if $(\Lambda_1, \alpha) \neq 0, 1, \dots, k - 1$.

(2.3.4) COROLLARY. *Assume that* $\Lambda_1, \dots, \Lambda_n$, κ *are such that* $(\Lambda_1, \alpha) \neq 0, 1, \dots, k-1$, *and the right-hand side of (2.3.3) is well-defined and not equal to zero. Then* $\{\Psi_D\}$ *form a basis of solutions of the KZ equation with values in* $\mathrm{Sing}(V)_k$, *where* D *ranges over admissible components.*

(2.4) Quasiclassical Asymptotics

Assume that $\Lambda_1, \dots, \Lambda_n$ are such that $-(\Lambda_1, \alpha), \dots, -(\Lambda_n, \alpha)$ are positive. Assume that κ is positive and tends to zero. We will compute asymptotics of the basis $\{\Psi_D\}$, see Corollary (2.3.4).

For an admissible domain $D(z)$, let $t_D(z)$ be the unique critical point of Φ in $D(z)$, see Theorem (1.2.1). The point $t_D(z)$ depends on z, $\Lambda_1, \dots, \Lambda_n$ and does not depend on κ.

By the method of steepest descent we have an asymptotic expansion

$$\Psi_D(z) = (2\pi\kappa)^{k/2} \cdot \Phi(t_D(z), z) \cdot \mathrm{Hess}_t(-S(t_D(z), z))^{-1/2} \cdot$$

$$\cdot (F(t_D(z), z) + O(\kappa)), \tag{2.4.1}$$

where $S = \kappa \ln \Phi$

$$F(t, z) = \sum_{f_I \in (V)_k} A_I(t, z) \cdot f_I,$$

the sum is over all monomials in $(V)_k$.

(2.4.2) THEOREM [RV]. *The vector* $F(t_D(z), z)$ *lies in* $\mathrm{Sing}(V)_k$. *For any* $l = 1, \dots, n$, *the vector* $F(t_D(z), z)$ *is an eigenvector of the operator* $H_l(z)$ *with eigenvalue* $\frac{\partial S}{\partial z_l}(t_D(z), z)$. *Moreover*

$$B(F(t_D(z), z), F(t_D(z), z)) = \mathrm{const.}\ \mathrm{Hess}_t(S(t_D(z), z)),$$

where B *is the Shapovalov form defined in Section (2.1), and* const. *does not depend on* z.

REMARK. Theorem (1.4.4) states that const. $= 1$ if $n = 2$.

(2.4.3) THEOREM. *We have the following two formulae.*

$$\prod_D \bar{\Phi}(t_D(z), z) = \prod_{D,g} c(D, g) \cdot$$

$$\cdot \prod_{i=0}^{k-1} \left(\frac{(i+1)^{\frac{(n-1)(i+1)}{\kappa}} \prod_{l=1}^n (-(\Lambda_l, \alpha) + i)^{\frac{-(\Lambda_l,\alpha)+i}{\kappa}}}{(-\sum_{l=1}^n (\Lambda_l, \alpha) + 2k - i - 2)^{\frac{-\sum_{l=1}^n (\Lambda_l,\alpha)+2k-i-2}{\kappa}}} \right)^{p_i} . \tag{2.4.4}$$

Here the first product is over admissible domains, the second product and p_i are explained in Theorem (2.3.2).

$$\det(A_I(t_D(z), z)) = \pm \prod_D \operatorname{Hess}_t(-S(t_D(z), z))^{1/2} \cdot$$

$$\cdot \prod_{f_I} \bar{b}(f_I)^{-1} \prod_{i=0}^{k-1} \left(\frac{(i+1)^{n+1} \prod_{l=1}^{n}(-(\Lambda_l, \alpha) + i)}{-\sum_{l=1}^{n}(\Lambda_l, \alpha) + 2k - i - 2} \right)^{p_i/2}. \qquad (2.4.5)$$

Here the rows (columns) of the determinant are numerated by admissible domains (admissible monomials). The first (second) product is over admissible domains (admissible monomials), and

$$\bar{b}(f_I) = \prod_{l=2}^{n} i_l! \prod_{j=0}^{i_l-1} (-(\Lambda_l, \alpha) + j).$$

To prove the theorem it is enough to write an asymptotic expansion for the rhs of (2.3.3) by the Stirling formula, for the lhs by formula (2.4.1), and then to compare the corresponding terms.

(2.4.6) COROLLARY. *Vectors $\{F(t_D(z), z)\}$ form a basis in* $\operatorname{Sing}(V)_k$ *if $-(\Lambda_1, \alpha), \dots, -(\Lambda_n, \alpha)$ are positive and $z_1, \dots, z_n$ are real pairwise different numbers.*

REMARKS.

1. There are integral formulae for solutions of the KZ equation associated with an arbitrary Kac–Moody algebra [SV]. Theorem (2.4.2) holds in this more general context [RV]. A conjectural analog of Theorem (2.3.3) is formulated in [V1]. It is plausible that the conjectural determinant formula in [V1] would imply that the eigenvectors given by the first terms of asymptotic expansions of solutions of the KZ equation for a Kac–Moody algebra generate a basis of the corresponding space of singular vectors.
2. It is plausible that there are analogs of formulae of Theorem (2.4.3) for an arbitrary configuration of hyperplanes, and these analogs are corollaries of the conjectural determinant formula in [V1].

(2.5) Norms of Eigenvectors

(2.5.1) THEOREM. *Under assumptions of Theorem (2.4.3) we have*

$$B(F(t_D(z), z), F(t_D(z), z)) = \operatorname{Hess}_t(S(t_D(z), z)).$$

COROLLARY. *For arbitrary $\Lambda_1, \dots, \Lambda_n$ and for arbitrary nondegenerate critical point $t = t(z)$ of the function $\Phi(t, z)$, we have*

$$B(F(t(z), z), F(t(z), z)) = \operatorname{Hess}_t(S(t(z), z)).$$

Proof of the Theorem. Let W_i, $i = 1, 2$, be sl_2 modules. Let $w_i \in W_i$ be a singular vector of weight $m_i \in \mathbb{C}$, that is, $ew_i = 0$ and $hw_i = m_i w_i$. For a nonnegative integer l, the vector

$$(w_1, w_2)_l :=$$

$$\sum_{p=0}^{l} (-1)^p \binom{l}{p} \frac{\prod_{j=0}^{l-1}(m_1 + m_2 + j + 2 - 2l)}{\prod_{j=0}^{p-1}(m_1 - j) \cdot \prod_{j=0}^{l-p-1}(m_2 - j)} f^p w_1 \otimes f^{l-p} w_2$$

is a singular vector in $W_1 \otimes W_2$ of weight $m_1 + m_2 - 2l$.

Let $V_1, \dots, V_n$ be sl_2 modules with highest vectors $v_1, \dots, v_n$ and highest weights $\Lambda_1, \dots, \Lambda_n$, respectively. Set $m_i = (\Lambda_i, \alpha)$. For any sequence of non-negative integers $I = (i_2, \dots, i_n)$, $i_2 + \cdots + i_n = k$, set

$$v_I = (\cdots((v_1, v_2)_{i_2}, v_3)_{i_3}, \dots, v_n)_{i_n}. \tag{2.5.2}$$

The vector v_I is a singular vector in $V = V_1 \otimes \cdots \otimes V_n$ of weight $m_1 + \cdots + m_n - 2k$.

(2.5.3) LEMMA. *Let $B = B_1 \otimes \cdots \otimes B_n$ be the Shapovalov form on V. Then*

$$B(v_I, v_I) = \prod_{l=2}^{n} b(m_1 + \cdots + m_{l-1} - 2(i_2 + \cdots + i_{l-1}), m_l; i_l)$$

where $b(\alpha, \beta; i)$ is defined in Section (1.3).

The lemma easily follows from Lemma (1.3.6).

For any $l = 2, \dots, n$, set

$$G_l = \sum_{i<l} \Omega_{i,l}.$$

(2.5.4) LEMMA [RV]. *For every l and $I = (i_2, \dots, i_n)$, the iterated vector v_I is an eigenvector of G_l with eigenvalue*

$$\lambda(m_1 + \cdots + m_{l-1} - 2(i_2 + \cdots + i_{l-1}), m_l; i_l)$$

where $\lambda(a, b; i) = \frac{1}{2}ab - i(a + b) + i(i - 1)$.

Under assumptions of Theorem (2.4.3) assume that $z_j = s^j$, $j = 1, \dots, n$, and s tends to $+\infty$.

For any $I = (i_2, \dots, i_n)$, $i_2 + \cdots + i_n = k$, let

$$D_I(z) = \{t \in \mathbb{R}^k \,|\, z_1 < t_1 < \cdots < t_{i_2} < z_2 < \cdots$$
$$< z_{n-1} < t_{i_2 + \cdots + i_{n-1} + 1} < \cdots < t_k < z_n\}$$

be the admissible domain corresponding to this sequence. Let $t_I(z)$ be the critical point of Φ in $D_I(z)$. For any $l = 2, \dots, n$, introduce a function

$$S_l = -\sum_{j=1}^{i_l} (\alpha_l \ln t_j + m_l \ln(t_j - 1)) + 2 \sum_{1 \leqslant i < j \leqslant i_l} \ln(t_i - t_j), \qquad (2.5.5)$$

where $\alpha_l = m_1 + \dots + m_{l-1} - 2(i_2 + \dots + i_{l-1})$. Let t^l be a critical point of S_l.

(2.5.6) LEMMA. *We have*

1.

$$F(t_I(z), z) = s^{d(I)}(v_I + \mathcal{O}(s^{-1})),$$

where $d(I)$ is some integer, see [RV].

2.

$$\mathrm{Hess}(-S(t_I(z), z)) = s^{c(I)} \left(\prod_{j=2}^{n} \mathrm{Hess}(-S_j(t^j)) + \mathcal{O}(s^{-1}) \right),$$

where $c(I)$ is some integer.

3. *For any $l = 2, \dots, n$, the operator $H_l(z) = \Sigma_{j \neq l} \Omega_{j,l}/(z_j - z_l)$ has the following asymptotics:*

$$H_l(z(s)) = s^{-l}(\Omega_{1,l} + \dots + \Omega_{l-1,l} + \mathcal{O}(s^{-1})).$$

Proof. Make a change of variables: $t_j = s^l u_j$ if $i_2 + \dots + i_{l-1} < j < i_2 + \dots + i_l$. Then we have

$$S(t(u)) = A \ln s + S_2(u) + \dots + S_n(u) + \mathcal{O}(s^{-1}) \qquad (2.5.7)$$

for some number A. This formula and the explicit formula for $F(t, z)$ imply the lemma.

By Theorem (1.4.4) and Lemma (2.5.3) we have

$$B(v_I, v_I) = \prod_{j=2}^{n} \mathrm{Hess}(S_j(t^j)).$$

Now Theorem (2.4.3) implies equality $2d(I) = c(I)$ and Theorem (2.5.1).

REMARK. Using part 3 of Lemma (2.5.6) and Lemma (2.5.4) we can compute asymptotics of eigenvalues of the operators $H_1(z), \dots, H_n(z)$ on vectors $\{F(t_D(z), z)\}$ for $z_1 \ll z_2 \ll \dots \ll z_n$. This computation shows that the eigenvalues separate the vectors. This means that the vectors are pairwise orthogonal with respect to the Shapovalov form [RV].

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