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BJORN POONEN

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Local height functions and the Mordell-Weil theorem for Drinfeld modules

BJORN POONEN

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720

Abstract. We prove an analogue for Drinfeld modules of the Mordell-Weil theorem on abelian varieties over number fields. Specifically, we show that if ϕ is a Drinfeld A-module over a finite extension L of the fraction field of A, then L considered as an A-module via ϕ is the direct sum of a free A-module of rank $\aleph_0$ with a finite torsion module. The main tool is the canonical global height function defined by Denis. By developing canonical local height functions, we are also able to show that if ϕ is defined over the ring of S-integers O_S in L, then O_S and L/O_S considered as A-modules via ϕ also are each isomorphic to the direct sum of a free A-module of rank $\aleph_0$ with a finite torsion module. If M is a nontrivial finite separable extension of L, then the quotient module M/L as well is isomorphic to the direct sum of a free A-module of rank $\aleph_0$ with a finite torsion module. Finally, the original result holds if L is replaced by its perfection.

1. Introduction

For abelian varieties over a number field, there is the well-known Mordell-Weil theorem, which states that the group of rational points is a finitely generated abelian group, and hence is isomorphic to the direct sum of its torsion subgroup with a free abelian group $\mathbb{Z}^r$ of some finite rank r. This paper studies the analogous question for Drinfeld modules. As usual, let K be a global function field, fix a nontrivial place ∞ of K and let A be the ring of elements of K which are integral away from ∞. If ϕ is a Drinfeld A-module defined over a finite extension L of the fraction field K of A, then L becomes an A-module via ϕ, and one can ask for a description of this A-module. (Complete definitions will be given in the next section.)

This A-module, which we call $\phi(L)$, is never finitely generated, as follows from the remark at the beginning of the proof of Theorem 5 in [2]. (So the most obvious analogue of the Mordell-Weil theorem is false.) Our main result is that in fact $\phi(L)$ is isomorphic to the direct sum of a free A-module of rank $\aleph_0$ with a finite torsion module. This makes it impossible to define an interesting "Mordell-Weil rank" for Drinfeld modules based on the A-module structure of $\phi(L)$ alone. On the other hand, it reveals a close analogy between Drinfeld modules and the multiplicative group $\mathbb{G}_m$ over number fields, as we will discuss in Section 7.

If S is a finite nonempty set of places of L (which we identify with nonarchimedean valuations), the *ring of S-integers in L* is

$$O_S = \{x \in L \,|\, v(x) \geqslant 0 \text{ for all places } v \text{ of } L \text{ not in } S\}.$$

Any Drinfeld A-module ϕ over L can be defined over O_S for some S. (See Lemma 2.) Then this O_S becomes an A-module $\phi(O_S)$ as well, and we can show that $\phi(O_S)$ and $\phi(L)/\phi(O_S)$ also are each isomorphic to the direct sum of a free A-module of rank $\aleph_0$ with a finite torsion module. If M is a nontrivial finite separable extension of L, then $\phi(L)$ is a submodule of $\phi(M)$, and we show that $\phi(M)/\phi(L)$ is the direct sum of a free A-module of rank $\aleph_0$ with a finite torsion module. If L^{perf} is the perfection of L, then $\phi(L^{\mathrm{perf}})$ also has this structure.

The method of proof is as follows. First we develop a theory of canonical local height functions for Drinfeld modules, building on the work of Denis on canonical global height functions [3], and we use these height functions to show that the A-modules in question are of rank $\aleph_0$ and are tame, meaning that every submodule of finite rank is finitely generated. (In fact, for the module structure of L alone, we only need global height functions, but for O_S and L/O_S we really need the local height functions. In any case, it seems likely that the local height functions defined here will have other applications as well.) The proof of tameness is similar to the proof of the Mordell-Weil theorem for abelian varieties given the weak Mordell-Weil theorem. To complete the proof, we classify all tame modules of rank $\aleph_0$ over a Dedekind domain.

The results of this paper can undoubtedly be generalized to certain higher dimensional t-modules, probably to the same class of t-modules for which Denis [3] is able to define his canonical global height function, among others those in which the action of t as an endomorphism of $\mathbb{G}_a^n$ is of the form

$$a_0 + a_1\tau + \cdots + a_d\tau^d$$

where $a_i \in M_n(L)$, τ denotes the Frobenius endomorphism on $\mathbb{G}_a^n$, and a_d is an *invertible* n by n matrix. We have restricted the discussion to Drinfeld modules for simplicity.

2. Review of Drinfeld modules

All the material of this section can be found in [6] or in Drinfeld's original paper [4]. We use the following notation throughout our paper (except in Lemma 3 and in the Appendix, where we generalize by allowing A to be any Dedekind domain and K its fraction field):

$\mathbb{F}_q$ = the field of q elements ($q = p^m$ for some prime p)

K = a global function field with field of constants $\mathbb{F}_q$ (i.e., a finite extension of $\mathbb{F}_q(t)$ in which $\mathbb{F}_q$ is algebraically closed)

∞ = a fixed nontrivial place of K

A = the set of elements of K which are integral at all places except possibly ∞ (this is a Dedekind ring with field of fractions K)

$|\ |$ = the absolute value associated with ∞, normalized so that $|a| = \#(A/a)$ for nonconstant $a \in A$

Let L be an *A-field*, that is, a field equipped with a ring homomorphism $\iota : A \to L$. Let $\mathbb{G}_a$ be the additive group scheme over L. The ring $\text{End}_L\,\mathbb{G}_a$ of endomorphisms of $\mathbb{G}_a$ as a group scheme over L is a twisted polynomial ring $L\{\tau\}$ generated over L by the pth-power Frobenius morphism τ, with the relation $\tau\alpha = \alpha^p\tau$ for all $\alpha \in L$. Each twisted polynomial represents a polynomial map on $\mathbb{G}_a$, and we define the degree of a twisted polynomial to be the degree of this polynomial map. (For example, the twisted polynomial $2\tau^2 + 3$ represents the map $x \mapsto 2x^{p^2} + 3x$, and hence $\deg(2\tau^2 + 3) = p^2$.) By convention $\deg 0 = 0$. Let $D : \text{End}_L\,\mathbb{G}_a \to L$ be the map taking an endomorphism to its derivative at zero; explicitly, $D : L\{\tau\} \to L$ takes a twisted polynomial to its constant term. Then a *Drinfeld A-module over L* is a ring homomorphism

$$\phi : A \to \text{End}_L\,\mathbb{G}_a = L\{\tau\}$$

$$a \mapsto \phi_a$$

such that $D \circ \phi = \iota$, and which is not the trivial homomorphism sending each $a \in A$ to the constant polynomial $\iota(a) \in L\{\tau\}$. Informally, one can think of a Drinfeld A-module as the additive group of L with an A-module structure where each $a \in A$ acts as a polynomial map $\phi_a \in L[x]$ (with a few additional conditions).

Drinfeld showed that for each Drinfeld A-module ϕ there is a unique positive integer r such that $\deg \phi_a = |a|^r$ for all $a \in A$. This integer is called the *rank* of ϕ. If ϕ' is another Drinfeld A-module defined over L, a *morphism* from ϕ to ϕ' is an element $u \in \text{End}_L\,\mathbb{G}_a$ such that $u\phi_a = \phi'_a u$ for all $a \in A$. By taking degrees, we see that if a nonzero morphism from ϕ to ϕ' exists, then ϕ and ϕ' have the same rank.

3. The canonical local function associated with a Drinfeld module

Let L be an A-field which is also a local field; i.e. it is complete with respect to a discrete valuation v and the residue field is finite. For $x \in L$, let $\tilde{v}(x) = \min\{0, v(x)\}$.

LEMMA 1. *If $d > 0$ and $f(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_0 \in L[x]$, then $\tilde{v}(f(x)) - d\tilde{v}(x)$ is bounded. Also $\tilde{v}(f(x)) = d\tilde{v}(x) + v(c_d)$ when $\tilde{v}(x)$ is sufficiently negative.*

Proof. If $v(x)$ is sufficiently negative, then $v(c_d x^d) < v(c_i x^i)$ for all $i < d$, so

$$v(f(x)) = v(c_d x) = dv(x_d) + v(c_d),$$

which is negative if $v(x)$ is sufficiently negative. For the other x's, the lower bound on $v(x)$ gives a lower bound on $v(f(x))$ by the triangle inequality. $\qquad\square$

Let ϕ be a Drinfeld A-module over L. Fix $a \in A \backslash \mathbb{F}_q$. (We will show later that the choice of a is inconsequential.) Then $|a| > 1$, since otherwise a would be integral at all places (including infinity), forcing it to be in $\mathbb{F}_q$.

The next proposition defines a nonpositive function $V: L \to \mathbb{R}$ associated with ϕ, which behaves like a canonical local height function (except that it is nonpositive, so in the next section we scale it by a negative constant). The definition is modeled on Tate's definition of the (Néron-Tate) canonical global height function for abelian varieties, as was Denis' definition of the canonical global height function for a Drinfeld module.

PROPOSITION 1.

(1) *For $x \in L$, the limit*

$$V(x) = \lim_{n \to \infty} (\deg \phi_{a^n})^{-1} \tilde{v}(\phi_{a^n}(x))$$

   exists.
(2) $V(x) - \tilde{v}(x)$ *is bounded.*
(3) *If $v(x)$ is sufficiently negative, then $V(x) = v(x) + v(c)/(d-1)$, where $d = \deg \phi_a$ and c is the leading coefficient of ϕ_a considered as a polynomial map.*
(4) $V(x \pm y) \geqslant \min\{V(x), V(y)\}$ *and* $V(-x) = V(x)$.
(5) *If $x_1, \ldots, x_n \in L$ and there is an i such that $V(x_i) < V(x_j)$ for all $j \neq i$, then $x_1 + \cdots + x_n \neq 0$.*

Proof. First note that $d = |a|^r > 1$, where r is the rank of ϕ, and $\deg \phi_{a^n} = d^n$. Let M be the bound on $\tilde{v}(\phi_a(x)) - d\tilde{v}(x)$ given by Lemma 1. Then

$$|(\deg \phi_{a^{n+1}})^{-1} \tilde{v}(\phi_{a^{n+1}}(x)) - (\deg \phi_{a^n})^{-1} \tilde{v}(\phi_{a^n}(x))|$$
$$= d^{-(n+1)} |\tilde{v}(\phi_a(\phi_{a^n}(x))) - d\tilde{v}(\phi_{a^n}(x))|$$
$$\leqslant d^{-(n+1)} M.$$

Thus the limit in (1) exists, and (2) follows as well, since $\sum_{n=0}^{\infty} d^{-(n+1)}M$ is finite.

If $v(x)$ is sufficiently negative, then by Lemma 1 and induction on n we see that for all $n \geqslant 0$, $\tilde{v}(\phi_{a^n}(x))$ is at least as negative as $v(x)$, and that

$$\tilde{v}(\phi_{a^{n+1}}(x)) = d\tilde{v}(\phi_{a^n}(x)) + v(c).$$

This can be rewritten

$$\tilde{v}(\phi_{a^{n+1}}(x))/d^{n+1} = \tilde{v}(\phi_{a^n}(x))/d^n + v(c)/d^{n+1},$$

so

$$\tilde{v}(\phi_{a^N}(x))/d^N = \tilde{v}(\phi_{a^0}(x))/d^0 + \sum_{i=1}^{N} v(c)/d^i$$

$$= v(x) + \sum_{i=1}^{N} v(c)/d^i.$$

Letting $N \to \infty$, and summing the infinite geometric series gives (3).

The first half of (4) follows by applying $\min\{0, \cdot\}$ to

$$v(\phi_{a^n}(x \pm y)) = v(\phi_{a^n}(x) \pm \phi_{a^n}(y)) \geqslant \min\{v(\phi_{a^n}(x)), \ v(\phi_{a^n}(y))\},$$

dividing by $\deg \phi_{a^n}$ and letting $n \to \infty$. The second half of (4) follows in the same way.

Finally, (5) is proved from (4) in the same way that it is proved for valuations. ∎

The usefulness of V springs from the following.

PROPOSITION 2. *Let ϕ' be another Drinfeld A-module over L, with corresponding function V' obtained from Proposition 1, and let $u: \phi \to \phi'$ be a morphism of Drinfeld modules. Then for all $x \in L$,*

$$V'(u(x)) = (\deg u)V(x).$$

Proof. If $u = 0$, both sides are zero, so assume $u \neq 0$. Then ϕ and ϕ' have the same rank r, and $\deg \phi_{a^n} = |a^n|^r = \deg \phi'_{a^n}$. Let M be the bound on $\tilde{v}(u(x)) - (\deg u)\tilde{v}(x)$ given by Lemma 1. Then

$$|\tilde{v}(\phi'_{a^n}(u(x))) - (\deg u)\tilde{v}(\phi_{a^n}(x))| = |\tilde{v}(u(\phi_{a^n}(x))) - (deg \ u) \ \tilde{v}(\phi_{a^n}(x))|$$

$$\leqslant M.$$

Now divide by $\deg \phi_{a^n}$ (which equals $\deg \phi'_{a^n}$), and let $n \to \infty$ to get

$$|V'(u(x)) - (\deg u)V(x)| \leqslant 0. \qquad \square$$

COROLLARY 1. *If $b \in A$,*

$$V(\phi_b(x)) = (\deg \phi_b)V(x)$$

for all $x \in L$.

PROPOSITION 3. *V is independent of the choice of $a \in A \backslash \mathbb{F}_q$.*

  Proof. Let V be defined as before using $a \in A \backslash \mathbb{F}_q$, and suppose b is another element of $A \backslash \mathbb{F}_q$. Since $\deg \phi_{b^n} \to \infty$ as $n \to \infty$, and $V - \tilde{v}$ is bounded, we have

$$\lim_{n \to \infty} (\deg \phi_{b^n})^{-1} \tilde{v}(\phi_{b^n}(x)) = \lim_{n \to \infty} (\deg \phi_{b^n})^{-1} V(\phi_{b^n}(x))$$

$$= V(x)$$

by Corollary 1. $\qquad \square$

  Let $O = \{x \in L \mid v(x) \geqslant 0\}$ be the valuation ring of L, let $\pi \in O$ be a uniformizing parameter (i.e., $v(\pi) = 1$), and let $l = O/\pi$ be the residue field. We say that the Drinfeld module ϕ is defined over O if for each $a \in A$, all coefficients of ϕ_a belong to O. (This definition is slighty non-standard: usually one also requires the leading coefficient of ϕ_a to be a unit for each $a \in A$.)

PROPOSITION 4. *Suppose ϕ is defined over O. Then*

  (1) *$V(x + y) = V(x)$ whenever $x \in L$ and $y \in O$. In other words, V induces a function on L/O.*
  (2) *If C is a real constant, only finitely many elements $\bar{x}$ of L/O satisfy $V(\bar{x}) \geqslant C$.*
  (3) *$V(x) = 0$ if and only if $\phi_b(x) \in O$ for some nonzero $b \in A$.*
  (4) *If in addition, for some $a \in A \backslash \mathbb{F}_q$, the leading coefficient of ϕ_a is a unit of O, then $V(x) = \tilde{v}(x)$ for all $x \in L$.*

  Proof. If $y \in O$, then $\phi_{a^n}(y) \in O$, so by definition of V, $V(y) = 0$. By (4) in Proposition 1,

$$V(x + y) \geqslant \min\{V(x), V(y)\} = V(x),$$

since $V(x) \leqslant 0$ for any $x \in L$. The same argument with x and y replaced by $x + y$ and $-y$ shows $V(x) \geqslant V(x + y)$, so $V(x + y) = V(x)$, proving (1).

  Now V and $\tilde{v}$ are both functions defined on L/O and they differ by a bounded amount by (2) in Proposition 1, so to prove part (2), it suffices to

show that for each constant C, there are finitely many $\bar{x} \in L/O$ such that $\tilde{v}(\tilde{x}) \geqslant C$. This is equivalent to showing that $\pi^{-n}O/O$ is finite for each $n \geqslant 1$. This is finite, because it has a composition series as an O-module

$$\pi^{-n}O/O \supset \pi^{-(n-1)}O/O \supset \cdots \supset O/O,$$

in which each quotient is isomorphic to the residue field l, which was assumed to be finite. This proves (2).

If $\phi_b(x) \in O$ for some nonzero $b \in A$, then $V(\phi_b(x)) = 0$. On the other hand, by Corollary 1, $V(\phi_b(x)) = (\deg \phi_b)V(x)$ and $\deg \phi_b \neq 0$ since ϕ is injective (see Proposition 2.1 in [4]), so $V(x) = 0$. Conversely, if $V(x) = 0$, then $V(\phi_b(x)) = (deg\ \phi_b)V(x) = 0$ for all $b \in A$. But A is infinite, and $\{y \in L \mid V(y) = 0\}$ consists of only finitely many cosets of O, so some $\phi_b(x)$ and $\phi_{b'}(x)$ belong to the same coset. Then $\phi_{b-b'}(x) \in O$. This proves (3).

Finally, to prove (4), notice that for any polynomial $\phi_a \in O[t]$ whose leading coefficient is a unit,

$$\tilde{v}(\phi_a(x)) = (\deg \phi_a)\tilde{v}(x)$$

for all $x \in L$. Hence by induction on n,

$$\tilde{v}(\phi_{a_n}(x)) = (\deg \phi_{a_n})\tilde{v}(x)$$

for all $x \in L$. Divide by $\deg \phi_{a_n}$ and let $n \to \infty$ to get $V(x) = \tilde{v}(x)$. $\square$

4. Local and global heights

For the next three sections, L will be a finite extension of K. We make L an A-field using the inclusion maps $A \subset K \subset L$. Let ϕ be a Drinfeld A-module over L.

Let v be a place of L (by place, we mean a nontrivial place). We normalize the valuation v to take values in $\mathbb{Z} \cup \infty$. The completion L_v of L at v is a local field, and we can consider ϕ as a Drinfeld A-module over L_v. In particular, for each v we get a function V_v as in the previous section. We define the *canonical local height* on L associated with ϕ and v to be the real valued function

$$\hat{h}_v(x) = -[L:K]^{-1}d(v)V_v(x),$$

where $d(v)$ is the degree of the residue field of v over $\mathbb{F}_q$. Since $\hat{h}_v$ is simply a constant multiple of V_v, all the results of the previous section concerning V can be translated into results about $\hat{h}_v$.

On the other hand, Denis has defined a canonical global height function $\hat{h}$ associated with ϕ. (Actually, his definition is given for certain higher dimensional t-modules as well, but for the case $A = \mathbb{F}_q[t]$ only. There is no problem with extending the definition to other A's as well.) If $a \in A \backslash \mathbb{F}_q$, then (a restatement of) his definition is

$$\hat{h}(x) = \lim_{n \to \infty} (\deg \phi_{a^n})^{-1} h(\phi_{a^n}(x)),$$

where $h(x)$ denotes the Weil height on $\mathbb{A}^1(\bar{K})$. (If x belongs to a finite extension E of K, then

$$h(x) = -[E:K]^{-1} \sum_{\text{places } w \text{ of } E} d(w) \min\{0, w(x)\},$$

where $d(w)$ denotes the degree of the residue field of w over $\mathbb{F}_q$, and $w(x)$ is normalized to take values in $\mathbb{Z} \cup \infty$.)

We recall some properties (due to Denis) of this global height function for later use.

PROPOSITION 5.

(1) *If $a \in A$, and $x \in \bar{K}$, then*

$$\hat{h}(\phi_a(x)) = (\deg \phi_a)\hat{h}(x).$$

(2) *If C is a real constant, there are only finitely many $x \in L$ with $\hat{h}(x) \leqslant C$.*
(3) $\hat{h}(x) = 0$ *if and only if $\phi_a(x) = 0$ for some nonzero $a \in A$. By the previous statement, the number of such x's belonging to L is finite.*
(4) $\hat{h}(x \pm y) \leqslant \hat{h}(x) + \hat{h}(y)$.

Proof. See [3]. $\square$

The last goal of this section is to relate our local height functions with Denis' global height function. The following lemma will allow us to replace sums over all places of L with sums over a finite number of places.

LEMMA 2. *A Drinfeld A-module ϕ over L is defined over the valuation ring O_v corresponding to v for all but finitely many places v. In other words, ϕ is defined over the ring of S-integers O_S for some finite set of places S.*

Proof. The ring A is finitely generated. (In fact, if $a \in A \backslash \mathbb{F}_q$, then A is a finitely generated module over $\mathbb{F}_q[a]$.) If $a_1, \ldots, a_n$ are a set of generators, and v corresponds to a prime not occurring in the denominators of $\phi_{a_1}, \ldots, \phi_{a_n}$, then that prime will not occur in the denominators of ϕ_a for any $a \in A$, since ϕ_a can be expressed as a sum of compositions of the ϕ_{a_i}'s. This holds for all but finitely many v. $\square$

PROPOSITION 6. *If $x \in L$, then $\hat{h}_v(x) = 0$ for all but finitely many places v of L, and*

$$\hat{h}(x) = \sum_{\text{places } v \text{ of } L} \hat{h}_v(x).$$

Proof. Let S be the set of places v such that either $v(x) < 0$ or ϕ is not definable over the corresponding valuation ring. By Lemma 2, S is finite. If $v \notin S$, then Proposition 4 shows $V_v(x) = 0$ and hence $\hat{h}_v(x) = 0$, proving the first claim.

Now $v(\phi_{a^n}(x)) \geq 0$ if $v \notin S$, so

$$h(\phi_{a^n}(x)) = -[L:K]^{-1} \sum_{v \in S} d(v) \min\{0, v(\phi_{a^n}(x))\}$$

If we now divide by $\deg \phi_{a^n}$ and let $n \to \infty$, we obtain

$$\hat{h}(x) = -[L:K]^{-1} \sum_{v \in S} d(v) V_v(x)$$

$$= \sum_{v \in S} \hat{h}_v(x)$$

$$= \sum_{\text{all places } v \text{ of } K} \hat{h}_v(x),$$

since we just showed that $\hat{h}_v(x) = 0$ when $v \notin S$. $\square$

5. A Mordell-Weil type theorem

For the next two sections we retain the assumption that ϕ is a Drinfeld A-module over a finite extension L of K. The additive group of L becomes an A-module by letting each $a \in A$ act as the polynomial map ϕ_a. We will use the notation $\phi(L)$ to denote this A-module. Our goal is to characterize $\phi(L)$ (as an A-module).

By the rank of an A-module M we mean the dimension of the K-vector space $M \otimes_A K$, which is some cardinal number. The *torsion submodule* of M is

$$M_{\text{tors}} = \{x \in M \mid ax = 0 \text{ for some } a \in A\},$$

which is also the kernel of the natural map $M \to M \otimes_A K$. An A-module is *tame* if every submodule of finite rank is finitely generated as an A-module. We will show that $\phi(L)$ is not finitely generated, but at least it is tame.

Some parts of the proof are similar to the proof of the Mordell-Weil theorem for abelian varieties. What makes a direct reproduction of the argument impossible is the failure of the "weak Mordell-Weil theorem," which for abelian varieties says that if G is the group of rational points and $n \geq 1$, then G/nG is finite. To prove the tameness in our situation, we use the following as a substitute.

LEMMA 3. *Let A be any Dedekind domain, let M be an A-module of finite rank, and suppose the torsion submodule M_{tors} is a finitely generated A-module. Then for any nonzero ideal $I \subset A$, M/IM is a finitely generated A-module. In particular, if a is a nonzero element of A and A/a is finite, then the finitely generated A/a-module M/aM must be finite.*

Proof. If M' is the image of M in $M \otimes_A K$, then we have an exact sequence

$$0 \to M_{\text{tors}} \to M \to M' \to 0.$$

Tensoring with A/I yields a right exact sequence

$$M_{\text{tors}}/IM_{\text{tors}} \to M/IM \to M'/IM' \to 0,$$

and $M_{\text{tors}}/IM_{\text{tors}}$ is finitely generated, so it suffices to show M'/IM' is finitely generated. Hence without loss of generality we may assume M is a sub-A-module of K^r for some $r \geq 0$.

If we knew that for given nonzero ideals I, J, the result held for all M, then for any sub-A-module $M \subset K^r$, M/JM and $JM/I(JM)$ would be finitely generated, so M/IJM would be finitely generated too. Thus we can reduce to the case where I is a nonzero prime ideal $\mathfrak{p}$ of A.

In fact we claim that $\dim_{A/\mathfrak{p}} M/\mathfrak{p}M \leq r$. It suffices to show that if $m_1, \ldots, m_n$ are elements of M whose images in $M/\mathfrak{p}M$ are independent over $A/\mathfrak{p}$, then $m_1, \ldots, m_n$ are independent over K as well. Suppose not; i.e. suppose that $\alpha_1 m_1 + \cdots + \alpha_n m_n = 0$ for some $\alpha_1, \ldots, \alpha_n \in K$. By multiplying by some power of a uniformizing parameter for $\mathfrak{p}$, we may assume that $v(\alpha_i) \geq 0$ for all i, with equality for at least one i. Then reduction modulo $\mathfrak{p}$ shows that the images of $m_1, \ldots, m_n$ are not independent over $A/\mathfrak{p}$.

Thus $M/\mathfrak{p}M$ can be generated by less than or equal to r elements as an $A/\mathfrak{p}$-vector space, or equivalently as an A-module, as desired. $\square$

LEMMA 4. *$\phi(L)$ is a tame A-module.*

Proof. Suppose M is a submodule of $\phi(L)$ of finite rank. We must show M is finitely generated. By (3) in Proposition 5, the torsion submodule of $\phi(L)$ is finite, so the same is true for M. Thus we may apply Lemma 3 to deduce that M/aM is finite for any nonzero $a \in A$.

Pick $a \in A \backslash \mathbb{F}_q$. Then $\deg \phi_a \geqslant 2$. Let $S \subset M$ be a set of representatives for M/aM. Let $C = \max_{s \in S} \hat{h}(s)$. Let T be the union of S with $\{x \in L \,|\, \hat{h}(x) \leqslant C\}$. By (2) in Proposition 5, T is finite.

We claim that T generates M as an A-module. Let N be the submodule generated by T. If $N \neq M$, then by (2) in Proposition 5, we can pick $m_0 \in M \backslash N$ with $\hat{h}(m_0)$ minimal. Since S is a set of representatives for M/aM, we can write $m_0 = s + \phi_a(m)$ for some $s \in S$ and $m \in M$. Moreover $m \notin N$, since otherwise $m_0 \in N$ also. Then

$$
\begin{aligned}
2\hat{h}(m_0) &\leqslant 2\hat{h}(m) && \text{(by the minimality of } m_0) \\
&\leqslant (\deg \phi_a)\hat{h}(m) \\
&= \hat{h}(\phi_a(m)) && \text{(by (1) in Proposition 5)} \\
&= \hat{h}(m_0 - s) \\
&\leqslant \hat{h}(m_0) + \hat{h}(s) && \text{(by (4) in Proposition 5)} \\
&\leqslant \hat{h}(m_0) + C
\end{aligned}
$$

so $\hat{h}(m_0) \leqslant C$, and $m_0 \in T \subset N$, contradicting the definition of m_0. Thus M is finitely generated, as desired. $\qquad\square$

THEOREM 1. *The A-module $\phi(L)$ is the direct sum of its torsion submodule, which is finite, with a free A-module of rank $\aleph_0$.*

Proof. First we compute the rank of $\phi(L)$. Since L is countable even as a set, the rank of $\phi(L)$ is at most $\aleph_0$. Suppose the rank is not $\aleph_0$; i.e. suppose it is finite. Then the tameness (Lemma 4) implies that $\phi(L)$ is finitely generated. Let Z be a finite set of generators. We can find a finite set S of places of L such that ϕ is defined over O_S (by Lemma 2), and which is large enough that $Z \subset O_S$. Then O_S is a proper submodule over $\phi(L)$ containing Z, contradicting the fact that Z generates all of $\phi(L)$.

Thus the rank of $\phi(L)$ is $\aleph_0$. Now applying Proposition 10 from the Appendix yields the desired result. (The finiteness of the torsion submodule is (3) in Proposition 5.) $\qquad\square$

There is nothing mysterious about the finite torsion submodule. For a given ϕ and L, it can be calculated effectively by bounding the Weil height of a torsion point, or by using reductions modulo various primes of L. By Theorem 10.15 in [7], every finitely generated torsion module M over a Dedekind domain A is isomorphic to a direct sum

$$
A/I_1 \oplus \cdots \oplus A/I_n
$$

where $I_1 \subset \cdots \subset I_n$ are nonzero ideals in A. Moreover, the I_j's are uniquely determined by M. In fact, I_j is the annihilator of the jth exterior power of M, as pointed out by Michael Rosen.

We conclude this section by remarking that the Drinfeld module structures of the algebraic closure $\bar{L}$ and separable closure L^{sep} are much easier to determine.

PROPOSITION 7. *Each of the A-modules $\phi(\bar{L})$ and $\phi(L^{\mathrm{sep}})$ is the direct sum of a K-vector space of dimension $\aleph_0$ with a torsion module isomorphic to $(K/A)^r$, where r is the rank of ϕ.*

Proof. If a is a nonzero element of A and $c \in \bar{L}$, then the equation $\phi_a(x) = c$ can be solved in $\bar{L}$. Moreover, if $c \in L^{\mathrm{sep}}$, then any solution x_0 lies in L^{sep}, because the polynomial $\phi_a(x) - c$ is separable (since its derivative is the nonzero constant a). This means that $\phi(\bar{L})$ and $\phi(L^{\mathrm{sep}})$ are divisible A-modules.

By Theorem 7 in [10], any divisible A-module is the direct sum of a K-vector space and the torsion submodule. The torsion submodule in both of our cases is $(K/A)^r$ by Proposition 2.2 in [4]. The K-vector space must have dimension at least $\aleph_0$, because even $\phi(L)$ has rank $\aleph_0$, by Theorem 1. On the other hand, both $\bar{L}$ and L^{sep} are countable even as sets, so the vector space must have dimension exactly $\aleph_0$. $\qquad\square$

6. Other module structure theorems

In this section we answer some related questions about modules arising from Drinfeld modules. (These were posed by David Goss.) Fix a finite set S of places of L such that ϕ is defined over the ring of integers O_S. (The existence of S is guaranteed by Lemma 2.) Then each polynomial map ϕ_a maps O_S into O_S, so we get a submodule $\phi(O_S)$ of $\phi(L)$. We want to describe the A-module structures of $\phi(O_S)$ and $\phi(L)/\phi(O_S)$.

For each place $v \notin S$, the inclusion of L in its completion L_v induces a group homomorphism $L/O_S \to L_v/O_v$ where $O_v = \{x \in L_v \mid v(x) \geqslant 0\}$, and as is well known, the map

$$L/O_S \to \bigoplus_{v \notin S} L_v/O_v$$

is an isomorphism. The Drinfeld module ϕ induces module structures on L_v and O_v as well, and it is clear that the isomorphism above respects these module structures. So we can understand $\phi(L)/\phi(O_S)$ by understanding $\phi(L_v)/\phi(O_v)$ for each $v \notin S$.

LEMMA 5. *The A-modules $\phi(O_S)$ and $\phi(L_v)/\phi(O_v)$ (for $v \notin S$) are tame.*

Proof. A submodule of a tame module is tame, so the tameness of $\phi(O_S)$ follows from Lemma 4.

The proof of the tameness of $\phi(L_v)/\phi(O_v)$ is the same as the proof of the tameness of $\phi(L)$, except using the function V_v (or if you prefer, $\hat{h}_v$) defined by Proposition 1, instead of $\hat{h}$. In particular we use Corollary 1 instead of (1) in Proposition 5, (2) in Proposition 4 instead of (2) in Proposition 5, Proposition 4 instead of (3) in Proposition 5 to show that the torsion submodule of $\phi(L_v)/\phi(O_v)$ is finite, and (4) in Proposition 1 instead of (4) in Proposition 5. □

Computing the ranks of these A-modules is now more difficult than for $\phi(L)$. We will use the following simple lemma.

LEMMA 6. *Suppose p is a prime number and $\alpha \in \mathbb{Q}$. Then for each $k \geqslant 1$, there exist positive integers $n_1, \ldots, n_k$ such that the rational numbers $p^i(\alpha - n_j)$ for i ranging over nonnegative integers and $1 \leqslant j \leqslant k$ are all distinct and negative.*

Proof. Choose $n_1, \ldots, n_k$ to be large consecutive integers. Then each ratio $(\alpha - n_j)/(\alpha - n_{j'})$ will be close to 1, and in particular will not be a power of p. □

LEMMA 7. *The A-modules $\phi(O_S)$ and $\phi(L_v)/\phi(O_v)$ (for $v \notin S$) have rank $\aleph_0$.*

Proof. First of all, the set O_S is countable since $O_S \subset L$, and L_v/O_v is countable as well since it is a direct summand of L/O_S, and L is countable. Hence the ranks of $\phi(O_S)$ and $\phi(L_v)/\phi(O_v)$ are at most $\aleph_0$. It will suffice in each case to exhibit for each $k \geqslant 1$ elements $x_1, \ldots, x_k$ of the module which are A-independent.

Let us consider $\phi(O_S)$ first. Pick a place w in S, and let V be the corresponding local function. Then $w(O_S)$ contains all sufficiently negative integers by the Riemann-Roch theorem, so by (3) in Proposition 1, there exists (a very negative) $\alpha \in \mathbb{Q}$ such that $V(O_S)$ contains $\alpha - n$ for every positive integer n. Choose $n_1, \ldots, n_k$ as in Lemma 6, and choose $x_i \in O_S$ such that $V(x_i) = \alpha - n_i$.

We claim that $x_1, \ldots, x_k$ are A-independent in $\phi(O_S)$. Suppose not. Then for some $a_1, \ldots, a_k \in A$,

$$\phi_{a_1}(x_1) + \cdots + \phi_{a_k}(x_k) = 0.$$

But

$$V(\phi_{a_i}(x_i)) = (\deg \phi_{a_i})V(x_i) \quad \text{(by Corollary 1)}$$
$$= (\deg \phi_{a_i})(\alpha - n_i),$$

and $\deg \phi_{a_i}$ is 0 or a power of p, so by the choice of n_i, the values $V(\phi_{a_i}(x_i))$ are all distinct and negative, after one throws out the i for which $a_i = 0$. By

(5) in Proposition 1, this forces $a_i = 0$ for all i. Thus $x_1, \ldots, x_k$ are A-independent, as desired.

We use virtually the same construction to find independent $x_1, \ldots, x_k$ in $\phi(L_v)/\phi(O_v)$. This time let V be the local function corresponding to v. As before, we can pick $\alpha \in \mathbb{Q}$ such that $V(L_v)$ contains $\alpha - n$ for every positive integer n. Choose $n_1, \ldots, n_k$ as in Lemma 6, and choose $x_i \in L_v$ such that $V(x_i) = \alpha - n_1$. Then the images of $x_1, \ldots, x_k$ in $\phi(L_v)/\phi(O_v)$ are A-independent for the same reason as before. □

LEMMA 8. *For all but finitely many places* v, *the* A-*module* $\phi(L_v)/\phi(O_v)$ *is torsion-free.*

Proof. Pick $a \in A \backslash \mathbb{F}_q$, and let V_v be the function on the local field L_v defined by Proposition 1. By Lemma 2, for all but finitely many places v, ϕ is defined over O_v and the leading coefficient of ϕ_a is a unit at v. By (3) and (4) in Proposition 4, for any such v, $\phi(L_v)/\phi(O_v)$ is torsion-free. □

THEOREM 2. *Each of the* A-*modules* $\phi(O_S)$, $\phi(L)/\phi(O_S)$, *and* $\phi(L_v)/\phi(O_v)$, *is the direct sum of a free* A-*module of rank* $\aleph_0$ *and a finite torsion module.*

Proof. For $\phi(O_S)$ and $\phi(L_v)/\phi(O_v)$, combine Lemmas 5 and 7 with Proposition 10 to obtain the result. (The finiteness of the torsion submodule of $\phi(O_S)$ follows from (3) in Proposition 5, and the finiteness of the torsion submodule of $\phi(L_v)/\phi(O_v)$ follows from Proposition 4.)

Now $\phi(L)/\phi(O_S)$ is the countable direct sum of modules $\phi(L_v)/\phi(O_v)$, each isomorphic to the direct sum of a free module of rank $\aleph_0$ and a finite torsion module, so $\phi(L)/\phi(O_S)$ is the direct sum of a free A-module of rank $\aleph_0 \cdot \aleph_0 = \aleph_0$ and a torsion module. This torsion module is finite as well, because of Proposition 8. □

Next we describe the A-module structure of $\phi(M)/\phi(L)$ where M is a finite extension of L.

LEMMA 9. *If* M *is a finite separable extension of* L, *then* $\phi(M)/\phi(L)$ *is tame.*

Proof. Since a submodule of a tame module is tame, we may without loss of generality enlarge M to assume M is a Galois extension of L. Let $\sigma_1, \ldots, \sigma_n$ be all the elements of $\mathrm{Gal}(M/L)$. Each σ_i commutes with the action of A, so we have a homomorphism of A-modules

$$\phi(M) \to \phi(M) \oplus \cdots \oplus \phi(M)$$

$$x \mapsto (\sigma_1 x - x, \ldots, \sigma_n x - x)$$

whose kernel is $\phi(L)$ by Galois theory. Now $\phi(M)/\phi(L)$ is isomorphic to the image, which is a submodule of a finite direct sum of tame modules, and is hence tame. □

LEMMA 10. *If M is a nontrivial finite separable extension of L, then $\phi(M)/\phi(L)$ has rank $\aleph_0$.*

Proof. Since we know from Theorem 1 that $\phi(M)$ has rank $\aleph_0$, the rank of $\phi(M)/\phi(L)$ is at most $\aleph_0$. So it will suffice to construct for each $k \geqslant 1$, elements $x_1, \ldots, x_k \in \phi(M)$ whose images in $\phi(M)/\phi(L)$ are A-independent.

Using Lemma 2, choose a finite set of places S of L such that ϕ is defined over O_S. Pick $a \in A \backslash \mathbb{F}_q$ and enlarge S if necessary so that the leading coefficient of ϕ_a is a unit at all places outside S. Then by (4) in Proposition 4, if w is any place of M lying above a place of L not in S, the local function V_w on M equals $\tilde{w}$.

By the Čebotarev Density Theorem for function fields [8], there exist primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ of L outside S which split completely in M. For $1 \leqslant i \leqslant k$, let $\mathfrak{q}_i$ and $\mathfrak{r}_i$ be two primes of M above $\mathfrak{p}_i$. Next, for each i, use an approximation theorem to find $x_i \in M$ such that x_i is integral at all the $\mathfrak{q}_j$'s and $\mathfrak{r}_j$'s except $\mathfrak{q}_i$, where it is not integral.

Suppose we have a dependence relation for the images of $x_1, \ldots, x_k$ in $\phi(M)/\phi(L)$; i.e., there exist $a_1, \ldots, a_k \in A$ and $y \in L$ not all zero such that

$$\phi_{a_1}(x_1) + \cdots + \phi_{a_k}(x_k) = y.$$

Since everything on the left side is integral at $\mathfrak{r}_i$, so is y. Since $y \in L$, this is the same as saying y is integral at $\mathfrak{p}_i$, so y is integral at $\mathfrak{q}_i$ as well. Now everything in the dependence equation is integral at $\mathfrak{q}_i$ except possibly $\phi_{a_i}(x_i)$, so this is integral also. In other words, if w and V_w are the valuation and local function on M corresponding to $\mathfrak{q}_i$, then since $V_w = \tilde{w}$, we get $V_w(\phi_{a_i}(x_i)) = 0$, whereas $V_w(x_i) = \tilde{w}(x_i) < 0$. By Corollary 1, this forces $a_i = 0$. This holds for each i, so the images of $x_1, \ldots, x_k$ in $\phi(M)/\phi(L)$ are A-independent, as desired. $\square$

THEOREM 3. *If M is a nontrivial finite separable extension of L, then $\phi(M)/\phi(L)$ is the direct sum of a free A-module of rank $\aleph_0$ and a finite torsion module.*

Proof. Lemmas 9 and 10 allow us to apply Proposition 10 from the Appendix. The torsion submodule is finitely generated (since it is tame), and hence finite, since all ideals of A have finite index. $\square$

REMARK. Theorem 3 fails miserably if we do not require M to be separable over L. For example, if $M = L^{1/p}$, which is a purely inseparable extension of L of degree p, the A-module structure of $\phi(M)/\phi(L)$ is the same as the usual A-module structure of M/L, because any positive power of the Frobenius τ acts as zero on M/L, so that a twisted polynomial $a_0 + a_1\tau + \cdots + a_d\tau^d$ acts only by its constant term. And of course, in the usual A-module structure, M/L is simply a finite-dimensional vector space over the quotient field K of A.

THEOREM 4. *If L^{perf} is the perfection of L, then $\phi(L^{\mathrm{perf}})$ is the direct sum of a free A-module of rank $\aleph_0$ and a finite torsion module.*

Proof. The rank of $\phi(L^{\mathrm{perf}})$ is at least that of $\phi(L)$, which is $\aleph_0$, and in fact must equal $\aleph_0$, since L^{perf} is countable as a set. Suppose M is a submodule of $\phi(L^{\mathrm{perf}})$ of finite rank. Let $m_1, m_2, \ldots, m_r \in M$ be a basis for the K-vector space $M \otimes_A K$. Then $m_1, m_2, \ldots, m_r$ and the submodule N they generate lie in L^{1/p^n} for some $n \geq 1$. Each element of M is a root of a separable polynomial $\phi_a(x) - m$ for some $m \in N$, $a \in A$. But L^{perf} is purely inseparable over L^{1/p^n}, so all of M must lie in L^{1/p^n}. Finally, L^{1/p^n} is just another global field, so $\phi(L^{1/p^n})$ is tame, and hence M is finitely generated. Thus $\phi(L^{\mathrm{perf}})$ is tame and we may apply Proposition 10.   □

7. Comparison with $\mathbb{G}_m$ over number fields

(Most of the material in this section is due to David Goss.) Let F be a number field with ring of integers $\mathcal{O} := \mathcal{O}_F$. Then the structure of F^* as an abstract group ($\mathbb{Z}$-module) is exactly analogous to the A-module structure of L given by a Drinfeld A-module over L, as described by Theorem 1. Specifically, there is the following result, which is well known although there seems to be no good reference.

PROPOSITION 8. *As an abstract abelian group F^* is isomorphic to the product of its torsion subgroup (i.e., the finite group of roots of unity in F) and a free abelian group of rank $\aleph_0$.*

Proof. Let $\mathfrak{P}$ be the group of principal fractional ideals. This is a subgroup of finite index in the free abelian group of divisors of F. Thus $\mathfrak{P}$ also is free of rank $\aleph_0$. Now let U be the units of $\mathcal{O}$. Then there is the standard exact sequence:

$$0 \to U \to F^* \to \mathfrak{P} \to 0.$$

The freeness of the group on the right now tells us that this sequence splits. Thus

$$F^* \simeq U \times \mathfrak{P}.$$

The result now follows from Dirichlet's unit theorem.   □

An obvious question is whether one can also obtain a proof of Theorem 1 in the same fashion as Proposition 8. This would be amazing since it would entail finding the analogue of the divisor group in the Drinfeld theory as well as the class group perhaps. It would also entail finding an

analogue of the unit group, and thus (hopefully) a canonical finitely generated submodule of the rational points, which would again be very important.

We can state analogues of Theorems 2 and 3 for the multiplicative groups of number fields as well, although sometimes the results are not entirely similar. (Again, these may be well known.)

PROPOSITION 9. *Let S be a finite set of places of F including the infinite ones, and let $\mathcal{O}_S$ be the ring of S-integers in F.*

(1) *$\mathcal{O}_S^*$ is a finitely generated abelian group.*
(2) *$F^*/\mathcal{O}_S^*$ is a free abelian group of rank $\aleph_0$.*
(3) *If E is a finite extension of F, then E^*/F^* is isomorphic to the product of a free abelian group of rank $\aleph_0$ with a finite torsion group.*

Sketch of proof. Part (1) is the Dirichlet S-unit theorem. Part (2) follows by consideration of divisors, as in the proof of Proposition 8. Part (3) can be proved in the same way as Theorem 3. (For the proof analogous to that of Lemma 10, choose primes $\mathfrak{p}_i$ of F which split completely in E, let $\mathfrak{q}_i$, $\mathfrak{r}_i$ be two primes of E above $\mathfrak{p}_i$, and choose $x_i \in E^*$ to be a unit at all the $\mathfrak{q}_j$'s and $\mathfrak{r}_j$'s, except not at $\mathfrak{q}_i$. Then considering the valuations corresponding to $\mathfrak{q}_i$ and $\mathfrak{r}_i$ shows that

$$x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \in F^*$$

only if each integer n_i is zero.) $\qquad\qquad\square$

In fact, the proof of Theorem 3 was inspired by the corresponding proof for multiplicative groups of number fields rather than the other way around.

Appendix: classification of tame modules of rank $\aleph_0$ over a Dedekind domain

Throughout this section, A is an arbitrary Dedekind domain, and K is its field of fractions. Recall that the rank of an A-module M is the dimension of the K-vector space $M \otimes_A K$, and that an A-module is called tame if every submodule of finite rank is finitely generated as an A-module. The goal of this section is to prove the following, which can be considered an extension of the classification theorem for finitely generated modules over a Dedekind domain.

PROPOSITION 10. *Every tame A-module M of rank $\aleph_0$ is isomorphic to the direct sum of its torsion submodule M_{tors} with a free A-module of rank $\aleph_0$.*

The special case where A is a principal ideal domain and M is torsion-free occurs as Exercise 52 in [9]. The only difficulty that arises from

generalizing to Dedekind domains is the elimination of non-principal fractional ideals, which is handled by Lemma 12. This lemma is well known, and in fact much more general results are known [1]. Nevertheless, for the case at hand, there is a beautiful proof which is not long, so we will give it anyway.

LEMMA 11. *If M is a torsion-free tame A-module of rank $\aleph_0$, then*

$$M \cong I_1 \oplus I_2 \oplus \cdots$$

where each I_i is a projective A-module of rank 1 (i.e., a fractional ideal).
  Proof. Let

$$0 = V_0 \subset V_1 \subset V_2 \cdots$$

be a full flag of K-vector spaces in $M \otimes_A K$; i.e.

$$\dim V_i = i \quad \text{and} \quad \bigcup_{i=1}^{\infty} V_i = M \otimes_A K.$$

Let $M_i = M \cap V_i$, so M_i is a finitely generated A-module with

$$M_i \otimes_A K = V_i \quad \text{and} \quad \bigcup_{i=1}^{\infty} M_i = M.$$

It suffices to construct for each $i \geq 1$ a projective A-module $I_i \subseteq M_i$ of rank 1 such that $M_i = M_{i-1} \oplus I_i$.
  We have an injection

$$M_i/M_{i-1} \hookrightarrow V_i/V_{i-1} \cong K.$$

Here M_i/M_{i-1} is embedded as a finitely generated sub-A-module of K, and it is nonzero (since $M_i \otimes_A K = V_i$), so M_i/M_{i-1} is a projective A-module of rank 1. Since it is projective, the exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$$

splits to give $M_i \simeq M_{i-1} \oplus I_i$, where I_i is a submodule of M_i projecting isomorphically onto M_i/M_{i-1}. Thus

$$M = \bigcup_{i=1}^{\infty} M_i = I_1 \oplus I_2 \oplus \cdots \qquad \square$$

LEMMA 12. *If $I_1, I_2, \ldots$ are fractional ideals of A, then*

$$I_1 \oplus I_2 \oplus \cdots \oplus \simeq A \oplus A \oplus \cdots$$

as A-modules.

Proof. Since the isomorphism type of the direct sum of two fractional ideals I, J is determined by the ideal class of IJ, we may replace $I_2 \oplus I_3$ on the left by the isomorphic A-module $I_1^{-1} \oplus J_1$ where $J_1 = I_1 I_2 I_3$ (in the group of fractional ideals). Similarly replace $I_4 \oplus I_5$ with $J_1^{-1} \oplus J_2$ where $J_2 = J_1 I_4 I_5$, replace $I_6 \oplus I_7$ with $J_2^{-1} \oplus J_3$ where $J_3 = J_2 I_6 I_7$, etc. We get

$$I_1 \oplus (I_2 \oplus I_3) \oplus (I_4 \oplus I_5) \oplus (I_6 \oplus I_7) \oplus \cdots$$

$$\cong I_1 \oplus (I_1^{-1} \oplus J_1) \oplus (J_1^{-1} \oplus J_2) \oplus (J_2^{-1} \oplus J_3) \oplus \cdots$$

$$\cong (A \oplus A) \oplus (A \oplus A) \oplus (A \oplus A) \oplus \cdots$$

as desired. □

Proof of Proposition 10. If M is a tame A-module of rank $\aleph_0$ then M/M_{tors} is a torsion-free A-module of rank $\aleph_0$, so by Lemmas 11 and 12, M/M_{tors} is a free A-module of rank $\aleph_0$. Because it is free, the exact sequence

$$0 \to M_{\mathrm{tors}} \to M \to M/M_{\mathrm{tors}} \to 0$$

splits and gives the desired result. □

REMARK. Proposition 10 fails if you replace $\aleph_0$ by a finite cardinal or by an uncountable one. It is of course possible to have finitely generated torsion-free modules which are not free, if A is not a principal ideal domain. In the other direction, an infinite direct *product* of copies of $\mathbb{Z}$ is a torsion-free tame $\mathbb{Z}$-module of uncountable rank, but it is not free (see Theorem 19.2 in [5].)

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