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Hodge theory and deformations of maps

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The purpose of this paper is to establish and apply a general principle (Theorem 1.1, Section 1) which serves to relate Hodge theory and deformation theory. This principle, which we state in a somewhat technical abstract-nonsense framework, takes on two dual forms which say roughly, respectively, that:

(0.1) *for a functorial homomorphism*

$$\tau: K \to L^1$$

from a Hodge-type group to an infinitesimal deformation group, $\mathrm{im}(\tau)$ *consists of unobstructed deformations;*

(0.2) *for a functorial homomorphism*

$$\pi: L^2 \to K$$

from an obstruction group to a Hodge-type group, "actual" obstructions lie in $\ker(\pi)$.

A familiar example of (0.1) is the case of deformations of manifolds with trivial canonical bundle, treated from a similar viewpoint in [R1]; a familiar example of (0.2) is Bloch's semi-regularity map [B], treated from a similar viewpoint in [R3]. Our principle may be viewed as an amalgamation and abstraction of parts of [R1] and [R3].

We will apply the above principle to deformations of maps of complex manifolds. After recalling and developing in Section 2 some general formalism concerning such deformations, we develop, in Sections 3 and 4, two instances of a map π as in (0.2) in the respective cases of generic immersions and fibre spaces. It may be noted that even in the case of an

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embedding our map π does *not* reduce to Bloch's semi-regularity map but rather has the latter as a component (cf. Remark 3.2.1); in particular our map has smaller kernel than Bloch's, leading to a better consequence of (0.2) (cf. Remark 3.4.1).

Some applications given in Sections 3 and 4 are to the question of *stability*, i.e. of identifying, given a generic immersion $f: Y \to X$ or a fibre space $f: X \to Y$, the locus of deformations of X to which f extends. We will show, in particular, that under favorable conditions the latter locus may be identified Hodge-theoretically (Corollaries 3.2, 4.2); in the case of an embedding f this may be considered as a special case of an infinitesimal form of the Generalized Hodge Conjecture. A particularly nice case is that of q-Lagrangian submanifolds of a q-symplectic manifold, in which we give a result (Corollary 3.4) generalizing work of Voisin [V].

In the final Section 5 we systematically apply the foregoing to the problem of moving curves on a manifold, a problem made popular in recent years by Mori theory. We will give a variety of conditions under which a curve is forced to move. As a final application, we describe the structure of manifolds X such that Ω_X^2 is spanned off a finite set: namely we prove (Theorem 5.3) that either X is fibred by tori or the canonical bundle K_X is ample. This is in complete analogy with the known case of 1-forms (Ueno's conjecture, cf. [R4]). It is interesting to note that the proof involves applications of both (0.1) and (0.2) (at two different points).

We begin in Section 1 by setting up an abstract categorical framework for deformation theory, one which seems best suited for stating (let alone proving) our general principle; it is shown in Section 2 that deformations of maps of manifolds fit into this framework, which is a variant of the usual one (cf. [S]); perhaps the main novelty here over [S] is the systematic use of "canonical elements" to describe infinitesimal deformations, in generalization of the method of [R1] and [R2]. For a different application of the results of Section 1, see [R6].

Convention. In this paper 'Kähler manifold' should be understood in the cohomological sense, i.e. a manifold whose Hodge-de Rham spectral sequence degenerates at E_1.

1. Abstract nonsense

Let $\mathscr{A}$ be the category of Artin local k-algebras with residue field k. In order to use geometric language, we may, by the assignment $R \to S = \mathrm{Spec}(R)$, identify the opposite category $\mathscr{A}^{op}$ with a category of (1-point) schemes over k. We may also, from time to time, refer to an element $R \in \mathscr{A}$ as (R, M) or (R, M, I) where M is the maximal ideal and

$I = \mathrm{Ann}(M) \subset R$. To category such as $\mathscr{A}$ we may associate 2 other categories: $\mathscr{A}$-$\mathscr{H}$om whose objects are morphisms of $\mathscr{A}$ and whose morphism are commutative squares in $\mathscr{A}$; and $\mathscr{A}$-$\mathscr{M}$od, whose objects are pairs (R, A) where $R \in \mathscr{A}$, and A is an R-module; and whose morphisms are defined by

$$\begin{aligned} &\mathrm{Hom}((R_1, A_1), (R_2, A_2)) \\ &\quad = \{(f, g)\colon f \in \mathrm{Hom}_{\mathscr{A}}(R_1, R_2),\ g \in \mathrm{Hom}_Z(A_1, A_2),\ g(ra) \\ &\qquad = f(r)g(a),\ \ r \in R_1,\ a \in A_1\}. \end{aligned}$$

Now consider a category $\mathscr{B}/\mathscr{A}$, i.e. a small category together with a contravariant functor $\Phi\colon \mathscr{B} \to \mathscr{A}$; we will sometimes write an object X of $\mathscr{B}$ as X/R or X/S indicating that $R = \Phi(X)$ or $S = \mathrm{Spec}(\Phi(X))$. We will assume that $\mathscr{B}$ has an unique object X^0/k. Define another category $\tilde{\mathscr{B}}$ as the fibre product

$$\begin{tikzcd} \tilde{\mathscr{B}} \arrow[r] \arrow[d] & \mathscr{A}\text{-}\mathscr{H}\mathrm{om} = \mathscr{A}^{op}\text{-}\mathscr{H}\mathrm{om} \arrow[d, "q"] \\ \mathscr{B} \arrow[r, "\Phi"] & \mathscr{A} = \mathscr{A}^{op} \end{tikzcd} \qquad q(S_2 \to S_1) = S_1$$

We will say that the category $\mathscr{B}/\mathscr{A}$ is *base-changing* if it comes equipped with a functor $\tilde{\mathscr{B}} \to \mathscr{B}$-$\mathscr{H}$om such that the following diagram commutes

$$\begin{tikzcd} & \mathscr{B}\text{-}\mathscr{H}\mathrm{om} \arrow[dl, "q'"'] \arrow[dr, "\Phi'"] & \\ \mathscr{B} & \tilde{\mathscr{B}} \arrow[l] \arrow[u] \arrow[r] & \mathscr{A}^{op}\text{-}\mathscr{H}\mathrm{om} \end{tikzcd}$$

where $q'(X_2 \to X_1) = X_1$ and Φ' is the evident functor induced by Φ; more concretely, what this means is that any X_1/S_1 and $S_2 \to S_1$ may be functorially completed to a "base change" diagram

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ S_2 \arrow[r] & S_1 \end{tikzcd}$$

In this case we will use the notation $X_2/S_2 = X_1 \times_{S_1} S_2$, $X_2/R_2 = X_1 \times_{R_1} R_2$, $S_i = \mathrm{Spec}(R_i)$.

Note that to a base-changing category $\mathscr{B}/\mathscr{A}$ we may associate a (covariant) functor

$$F: \mathscr{A} \to \mathscr{S}\text{ets}$$

$$F(R) = \Phi^{-1}(R)$$

and we will say that $\mathscr{B}$ is a *realization* of F. In what follows we will assume that F admits a hull in the sense of Schlessinger [S], in which case we will say that $\mathscr{B}$ is *hullable*. Let $\hat{\mathscr{B}} = \{(X/R, R')\colon X/R \in \mathscr{B}, \mathscr{R} = \mathscr{R}'/\mathscr{I}_{\mathscr{R}'} = \mathscr{R}'/\mathscr{I}'\}$.

Now given $\mathscr{B}/\mathscr{A}$ as above, by an *admissible $\mathscr{B}$-module*, we mean a functor $L\colon \hat{\mathscr{B}} \to \mathscr{A}\text{-}\mathscr{M}\text{od}$ compatible with Φ, together with the data, for all $(R, M, I) \in \mathscr{A}$ and $(X/R, R') \in \hat{\mathscr{B}}$, of a functorial exact diagram

$$\begin{array}{ccccccc}
L(X, R') \bigotimes_R I & \longrightarrow & L(X, R') & \longrightarrow & L(X, R') \otimes R/I & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & \\
L(X^0/k, k[I']) & \xrightarrow{f} & L(X, R')) & \xrightarrow{g} & L(X \times_R (R/I), R) & &
\end{array}$$

where the top row is the obvious thing and g is the map induced by base-change, plus a group $L(X^0)$ with functorial isomorphisms, for all $(R, M, I = M) \in \mathscr{A}, L(X^0/k, R) = L(X^0) \otimes M$. L is said to be *right* (resp. *left*)-exact if g is always surjective (resp. f is always injective). By a *∂-pair of* $\mathscr{B}$-modules we mean a pair (L^1, L^2) of admissible $\mathscr{B}$-modules together with the data, for all $(R, M, I) \in \mathscr{A}$ and $X/R \in \mathscr{B}$, of a functorial exact sequence

$$L^1(X^0) \otimes I \to L^1(X, R') \to L^1(X \times_R (R/I), R) \to L^2(X^0) \otimes I \to L^2(X, R')$$
$$\to L^2(X \times_R (R/I), R).$$

By a *linearization* of $\mathscr{B}/\mathscr{A}$ we mean a ∂-pair (L^1, L^2) of admissible $\mathscr{B}$-modules together with the data, for all $(R, M, I) \in \mathscr{A}, J = \mathrm{Ann}(M/I) \subset R$ and $\bar{X}/(R/I) \in \mathscr{B}$, of a canonical element

$$\alpha = \alpha_{\bar{X}} \in L^1(\bar{X} \times_{R/I} R/J, R/I)$$

with the property that $\bar{X}$ comes via base-change from some X/R iff α lies in the image of the canonical map

$$L^1(\bar{X}, R) \to L^1(\bar{X} \times_{R/I} (R/J), R/I).$$

Note that given a linearized base-changing category $\mathscr{B}/\mathscr{A}$, an admissible module $L' \subset L^1$ gives rise to a subcategory $\mathscr{B}' \subset \mathscr{B}$ defined "inductively" by the conditions that $X^0/k \in \mathscr{B}'$ and that given $X/R \in \mathscr{B}$ such that $X \times_R (R/I) \in \mathscr{B}'$, $X/R \in \mathscr{B}'$ iff the canonical element

$$\alpha_X \in L'(X \times (R/I), R) \subseteq L^1(X \times (R/I), R).$$

Next, by an *exact triple* of the linearized base-changing categories over $\mathscr{A}$ we mean a triple $(\mathscr{B}_i, L_i^1, L_i^2)$ of such, together with functors

$$\mathscr{B}_1 \to \mathscr{B}_2 \to \mathscr{B}_3,$$

an extension of (L_1^1, L_1^2) as ∂-pair to $\mathscr{B}_2$, still denoted (L_1^1, L_1^2), and an exact sequence of $\mathscr{B}_2$-modules

$$L_1^1 \to L_2^1 \to L_3^1 \xrightarrow{\partial} L_1^2 \to L_2^2 \to L_3^2,$$

where (L_3^1, L_3^2) are viewed as $\mathscr{B}_2$-modules in the obvious way, such that the respective canonical α-elements are compatible.

Now we can finally state our result:

THEOREM 1.1. *Let $(\mathscr{B}_i, L_i^1, L_i^2)$, $i = 1, 2, 3$ be an exact triple of linearized base-changing $\mathscr{A}$-categories, realizing respectively the functors $F_i: \mathscr{A} \to \mathscr{S}ets$. Let Z_i, $i = 1, 2, 3$, be a formal scheme which is a hull for F_i and $Z_1 \xrightarrow{\beta_1} Z_2 \xrightarrow{\beta_1} Z_3$ the induced maps. Also put $l_i^j = \dim_k(L_i^j(X_i^0))$ where $X_i^0/k \in \mathscr{B}_i$ is the unique object.*

(i) *Suppose K is a right-exact $\mathscr{B}_3$-module with a natural transformation*

$$\tau: K \to L_3^1$$

and put $\sigma = rk_k(\tau: K(X_3^0) \to L_3^1(X_3^0))$. Then there is a smooth σ-dimensional formal subscheme $Z_3^0 \subset Z_3$ corresponding to $\mathrm{im}(\tau)$, such that $Z_2^0 := \beta_2^{-1}(Z_3^0)$ has embedding dimension $\leqslant \sigma + l_1^1$ and is defined by at most l_1^2 equations.

(ii) *Suppose K is a left-exact $\mathscr{B}_2$-module with a natural transformation*

$$\pi: L_1^2 \to K$$

and put $\rho = rk_k(\pi: L_1^2(X_1^0) \to K(X_1^0))$. Suppose $Z_3^0 \subset Z_3$ is a formal subscheme corresponding to a subfunctor of

$$\ker(\pi \circ \partial: L_3 \to K) \subset L_3^1.$$

Then the natural map $Z_2^0 = \beta_2^{-1}(Z_3^0) \to Z_3^0$ *factors*

$$\begin{tikzcd} & Z_3^0 \times B \arrow[d] \\ Z_2^0 \arrow[ur, "i"] \arrow[r, "\beta_2"] & Z_3^0 \end{tikzcd}$$

where B *is smooth of dimension* l_1^1 *and* i *is an embedding onto a formal subscheme defined by at most* $l_1^2 - \rho$ *equations.*

Proof. Cases (i) and (ii) are essentially dual to each other; moreover the proof of case (ii) is almost word-for-word the same as that of Theorem 1 in [R3], which is its special case concerning deformations of embeddings and Bloch's semi-regularity map (cf. §2, 3). For completeness' sake, we will sketch the proof of case (i), which in turn is similar to that of Theorem 1 in [R1]. Take $(R, M, I) \in \mathscr{A}$, $J = \mathrm{Ann}(M/I)$, and suppose we have

$$\begin{array}{ccc} \bar{X}_2/(R/I) & \mapsto & \bar{X}_3/(R/I) \\ \mathscr{B}_2 & \to & \mathscr{B}_3 \end{array}$$

such that $\bar{X}_3$ belongs to the subcategory $\mathscr{B}_3^0 \subset \mathscr{B}_3$ corresponding to Z_3^0, i.e. such that the canonical element

$$\bar{\alpha}_3 = \alpha_{\bar{X}_3} \in L_3^1(\bar{X}_3 \underset{R/I}{\times} (R/J), R/I)$$

lies in the image of

$$K(\bar{X}_3 \underset{R/I}{\times} (R/J), R/I) \to L_3^1(\bar{X}_3 \underset{R/I}{\times} (R/J), R/I).$$

Then the right-exactness of K yields immediately that $\bar{\alpha}_3$ lifts to an element $\alpha_3 \in \mathrm{im}(K(\bar{X}_3, R) \to L_3^1(\bar{X}_3, R))$, hence $\bar{X}_3$ lifts to some $X_3/R \in \mathscr{B}_3^0$. This shows that Z_3^0 is smooth, and to complete the proof it remains to investigate the obstruction to lifting $\bar{X}_2$ to some $X_2/R \in \mathscr{B}_2$ which maps to X_3, or what is the same, the obstruction to lifting the canonical element

$$\bar{\alpha}_2 = \alpha_{\bar{X}_2} \in L_2^1(\bar{X}_2 \underset{R/I}{\times} (R/J), R/I).$$

to an element $\alpha_2 \in L_2^1(\bar{X}_2, R)$ which maps to a_3. Now from the commutative diagram

$$
\begin{array}{ccccc}
L_1^1(\bar{X}_2, R) & \longrightarrow & L_2^1(\bar{X}_2, R) & \longrightarrow & L_3^1(\bar{X}_3, R) \\
\downarrow & & \downarrow & & \downarrow \\
L_1^1(\bar{X}_2 \times (R/J), R/I) & \longrightarrow & L_2^1(\bar{X}_2 \times (R/J), R/I) & \longrightarrow & L_3^1(\bar{X}_3 \times (R/J), R/I) \\
\downarrow & & \downarrow & & \downarrow \\
I \otimes L_1^2(X_2^0) & \longrightarrow & I \otimes L_2^2(X_2^0) & \longrightarrow & I \otimes L_3^2(X_3^0)
\end{array}
$$

it is clear that the latter obstruction lies in $I \otimes L_1^2(X_2^0)$. □

2. Generalities on deformations of maps

In [R5], we described a general formalism for studying deformations of maps. Our purpose here is to revisit and elaborate on this in the relatively benign case of maps of manifolds, fitting it, in particular, into the abstract framework of Section 1. We begin with some topology. Let

$$f: X \to Y$$

be a continuous map of topological spaces. As in [R5], we associate to f a Grothendieck topology $\tau(f)$ whose open sets are the pairs (U, V) such that $U \subset X$ and $V \subset Y$ are open and $f(U) \subset V$, and whose notion of covering is the obvious one. Note that we have a commutative diagram of continuous maps of Grothendieck topologies

$$
\begin{tikzcd}
\tau(f) \arrow[r, shift left, "j"] & Y \arrow[l, shift left, "\pi"] \\
X \arrow[u, "i"] \arrow[ur, "f"'] &
\end{tikzcd}
\tag{2.1}
$$

where i, j, π are defined, respectively, by

$$i^{-1}(U, V) = U, j^{-1}(U, V) = V, \pi^{-1}(V) = (f^{-1}(V), V).$$

For sheaves E on X and F on Y, we define sheaves E^+, F_+, $F_\#$ on $\tau(f)$ by

$$E^+ = i_*(E), \ F_+ = j_*(F).$$

$$F_\# = \text{sheafification of } (U, V) \to \begin{cases} F(V) & U \neq \varnothing \\ 0 & U = \varnothing \end{cases}.$$

Note that we have

$$H^i(X, E) = H^i(\tau(f), E^+),\ H^i(Y, F) = H^i(\tau(f), F_+),$$

and that there is a canonical diagram of sheaves on $\tau(f)$:

$$\begin{tikzcd} F_+ \arrow[r, "f^{-1}"] & (f^{-1}F)^+ \\ F_\# \arrow[ur, "f^\#"'] & \end{tikzcd}$$

where $f^\#$ is injective provided f is open. Put

$$Q_F = \mathrm{cok}(f^{-1}) = \mathrm{cok}(f^\#).$$

A small remark which will be useful later is

LEMMA 2.1. *Suppose f is open and $f_* f^{-1}F \xleftarrow{\sim} F$. Then we have*

$$H^0(Q_F) = 0,\ H^1(Q_F) = H^0(R^1 f_* f^{-1}F).$$

Proof. Consider the exact sheaf sequence on $\tau(f)$

$$0 \to F_\# \to (f^*F)^+ \to Q_F \to 0 \tag{2.2}$$

Applying π_* to (2.2), using the basic diagram (2.1) and our hypothesis on F, we conclude easily that

$$\pi_* Q_F = 0, \quad R^i\pi_* Q_F \simeq R^i f_* f^{-1}F, \qquad i > 0.$$

Hence the Leray spectral sequence for Q_F and π yield the Lemma.

Suppose now that our $f\colon X \to Y$ is a map of complex manifolds. We may then define on $\tau(f)$ a sheaf T_f of 'f-related pairs of germs of vector fields' by

$$\Gamma((U, V), T_f) = \{(\alpha, \beta) : \alpha \in \Gamma(U, T_X),\ \beta \in \Gamma(V, T_Y),\ df(\alpha) = f^{-1}(\beta) \mid U\}.$$

Note the exact sequence

$$0 \to T_f \to (T_X)^+ \bigoplus (T_Y)_+ \to (f^{-1}T_Y)^+ \to 0 \tag{2.3}$$

and its cohomology sequence

$$\cdots H^0(f^{-1}T_Y) \to T^1_f \to T^1_X \oplus T^1_Y \to H^1(f^{-1}T_Y) \to T^2_f \to \cdots \tag{2.4}$$

where $T^i_f := H^i(\tau(f), T_f)$ and as usual, $T^i_X = H^i(X, T_X)$, $T^i_Y = H^i(Y, T_Y)$.

Now denote by $\mathscr{B}_f$ (resp. $\mathscr{B}_X$, $\mathscr{B}_Y$) the category of deformations of f (resp. X, Y). Based on the results of [R7], we extend T_X^1, T_X^2 to functors on $\hat{\mathscr{B}}_X$ making up a ∂-pair, in the following way, assuming $H^0(\Theta_X) = 0$. Let $J^\cdot(T_X) = \lim J^\cdot_m(T_X)$ be the Jacobi complex associated to the Lie algebra T_X and $R^{\text{univ}} = \lim R_m^{\text{univ}}$ the ring of the universal formal deformation. Given $(X_1/R, R') \in \hat{\mathscr{B}}_X$, there is a Kodaira-Spencer homomorphism $R^{\text{univ}} \to R$ which endows $M_{R'}$ and hence $T(R') = M^*_{R'}$ with an R^{univ} – module structure, which obviously factors through R_m^{univ}, $m = \text{exponent}(R)$. Define

$$\begin{aligned} T_X^i(X_1/R, R') &= Hom_{R^{\text{univ}}}(T(R'), H^{i-1}(J^\cdot(T_X)) \\ &= Hom_{R_m^{\text{univ}}}(T^m(R'), H^{i-1}(J^\cdot_m(T_X))),\ i = 1, 2. \end{aligned}$$

The canonical element is just the Kodaira-Spencer map $TR \to H^0(J^\cdot(T_X))$. Similarly for T_Y^i and T_f^i.

PROPOSITION 2.2. *For any compact complex manifold X (resp. map $f: X \to Y$ of such), the category $\mathscr{B}_X/\mathscr{A}$ (resp. $\mathscr{B}_f/\mathscr{A}$) forms a linearized base-changing category, with associated ∂-pair (T_X^1, T_X^2) (resp. (T_f^1, T_f^2)).*

Proof. This is a classical save possibly for the part involving the canonical element α. As for the latter, the case of manifolds X is discussed in detail in [R7], while that of map f is similar (and is mentioned in [R2]).

EXAMPLE 2.2. A special case of Theorem 1.1 is when $\mathscr{B}_2 = \mathscr{B}_3 = \mathscr{B}_f$, $\mathscr{B}_1 = \mathscr{A}$ (the "trivial" $\mathscr{A}$-category). In case (i), assuming further that τ is an isomorphism, the conclusion is that Def(f) is smooth; in other words, whenever T_f^1 is right-exact (i.e., in the terminology of [R1], f satisfies the T^1-lifting property) then f is unobstructed. This is precisely Theorem 1 of [R1].

3. Generic immersions

We now consider a generic immersion (i.e. generically unramified map) $f: Y^m \to X^n$. Let N denote the normal sheaf of f, defined by the exact sequence on Y.

$$0 \to T_Y \to f^{-1}T_X \to N \to 0 \tag{3.1}$$

Combining (3.1) with (2.3), we obtain the following exact sequence on $\tau(f)$:

$$0 \to T_f \to (T_X)_+ \to N^+ \to 0 \tag{3.2}$$

and its cohomology sequence

$$\cdots\ H^0(N) \to T_f^1 \to T_X^1 \to H^1(N) \to T_f^2\ \cdots$$

Denote by $\mathscr{B}_{f/X}$ the category of deformations of f with fixed target X, and note as before that $H^0(N)$, $H^1(N)$ extend to a ∂-pair on $\mathscr{B}_{f/X}$ and $\mathscr{B}_X$, and that the following is clear.

PROPOSITION 3.1

(i) *$\mathscr{B}_{f/X}$ forms a linearized base-changing category with associated ∂-pair* $(H^0(N),\ H^1(N))$.
(ii) $\mathscr{B}_{f/X} \to \mathscr{B}_f \to \mathscr{B}_X$ *forms an exact triple.*

Our purpose now is to define a $\mathscr{B}_f$-module K together with a natural transformation

$$\Pi\colon H^1(N) \to K,$$

so that Theorem 1.1(ii), becomes applicable. We work in the derived categories of the topologies in question, identifying N via (3.1) with the (quasi-isomorphism class of) the complex having T_Y in degree -1, $f^{-1}T_X$ in degree 0, and otherwise zeros. Let C_j be the complex on $\tau(f)$ given by

$$\underset{(0)}{(\Omega_X^j)_+} \to \underset{(1)}{(\Omega_Y^j)^+}.$$

Consider the tensor-product complex $N^+ \otimes C_j$, given by

$$\underset{(-1)}{(T_Y)^+ \otimes (\Omega_X^j)_+} \to \underset{(0)}{(f^*T_X)^+ \otimes (\Omega_X^j)_+ \oplus (T_Y)^+ \otimes (\Omega_Y^j)^+} \to \underset{(1)}{(f^*T_X)^+ \otimes (\Omega_Y^j)^+},$$

and note the natural "contraction" map

$$N^+ \otimes C_j \to (\Omega_Y^{j-1})^+,$$

whence a map on cohomology

$$\mu_{ij}\colon H^1(N) \times \mathbb{H}^i(C_j) \to H^{i+1}(\Omega_Y^{j-1}).$$

Transposing, we obtain maps

$$\pi_{ij}\colon H^1(N) \to H^{i+1}(\Omega_Y^{j-1}) \otimes \mathbb{H}^i(C_j)^*, \quad 0 \leqslant i \leqslant m-1,\ 1 \leqslant j \leqslant m+1,$$

which we may assemble together as

$$\Pi = \bigoplus_{\substack{0 \leqslant i \leqslant m-1 \\ 1 \leqslant j \leqslant m+1}} \pi_{ij}.$$

Note that the spectral sequence of hypercohomology of C_j yields exact sequences

$$0 \to \mathrm{cok}(H^{j,i-1}(f)) \to \mathbb{H}^i(C_j) \to \ker(H^{j,i}(f)) \to 0$$

where $H^{j,i}(f)\colon H^{j,i}(X) \to H^{j,i}(Y)$ is the pullback map. In particular, standard Hodge theory (cf. [D]) implies for X Kähler that

$$K = \bigoplus_{\substack{0 \leq i \leq m-1 \\ 1 \leq j \leq m+1}} H^{i+1}(\Omega_Y^{j-1}) \otimes \mathbb{H}^i(C_j)^*$$

extends to a left (as well as right)-exact $\mathscr{B}_f$-module. Moreover, note the commutative diagram

$$\begin{array}{ccccccc} T_X^1 \times \mathbb{H}^i(C_j) & \to & T_X^1 \times \ker(H^{j,i}(f)) & \hookrightarrow & T_X^1 \times H^i(\Omega_X^j) & \to & H^{i+1}(\Omega_X^{j-1}) \\ \downarrow & & & & & & \downarrow \\ H^1(N) \times \mathbb{H}^i(C_j) & & & \to & & & H^{i+1}(\Omega_Y^{j-1}). \end{array} \tag{3.3}$$

This implies the following. Let

$$\varphi\colon T_X^1 \to \mathrm{End}_{(1,-1)}(\oplus H^i(\Omega_X^j))$$

be the natural map induced by cup-product, and $T_X^{1,0} \subset T_X^1$ the subgroup of elements leaving $\oplus \ker(H^{j,i}(f))$ invariant (via φ). Then $T_X^{1,0}$ extends to a functor on $\mathscr{B}_X$ which gives rise to a subcategory $\mathscr{B}_X^0 \subset \mathscr{B}_X$ which consists precisely of the deformations preserving $\ker(H^{\cdot,\cdot}(f))$ as sub-Hodge structure, and similarly a formal subscheme

$$\mathrm{Def}(X)^0 \subset \mathrm{Def}(X).$$

which may be thought of as a type of "infinitesimal Noether-Lefschetz locus". The following, then, is clear from the above discussion:

COROLLARY 3.2. *With the above notations, Theorem* 1.1, (*ii*) *is applicable to the exact triple* $\mathscr{B}_{f/X} \to \mathscr{B}_f \to \mathscr{B}_X$, *the subcategory* $\mathscr{B}_X^0 \subset \mathscr{B}_X$ *and the map* $\Pi\colon H^1(N) \to K$, *provided* X *is Kähler. In particular, the hull* $\mathrm{Def}(f/X)$ *of* f *has embedding dimension* $h^0(N)$ *and is defined by at most* $h^1(N)$-ρ *equations,* $\rho := rk_{\mathbb{C}}(\Pi)$. *The latter conclusion holds even if* X *is not Kähler.*

Proof. It remains to justify the last sentence. The point here is that the Kähler hypothesis on X is used only in controlling the Hodge theory of deformations of X, and these do not come into play in considering $\mathrm{Def}(f/X)$.

REMARK 3.2.1. Note that $C_{m+1} = \Omega_X^{m+1}$. Identifying $H^{m-1}(\Omega_X^{m+1})^* = H^{p+1}(\Omega_X^{p-1})$, $p = n - m$, it is easy to see that for f an embedding, $\pi_{m-1,m+1}$ coincides with the Kodaira-Spencer-Bloch semi-regularity map. Thus Corollary 3.2 constitutes both a refinement and a generalization of Theorem 1 of [R3].

In section 5, we will apply Corollary 3.2 systematically to curves. For now we give 2 other applications. First to an infinitesimal form of the *generalized Hodge conjecture*: this would say that in the above situation we should have

$$\mathscr{B}_X^0 = \mathrm{im}(\mathscr{B}_f \to \mathscr{B}_X) \tag{3.3}$$

In this regard, Corollary 3.2 yields the following.

COROLLARY 3.3. *In the situation of Corollary* 3.2, *suppose that*

$$\dim(Def(f/X)) = h^0(N)\text{-}h^1(N) + \rho$$

and X is Kähler. Then the infinitesimal generalized Hodge conjecture (3.3) *holds.*

Next, let us say that a manifold X^n is *q-symplectic* if it admits a holomorphic q-form ω such that the induced contraction map $\neg\omega$: $T_X \to \Omega_X^{q-1}$ is a bundle injection; in this case an immersion f: $Y^m \to X$ is said to be *q-lagrangian* if the pullback of ω on Y vanishes and $m + \binom{m}{q-1} = n$ (if $q = 2$ these become the usual notions of complex symplectic manifold and lagrangian submanifold). Note that for such Y, ω induces an *isomorphism* $N \xrightarrow{\sim} \Omega_Y^{q-1}$, and in particular the map π_{0q} is injective, hence so is Π. Hence Corollary 3.2 yields

COROLLARY 3.4. *Let X be a q-symplectic complex manifold and f: $Y \to X$ a q-lagrangian immersion. Then $Def(f/X)$ is smooth. If moreover X is Kähler then the infinitesimal generalized Hodge conjecture* (3.3) *holds.*

REMARK 3.4.1. For $q = 2$ this was proven by Voisin [V], who also points out that Bloch's semi-regularity map, i.e. $\pi_{m-1,m+1}$, need not be injective in this case, whence the advantage of using Π.

4. Fibre spaces

We now consider the case where our map f: $X^n \to Y^m$ of compact complex manifolds is a *fibre space*, i.e. f is proper, flat and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Combining the exact sequence (2.3) with (2.2) for $F = T_Y$, we obtain the exact sheaf sequence on $\tau(f)$

$$0 \to T_f \to (T_X)^+ \to Q_{T_Y} \to 0,$$

whose cohomology sequence reads, in view of Lemma 2.1,

$$0 \to T_f^1 \to T_X^1 \to H^0(R^1 f_* \mathcal{O}_X \otimes \mathcal{T}_Y) \to T_f^2 \to \cdots$$

Denote by $\mathscr{B}_{X\setminus f}$ the category of deformations of f with fixed source X. Note that in this case $\mathscr{B}_{X\setminus f} \simeq \mathscr{A}$, i.e. there are no nontrivial deformations of f fixing X, so that the natural map $\mathscr{B}_f \to \mathscr{B}_X$ is in fact an embedding, hence $\mathrm{Def}(f) \hookrightarrow \mathrm{Def}(X)$. As before, the pair of groups $(0, H^0(R^1 f_* \mathcal{O}_X \otimes T_Y))$ extends to a ∂-pair on $\mathscr{B}_{X\setminus f}$ and $\mathscr{B}_f$, and we have the following

PROPOSITION 4.1. *$\mathscr{B}_{X\setminus f} \to \mathscr{B}_f \to \mathscr{B}_X$ forms an exact triple.*

We proceed again to define a map

$$\Sigma: H^0(R^1 f_* \mathcal{O}_X \otimes \mathcal{T}_Y) \to K.$$

Define a sheaf C_j on $\tau(f)$ by the exact sequence

$$0 \to (\Omega_Y^j)_+ \to (\Omega_X^j)^+ \to C_j \to 0,$$

Note the commutative diagram

$$\begin{array}{ccccc} (\Omega_Y^j \otimes T_Y)_+ & \longrightarrow & (\Omega_Y^j) \otimes (f^* T_Y)^+ & \longrightarrow & (\Omega_Y^j)_+ \otimes \mathrm{Q}_{T_Y} \\ \downarrow & & \downarrow & & \downarrow \\ (\Omega_Y^{j-1})_+ & \longrightarrow & (\Omega_X^{j-1})^+ & \longrightarrow & C_{j-1} \end{array} \tag{4.1}$$

where the left vertical arrow is contraction and the right vertical arrow is deduced in an obvious way from the pairing $(\Omega_Y)_+ \otimes (f^* T_Y)^+ \to (\mathcal{O}_X)^+$ and the map $(\Omega_Y^{j-1})_+ \to (\Omega_X^{j-1})^+$.

The right vertical arrow of (4.1) then yields a pairing on cohomology

$$\mu_{ij}: H^0(R^1 f_* \mathcal{O}_X \otimes \mathcal{T}_Y) \otimes H^i(\Omega_Y^j) \to H^{i+1}(C_{j-1})$$

whence a map

$$\sigma_{ij}: H^0(R^1 f_* \mathcal{O}_X \otimes T_Y) \to H^{i+1}(C_{j-1}) \otimes H^{m-i}(\Omega_Y^{m-j}), \quad 0 \leqslant i \leqslant m, 1 \leqslant j \leqslant m,$$

and we may assemble these into

$$\Sigma = \oplus \sigma_{ij}: H^0(R^1 f_* \mathcal{O}_X \otimes T_Y) \to \bigoplus_{\substack{0 \leq i \leq m \\ 1 \leq j \leq m}} H^{i+1}(C_{j-1}) \otimes H^{m-i}(\Omega_Y^{m-j}) =: K$$

As before, we have an exact sequence

$$0 \to \operatorname{cok}(H^{j,i}(f)) \to H^i(C_j) \to \ker(H^{j,i+1}(f)) \to 0$$

where $H^{\cdot,\cdot}(f)$ is the pullback map, showing in particular that K yields a left-exact module on $\mathscr{B}_f$. And as before (3.3), μ_{ij} is compatible with the natural composite map

$$\begin{aligned} T_X^1 \otimes H^1(\Omega_Y^j) &\to T_X^1 \otimes H^1(\Omega_X^j) \to H^{i+1}(\Omega_X^{j-1}) \\ &\to \operatorname{Cok}(H^{j-1,i+1}(f)) \hookrightarrow H^{i+1}(C_{j-1}). \end{aligned}$$

and we may consider the subgroup $T_X^{1,0} \subset T_X^1$ leaving invariant $\oplus \operatorname{im}(H^{j,i}(f))$, and the associated subcategory $\mathscr{B}_X^0 \subset \mathscr{B}_X$ and subscheme $\operatorname{Def}(X)^0 \subset \operatorname{Def}(X)$, which evidently contains $\operatorname{Def}(f)$. In analogy with Corollary 3.2, we have

COROLLARY 4.2. *With the above notation, Theorem* 1.1(*ii*), *is applicable to the exact triple* $\mathscr{B}_{X\backslash f} \to \mathscr{B}_f \to \mathscr{B}_X$, *the subcategory* $\mathscr{B}_X^0 \subset \mathscr{B}_X$ *and the map* $\Sigma: H^0(R^1 f_* \mathcal{O}_X \otimes T_Y) \to K$. *In particular, as subscheme of* $Def(X)^0$, $Def(f)$ *is defined by at most* $h^0(R^1 f_* \mathcal{O}_X \otimes T_Y) - rk\Sigma$ *equations and if* Σ *is injective then* $Def(f) = Def(X)^0$.

REMARK 4.3. It is not hard to check that the target of $\sigma_{m,m}$ is just $H^{m-1,m+1}(X)$ and that the composite map

$$T_X^1 \to H^0(R^1 f_* \mathcal{O}_X \otimes T_Y) \xrightarrow{\sigma_{m,m}} H^{m-1,m+1}(X)$$

represents the obstruction to the cohomology class $[F]$ of a fibre of f remaining of type (m, m) under deformation of X; in particular, whenever $\sigma_{m,m}$ is itself injective, the "Noether-Lefschetz" type sublocus $Def(X)^{00} \subset Def(X)$ where $[F]$ has type (m, m) already coincides with $Def(f)$.

5. Moving curves

We now consider applications of the preceding results, in particular Corollary 3.2, to the problem of moving curves on complex manifolds. Let $f: Y \to X$ be a nonconstant map from a connected projective nonsingular curve of genus g to a complex n-manifold (we call this a curve of genus g on X) and let $\mathscr{H}$ be any component of $\mathrm{Def}(f/X)$ through $\{f\}$. Then Corollary 3.2 coupled with Riemann-Roch yields the basic estimate

$$\dim \mathscr{H} \geqslant -Y.K_X + (n-3)(1-g) + \rho_1 + \rho_2 \tag{5.1}$$

where $Y.K_X := \deg(f^*K_X)$ and $\rho_i = \mathrm{rk}(\pi_{0i})$. Note that in the present circumstances $H^1(N)$ is dual to $H^0(N^* \otimes \Omega_Y)$ and the maps π_{01}, π_{02} are respectively dual to the natural maps

$${}^t\pi_{01} : \ker(H^{0,1}(f)) \otimes H^0(\Omega_Y) \xrightarrow{f^*\otimes id} H^0(N^*) \otimes H^0(\Omega_Y) \to H^0(N^* \otimes \Omega_Y).$$

$${}^t\pi_{02} : H^0(\Omega_X^2) \xrightarrow{f^*} H^0(f^*\Omega_X^2) \to H^0(N^* \otimes \Omega_Y),$$

the last map coming from the exact sequence

$$0 \to \Lambda^2 N^* \to f^*\Omega_X^2 \to N^* \otimes \Omega_Y$$

We seek some conditions which imply that ρ_1 or ρ_2 is positive. To this end, note the following elementary observations:

(i) if $\eta \in \ker(H^{1,0}(f))$ is nonzero at some point of $f(Y)$, and $0 \neq \zeta \in H^0(\Omega_Y)$, then ${}^t\pi_{01}(\eta \otimes \zeta) \neq 0$;
(ii) if $\omega \in H^0(\Omega_X^2)$ is nondegenerate at some point of $f(Y)$, then ${}^t\pi_{02}(\omega) \neq 0$, hence $\rho_2 > 0$;
(iii) if Ω_X^2 is spanned at some point of $f(Y)$, then $H^0(\Omega_X^2)$ generically spans $N^* \otimes \Omega_Y$, hence $\rho_2 \geqslant n-1$;
(iv) in general, $\rho_2 \geqslant rk(f^*) - h^0(\Lambda^2 N^*)$; moreover if $n = 3$ then $\Lambda^2 N^*$ is a line bundle on Y of degree at most $v = Y.K_X - (2g-2)$, so its h^0 may be estimated.

For example, combining (ii) with (5.1) yields the following result, which was already noted in [R3] in the case of smooth curves.

COROLLARY 5.1. *Let X be a symplectic n-fold. Then any rational* (*resp. elliptic*) *curve in X moves in a family of dimension at least $n-2$* (*resp.* 1).

EXAMPLE 5.2. A rational curve on a symplectic 4-fold X must fill up either a divisor or a rational surface. Consideration of the case $X = \mathscr{H}ilb_2(S)$, S a $K3$ surface, shows that both possibilities can occur: indeed take Y of the form $s + Z$ where $Z \subset S$ is a rational curve and $s \in S$; if $s \notin Z$ then Y fills up $S + Z$, while if $s \in Z$ then Y fills up $\mathscr{H}ilb_2(Z) \sim \mathbb{P}^2$.

As another application, we consider the structure of manifolds carrying many 2-forms. To put matters in perspective, we recall first the well-known case of 1-forms: if X is a Kähler manifold such that Ω_X^1 is spanned, then the Albanese mapping

$$a: X \to Alb(X)$$

is an immersion, and it is well-known ([U], Thm. 10.9, p. 120) that either X is fibred by tori, or else it is of general type, in which case it is known that the canonical bundle K_X is ample [R4].

THEOREM 5.3. *Let X be a Kähler manifold of dimension $n \geqslant 3$ such that Ω_X^2 is spanned outside a finite set. Then either X is fibred by tori or its canonical bundle K is ample.*

Proof. Note first that $(n-1)K = \det(\Omega_X^2)$ has finite base locus and it follows by some classical work of Zariski [Z] (kindly pointed out to me by M. Reid) that mK is free for large m, so that we have a stable pluricanonical morphism

$$\varphi = \varphi_{|mK|} : X \to Z \hookrightarrow \mathbb{P}^N.$$

We begin with the case $n = 3$. Suppose first that X has Kodaira dimension $\kappa = 0$. As $2K$ is effective, $2K$ must be trivial, so that we can write

$$X = \tilde{X}/\tau$$

where τ is a fixed-point-free involution and $K_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$. As $T_{\tilde{X}} = \Omega_{\tilde{X}}^2 \otimes K_{\tilde{X}}^{-1} = \Omega_{\tilde{X}}^2$ is spanned outside a finite set, it follows from Claim 5.3.0 below that $\tilde{X}$ must be a torus; moreover as $H^0(\Omega_X^2) = H^0(\Omega_{\tilde{X}}^2)^\tau$ is (at least) 3-dimensional, τ must be a pure translation, so that X itself is a complex torus.

CLAIM 5.3.0. *Let X be a non-uniruled compact Kähler manifold with generically spanned tangent bundle. Then X is a complex torus.*

Proof. Let $\mathrm{Aut}^\circ(X)$ denote the identity component of the holomorphic automorphism group of X and $\mathrm{Alb}(X)$ its Albanese torus. There is a natural map

$$\alpha : \mathrm{Aut}^\circ(X) \to \mathrm{Aut}^\circ(\mathrm{Alb}(X)) = \{\text{translations}\} = \mathrm{Alb}(X).$$

By results of Fukiji [F] and Lichnérowicz [L], $\ker\alpha$ is a reductive linear algebraic group. As X is not unifuled, $\ker\alpha$ must be trivial, i.e. α is injective.

As $\mathrm{Lie}(\mathrm{Aut}^\circ(X)) = H^\circ(T_X)$ generically spans T_X, it follows that there is a subgroup of the translation group on $\mathrm{Alb}(X)$ acting on X with Zariski open orbit, hence X itself is a torus (and so $X = \mathrm{Alb}(X)$).

Suppose next that $\kappa = 1$, and let S be a generic fibre of φ. Then $\kappa(S) = 0$ by Iitaka's fibration theorem ([U] Thm. 5.10, p. 58) and moreover the 2-forms on X yield a nonzero 2-form on S, so that S must be an abelian or $K3$ surface. In the former case there is nothing to prove; in the latter case S must contain a rational curve Y, and by (5.1) and observation (iii) above Y must move on X in a 2-parameter family, which is obviously impossible.

If $\kappa = 2$ then φ is an elliptic fibration, so again there is nothing to prove. It remains to prove that if $\kappa = 3$ then K is ample. If not then, as mK is free, there is a curve $Y \to X$ with $Y.K = 0$. As above, Y must move in a 2-dimensional family and fill up a surface E such that $\varphi(E)$ is a point, and in particular $K \otimes \mathcal{O}_E$ is torsion. Let

$$g: \tilde{E} \to E$$

be the normalization, and $\tilde{E}^0 \subset \tilde{E}$ the smooth part. Then we have an inclusion

$$\omega_{\tilde{E}^0} \hookrightarrow g^*\omega_E|_{\tilde{E}^0} = g^*\mathcal{O}_E(E) \otimes K|_{\tilde{E}^0}.$$

But as $\mathcal{O}_E(E)$ must be negative, we have

$$H^0(g^*\mathrm{O}_E(E) \otimes K) = 0$$

hence by normality

$$H^0(g^*\mathrm{O}_E(E) \otimes K|_{\tilde{E}^0}) = 0,$$

so that $H^0(\omega_{\tilde{E}^0}) = H^0(\Omega^2_{\tilde{E}^0}) = 0$, contradicting the existence of many 2-forms on X and completing the proof in case $n = 3$.

Suppose now $n \geqslant 4$. If $0 < \kappa(X) < n$, we may apply induction to a generic fibre of the stable pluricanonical morphism of X. Suppose $\kappa(X) = 0$. As above, we may assume K is trivial, and consider a de Rham decomposition of X. As Ω^2_X is generically spanned, clearly X cannot have an irreducible symplectic factor of dimension $\geqslant 4$ nor an $SU(n \geqslant 3)$ factor, and we may argue as above to see X has no $K3$ factor either. Thus X is a torus and we are done. Hence we may assume X is of general type, and it will suffice to derive a contradiction from the existence of a curve $f: Y \to X$ with $K \,.\, Y = 0$ (actually, the argument we give for this works for

$n = 3$ as well, but is less elementary than the above). Put $E = f^*T_X$, $F = \Lambda^2 E^* = f^*\Omega_X^2$. We claim first that F is trivial. This follows from

LEMMA 5.4. *Let F be a generically spanned vector bundle of degree* 0 *on a curve. Then F is trivial.*

Proof. By hypothesis, every quotient bundle of F must have nonnegative degree, and this implies that F cannot have a nonzero section vanishing somewhere, so that $h^0(F) = rk(F)$ and we have an exact sequence

$$0 \to H^0(F) \otimes \mathcal{O} \to F \to (\text{torsion}) \to 0.$$

Comparing degrees, we see that (torsion) $= 0$. □

I claim next that our bundle E must be semi-stable: indeed if $E_0 \subset E$ were a (locally split) subbundle of positive degree, then either E_0 or E/E_0 must have rank $\geqslant 2$, hence either $\deg(\Lambda^2 E_0) > 0$ or $\deg(\Lambda^2(E/E_0)) < 0$. Since these are respectively a subbundle and a quotient bundle of the trivial bundle $\Lambda^2 E$, we have a contradiction.

Using the theorem of Narasimhan and Seshadri [NS], our semi-stable bundle E is S-equivalent to a bundle which arises from a unitary representation

$$\rho\colon \pi_1(Y) \to U(n).$$

By assumption, the composite homomorphism

$$\pi_1(Y) \to U(n) \xrightarrow{\Lambda^2} U(n(n-1)/2)$$

is trivial, and as $\ker(\Lambda^2) = \{\pm 1\}$ it follows that $\mathrm{im}(\rho) \subseteq \{\pm 1\}$. By composing f with a 2:1 covering of Y, we may assume ρ is trivial. This means that E possesses a filtration

$$E_0 = 0 \subset E_1 \subset \cdots \subset E_n = E$$

such that $E_i/E_{i-1} = \mathcal{O}_Y$, $i = 1, \ldots, n$.

LEMMA 5.5. *Let E be a vector bundle on a curve such that $\Lambda^2 E$ is trivial and E/E' is trivial for some proper subbundle $E' \subset E$. Then E is trivial.*

Proof. We may assume $\mathrm{rk}E' = \mathrm{rk}E - 1$. Note that $E' = E' \otimes (E/E')$ is a quotient of $\Lambda^2 E$, hence E' is trivial by Lemma 5.4. Now take an arbitrary rank-1 trivial subbundle $M \subseteq E'$. Then from the exact sequence

$$0 \to M \otimes (E/M) \to \Lambda^2 E \to \Lambda^2(E/M) \to 0$$

and Lemma 5.4 again, it follows that $\Lambda^2(E/M)$ is trivial, hence so is $M \otimes (E/M) = E/M$ hence $E/M \to E/E'$ splits. This implies that the extension element $[E] \in H^1((E/E')^* \otimes E') = H^1(E')$ goes to zero in $H^1((E/E')^* \otimes (E'/M)) = H^1(E'/M)$. M being arbitrary it follows that $[E] = 0$ so E is trivial. □

Now a section of the trivial bundle $E = f^*T_X$ yields a first order deformation $\tilde{f}$ of f, and by a similar argument as above we see that $\tilde{f}^*T_X$ is trivial as well etc. (use the fact that a deformation of a trivial bundle E inducing a trivial deformation on $\Lambda^2 E$ and $\det(E)$ is trivial). Therefore by applying Theorem 1.1(i) to $\mathrm{Def}(f/X)$ we see that f has unobstructed deformations whose images fill up X, which is a contradiction. □

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