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Construction of some classes in the cohomology of Siegel modular threefolds

Dedicated to Frans Oort on the occasion of his 60th birthday

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Introduction

In a previous paper [6] we computed the Betti numbers and L-functions for several Siegel modular threefolds. In some of these cases we found that the L-function is a product of two Hecke L-series attached to characters of an imaginary quadratic fields. This indicates that the underlying Galois representation should be a product of two abelian representations and indeed that the "motive"associated to H^3 of the variety should be a product of two CM-motives. The purpose of this paper is to show that this is in fact the case.

We show this by constructing classes in $H^3_{\mathbb{Q}}$ which lie in the $(1,2),(2,1)$ part of the Hodge structure. These classes are cohomology classes of certain hyperbolic threemanifolds which are naturally embedded in the Siegel threefold (these submanifolds also play important roles in the work of Andrianov [2] and Shintani [7]). The main problem is to show that these cohomology classes are non-zero. The usual method for proving that cohomology classes in arithmetic quotients of symmetric spaces are non-zero is to integrate automorphic forms over the class ([4]). In this case the integral vanishes and so this method can not be applied. In fact, for Siegel modular forms the non-vanishing of the integral is an indication that the form is a lift from an elliptic modular form (see [7] and the appendix in [6]). Here we instead use a modification of an idea of Millson–Raghunatan ([5]). We intersect the class with one of its conjugates by an element in $\mathrm{Sp}_4(\mathbb{Z})$ and show that the intersection is non-zero.

In the first three sections we prove some general results about these submanifolds, giving a moduli theoretic description of them and describe how they compactify in the toroidal compactification of the Siegel modular threefold. The main result is that the closure in the toroidal compactification is a smooth, orientable manifold and that two distinct submanifolds of this type only has finitely many points of intersection in the interior.

In the last section we apply some of these results to our specific case and using the very explicit description of the Siegel modular threefold as a complete intersection in $\mathbb{P}^7$ we construct two submanifolds with the property that all the intersection points have the same multiplicity and hence the intersection product is non-zero.

1. Manifolds of Picard type in Siegel threefolds

We fix a lattice $V_{\mathbb{Z}} = \mathbb{Z}^4$ equipped with the standard alternating form ψ given by the matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ in the standard basis e_1, e_2, e_3, e_4. Consider $\bigwedge^2 V_{\mathbb{Z}} \cong \mathbb{Z}^6$ with standard basis $\{e_i \wedge e_j\}_{i<j}$ ordered lexicographically. This lattice comes with a symmetric bilinear form ϕ defined by $e_i \wedge e_j \wedge e_k \wedge e_l = \phi(e_i \wedge e_j, e_k \wedge e_l) e_1 \wedge e_2 \wedge e_3 \wedge e_4$. The matrix of ϕ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence is an even, unimodular pairing of signature $(+3, -3)$. In the following it will be convenient sometimes to identify $\bigwedge^2 V_{\mathbb{Z}}$ with the space $L_{\mathbb{Z}}$ of skew-symmetric 4×4 integral matrices

$$\begin{pmatrix} \begin{matrix} 0 & a \\ -a & 0 \end{matrix} & B \\ -{}^tB & \begin{matrix} 0 & c \\ -c & 0 \end{matrix} \end{pmatrix}$$

by identifying $e_i \wedge e_j$ with the matrix with 1 as (i, j)'th entry and -1 as the (j, i)'th entry and 0's everywhere else. Under this identification we have $\phi(M, M) = 2ac - 2\det(B)$ where M is a matrix as above. The alternating form ψ defines a linear functional on $\bigwedge^2 V_{\mathbb{Z}}$ by $e_i \wedge e_j \mapsto \psi(e_i, e_j)$. In terms of skew-symmetric matrices this functional is given by $M \mapsto \operatorname{tr}(B)$. We let $P_{\mathbb{Z}}$ denote the kernel. The symmetric form ϕ restricted to $P_{\mathbb{Z}}$ has signature $(3, -2)$ and discriminant 2.

Let $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$ be a subgroup of finite index which we assume to be torsion free. It is known that Γ contains a congruence subgroup $\Gamma(n)$. By our assumption on Γ we have $n \geqslant 3$. Consider a primitive (i.e. not divisible) element $v \in P_{\mathbb{Z}}$ with $\phi(v, v) < 0$ and n prime to $\phi(v, v)$.

Consider the following data: $(A, \mathcal{L}, x, \alpha)$ where $(A, \mathcal{L})$ is a principally polarized abelian surface over $\mathbb{C}$, x is a primitive element in the primitive cohomology $P^2(A, \mathbb{Z})$ such that $x \cup x = \phi(v, v)$. The Hodge structure h on $P^2(A, \mathbb{Z})$ induces a decomposition, orthogonal w.r.t. the cup-product $P^2(A, \mathbb{R}) = P_h^+ \oplus P_h^-$ where P_h^+

is a positive definite 3-dimensional subspace and P_h^- is a negative definite oriented 2-dimensional subspace. We require that $x \in P_h^-$. Finally α is a $\Gamma/\Gamma(n)$ equivalence class of isomorphisms $\alpha\colon V_{\mathbb{Z}/n} \to H^1(A, \mathbb{Z}/n)$ compatible with the pairings (the Weil pairing on $H^1(A, \mathbb{Z}/n)$) such that $\bigwedge^2 \alpha : L_{\mathbb{Z}/n} \to \bigwedge^2 H^1(A, \mathbb{Z}/n) = H^2(A, \mathbb{Z}/n)$ maps v to x. Remark that $\bigwedge^2 \alpha$ maps $P_{\mathbb{Z}/n}$ to $P^2(A, \mathbb{Z}/n)$. The notion of isomorphism between two such objects is obvious and we define $\mathcal{A}_{2,\Gamma,v}$ to be the set of isomorphism classes.

We shall give another description of this set. Let M be the skew-symmetric matrix corresponding to v and let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Put $N = M\ J$ then we have $N^2 = \frac{\phi(M,M)}{2} I_4$. If we let $d = \frac{\phi(M,M)}{2} < 0$ then N makes $V_{\mathbb{Z}}$ a module over the order $\mathcal{O}$ spanned by $1, \sqrt{d}$ in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$. Let $h : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G\,\mathrm{Sp}(V_{\mathbb{R}})$ be a polarized Hodge structure of type $(1,0),(0,1)$ on $V_{\mathbb{Z}}$. The group $G\,\mathrm{Sp}(V_{\mathbb{R}})$ acts on $L_{\mathbb{R}}$ by $g \in G\,\mathrm{Sp}(V_{\mathbb{R}}) : M \mapsto g\ M\ {}^tg$. This action preserves the linear funtional and hence stabilizes $P_{\mathbb{R}}$. In fact it defines an isomorphism $\mathrm{Sp}(V_{\mathbb{Z}})/\{\pm I\} \simeq \mathrm{SO}^0(P_{\mathbb{Z}})$ where $\mathrm{SO}^0(V_{\mathbb{Z}})$ denotes the intersection with the connected component $\mathrm{SO}^0(P_{\mathbb{R}})$ = the subgroup of elements of spinor norm 1 (see [9]). Via this representation h defines a Hodge structure of type $(2,0),(1,1),(0,2)$ on $P_{\mathbb{Z}}$. Let $C = h(i) \in \mathrm{Sp}(V_{\mathbb{R}})$ denote the Weil element for h then $\phi(x, Cy)$ is symmetric and positive definite. The Weil element for the Hodge structure induced on $P_{\mathbb{Z}}$ is given by $\mathcal{C} : M \mapsto C\ M\ {}^tC$ and the condition that $M \in P_h^-$ is equivalent to $\mathcal{C}(M) = -M$. It follows that we have $N\ C = M\ J\ C = M\ {}^tC^{-1}\ J = -C\ M\ J = -C\ N$ i.e. C and N anti-commute. Also ${}^tN\ J\ N = {}^tJ\ {}^tM\ J\ M\ J = dJ$ so $\psi(Nx, Ny) = d\psi(x,y)$ and since $J\ N = J\ M\ J$ is skew-symmetric we have $\psi(x, Nx) = 0$

Assume on the other hand that N is an integral, primitive 4×4 matrix satisfying $N^2 = d < 0$, $\psi(Nx, Ny) = d\psi(x,y)$ and $\psi(x, Nx) = 0$. Then the matrix $M = -N\ J \in P_{\mathbb{Z}}$ is primitive and $\phi(M,M) = 2d$. If h is a polarized Hodge structure on $V_{\mathbb{Z}}$ as above then $M \in P_h^-$ if and only if N anti-commutes with the Weil element.

It follows from this that we can describe the set $\mathcal{A}_{2,\Gamma,v}$ as the set of isomorphism classes of objects $(A, \mathfrak{L}, \gamma, \alpha)$ where $(A, \mathfrak{L})$ is a principally polarized abelian surface, $\gamma\colon \mathcal{O} \to \mathrm{End}(H^1(A, \mathbb{Z}))$ is an $\mathcal{O}$-module structure on $H^1(A, \mathbb{Z})$ such that the Weil element satisfies the condition $C\gamma(a) = \gamma(a^\sigma)C$ i.e. C is conjugate linear with respect to the $\mathcal{O}$ module structure. Finally α is a $\Gamma/\Gamma(n)$ equivalence class of $\mathcal{O}$ module isomorphisms $V_{\mathbb{Z}/n} \to H^1(A, \mathbb{Z}/n)$ compatible with the symplectic forms.

LEMMA 1.1. *Let $G_v \subset \mathrm{Sp}(V_{\mathbb{Z}})$ be the stabilizer of v i.e.*

$$G_v = \{g \in \mathrm{Sp}(V_{\mathbb{Z}}) | g\ M\ {}^tg = M\}.$$

Then $G_v(\mathbb{Q}) \simeq \mathrm{SL}_2(K)$.

Proof. We define $\lambda_M(x,y) = \frac{\phi(x,y)}{2} + \frac{\phi(x,Ny)}{2\sqrt{d}} \in K$ then λ_M is a non-degenerate K-bilinear alternating form on $V_{\mathbb{Q}}$. Now assume $g \in G_v$ i.e. $g\ M\ {}^tg = M$. We get $g\ N = M\ {}^tg^{-1}\ J = M\ J\ g = N\ g$ thus g is K-linear, hence $g \in \mathrm{GL}_2(K)$. It is clear that g preserves the alternating form λ_M and hence is in $\mathrm{SL}_2(K)$. Conversely if $g \in \mathrm{GL}(V_{\mathbb{Q}})$ is in $\mathrm{SL}_2(K)$ then g commutes with N and since it has determinant 1 as a K-linear map it preserves λ_M it follows easily that $g \in \mathrm{Sp}(V_{\mathbb{Q}})$ and hence the computation above shows that $g \in G_v$. □

LEMMA 1.2. *The set of polarized Hodge structures h on $V_{\mathbb{Z}}$ as above, for which $v \in P_h^-$ is parametrized by the symmetric space* $\mathrm{SL}_2(\mathbb{C})/\mathcal{K}$ *where $\mathcal{K}$ is a maximal compact subgroup i.e. $\mathcal{K}$ is conjugate to* $\mathrm{SU}(2) \subset \mathrm{SL}_2(\mathbb{C})$.

Proof. We first show that $G_v(\mathbb{R})$ acts transitively on the set of these Hodge structures. So let h and h' be two such Hodge structures. We can find an element $g \in \mathrm{Sp}(V_{\mathbb{R}})$ such that $h' = ghg^{-1}$. We want to show that we can choose g such that $g\ v = v$. Let $w = g^{-1}v$ and let C and C' be the Weil elements for h and h' respectively. Then we have $-M = C'M^tC' = gCg^{-1}M^tg^{-1^t}C^tg$ thus $Cg^{-1}M^tg^{-1}C = -g^{-1}M^tg^{-1}$ which shows that $w \in P_h^-$.

Assume first that v, w are linearly independent then they span P_h^-. Consider the isomorphism $\eta \in \mathrm{GL}_2(P_h^-)$ defined by $v \mapsto w$ and $w \mapsto -v$. Then $\eta \in \mathrm{SO}(P_h^-)$. Since $\mathrm{SO}(P_h^-)$ is connected we get that $\kappa = \eta \perp \mathrm{id} \in \mathrm{SO}^0(P_{\mathbb{R}})$. Viewing this as an element of $P\,\mathrm{Sp}(V_{\mathbb{R}})$ (via the isomorphism $P\,\mathrm{Sp}(V_{\mathbb{R}}) \simeq \mathrm{SO}^0(P_{\mathbb{R}})$) it lies in the stabilizer of h. Since $g\kappa v = v$ and $h' = g\kappa h(g\kappa)^{-1}$ we are done in this case.

If v, w are linearly dependent we must have $w = \pm v$ since $\phi(v,v) = \phi(w,w)$. If $w = v$ there is nothing to prove. If $w = -v$ we compose g with $-\mathrm{id}_{P_h^-} \perp \mathrm{id}_{P_h^+} \in \mathrm{SO}^0(P_{\mathbb{R}})$.

In order to finish the proof we need to show that there does in fact exist a Hodge structure h_o such that $v \in P_{h_o}^-$. But if we choose $w \in v^\perp$ with $\phi(w,w) < 0$, which is possible since $v^\perp$ has signature $(3,-1)$, then v, w span a negative definite oriented plane. This corresponds to a unique Hodge structure h_0 on V such that $P_{h_o}^- = \mathrm{Span}\{v, w\}$. Finally $\mathrm{Stab}(h_o) = (\mathrm{SO}(P_{h_o}^-) \times \mathrm{SO}(P_{h_o}^+))^0$ which is a maximal compact subgroup of $\mathrm{SO}^0(P_{\mathbb{R}})$. □

LEMMA 1.3. *G_v is isomorphic to* $\mathrm{SL}_2(\mathcal{O})$.

Proof. By a result of Humbert ([9]) any two primitive elements $v, v' \in P_{\mathbb{Z}}$ such that $\phi(v,v) = \phi(v',v')$ are conjugate under $\mathrm{Sp}(V_{\mathbb{Z}})$. It follows that we can find $g \in \mathrm{Sp}(V_{\mathbb{Z}})$ such that

$$gM^tg = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & d & 0 \\ 0 & -d & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $gNg^{-1} = \begin{pmatrix} U & 0 \\ 0 & {}^tU \end{pmatrix}$ with $U = \begin{pmatrix} 0 & -1 \\ -d & 0 \end{pmatrix}$ and so we can find a symplectic basis $\{f_1, f_2, f_3, f_4\}$ such that the two subspaces $E_1 = \mathrm{Span}\{f_1, f_2\}$ and $E_2 = \mathrm{Span}\{f_3, f_4\}$ are stable under N. The restrictions of N to E_1, E_2 are given by the matrices U and ${}^tU$ respectively. The maps $\mathcal{O} \to E_i$ given by $1 \mapsto -f_2, \sqrt{d} \mapsto f_1$ and $1 \mapsto f_3, \sqrt{d} \mapsto -f_4$ define $\mathcal{O}$-linear isomorphisms. Thus $V_{\mathbb{Z}}$ is a free $\mathcal{O}$ module spanned by f_2, f_3 and the lemma follows. □

Let $\Gamma_v = \Gamma \cap G_v$.

PROPOSITION 1.1. $\mathcal{A}_{2,\Gamma,v} \simeq \Gamma_v \backslash \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$.

Proof. Let $(A, \mathfrak{L}, x, \alpha)$ represent an element in $\mathcal{A}_{2,\Gamma,v}$. Since any two alternating forms with determinant 1 over $\mathbb{Z}$ are isomorphic we can find a symplectic isomorphism $\delta \colon V_{\mathbb{Z}} \to H^1(A, \mathbb{Z})$. Let $u = (\wedge^2\delta)^{-1}x$. Again by Humbert's result u and v are conjugate under $\mathrm{Sp}(V_{\mathbb{Z}})$ and hence we can change δ if necessary so that $\wedge^2 u$ maps v to x.

By strong approximation the reduction map $G_v(\mathbb{Z}) \to G_v(\mathbb{Z}/n)$ is surjective and hence we can also assume that $\delta \equiv \alpha \bmod n$. Now fix a Hodge structure h_o on $V_{\mathbb{Z}}$ as above such that $v \in P^-_{h_o}$. This exists by Lemma 1.2. Let $\mathcal{K} = \mathrm{Stab}(h_o) \simeq \mathrm{SU}(2)$. The Hodge structure on $H^2(A, \mathbb{Z})$ pulls back to a Hodge structure h on $V_{\mathbb{Z}}$ such that $v \in P^-_h$. Again by Lemma 1.2 we can find $g \in G_v(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{C})$ such that $h = gh_og^{-1}$. We associate to $(A, \mathfrak{L}, x, \alpha)$ the double coset $\Gamma_v g\mathcal{K} \in \Gamma_v \backslash G_v(\mathbb{R})/\mathcal{K}$.

The (easy) proof that this construction is independent of the choices and defines a bijection is left to the reader. □

REMARK. It is easy to see that if v is represented by the skew-symmetric matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & d & 0 \\ 0 & -d & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

then $G_v(\mathbb{Z}) \subset \mathrm{Sp}_4(\mathbb{Z})$ consists of matrices of the form

$$\begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ db_1 & a_1 & b_2 & da_2 \\ da_3 & b_3 & a_4 & db_4 \\ b_3 & a_3 & b_4 & a_4 \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}).$$

Thus for any primitive v, G_v is conjugate in $\mathrm{Sp}_4(\mathbb{Z})$ to a group of this form.

Let $\mathfrak{S}_2$ be the Siegel upper half-space of degree 2. If h_o is a Hodge structure as above then $\mathfrak{S}_2 = \mathrm{Sp}(V_{\mathbb{R}})/\mathrm{Stab}(h_o)$. Now $\mathfrak{H}_v = G_v(\mathbb{R})/\mathcal{K} \simeq \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$ is hyperbolic 3-space $\mathbb{H}_3$ and we have an isometric embedding $\mathfrak{H}_v \subset \mathfrak{S}_2$ identifying $\mathfrak{H}_v$ with a copy of $\mathbb{H}_3$ embedded as a totally geodesic submanifold of $\mathfrak{S}_2$.

Let N be the matrix from before. Since $N^2 = dI_4$ and $N \in G\,\mathrm{Sp}(V_{\mathbb{Q}})$, conjugation by N induces an involution on $\mathrm{Sp}(V_{\mathbb{Q}})$. Clearly $G_v = \mathrm{Sp}(V_{\mathbb{Z}}) \cap \mathrm{Sp}(V_{\mathbb{Q}})^N$. We also have

LEMMA 1.4. *$\mathfrak{H}_v$ is the set of fixed points for the anti-holomorphic involution $\tau \mapsto N\bar{\tau}$ on $\mathfrak{S}_2$.*

Proof. We can assume that $M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & d & 0 \\ 0 & -d & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$. Let $U = \begin{pmatrix} 0 & -1 \\ -d & 0 \end{pmatrix}$ then $N = \begin{pmatrix} U & 0 \\ 0 & {}^tU \end{pmatrix}$. Now $g \in G_v$ if and only if $gM{}^tg = M$ and since we also have $gJ{}^tg = J$ we get $M = gM{}^tg = gMJ^{-1}g^{-1}J$ hence $Ng = gN$. Let $\tau_o = \begin{pmatrix} i & 0 \\ 0 & -di \end{pmatrix} \in \mathfrak{S}_2$ then $U\tau_o = \bar{\tau}_o{}^tU$. We use τ_o as base point so $g \mapsto g\tau_o$ defines an isomorphism $\mathrm{Sp}(V_{\mathbb{R}})/\mathrm{Stab}(\tau_o) \simeq \mathfrak{S}_2$. Thus we get $\mathfrak{H}_v = G_v(\mathbb{R})\tau_o$ i.e. $\mathfrak{H}_v = \left\{(A\tau_o + B)(C\tau_o + B)^{-1} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_v(\mathbb{R})\right\}$. Now let $g \in G_v(\mathbb{R})$. Then $UAU^{-1} = A, UB{}^tU^{-1} = B,{}^t UCU^{-1} = C$ and ${}^tUD{}^tU^{-1} = D$.

If $\tau = (A\tau_o + B)(C\tau_o + D)^{-1} \in \mathfrak{H}_v$ then

$$\begin{aligned}
\tau &= (UAU^{-1}\tau_o + UB{}^tU^{-1})({}^tUCU^{-1}\tau_o +{}^t UD{}^tU^{-1})^{-1} \\
&= U(AU^{-1}\tau_o + B{}^tU^{-1})(CU^{-1}\tau_o + D{}^tU^{-1})^{-1}{}^tU^{-1} \\
&= U(A\bar{\tau}_o + B){}^tU^{-1}((C\bar{\tau}_o + D){}^tU^{-1})^{-1}{}^tU^{-1} \\
&= U\bar{\tau}{}^tU^{-1} \\
&= N\bar{\tau}.
\end{aligned}$$

□

PROPOSITION 1.2. *The map $\Gamma_v\backslash\mathfrak{H}_v \to \Gamma\backslash\mathfrak{S}_2$ is a closed immersion.*

Proof. It suffices to show that the map is injective. Assume $\tau_1, \tau_2 \in \mathfrak{H}_v$ and assume that there exists $g \in \Gamma$ such that $\tau_2 = g\tau_1$. This implies that $\tau_2 = N^{-1}gN\tau_1$ and since Γ acts freely on $\mathfrak{S}_2$ by assumption, we get $N^{-1}gN = g$ i.e. $g \in \Gamma \cap G_v = \Gamma_v$. □

DEFINITION 1.1. We let W_v denote the image. Thus W_v is a manifold of Picard type, that is a quotient of hyperbolic threespace by the action of a subgroup of finite index in $\mathrm{SL}_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers in an imaginary quadratic field.

2. Compactifications

In this section we study the closure of W_v in the toroidal compactification of $\Gamma\backslash\mathfrak{S}_2$. We show that this gives a smooth compactification of W_v.

THEOREM 2.1. *Let $\widetilde{W_v}$ denote the closure of W_v in the toroidal compactification Y_Γ of $\Gamma\backslash\mathfrak{S}_2$. Then $\widetilde{W_v}$ is an orientable smooth manifold. Let B be a 2-dimensional boundary component so B is an elliptic modular surface. Then the intersection $B \cap \widetilde{W_v}$ is a copy of S^1 contained entirely within one component of a singular fiber in the elliptic pencil on B and does not meet any of the other components of the fiber.*

Proof. We may assume that v corresponds to the primitive matrix $M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & d & 0 \\ 0 & -d & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$ so $\mathfrak{H}_v$ is the subset fixed by $\tau \mapsto \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix} \bar{\tau} \begin{pmatrix} 0 & 1 \\ d^{-1} & 0 \end{pmatrix}$.
Thus we have

$$\mathfrak{H}_v = \left\{ \tau = \begin{pmatrix} t & r \\ r & d\bar{t} \end{pmatrix} \mid t \in \mathbb{C}, \operatorname{Im} t > 0, r \in \mathbb{R} \right\}.$$

We follow the description of the toroidal compactification given in [9]. Consider the standard neighborhood of the 2-dimensional boundary component

$$V(q) = \left\{ \tau = \begin{pmatrix} t & z \\ z & w \end{pmatrix} \in \mathfrak{S}_2 \mid \operatorname{Im} t \operatorname{Im} w - (\operatorname{Im} z)^2 > q \operatorname{Im} t \right\}$$

and the standard neighborhood of the 1-dimensional boundary component

$$W(q) = \left\{ \tau = \begin{pmatrix} t & z \\ z & w \end{pmatrix} \in \mathfrak{S}_2 \mid \operatorname{Im} \tau > qI \right\}.$$

We have $\mathfrak{H}_v \cap V(q) = \left\{ \tau = \begin{pmatrix} t & r \\ r & d\bar{t} \end{pmatrix} \mid \operatorname{Im} t > -q/d \right\} \subset \mathfrak{H}_v \cap W(-q/d)$. For $q \gg 0$ we have $gW(q) \cap W(q) \neq \emptyset$ for $g \in \operatorname{Sp}(V_{\mathbb{Z}})$ if and only if g is in the standard parabolic subgroup $Q = \left\{ \begin{pmatrix} u & b \\ 0 & {}^t u^{-1} \end{pmatrix} \in \operatorname{Sp}(V_{\mathbb{Z}}) \mid u \in \operatorname{GL}_2(\mathbb{Z}) \right\}$. It follows from this and from Proposition 1.2 that $g(\mathfrak{H}_v \cap W(q)) \cap (\mathfrak{h}_v \cap W(q)) \neq \emptyset$ if and only if $g \in Q \cap \Gamma_v$.

We first assume that $\Gamma = \Gamma(n)$. Since $n \geqslant 3$ it is clear from the description of G_v given in the previous section that $Q \cap \Gamma_v$ consists of matrices of the form $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$ with ${}^t B = B$ and $B \equiv 0 \bmod n$ and hence $Q \cap \Gamma_v$ acts by translations $\tau \mapsto \tau + B$ on $\mathfrak{H}_v \cap W(q)$ for $q \gg 0$. Let $\mathbf{e}(z)$ denote $\exp(2\pi i z)$ and let T denote the 3-dimensional torus of 2×2 symmetric matrices $\begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{pmatrix}$, $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}^*$ under componentwise multiplication. Then the map $\mathbf{e}\colon \mathfrak{H}_v \cap W(q) \to T$ defined by $\tau = \begin{pmatrix} t & r \\ r & d\bar{t} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{e}(t/n) & \mathbf{e}(r/n) \\ \mathbf{e}(r/n) & \mathbf{e}(d\bar{t}/n) \end{pmatrix}$ defines an embedding $Q \cap \Gamma_v \backslash \mathfrak{H}_v \cap W(q)$ into T.

Consider the cone $\mathcal{Y} \subset \mathbb{R}^3$ of semi-positive definite symmetric 2×2 matrices and consider the polyhedral cone C spanned by the three matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The group $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathcal{Y}$ by $g\colon \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto g \begin{pmatrix} a & b \\ b & c \end{pmatrix} {}^t g$. For each $g \in \mathrm{GL}_2(\mathbb{Z})$ let C_g be the polyhedral cone gC. Then the collection of cones $\{C_g\}$ forms a polyhedral decomposition of $\mathcal{Y}$. We use this decomposition to construct a toroidal embedding of the torus T. Let $g \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} {}^t g = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$, $g \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} {}^t g = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}$, $g \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} {}^t g = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix}$ this is clearly a basis of C_g. We use this to define an embedding $\psi_g : T \simeq (\mathbb{C}^*)^3 \subset \mathbb{C}^3 = \mathbb{C}^3_g$ by

$$\begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{pmatrix} \mapsto (\zeta_1^{a_1}\zeta_2^{b_1}\zeta_3^{c_1}, \zeta_1^{a_2}\zeta_2^{b_2}\zeta_3^{c_2}, \zeta_1^{a_3}\zeta_2^{b_3}\zeta_3^{c_3}).$$

For each pair of elements $g, h \in \mathrm{GL}_2(\mathbb{Z})$ we define $\phi_{g,h} : (\mathbb{C}^*)^3 \simeq (\mathbb{C}^*)^3 = \psi_h \psi_g^{-1}$. Let $U_{g,h} \subset \mathbb{C}^3_g$ be the largest open set on which $\phi_{g,h}$ is defined so $\phi_{g,h}\colon U_{g,h} \to \mathbb{C}^3_h$. Now consider the disjoint union

$$\coprod_{g \in \mathrm{GL}_2(\mathbb{Z})} \mathbb{C}^3.$$

We define an equivalence relation by $(\underline{z}, g) \sim (\underline{w}, h)$, $\underline{z}, \underline{w} \in \mathbb{C}^3$ if and only if $\phi_{g,h}$ is defined at $\underline{z}$ and $\underline{w} = \phi_{g,h}(\underline{z})$. Let M denote the quotient, then M is a complex manifold and $T \subset M$. The coordinate axis in the $\mathbb{C}^3_g$'s give rise to a configuration of $\mathbb{P}^1$'s in M. A component in this configuration can meet another component only at 0 or ∞.

Now the image of $Q \cap \Gamma_v \backslash \mathfrak{H}_v \cap W(q)$ in $\mathbb{C}^3_I$ is the set

$$\left\{(z_1, z_2, (\overline{z_1} z_2)^{-d} z_2) \mid |z_1| < s = \exp\left(-\frac{2\pi}{n} q\right), |z_2| = 1\right\} = D_s^* \times S^1,$$

where D_s^* denotes the punctured disc of radius s. It follows that the interior of the closure of the image in M is the open solid torus $D_s \times S^1$. Gluing this torus onto $\Gamma_v \backslash \mathfrak{H}_v$ along the punctured torus we clearly obtain a smooth manifold. The boundary component $\{0\} \times S^1 \subset D_s \times S^1$ is the unit circle in the $\mathbb{P}^1$ corresponding to the coordinate axis $\mathbb{C} = \{(0, 0, z)\} \subset \mathbb{C}^3$ and hence does not contain 0 or ∞ consequently it does not meet any other component of the configuration of $\mathbb{P}^1$'s. The partial compactification of $\Gamma(n) \backslash \mathfrak{S}_2$ is obtained by dividing M by the congruence subgroup $\mathrm{GL}_2(\mathbb{Z})(n) \subset \mathrm{GL}_2(\mathbb{Z})$ where the action is given by $u \in \mathrm{GL}_2(\mathbb{Z})(n)$ acts through $\begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix}$. Since the only matrix of this form in Γ_v is the identity it follows that there are no further identifications taking place in $Q \cap \Gamma_v \backslash \mathfrak{h}_v \cap W(q)$ by passing to $\mathrm{GL}_2(\mathbb{Z})(n)\backslash M$. Since $\mathrm{Sp}(V_{\mathbb{Z}})$ acts transitively on the set of boundary components it follows by homogeneity that $\widetilde{W}_v$ is a smooth manifold.

It follows immediately from this description and the Mayer-Vietoris exact sequence that $H^3(\widetilde{W_v}, \mathbb{Q}) \simeq H^3(W_v . \mathbb{Q}) = \mathbb{Q}$ so $\widetilde{W_v}$ is orientable and defines a class in $H^3(Y_\Gamma, \mathbb{Q})$.

In the general case $\Gamma(n) \subset \Gamma$ we first take the closure in $Y_{\Gamma(n)}$ and then take the quotient by the finite group $\Gamma_v/\Gamma(n)_v$. Since this group is orientation preserving and acts freely the quotient is a smooth, orientable manifold. □

3. Intersections

Let $v, w \in P_{\mathbb{Z}}$ with $\phi(v,v) < 0, \phi(w,w) < 0$. Consider the set

$$\{(\tau_1, \tau_2, g) \mid \tau_1 \in \mathfrak{h}_v, \tau_2 \in \mathfrak{H}_w, g \in \Gamma \text{ with } g\tau_1 = \tau_2\}.$$

The group $\Gamma_v \times \Gamma_w$ acts on this set by $(\alpha, \beta)\colon (\tau_1, \tau_2, g) \mapsto (\alpha\tau_1, \beta\tau_2, \beta g \alpha^{-1})$.

We have the following lemma due to Millson and Raghunathan [5].

LEMMA 3.1. *The set of orbits under this action is in* $1-1$ *correspondence with the intersection* $W_v \cap W_w \subset \Gamma\backslash\mathfrak{S}_2$.

Proof. Consider a point $[\tau] \in \Gamma\backslash\mathfrak{S}_2$. Then τ represents a point in the intersection if and only if there exist $\tau_1 \in \mathfrak{H}_v, \tau_2 \in \mathfrak{H}_w$ and $g_1, g_1 \in \Gamma$ such that $\tau = g_1\tau_1 = g_2\tau_2$ and thus $(\tau_1, \tau_2, g_2^{-1}g_1)$ represents an orbit. It is clear that any two elements in the same orbit define the same intersection point. Thus we have a surjective map from the set of orbits to the set of intersection points.

Assume that (τ_1, τ_2, g) and (τ_1', τ_2', g') define the same intersection point. Then there is $f \in \Gamma$ such that $\tau_1' = f\tau_1$. It follows from Lemma 1.2 that $f \in \Gamma_v$. Also $\tau_2' = g'fg^{-1}\tau_2$ and again Lemma 1.2 implies that $g'fg^{-1} \in \Gamma_w$. Now the element $(f, g'fg^{-1}) \in \Gamma_v \times \Gamma_w$ sends (τ_1, τ_2, g) to (τ_1', τ_2', g'). □

LEMMA 3.2. *Assume that there exists* $g \in \Gamma$ *such that* gv, w *are linearly dependent. Then* $W_v = W_w$.

Proof. Assume $gv = \lambda w$ with $\lambda \in \mathbb{Q}$. Consider a point $\tau \in \mathfrak{H}_w$ then $N_w\tau = \tau$ where $N_w = M_w J$. Now $\lambda M_w = g M_v {}^t g$ so we get $\lambda N_w = g N_v g^{-1}$. It follows that $g N_v g^{-1}\tau = (\lambda N_w)\tau = N_w\tau = \tau$. Thus $g^{-1}\tau \in \mathfrak{H}_v$ and hence the class $[\tau] \in \Gamma\backslash\mathfrak{S}_2$ lies in W_v. □

LEMMA 3.3. *Assume that* $W_v \neq W_w$. *Let* $(A, \mathfrak{L}, \alpha)$ *represent a point in the intersection* $W_v \cap W_w$. *Then* A *is a* CM*-type abelian variety.*

Proof. Since $(A, \mathfrak{L}, \alpha)$ represents an intersection point there are elements $x, y \in P^2(A, \mathbb{Z})$ such that $(A, \mathfrak{L}, x, \alpha) \in \mathcal{A}_{2,\Gamma,v} = W_v$ and $(A, \mathfrak{L}, y, \alpha) \in \mathcal{A}_{2,\Gamma,w} = W_w$. Thus $x, y \in P_A^-$ and $\wedge^2\alpha$ maps $\bar{v} \in P_{\mathbb{Z}/n}$ to $\bar{x} \in P^2(A, \mathbb{Z}/n)$ and $\bar{w}$ to $\bar{y}$. Since

v, w are linearly independent mod n it follows that x, y are linearly independent and so they span P_A^-. But this shows that the Hodge decomposition on $P^2(A)$ is defined over $\mathbb{Q}$ hence $H^{1,1}(A, \mathbb{C})$ is defined over $\mathbb{Q}$. Thus $NS(A)$ has rank 4 and so A is of CM-type. □

THEOREM 3.1. *Let v, w be as above then $W_v \cap W_w$ is a finite set.*

Proof. We shall first show that it suffices to prove it for some n with $\Gamma(n) \subset \Gamma$.

Let $\{g_1, \ldots, g_r\} \subset \Gamma$ be representatives of the cosets of $\Gamma(n)$ in Γ. Consider an intersection point represented by (τ_1, τ_2, g) with $\tau_1 \in \mathfrak{H}_v, \tau_2 \in \mathfrak{H}_w, g \in \Gamma$. Writing $g = g_i\gamma$ for some i with $\gamma \in \Gamma(n)$ we see that $(\tau_1, g_i^{-1}\tau_2, \gamma)$ defines a point of $W_v \cap W_{g_i^{-1}w}$. Clearly $v, g_i^{-1}w$ satisfy the hypothesis and so assuming the result is true for $\Gamma(n)$, $W_v \cap W_{g_i^{-1}w}$ is a finite set.

Now $(\tau_1, g_i^{-1}\tau_2, \gamma)$ and (τ_1, τ_2, g) represent the same point in $\Gamma \backslash \mathfrak{S}_2$ and hence $\bigcup_{g_i \in \Gamma/\Gamma(n)} W_v \cap W_{g_i^{-1}w} \subset \Gamma(n)\backslash\mathfrak{S}_2$ maps onto $W_v \cap W_w \subset \Gamma\backslash\mathfrak{S}_2$.

Let $d = \phi(v, v), d' = \phi(w, w)$ and choose n such that $n > |\phi(v, w)| + \sqrt{dd'}$ and $\Gamma(n) \subset \Gamma$. Let (τ_1, τ_2, γ) represent a point in $W_v \cap W_w \subset \Gamma(n)\backslash\mathfrak{S}_2$ so $v \in P_{\tau_1}^-$ and $w \in P_{\tau_2}^-$ (to be consistent with the previous notation we should really write $P_{h_\tau}^-$ where h_τ denotes the Hodge structure associated to $\tau \in \mathfrak{S}_2$), $\gamma \in \Gamma(n)$. Since $\gamma\tau_1 = \tau_2$ we have $\gamma(v), w \in P_{\tau_2}^-$ and so they span a negative definite plane. Since $\gamma \in \Gamma(n)$ we can write $\gamma v = v + nu$ for some $u \in P_{\mathbb{Z}}$. It follows that $\phi(\gamma v, \gamma v)\phi(w, w) - (\phi(v, w) + n\phi(u, w))^2 > 0$ and since $\phi(\gamma v, \gamma v) = \phi(v, v)$ this is equivalent to $n^2\phi(u, w)^2 + 2n\phi(u, v)\phi(v, w) + \phi(v, w)^2 - dd' < 0$. Thus $-\phi(v, w) - \sqrt{dd'} < n\phi(u, w) < -\phi(v, w) + \sqrt{dd'}$. If $\phi(u, w) \neq 0$ then $|\phi(u, w)| \geqslant 1$ and it follows that $n \leqslant |\phi(v, w)| + \sqrt{dd'}$ which is a contradiction. Thus $\phi(u, w) = 0$ and hence that the map defined by $v \mapsto \gamma v, w \mapsto w$ is an isometry, in particular v, w also span a negative definite sublattice. It follows that a necessary condition for $W_v \cap W_w \neq \emptyset$ is that $\phi(v, v)\phi(w, w) - \phi(v, w)^2 > 0$ or equivalently that $|\phi(v, w)| < \sqrt{dd'}$ hence if we choose $n > 2\sqrt{dd'}$ the previous argument applies. Now let $g \in \Gamma$ and assume that $W_v \cap W_{g^{-1}w} \neq \emptyset$ in $\Gamma(n)\backslash\mathfrak{S}_2$. It follows that $v, g^{-1}w$ span a negative definite plane hence $|\phi(v, g^{-1}w)| < \sqrt{dd'}$. Define $T_i = \operatorname{Span}_{\mathbb{Z}}\{v, g_i^{-1}w\}$ and let $T_i^* = \{y \in T_i \otimes \mathbb{Q} \mid \phi(x, y) \in \mathbb{Z}\ \forall x \in T_i\}$. Then T_i^* is a lattice and T_i^*/T_i is a finite group. Let (τ_1, τ_2, γ) represent a point in $W_v \cap W_{g^{-1}w}$. Then $T_{\tau_2} = P_{\tau_2}^- \cap P_{\mathbb{Z}}$ has rank 2 and the map $\pi\colon v \mapsto \gamma v, g_i^{-1}w \mapsto g^{-1}w$ is an isometry $T_i \otimes \mathbb{Q} \to T_{\tau_2} \otimes \mathbb{Q}$. We have $T_i \subset \pi^{-1}T_{\tau_2}$ and if $x \in T_i$ and $y \in \pi^{-1}T_{\tau_2}$ then $\phi(x, y) = \phi(\pi x, \pi y) \in \mathbb{Z}$ since both πx and $\pi y \in P_{\mathbb{Z}}$. It follows that $T_i \subset \pi^{-1}T_{\tau_2} \subset T_i^*$ and so T_{τ_2} corresponds uniquely to a subgroup of T_i^*/T_i. This shows that there are only finitely many isometry classes occurring as T_{τ_2} of points in $W_v \cap W_{g_i^{-1}w}$.

Let $(A, \mathfrak{L}, \alpha)$ represent a point in the intersection. Then A is of CM-type by Lemma 3.3 and by a theorem of Shioda and Mitani [8] the isomorphism class of a

CM-type abelian surface is uniquely determined by the isometry class of the lattice of transcendental cycles $T_A = P_A^- \cap P_{\mathbb{Z}}$. Thus there are only a finite number of isomorphism classes of A's occurring in the intersection and since each abelian surface has only finitely many principal polarizations and finitely many level n-structures we get that $W_v \cap W_{g_i^{-1}w} \subset \Gamma(n)\backslash\mathfrak{S}_2$ is finite which proves the theorem □

4. Example

In this section we consider one of the Siegel modular threefolds whose L-functions were computed in [6]. We shall show that the $(1,2),(2,1)$ part of H^3 is spanned by cohomology classes of the type considered in the previous sections.

In [6] we found equations and computed the L-functions for various types of Siegel modular threefolds. Here we shall consider the type for which H^3 is 4-dimensional and $h^{0,3} = h^{1,2} = h^{2,1} = h^{3,0} = 1$. The Satake compactification is a complete intersection $Y \subset \mathbb{P}^7$ and we can take the equations to be

$$\begin{aligned} Y_0^2 &= 2(X_0X_1 + X_2X_3), \\ Y_1^2 &= 2(X_0X_2 + X_1X_3), \\ Y_2^2 &= 2(X_0X_3 + X_1X_2), \\ Y_3^2 &= 2(X_0X_3 - X_1X_2). \end{aligned}$$

This is realized as a Siegel modular threefold through the map $\mathfrak{S}_2 \to \mathbb{P}^7$ given by theta constants

$$\tau \mapsto (\theta_{0100}(\tau), \theta_{1000}(\tau), \theta_{1100}(\tau), \theta_{1111}(\tau), \Theta_{00}(\tau), \Theta_{01}(\tau), \Theta_{10}(\tau), \Theta_{11}(\tau)),$$

where $\Theta_{ab}(\tau) = \theta_{ab00}(2\tau)$. The equations are just the Riemann theta relations. Let Γ denote the stabilizer of this map so it defines an embedding $\Gamma\backslash\mathfrak{S}_2 \to \mathbb{P}^7$.

We consider an anti-holomorphic involution on $\mathfrak{S}_2$ defined by $\sigma(\tau) = \sigma\left(\begin{pmatrix} t & z \\ z & w \end{pmatrix}\right) = \begin{pmatrix} -\bar{w} & \bar{z} \\ \bar{z} & -\bar{t} \end{pmatrix}$. This is the involution associated to the skew-symmetric matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The fixed point set is given by $\mathfrak{H}_v = \left\{\begin{pmatrix} t & r \\ r & -\bar{t} \end{pmatrix} \mid r \in \mathbb{R}, t \in \mathfrak{H}\right\}$. We can identify the fixed point set with hyperbolic space $\mathbb{C}\times\mathbb{R}_{>0}$ by $(x+iy, s) \mapsto \begin{pmatrix} x+is & y \\ y & -x+is \end{pmatrix}$.

In [6] (Appendix) we proved the following transformation formula: $\theta_{abcd}(\sigma\tau) = \overline{\theta_{badc}(\tau)}$. Thus σ descends to an involution on Y given by

$(Y_0, Y_1, Y_2, Y_3, X_0, X_1, X_2, X_3) \mapsto (\overline{Y_1}, \overline{Y_0}, \overline{Y_2}, \overline{Y_3}, \overline{X_0}, \overline{X_2}, \overline{X_1}, \overline{X_3})$. Let W_v be the closure of the image of $\mathfrak{H}_v$ so $W_v = Y^\sigma$. We can describe W_v as the set of points

$$W_v = \left\{(Y_0, \overline{Y_0}, y_2, y_3, x_0, X_1, \overline{X_0}, x_3) \in Y \mid y_2, y_2, x_0, x_3 \in \mathbb{R}\right\}.$$

To see this assume $(Y_0, Y_1, Y_2, Y_3, X_0, X_1, X_2, X_3) \in Y$ is a fixed point i.e. there exists $\lambda \in \mathbb{C}^*$ such that

$$\lambda(Y_0, Y_1, Y_2, Y_3, X_0, X_1, X_2, X_3) = (\overline{Y_1}, \overline{Y_0}, \overline{Y_2}, \overline{Y_3}, \overline{X_0}, \overline{X_2}, \overline{X_1}, \overline{X_3}).$$

Since at least one of the X's has to be $\neq$ 0 it follows that $|\lambda| = 1$ i.e. $\lambda = e^{i\theta}$ and so $e^{i\frac{\theta}{2}}(Y_0, Y_1, Y_2, Y_3, X_0, X_1, X_2, X_3) = \overline{e^{i\frac{\theta}{2}}(Y_1, Y_0, Y_2, Y_3, X_0, X_2, X_1, X_3)}$.

Consider the matrix $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z})$. The transformation formulas for theta constants ([3]) show that C descends to an automorphism δ of Y given by $\delta\colon (Y_0, Y_1, Y_2, Y_3, X_0, X_1, X_2, X_3) \mapsto (Y_0, Y_1, Y_2, -iY_3, X_1, X_0, X_3, X_2)$. C maps v to $w = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & -2 & 0 \end{pmatrix}$ so $W_w = \delta(W_v)$ and W_w is the set of fixed points under the involution $\delta\sigma\delta^{-1}$. It follows that

$$W_w = \left\{(Y_0, \overline{Y_0}, y_2, -iy_3, X_0, x_1, x_2, \overline{X_0}) \in Y \mid y_2, y_3, x_1, x_2 \in \mathbb{R}\right\}.$$

Remark that $\mathfrak{H}_v \cap \mathfrak{H}_w = \emptyset$ so the results of Section 3 show that all the intersection points must lie on the boundary. The intersection can then be described as

$$W_v \cap W_w = \left\{(Y_0, \overline{Y_0}, y_2, 0, x_0, x_1, x_1, x_0) \in Y \mid y_2, x_0, x_1 \in \mathbb{R}\right\}.$$

Since the Y_3-coordinate is 0 it follows that $x_0^2 - x_1^2 = 0$ hence $x_0 = \pm x_1$. Substituting this into the other equations we get $Y_0 = \pm 2x_0$ or $Y_0 = \pm 2ix_0$ and $y_2 = \pm 2x_0$. It follows that the intersection consists of the following 8 points

$$\begin{aligned}
P_1 &= (2, 2, 2, 0, 1, 1, 1, 1), \\
P_2 &= (2, 2, -2, 0, 1, 1, 1, 1), \\
P_3 &= (-2, -2, 2, 0, 1, 1, 1, 1), \\
P_4 &= (-2, -2, 2, 0, 1, 1, 1, 1), \\
P_5 &= (2i, -2i, 2, 0, 1, -1, -1, 1), \\
P_6 &= (2i, -2i, -2, 0, 1, -1, -1, 1), \\
P_7 &= (-2i, 2i, 2, 0, 1, -1, -1, 1), \\
P_8 &= (-2i, 2i, -2, 0, 1, -1, -1, 1).
\end{aligned}$$

The singularities of Y were described in [6] where also a resolution was constructed. Using this we find that 6 of the singular points lie on W_v ($(0,0,0,0,1,0,0,0)$, $(0,0,0,0,0,0,0,1)$, $(0,0,\pm\sqrt{2},\pm\sqrt{2},1,0,0,1)$) and none of these are in the intersection. Let $\widetilde{Y}$ denote the resolution and let $\widetilde{W_v}$ and $\widetilde{W_w}$ be the closures in $\widetilde{Y}$ so $\widetilde{W_v}$ and $\widetilde{W_w}$ are smooth manifolds in the smooth threefold $\widetilde{Y}$. The intersection consists of the 8 points above. Also remark that these points are all in the boundary of the Satake compactification, this follows from the description of the boundary given in [6].

Consider automorphisms of Y defined by

$$\alpha\colon (Y_0,Y_1,Y_2,Y_3,X_0,X_1,X_2,X_3) \mapsto (Y_0,Y_1,-Y_2,Y_3,X_0,X_1,X_2,X_3),$$
$$\beta\colon (Y_0,Y_1,Y_2,Y_3,X_0,X_1,X_2,X_3) \mapsto (iY_0,-iY_1,Y_2,Y_3,X_0,-X_1,-X_2,X_3).$$

Let G be the group of automorphisms generated by α,β then $G \simeq \mathbb{Z}/2\times\mathbb{Z}/4$ and the set $\{P_1,P_2,P_3,P_4,P_5,P_6,P_7,P_8\}$ is the orbit of P_1 under G.

One checks easily that G commutes with both σ and $\delta\sigma\delta^{-1}$ and hence $G \subset G_v(\mathbb{R})\cap G_w(\mathbb{R})$ so G acts on both $\widetilde{W_v}$ and $\widetilde{W_w}$. The orientations are given by the classes of the hyperbolic volumes which are invariant under $G_v(\mathbb{R})$ and $G_w(\mathbb{R})$ and hence G acts by orientation preserving automorphisms.

Fix an orientation on $\widetilde{W_v}$ and choose the orientation on $\widetilde{W_w}$ such that δ is orientation preserving. Remark that δ fixes P_1 and so the intersection multiplicity at P_1 can be computed as follows: let ω be the volume form on $\widetilde{Y}$ defined by the complex structure. Let e_1,e_2,e_3 be a positive basis of $T_{P_1}W_v$ then $\delta_*e_1,\delta_*e_2,\delta_*e_3$ is a positive basis of $T_{P_1}W_w$ and the intersection multiplicity is given by $m(P_1)\omega_{P_1} = e_1\wedge e_2\wedge e_3\wedge\delta_*e_1\wedge\delta_*e_2\wedge\delta_*e_3$. Let $g\in G$ then we have $g_*\omega_{P_1} = \omega_{g(P_1)}$ and g_*e_1,g_*e_2,g_*e_3 and $g_*\delta_*e_1,g_*\delta_*e_2,g_*\delta_*e_3$ are both positive bases for $T_{g(P_1)}W_v$ and $T_{g(P_1)}W_w$ respectively. Hence $m(g(P_1))\omega_{g(P_1)} = g_*e_1\wedge g_*e_2\wedge g_*e_3\wedge g_*\delta_*e_1\wedge g_*\delta_*e_2\wedge g_*\delta_*e_3 = g_*(e_1\wedge e_2\wedge e_3\wedge\delta_*e_1\wedge\delta_*e_2\wedge\delta_*e_3) = g_*(m(P_1)\omega_{P_1}) = m(P_1)\omega_{g(P_1)}$ hence $m(g(P_1)) = m(P_1)$ for all $g\in G$. It follows that $\left[\widetilde{W_v}\right]\cup\left[\widetilde{W_w}\right] = \sum_{g\in G} m(g(P_1)) = 8m(P_1) = \pm 8$. This proves that the cohomology classes $\left[\widetilde{W_v}\right]$ and $\left[\widetilde{W_w}\right]$ are linearly independent in $H^3(\widetilde{Y},\mathbb{Q})$.

Next we show that these classes lie in the $(1,2),(2,1)$ part of the Hodge structure. Let Y' denote the toroidal compactification. It suffices to prove that they are in the $(1,2),(2,1)$ part of $H^3(Y')$. There is a unique holomorphic 3-form on $\widetilde{Y}$. Its pull-back to Y' is given by the weight 3 cusp form for Γ, $f = \theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011}\theta_{0110}\theta_{1001}$. Let $\mathcal{D}$ be a fundamental domain for $\Gamma_v\subset\mathfrak{H}_v$. It suffices to show that $\int_{\mathcal{D}} f\,\mathrm{d}\tau = 0$. Now consider the matrix $A = \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&1&0\\ 0&0&0&-1\end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z})$. The action of A on $\mathfrak{S}_2$ commutes with σ so $\mathfrak{H}_v$ is stable under A. The restriction of f to $\mathfrak{h}_v$ is given by $f|_{\mathfrak{h}_v} = \theta_{0000}\theta_{0011}|\theta_{0001}|^2|\theta_{0110}|^2$. The transformation rules for theta functions (e.g. [3]) show that $f|_{\mathfrak{h}_v}(A\tau) = -f|_{\mathfrak{H}_v}(\tau)$. Thus f is an odd function with respect to the orientation preserving involution A and hence the integral vanishes.

Alternatively consider the involution defined by

$$\eta\colon (Y_0, Y_1, Y_2, Y_3, X_0, X_1, X_2, X_3) \mapsto (Y_1, Y_0, Y_2, -Y_3, X_0, X_2, X_1, X_3).$$

The rational 3-form $\Lambda = \frac{dX_1 dX_2 dX_3}{Y_0 Y_1 Y_2 Y_3}$ extends to a holomorphic 3-form on $\widetilde{Y}$. Since $\eta^*\Lambda = -\Lambda$, η acts by -1 on $H^{0,3}(\widetilde{Y}) \oplus H^{3,0}(\widetilde{Y})$. Remark that η commutes with σ so η defines an involution on $\widetilde{W_v}$ preserving the orientation. Hence η^* fixes $\left[\widetilde{W_v}\right]$. It follows that $H^{2,1} \oplus H^{1,2} = H^3(\widetilde{Y}, \mathbb{Q})^{\eta^*=1} \otimes \mathbb{C}$ and $H^{0,3} \oplus H^{3,0} = H^3(\widetilde{Y}, \mathbb{Q})^{\eta^*=-1} \otimes \mathbb{C}$. Thus the motive $H^3(\widetilde{Y})$ splits in the sense that it is a direct sum both as a Hodge structure and as a Galois representation.

References

[1] Ash, A., Mumford, D., Rapoport, M. and Tai, Y.: *Smooth compactifications of locally symmetric varieties*. Math. Sci. Press, Brookline, Mass., 1975.

[2] Andrianov, A. N.: Euler Products that correspond to Siegel Modular Forms of genus two. *Russian Math. Surveys*, 29, 1974.

[3] Igusa, J-I.: On Siegel modular forms of degree two (II). *Am. J. of Math.*, **86**:392–412, 1964.

[4] Kudla, S. and Millson, J.: Geodesic cycles and the Weil representation, I. Quotients of hyperbolic space and Siegel modular forms. *Comp. Math.*, **45**:207–271, 1982.

[5] Millson, J. and Raghunatan, M. S.: Geometric construction of cohomology for arithmetic groups I. *Proc. Indian Acad. Sci. (Math. Sci.)*, **90**:103–123, April 1981.

[6] Nygaard Niels, O. and Geeemen Bert van: On the Geometry and Arithmetic of some Siegel Modular Threefolds. *Journal of Number Theory, to appear,* 1995.

[7] Shintani, T.: On construction of holomorphic cusp forms of half integral weight. *Nagoya Math. Jour.*, **58**, 1975.

[8] Shioda, T. and Mitani, N.: Singular abelian surfaces and binary quadratic forms. In H. Popp, editor, *Classification of Algebraic Varieties and Compact Complex Manifolds,* pages 249–287, Springer Lecture Notes in Mathematics, No. 412, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[9] Van der Geer, G.: On the geometry of a Siegel modular threefold. *Math. Ann.*, **260**:317–350, 1982.