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Relative generic singularities of the Exponential Map

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Abstract. We investigate generic properties of the Exponential Map defined as $\mathrm{Exp}(v) = h_1^v$, for a vector field $v \in \Gamma(g)$ (where $\Gamma(g)$ denotes the Lipschitz sections of a subsheaf g of vector subspaces of the sheaf of all smooth vector fields on a smooth manifold M and h_t^v is the flow generated by v). We study restrictions of Exp to a suitable class of germs of submanifolds of M, and find necessary and sufficient conditions for a subsheaf $h \subset g$ such that for a generic vector field $v \in \Gamma(h)$ the singularities of the flow of v arise as singularities of the flow of a generic vector field belonging to $\Gamma(g)$. Applications of these results to Riemannian and sub-Riemannian geometry are presented and the context is chosen to include a theorem of A. Weinstein concerning the Riemannian Exponential Map.

1. Introduction

The main motivation of this paper lies in understanding the theorem of A. Weinstein [12]; in fact, the paper is just a slight generalization, with an easy direct proof, of that theorem. For a smooth n-dimensional manifold X, we consider the space $\mathcal{G}$ of all smooth complete Riemannian metrics on X, endowed with C^∞-Whitney topology. For each $g \in \mathcal{G}$ and $q \in X$, $\exp(g)|_q\colon T_qX \to X$ is the smooth map called the classical exponential map. To each $v \in T_qX$ it assigns the end point of the unique geodesic curve $\gamma\colon [0, 1] \to X$, $\gamma(0) = q$, $\dot\gamma(0) = v$. Let us write $g_{ij}(q) = \langle \frac{\partial}{\partial q_i}|_q, \frac{\partial}{\partial q_j}|_q\rangle$ for the entries of the matrix of the metric g and $g^{ij}(q)$ for the inverse matrix of $g_{ij}(q)$ then the function $H(q, p) = \frac{1}{2}\Sigma_{i,j=1}^n g^{ij}(q)p_ip_j$ on T^*X, defines a Hamiltonian vector field whose trajectories in T^*X project onto geodesics in X (cf. [2]). The exponential map is a Lagrangian map, i.e. $\mathrm{Exp}(g)_q = \pi_X \circ h_1^{vH}|_{T_qX}$, where h_t^{vH} is the flow of the Hamiltonian vector field defined by H and $h_i^{vH}|_{T_qX}$ is a Lagrangian immersion (cf. [1]). Recall that any germ of a Lagrangian immersion can be obtained in the above way by taking for H a suitable function on T^*X, (not necessarily quadratic with respect to p). By $\mathbf{h}$ we denote the space of quadratic Hamiltonians, and by $\mathbf{g}$ the space of all smooth Hamiltonians. It was stated by [12] (cf. [11]) that for a generic metric on X, i.e. for a generic $H \in \mathbf{h}$ the map $\mathrm{Exp}(g)|_q$ has only the singularities which are generic for Lagrangian maps, i.e. for a generic Hamiltonian $H \in \mathbf{g}$. Now we consider the problem in a more general way.

Let g be a subsheaf of vector subspaces of the sheaf of all smooth vector fields on a smooth manifold M. There are two natural questions to ask.

(1) Does g admit a subsheaf $h \subset g$ (we call it accessible) such that for a generic $v \in \Gamma(h)$ (where $\Gamma(h)$ denotes the Lipschitz sections of h) the singularities of the flow of v are the same as the singularities of the flow of a generic vector field $w \in \Gamma(g)$.

(2) What are necessary and sufficient conditions for the existence of such pairs g, h?

In attempting to answer these questions we define the Exp-map as $\mathrm{Exp}(v) = h_1^v$ for $v \in \Gamma(g)$, where h_t^v is the flow generated by v, and we study the restrictions of Exp to a suitable class of germs of submanifolds W. We impose some rather natural restrictions on g, e.g. we say assume that any vector field on X, which is a "piecewise section of g" can be approximated by sections of g. Then we find necessary and sufficient conditions, which answer our second question.

The paper is organized in three sections. In section 2 we formulate the problem and describe the assumed properties of the sheaf g. Then we give examples of sheaves satisfying these properties: the sheaf of all smooth vector fields, the sheaf of Hamiltonian vector fields and the sheaf of Hamiltonian vector fields with quadratic Hamiltonians. Section 3 contains the main results. We prove that the image of the r-jet of the Exp-map

$$E^r : \Gamma(g) \times W \to J^r(W, X)$$

is a submersive submanifold, and that a subspace h of g is accessible if and only if $E^r|_{\Gamma^*(h)\times W}$ is a submersion. The necessary condition (α-property) and the sufficient condition (β-property) for $E^r|_{\Gamma^*(h)\times W}$ to be a submersive map are found using the perturbation technique for the differential equation $\dot{x} = v(x)$. The last section of the paper contains applications to the Riemannian and sub-Riemannian cases, which were most interesting to us. As a consequence, a shorter proof of the standard genericity theorem for the Exp-map on a Riemannian manifold is presented (cf. [12, 5]) and an obstruction to the genericity of the Exp-map regarded as a family of Lagrangian maps is indicated. Analogous genericity results are obtained for sub-Riemannian Hamiltonians. In that case the image of the Exp-map is an isotropic submanifold and the generic properties of the sub-Riemannian Exp-map are reduced to those of isotropic submanifolds in the cotangent bundle.

2. Formulation of the problem

Let M be a locally trivial fiber bundle over X, $\pi: M \to X$. Let g be a subsheaf of vector subspaces of the sheaf of all smooth vector fields, $g \subset \Xi(M)$ on M. By

$\Gamma(g)$ we denote the space of Lipschitz sections of g over M. Let $v \in \Gamma(g)$ and let $t \to h_t^v: M \to M$ b a flow on M generated by v. Suppose that at each point $x \in M$ we are given a space of germs $\mathcal{M}_x$ of a class of submanifolds of M through x.

Let

$$\mathcal{M} = \bigcup_{x \in M} \mathcal{M}_x.$$

We shall assume that for every $v \in \Gamma(g)$ and $W_x \in \mathcal{M}_x$, $h_t^v(W_x) \in \mathcal{M}_{h_t^v(x)}$.

Let $W \in \mathcal{M}$, we define the Exp-map in the following way ([6]);

$$\mathrm{Exp}_v: W \to X, \quad \mathrm{Exp}_v = \pi \circ h_1^v|_W. \tag{1}$$

By $J^r = J^r(W, X)$ we denote the space of r-jets of smooth mappings $W \to X$. Let $\pi_r: J^r \to X$ denote the canonical projection onto the image space of the mapping. We have a natural map

$$E^r: \Gamma(g) \times W \to J^r(W, X),$$

we write also

$$E = E^r: \Gamma(g) \to C^\infty(W, J^r), \quad E^r(v); W \to J^r(W, X),$$

where $E^r(v)$ is the r-jet extention of Exp_v.

Let h be a sheaf of vector subspaces of g. Let A^r be a submanifold of the jet space $J^r(W, X)$.

DEFINITION 2.1. We say that $A^r \subset J^r(W, X)$ is *typical for* E^r if there is a residual subset $\Gamma'(g)$ of $\Gamma(g)$ such that for every $v \in \Gamma'(g)$ the corresponding jet-extention $E^r(v)$ is transversal to A^r.

In what follows we are interested in finding the subspaces of g which retain the typicality property for E^r.

DEFINITION 2.2. We say that the subsheaf $h \subset g$ is *accessible* if for every submanifold A^r, which is typical for E^r, there exists an open and dense subset $\Gamma'(h)$ in $\Gamma(h)$, such that for every $w \subset \Gamma'(h)$ the corresponding jet extention $E^r(w)$ is transversal to A^r.

In what follows we fix $x_0 \subset M$ with $\pi(x_0) = 0 \in X \cong R^n$. Let $W = W_{x_0} \in \mathcal{M}_{x_0}$. We denote

$$J^* = \pi_r^{-1}(R^n - \{0\}), \quad J_g^* = E^r(\Gamma(g))(W) \cap J^*;$$

clearly $\Gamma^*(g) = \{v \in \Gamma(g);\ v(x_0) \neq 0\}$, is an open subset of $\Gamma(g)$.

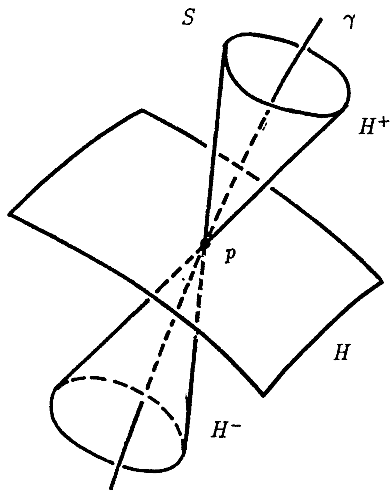


Fig. 1.

2.1. PROPERTIES OF THE SHEAF g

By g_{x_0} we denote the space of germs at x_0 of the sheaf g. Without loss of generality we introduce two important assumptions which have to be satisfied by our sheaf g.

PROPERTY 2.3. *If $w \in g_{x_0}$ and $v \in \Gamma(g)$, then*

$$(h_t^v)_* w \in g_{h_t^v(x_0)};$$

it follows that h_1^v induces an isomorphism $(h_1^v)_*\colon g_{x_0} \to g_{x_1}$, and $x_1 = h_1^v(x_0)$.

Let $v \in \Gamma^*(g)$; by γ we denote the integral curve of v starting at x_0.

PROPERTY 2.4. *Let us take a point p on the curve γ and a section w of g defined in a neighbourhood of p such that $j_p^r w = 0$. Then we assume that there exist:*

1. *a hypersurface H, separating M into two half-spaces H^+ and H^-, (as illustrated in Figure 1) transversal to γ at the point $p \in H \cap \gamma$, and*
2. *a family of vector fields $P_\epsilon(v, w) \in \Gamma^*(g)$, parametrized by $\epsilon \neq 0$, depending linearly on w with the following property*

$$P_\epsilon(v, w) \xrightarrow{\epsilon \to 0} \begin{cases} v & \text{on } H^- \\ v + w & \text{on } H^+ \end{cases}.$$

P_ϵ converges uniformly together with its derivatives up to order r, outside an open neighbourhood U of p and in a cone-like neighbourhood S of the curve γ,

$$S = \{x \in M;\ d(x, \gamma) < Cd(x, p)\}$$

for some positive constant C, and a metric $d(.,.)$ on M.

One can briefly state this assumption as follows:

> Every vector field on M, which is a “piecewise section of g” can be approximated by sections of g.
> Unless otherwise stated, in what follows, we assume both of these properties hold for our sheaf g.

Now we show how these assumptions work in some special situations.

EXAMPLE 2.5. Let g be the sheaf of all smooth vector fields on M. Then we can simply define

$$P_\epsilon(v, w) = v + \varphi_\epsilon w,$$

where φ_ϵ is a smooth function on M, such that

$$|D^a \varphi_\epsilon| \leqslant \frac{C_a}{\epsilon^a},$$

vanishing on H^-, varying on the strip of distance ϵ from H and equal 1 on the rest of H^+.

EXAMPLE 2.6. Let g be the sheaf of all Hamiltonian vector fields on $M = (T^*X, \omega_X)$, where ω_X is the Liouville symplectic form on the cotangent bundle T^*X. Let v, w be Hamiltonian vector fields with Hamiltonians H, K respectively, i.e. $\omega_X(v, \bullet) = -\mathrm{d}H$, $\omega_X(\omega, \bullet) = -\mathrm{d}K$. Let us denote the above correspondence of 1-forms and vector fields by J. We put

$$P_\epsilon(v, w) = Jd(H + \varphi_\epsilon K),$$

where φ_ϵ is defined as in Example 2.5.

EXAMPLE 2.7. Let g be the sheaf of Hamiltonian vector fields with Hamiltonians quadratic with respect to p: $H(q, p) = \Sigma_{ij} g^{ij}(q) p_i p_j((q, p)$ denote the standard Darboux coordinates on T^*X). Then the hypersurface $\mathcal{H}$ is defined by a smooth function L on X, $\mathcal{H} = \{(q, p): L(q) = 0\}$, and the function φ_ϵ depends only on q.

3. A transversality theorem

We start with a description of the image space of $(\mathrm{Exp})_*$. Let $g_x^{(r+1)}$ be the space of germs at x of vector fields in g vanishing at x together with all derivatives up to order r. By $J^r g_x$ we denote the jet-space of vector fields

$$J^r g_x = \frac{g_x}{g_x^{(r+1)}}.$$

By π^*g we shall denote the sheaf of TX-valued vector fields on M along π; i.e. the fields of the form

$$x \to \pi_* v(x),$$

where v is a section of g.

PROPOSITION 3.1.

$$J_g^* \textit{ is an immersive submanifold of } J^r$$

and the tangent space $T_z J_g^*$ *can be identified with*

$$\{j_{x_1}^r w;\ w \in \pi^* g_{x_1} | W_{x_1}\},$$

where $j_{x_1}^r w$ *is the* r*-jet of* w *and* z *is equal to the* r*-jet* $E^r(v)$.

Proof. As we remarked in Section 2

$$J_g^* = E^r(\Gamma^*(g))(W) \cap J^r.$$

We will show that $(E^r)_*$ has constant rank on $\Gamma^*(g)$.

Let $\xi \subset \Gamma(g)$ and let $t \to v + t\xi$ be a line in $\Gamma(g)$. Consider the curve $\gamma: t \to E^r(v + t\xi) \in C^\infty(W, J^r)$. The tangent vector to γ can be thought of as

$$\frac{d}{dt} E^r(v + t\xi)\bigg|_{t=0} = j^r \frac{d}{dt}\pi_*(y(x, 1, t))\bigg|_{t=0} = j^r \pi_*(u(x, 1)),$$

where $y(x, s, t)$ is a solution of the equation (with parameter t)

$$\partial_s y(x, s, t) = (v + t\xi)(y(x, s, t)). \tag{2}$$

We can view y as a perturbation of the solution of the equation $\dot{x} = v(x)$. Thus we can write y in the form

$$y(x, s, t) = y_0(x, s) + tu(x, s) + o(t).$$

From (2) we see that $u(x, s)$ satisfies the equation

$$\partial_s u(x, s) = Dv(y_0(x, s))u(x, s) + \xi(y_0(x, s)), \tag{3}$$

where $y_0(x, s)$ is a solution of the equation

$$\partial_s y_0(x, s) = v(y_0(x, s)). \tag{4}$$

We can also write $h_t^v(x) = y_0(x, t)$.

By Property 2.3 we have the following result.

LEMMA 3.2. *The linear part of the perturbation y is given by*

$$u(x,\ 1) = \int_0^1 (h^v_{1-t})_* \xi(h^v_t(x))\, \mathrm{d}t \in g_{x_1}, \tag{5}$$

Proof. We write

$$u(x,\ s) = \int_0^s (h^v_{s-t})_* \xi(h^v_t(x))\, \mathrm{d}t. \tag{6}$$

Obviously (6) satisfies equation (3):

$$\begin{aligned}\partial_s u(x,\ s) &= \xi(h^v_s(x)) + \int_0^s Dv(h^v_{s-t}(x))(h^v_{s-t})_* \xi(h^v_t(x))\, \mathrm{d}t \\ &= \xi(h^v_s(x)) + \int_0^s \left(\frac{\partial}{\partial s} h^v_s\right)_* (h^v_{-t})_* \xi(h^v_t(x))\, \mathrm{d}t \\ &= \xi(h^v_s(x)) + Dv(h^v_s(x)) \int_0^s (h^v_s)_* (h^v_{-t})_* \xi(h^v_t(x))\, \mathrm{d}t,\end{aligned}$$

where

$$\int_0^s (h^v_s)_* (h^v_{-t})_* \xi(h^v_t(x))\, \mathrm{d}t = u(x,\ s). \tag{7}$$

Q.E.D.

Now we prove that $T_z J^*_g \hookrightarrow J^r g$. Indeed by (7) we can approximate $u \in T_z J^* g$ by Riemann sums: each summand

$$(h^v_{-(1-t)})_* u(x,\ 1) = \xi(h^v_t(x)) \tag{8}$$

belongs, by Property 2.3, to $J^r g_{x_1}$. $J^r g_{x_1}$ is a vector subspace of the (finite dimensional) vector space of all jets of vector fields and therefore closed.

To prove that $J^r g \hookrightarrow T_z J^*_g$ we have to show that for every u there exists a ξ such that (6) is satisfied. This will follow from the fact that g has, by assumption, Property 2.4. First, we can assume that $u \in g^{(r)}_{x_1}$. We choose a suitable hypersurface H (cf. Property 2.4) intersecting transversally the trajectory γ of v starting at x_0. Then apply Property 2.4 putting $w = (h^v_{-\delta})_* u$, and $p = h^v_{1-\delta}(x_0)$ (cf. Figure 1). Let us denote by ξ_0 the vector field equal to v over H^- and $v + w$ over H^+. ξ_0 induces a flow $h^{\xi_0}_t$ which gives a smooth $h^{\xi_0}_1$ in a neighbourhood of x_0 and let $E^r(\xi_0)$ be its r-jet at x_0. By Property 2.4 we have an approximating family P_ϵ for which, for sufficiently small ϵ, we have

$$E^r(P_\epsilon(v,\ w)) = E^r(v) + \xi + o(|\xi|) + A_\epsilon,$$

(for more details see the proof of Theorem 3.8), where $A_\epsilon \overset{\epsilon\to 0}{\longrightarrow} 0$ uniformly. Since $\dim J^r g_x$ is independent of x, then $T_z J^*_g$ is independent of z, so J^*_g is an immersive submanifold. Q.E.D.

REMARK 3.3. We *conjecture* that the space J^*_g is a submanifold of $J(W, X)$.

The following corollary follows from the above proof.

COROLLARY 3.4. *The mapping* $E^r: \Gamma^*(g) \times W \to J^*_g$ *is a* submersion.

Let $h \subset g$, be a subspace of g. We introduce the following space of Lipschitz sections of h

$$\Gamma^*(h) = \{v \in \Gamma(h): v(x_0) \neq 0\}.$$

PROPOSITION 3.5. *A subspace h of g is an accessible subspace of the space g if and only if*

$$E^r|_{\Gamma^*(h)\times W}: \Gamma^*(h) \times W \to J^*_g$$

is a submersion.

Proof. First we prove that "only if" part. Let $E^r|_{\Gamma^*(h)\times W}$ be a submersion and let $A^r \subset J^r$ be a typical submanifold for E^r. Then by Thom-Abraham transversality theorem (cf. [6]), there exists an open and dense subset $\mathcal{A} \subset \Gamma^*(h)$ such that for every $a \in \mathcal{A}$ the mapping $E^r(a): W \to J^*_g$, is transversal to A^r. Thus the subspace h is accessible. To prove the "if" part, we note that the accessibility of h implies transversality of E^r to an arbitrary point from J^*_g. But this is exactly the submersivity of $E^r|_{\Gamma^*(h)\times W}$. Q.E.D.

REMARK 3.6. If $E^r|_{\Gamma^*(h)\times W}$ is a submersion then there exists a finite dimensional subspace h_0 of h, such that if $v \in \Gamma^*(h)$ and $t \to E^r(tv) \in J^*_g$ is a curve, then for every $t \neq 0$ we have

$$T_{E^r(tv)} J^*_g = \mathrm{Im}(E^r|_{\Gamma^*(tv+h_0)})_*|_{E^r(tv)}. \tag{9}$$

So the mapping $E^r: \Gamma^*(tv + h_0) \to J^*_g$ is a submersion.

Let $v, \xi \in \Gamma^*(g)$. We introduce the following iterated bracket of v and ξ:

$$[v, \xi]_i = \underbrace{[v, [, \ldots, [v, \xi], \ldots,]]}_{i\times} \in \Gamma^*(g), \tag{10}$$

where $[v, \xi]_0 = \xi$, $[v, \xi]_1 = [v, \xi]$.

Let h be a subsheaf of g.

DEFINITION 3.7. We say that h satisfies the *α-property* if for almost all $v \in h$, for every $w \in g_{x_0}^{(k)}$, $k \leqslant r$, $x_0 \in M$, and every $W_{x_0} \in \mathcal{M}$ there exists an $l \in \mathbf{N} \cup \{0\}$ and germs of vector fields $\xi_0, \xi_1, \ldots, \xi_l \in h_{x_0}$, such that

$$\pi_*([v, \xi_0]_j + [v, \xi_1]_{j-1} + \cdots + \xi_j)|W_{x_0} \in \pi^*(g)_{x_0}^{(k+1)}|W_{x_0} \qquad (\alpha_j)$$

$$\left(\text{i.e. } j_{x_0}^k\left(\pi_* \sum_{i=0}^{j}[v, \xi_i]_{j-i}|W_{x_0}\right) = 0\right), \quad \text{for } j = 0, \ldots, l-1,$$

and

$$\pi_*([v, \xi_0]_l + [v, \xi_1]_{l-1} + \cdots + \xi_l - w)|W_{x_0} \in \pi^*(g)_{x_0}^{(k+1)}|W_{x_0} \qquad (\alpha_l)$$

$$\left(\text{i.e. } j_{x_0}^k\left(\pi_* \sum_{i=0}^{l}[v, \xi_i]_{l-i}|W_{x_0}\right) = j_{x_0}^k(\pi_* w|W_{x_0}\right).$$

By $\pi^*(g)_{x_0}^{(k+1)}$ we denote the space, defined by g, of germs of sections of the induced bundle π^*TX, with zero k-jet at x_0.

THEOREM 3.8. *Let g be the sheaf of analytic vector fields. Let $E^r|_{\Gamma^*(h)\times\mathcal{M}}$ be a submersive map. Then h satisfies the α-property.*

Proof. We know that for some finite dimensional subspace h_0, $h_0 \subset h$, $E^r|_{\Gamma^*(tv+h_0)\times\mathcal{M}}$ is submersive. Let B_0 be a closed ball in h_0. Then $E^r(tv + B_0)$ contains some neighbourhood of $E^r(tv)$. Making use of the assumption of analyticity of g we have that $E^r(tv+B_0)$ is an analytic subset of J_g^*. Thus we immediately obtain that (see [7]) there exists an $N \in \mathbf{N}$ such that for every $w \in \Gamma^*(g)$

$$E^r(t(v + t^{N-1}w)) \in E^r(tv + B_0). \qquad (11)$$

Thus for every t there exists $\xi \in B_0$ such that

$$E^r(t(v + t^{N-1}w)) = E^r(t(v + \xi)). \qquad (12)$$

Using Puiseux theorem we can assume that ξ is a convergent fractional power series:

$$\xi = \xi(t^{1/m})$$

(depending also on w).

We notice that $E^r(t(v + \xi))$ is the r-jet at x_0 of the mapping $\pi(z(x, 1, t))$, where $z(x, s, t)$ satisfies the equation

$$\partial_s z(x, s, t) = [t(v + \xi)](z(x, s, t)). \qquad (13)$$

Let us denote $u(x, t) = z(x, 1, t)$. Inserting $\bar{s} = st$ into (13) we find that $u(x, \bar{s})$ (which obviously depends also on ξ), satisfies

$$\partial_{\bar{s}} u(x, \bar{s}) = (v + \xi)(u(x, \bar{s})); \tag{14}$$

thus concluding, we see that: $E^r(t(v + \xi))$ is the r-jet at x_0 of $u(x, t)$, where $u(x, \bar{s})$ is the solution of (14).

Let us look on (14) as a perturbation of the equation

$$\partial_{\bar{s}} y_0(x, \bar{s}) = v(y_0(x, \bar{s})),$$

which is clearly satisfied by $y_0(x, \bar{s}) = h^v_{\bar{s}}(x)$. We linearize (14); it is easy to see that the linear (with respect to ξ) term $y = y_\xi(x, \bar{s})$ satisfies

$$\partial_{\bar{s}} y = Dv(y_0)y + \xi(\bar{s}, y_0), \tag{15}$$

where $u(x, \bar{s}) = y_0(x, \bar{s}) + y_\xi(x, \bar{s}) + \{\text{terms of order} \geqslant 2 \text{ with respect to } \xi\}$ and $y(x, 0) \equiv 0$, $y_0(x, 0) = x$.

Now we expand ξ with respect to $\bar{s}$

$$\xi = \xi_0 + \xi_1 \bar{s} + \cdots,$$

where $\xi_i = \xi_i(x)$. Then from (15) the r-jet (with respect to x) of the linear term of the expansion of y with respect to $\bar{s} = t$ is

$$t j^r_{x0} \xi_0(x).$$

That is just the right hand side of the equation (12). If $j^r_{x0}\xi_0(x) \neq 0$ then similar arguments applied to the left hand side of (12) give the equality

$$t^N j^r_{x0} w = t j^r_{x0} \xi_0. \tag{16}$$

So the α-property is satisfied for $k = r$ and $l = 0$, if we put $N = 1$.

If $j^r_{x0}\xi_0 = 0$, then we have to consider the second term of the expansion of y with respect to $\bar{s}$. To do so we differentiate both sides of (15) with respect to $\bar{s}$, and put $\bar{s} = t = 0$.

Now we have

$$\partial^2_{\bar{s}} y|_{\bar{s}=0} = D\xi_0(x)v(x) + \xi_1(x). \tag{17}$$

Since $j^r_{x0}\xi_0 = 0$, $j^r_{x0}[v, \xi_0] = j^r_{x0} D\xi_0 v$, so we get

$$j^r_{x0} \partial^2_{\bar{s}} y|_{\bar{s}=0} = j^r_{x0}([v, \xi_0] + \xi_1). \tag{18}$$

and

$$j^r_{x_0}y(x,\, t) = \left(\frac{t^2}{2}\right) j^r_{x_0}([v,\, \xi_0] + \xi_1) + o(t^2). \tag{19}$$

Comparing both sides of (12) we obtain the α-property satisfied for $k = r$ and $l = 1$, provided $j^r_{x_0}([v,\, \xi_0] + \xi_1) \neq 0$.

If $j^r_{x_0}([v,\, \xi_0] + \xi_1) = 0$ then we continue the above procedure. Finally we arrive at the following statement:

Let $l \in \mathbf{N}$ be the smallest number for which

$$j^r_{x_0}\left(\sum_{i=0}^{l}[v,\, \xi_i]_{l-i}\right) \neq 0.$$

Then

$$j^r_{x_0}y(x_0,\, t) = \left(\frac{t^l}{l!}\right) j^r_{x_0}\left(\sum_{i=0}^{l}[v,\, \xi_i]_{l-i}\right) + o(t^l).$$

Passing to vector fields along π we see that this statement ends the proof of Theorem 3.8, i.e. we have got that the α-property is a necessary condition for the submersivity of $E^r|_{\Gamma^*(h)\times\mathcal{M}}$. Q.E.D.

Now we are going to state the corresponding sufficient condition. For this purpose we assume: at each fiber h_x of h, equipped with the inverse limit topology ($h_x = \varprojlim_{U\ni x} h_U$, where U denotes an open neighbourhood of x), there is a distinguished open set $h^0_x \subset h_x$. We take v to be a Lipschitz section of h^0 and write $v \in \Gamma(h^0)$, i.e. for any $x \in M$, $v(x) \in h^0_x$.

DEFINITION 3.9. We say that h satisfies the *β-property* if for every $v \in \Gamma(h^0)$, for every $w \in g^{(k)}_{x_0}$, $k \leqslant r$, $x_0 \in M$, and every $W_{x_0} \in \mathcal{M}$ there exists an $l \in \mathbf{N}$ and the germ of a vector field $\xi \in h^0_{x_0}$, such that

$$\pi_*([v,\, \xi]_j|W_{x_0}) \in \pi^*(g)^{(k+1)}_{x_0}|W_{x_0} \tag{β_j}$$

$$\text{(i.e. } j^k_{x_0}(\pi_*[v,\, \xi]_j|W_{x_0}) = 0), \quad \text{for } j = 0, \ldots, l-1,$$

and

$$\pi_*([v,\, \xi]_l - w)|W_{x_0} \in \pi^*(g)^{(k+1)}_{x_0}|W_{x_0} \tag{β_l}$$

$$\text{(i.e. } j^k_{x_0}(\pi_*([v,\, \xi]_l - w)|W_{x_0}) = 0).$$

THEOREM 3.10. *Let a subsheaf h of g satisfy the (β)-property. Then $E^r|_{\Gamma^*(h)\times\mathcal{M}}$ is a submersion.*

Proof. Let $v \in \Gamma^*(h)$, and h_t^v be the flow corresponding to v. We denote

$$g_{x_1} = \bigoplus_{k\leqslant r} \frac{g_{x_1}^{(k)}}{g_{x_1}^{(k+1)}} \oplus g^{(r+1)}, \quad x_1 = h_1^v(x_0),$$

and by

$$pr_k\colon g_{x_1}^{(k)} \to \frac{g_{x_1}^{(k)}}{g_{x_1}^{(k+1)}}$$

we denote the canonical projection. We choose $w_{ki} \in g_{x_1}^{(k)}$, $i \in I_k \subset \mathbf{N}$, such that $\{pr_k(w_{ki})\}_{i\in I_k}$ form a basis of the vector space $g_{x_1}^{(k)}/g_{x_1}^{(k+1)}$. The set of all elements of the form $\{pr_k(w_{ki}),\ 0 \leqslant k \leqslant r,\}_{i\in I_k}$ gives a basis of the space $g_{x_1}/g_{x_1}^{(r+1)}$.

Take any element $\bar{w} = w_{ki}$ of our basis. Let $w = \pi_*\bar{w}|_{W_{x_1}}$. We show that for $\lambda \in \mathbf{R}$ sufficiently close to zero, there exists a $\xi_\lambda \in \Gamma^*(h)$, such that the tangent vector $(d/dt)(E^r(v+t\xi_\lambda))|_{t=0}$ to the curve $t \to E^r(v+t\xi_\lambda)$ is equal to $\lambda w + o(\lambda)$. In fact, let ξ be a section of h, defined in a neighbourhood U of x_1 such that

$$\pi_*([v,\ \xi]_l - \bar{w})|_{W_{x_1}} \in \pi^*(g)_{x_1}^{(k+1)}|_{W_{x_1}},$$

where

$$\pi_*([v,\ \xi])_j|_{W_{x_1}} \in \pi^*(g)_{x_1}^{(k+1)}|_{W_{x_1}}, \quad \text{for } j = 0,\ldots,\ l-1.$$

Let us take a $\delta > 0$. We consider $(h_{-\delta}^v)_*(\xi) \in h_{x_\delta}$, $x_\delta = h_{1-\delta}^v(x_0)$, i.e. the vector field $\xi \in g_{x_1}^{(k+1)}$, moved to the point x_δ. We assume that δ is so small that $x_1 \in h_{-\delta}^v(U)$. Let us take a suitable hypersurface H transversal to the trajectory $\gamma\colon [0,\ 1] \ni\!\!\to h_t^v(x_0)$ at the point x_δ, (cf. Property 2.4). We consider two parts H^- and H^+ of an open neighbourhood of the trajectory (see Figure 2 below).

Now we consider the following (cf. Property 2.4),

$$\xi_0 = \begin{cases} v & \text{on } H^- \\ v+\xi & \text{on } H^+ \end{cases}.$$

Let

$$\mathcal{E}_t\left(v + \begin{cases} 0 & \text{on } H^- \\ \xi & \text{on } H^+ \end{cases}\right)$$

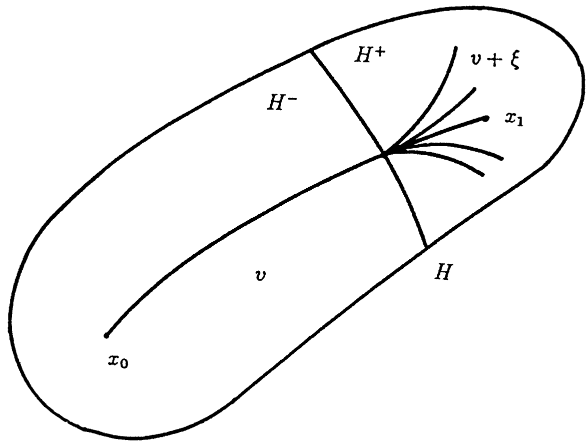


Fig. 2.

denote the flow induced by ξ_0 on M. By the transversality of H to γ we know that

$$\mathcal{E}_1\left(v+\begin{cases}0 & \text{on} \quad H^-\\ \xi & \text{on} \quad H^+\end{cases}\right)$$

is smooth in a neighbourhood of x_0. Thus we can have its r-jet,

$$\mathcal{E}^r\left(v+\begin{cases}0 & \text{on} \quad H^-\\ \xi & \text{on} \quad H^+\end{cases}\right).$$

As before, we denote by

$$E^r\left(v+\begin{cases}0 & \text{on} \quad H^-\\ \xi & \text{on} \quad H^+\end{cases}\right)$$

the r-jet of $\mathcal{E}_1(\xi_0)$ at x_0.

We compute the linear terms of

$$E^r\left(v+\begin{cases}0 & \text{on} \quad H^-\\ \xi & \text{on} \quad H^+\end{cases}\right)$$

with respect to ξ. Essentially we repeat the proof of Theorem 3.8, where we studied the equation (15). In the present case we obtain that the solution of (15) has the expansion

$$j^k\pi_* y(x,\,\delta) = \left(\frac{\delta^l}{l!}\right) j^k\pi_*([v,\,\alpha]) + o(\delta^l) = \left(\frac{\delta^l}{l!}\right) w + o(\delta^l),$$

which proves that

$$E^{\tau}\left(v+\begin{cases}0 & \text{on } H^{-}\\ \xi & \text{on } H^{+}\end{cases}\right)=E^{\tau}(v)+\left(\frac{\delta^{l}}{l!}\right)w+o(\delta^{l}).$$

The field ξ is $(k+1)$-flat at x_δ so we use the Property 2.4 and write the approximating family $P_\epsilon(v, \xi)$. Obviously we have

$$E^{\tau}(P_\epsilon(v,\xi))=E^{\tau}\left(v+\begin{cases}0 & \text{on } H^{-}\\ \xi & \text{on } H^{+}\end{cases}\right)+E_\epsilon(v,\xi),$$

where $E_\epsilon \overset{\epsilon\to 0}{\longrightarrow} 0$.

Then for sufficiently small ϵ we have

$$E^{\tau}(P_\epsilon(v,\xi))=E^{\tau}(v)+\left(\frac{\delta^{l}}{l!}\right)w+o(\delta^{l})+A_\epsilon,$$

where $A_\epsilon \overset{\epsilon\to 0}{\longrightarrow} 0$. We take ϵ so small that $|A_\epsilon| < o(\delta^l)$, $\lambda = \frac{\delta^l}{l!}$ and $\xi_\lambda = P_\epsilon(v, \xi) - v$. Q.E.D.

Our results can be briefly recapitulated as follows.

COROLLARY 3.11. *If h is accessible then h satisfies the α-property. If h satisfies the β-property then h is accessible.*

4. Genericity of Exp for Riemannian and sub-Riemannian metrices

Let M be the cotangent bundle; $M = T^*X$, $\dim X = n$, with $\pi = \pi_X : T^*X \to X$ the canonical bundle projection. By g we denote the sheaf of Hamiltonian vector fields on T^*X. Let h be a subsheaf of g and let $\mathcal{M}$ denote a class of germs of submanifolds of M; unspecified for the moment by $\bar{g}$ and $\bar{h}$ we denote the corresponding sheaves of local Hamiltonians on T^*X; thus $\bar{g}$ is the sheaf of germs of functions f, such that Jdf is a section of $\bar{g}$. For $W \in \mathcal{M}$ and $\bar{p}_0 \in W$ we denote by $\mathcal{F}_{W_{\bar{p}_0}}$ the space of germs of functions on W, at p_0; $\mathcal{F}^{(k)}_{W_{\bar{p}_0}}$ are the germs vanishing up to order $k-1$ at $\bar{p}_0$.

Let $v_{f_1}, v_{f_2} \in \Gamma^*(g)$, $v_{f_i} = Jdf_i$, where f_1, f_2 are the corresponding Hamiltonians. We have

$$[v_{f_1}, v_{f_2}] = v_{\{f_1,f_2\}},$$

where $\{.,.\}$ denotes the standard Poisson bracket on T^*X. In the Darboux coordinates (q, p) on T^*X we have, of course

$$(\pi_X)_* v_f = \partial_p f,$$

where by $\partial_p f$ we denote the n-tuple $(\partial_{p_1} f, \ldots, \partial_{p_n} f) \in (\bar{g}_{\bar{p}_0})^n$.

PROPOSITION 4.1. *The α-property for the subsheaf $h \subset g$ is equivalent to the following condition:*

(α): For almost all $f \in \bar{h}_{\bar{p}_0}$, for every $\eta \in \bar{g}^{(k)}_{\bar{p}_0}$, $k \leqslant r$ and $W_{\bar{p}_0} \in \mathcal{M}_{\bar{p}_0}$, there exist: an $l \in \mathbf{N} \cup \{0\}$ and $f_0, f_1, \ldots, f_l \in \bar{h}_{\bar{p}_0}$ such that

$$\left.\sum_{i=0}^{j} \partial_p \{f, f_i\}_{j-i}\right|_{W_{\bar{p}_0}} \in (\mathcal{F}^{(k+1)}_{W_{\bar{p}_0}})^n,$$

$$\left(i.e.\ j^k_{\bar{p}_0} \left(\sum_{i=0}^{j} \partial_p \{f, f_i\}_{j-i} \right)\Bigg|_{W_{\bar{p}_0}} = 0 \right),$$

for $j = 0, \ldots, l-1$, and

$$\left.\left(\partial_p \eta - \sum_{i=0}^{l} \partial_p \{f, f_i\}_{l-i} \right)\right|_{W_{\bar{p}_0}} \in (\mathcal{F}^{(k+1)}_{W_{\bar{p}_0}})^n.$$

$$\left(i.e.\ j^k_{\bar{p}_0} \partial_p \eta |W_{\bar{p}_0} = j^k_{\bar{p}_0} \sum_{i=0}^{l} \partial_p \{f, f_i\}_{l-i} \Bigg|_{W_{\bar{p}_0}} \right).$$

By a straightforward modification of the above α-property we obtain the β-property expressed in terms of Hamiltonians.

Let $\mathcal{L}_{\bar{p}}$, $\bar{p} \in T^*X$ denote the space of germs at $\bar{p}$ of Lagrangian submanifolds in (T^*X, ω_X). If $v \in \Gamma(g)$ then the class $\mathcal{L} = \cup_{\bar{p} \in T^*X} \mathcal{L}_{\bar{p}}$ is obviously preserved by the flow of symplectomorphisms h^v_t. From now on we take $\mathcal{L}$ as $\mathcal{M}$.

Let us fix a germ at $\bar{p}$ of a Lagrangian submanifold $W_{\bar{p}} \in \mathcal{L}_{\bar{p}}$. We shall study $E|_{W_{\bar{p}}}$, or more formally

$$\mathrm{Exp}_v |_{W_{\bar{p}}} = \pi_X \circ h^v_1 |_{W_{\bar{p}}}, \quad q = \pi_X(\bar{p}),$$

(in the most interesting and classical case $W_{\bar{p}}$ is the germ of the fibre $T^*_q X$). We notice that in standard terminology $h^v_1|_{W_{\bar{p}}}$ is a Lagrangian embedding and $\mathrm{Exp}_v |_{W_{\bar{p}}}$ is the corresponding Lagrangian projection.

We shall now discuss the genericity property of the Exp-map in the Riemannian geometry.

Let $h \subset g$ denote the subsheaf of Hamiltonian vector fields with quadratic Hamiltonians with respect to p. We look on $\Gamma(h)$ as the space of geodesic vector fields on X with the families of quadratic nondegenerate forms on T^*X playing the role of nondegenerate Hamiltonians. In what follows we assume $\det(h^{ij}) \neq 0$

for our Hamiltonian vector fields $v_H \in \Gamma(h^0)$, $H = \Sigma h^{ij} p_i p_j$, (see Definition 3.9).

THEOREM 4.5. *Let $\bar{p} \in T^*V$. There exists an open and dense set of Riemannian metrics $h' \subset \Gamma(\bar{h})$, such that for every $f \in h'$, the Exp-map $\mathrm{Exp}_{v_f}|_{W_{\bar{p}}}$ has only the singularities appearing in generic Lagrangian projections.*

Proof. First we prove the accessibility of h. It is enough to check the sufficient condition, i.e. the β-property for our sheaf h; Without loss of generality we take $\bar{p}_0 = (0, p_0)$; let $W_{\bar{p}_0} = W_{p_0}$ be an element of $\mathcal{L}_{\bar{p}_0}$. We denote $\mathcal{F}_{p_0} = \mathcal{F}_{W_{p_0}}$ the space of germs at p_0 of analytic functions on W_{p_0}. Thus the β-property reads as follows:

Let $f \in \bar{h}_{\bar{p}_0}$ be a nondegenerate Hamiltonian at $q = 0$. Let $k \in \mathbf{N}$. For every $\bar{\eta} \in g_{\bar{p}_0}^{(k)}$, there exists an $l \in \mathbf{N} \cup \{0\}$ and $f_0 \in \bar{h}_{\bar{p}_0}$ such that

(β_j):

$$\partial_p \{f, f_0\}_j|_{W_{p_0}} \in (\mathcal{F}_{p_0}^{(k+1)})^n, \quad \text{for } j = 0, \ldots, l-1$$

and

(β_l):

$$(\partial_p \eta - \partial_p \{f, f_0\}_l)|_{W_{p_0}} \in (\mathcal{F}_{p_0}^{(k+1)})^n.$$

We write

$$f = \sum_{ij} g^{ij}(q) p_i p_j.$$

Let us take

$$f_0 = \sum_{ij} h^{ij}(q) p_i p_j \in \bar{g}_{\bar{p}_0}^{(N)}. \tag{20}$$

Then

$$\{f, f_0\}_l = \partial_{v_f}^l f_0 = \sum_{ijkrs} \left(g^{ij}(q) p_j \frac{\partial}{\partial q^i} - g_{,k}^{ij}(q) p_i p_j \frac{\partial}{\partial p_k} \right)^l h^{rs}(q) p_r p_s.$$

Let $i_{W_{p_0}}$ be a Lagrangian embedding of W_{p_0} into T^*X. First assume that $i_{W_{p_0}}(W_{p_0})$ can be described by

$$q = \phi(p), \quad \phi(p) = (\phi_1(p), \ldots, \phi_n(p)). \tag{21}$$

Then the (β_l)-condition can be written in the form

$$\partial_p \eta(q, p)|_{\{q=\phi(p)\}}$$

$$= \partial_p \sum_{i_1,\dots,j_{l+2}} g^{i_3 j_3}(q)\cdots g^{i_{l+2} j_{l+2}}(q) h^{i_1 i_2}_{,j_3\cdots j_{l+2}}(q) p_{i_1}\cdots p_{i_{l+2}}|_{\{q=\phi(p)\}},$$

for the lowest degree terms in (q, p).

Let us define $I = \langle \phi_1,\dots, \phi_n\rangle$, the ideal in $\mathcal{F}_{p_0}(= \mathcal{F}_{W_{\bar{p}_0}})$ generated by $\phi_1(p),\dots, \phi_n(p)$. We consider $\mathcal{F}_{p_0}$ as a graded ring with respect to I^s, $s \in \mathbf{N}$. Let $\eta \in \bar{g}_{p_0}$. We put

$$w = \partial_p \eta|_{q=\phi(p)} \in (\mathcal{F}^{(N)}_{p_0})^n.$$

Let s be the biggest integer such that

$$w \in (I^s)^n. \tag{22}$$

The matrix $g^{ij}(0)$ is invertible, so we can find $l \in \mathbf{N} \cup \{0\}$ and $f_0 = \Sigma_{ij} h^{ij}(q) p_i p_j$ (i.e. a matrix $h^{ij}(q)$), such that

$$\sum_{i_1,\dots,j_{\kappa+2}} g^{i_3 j_3}(\phi(p))\cdots g^{i_{\kappa+2} j_{\kappa+2}}(\phi(p)) h^{i_1 i_2}_{,j_3\cdots j_{\kappa+2}}$$
$$\times(\phi(p))\partial_p(p_{i_1}\cdots p_{i_{\kappa+2}}) \in (I^{s+1})^n,$$

for $\kappa = 0,\dots, l-1$, and

$$\sum_{i_1,\dots,j_{l+2}} g^{i_3 j_3}(\phi(p))\cdots g^{i_{l+2} j_{l+2}}(\phi(p)) h^{i_1 i_2}_{,j_3\cdots j_{l+2}}$$
$$\times(\phi(p))\partial_p(p_{i_1}\cdots p_{i_{l+2}}) - w \in (I^{s+1})^n.$$

But from (22) we have $I^{s+1} \subset \mathcal{F}^{(N+1)}_{p_0}$. Thus we obtain the (β)-condition, i.e.

$$\partial_p\{f, f_0\}_\kappa|_{q=\phi(p)} \in (\mathcal{F}^{(N+1)}_{p_0})^n, \quad \text{for } \kappa = 0,\dots, l-1,$$

and

$$\partial_p\{f, f_0\}_l|_{q=\phi(p)} - w \in (\mathcal{F}^{(N+1)}_{p_0})^n.$$

Let us discuss briefly what modifications should be done if (21) is not satisfied. Suppose for a moment that W_{p_0} is transversal to the fibres of T^*X. In this case we write $i_{W_{p_0}}(q) = (q, \psi(q))$ and the (β)-condition is satisfied immediately. In fact, for any $w = (w^1,\dots, w^n) \in (\mathcal{F}^{(N)}_{p_0})^n$ (elements of $\mathcal{F}_{p_0}$ are parametrized by q) there exists an $f_0 = \Sigma_{ij} h^{ij}(q) p_i p_j$ such that

$$w^k(q) = \sum_i h^{ik}(q)\psi_i(q)(\mathrm{mod}(\mathcal{F}^{(N+1)}_{p_0})).$$

This is obvious; taking $p_0 = (1, 0, \dots, 0)$ we find $h^{1k}(q) = w^k(q)$.

The most general "IJ"-case, $I \cup J = \{1, \dots, n\}$, $I \cap J = \emptyset$, where

$$i_{W_{p_0}}(q_I, p_J) = (q_I, \phi_J(q_I, p_J), \psi_I(q_I, p_J), p_J)$$

can be treated in a similar way by mixing both methods. Q.E.D.

REMARK 4.3. A slightly stronger genericity result is true also for the map $h_1^v|_{W_{\bar{p}_0}}: W_{\bar{p}_0} \to T^*X$. In this case the (β)-condition is fulfilled and one can state the result as follows:

Let $\bar{p} \in T^*X$. There exists an open and dense subset h' of quadratic hamiltonians h such that for every $f \in \Gamma(h')$ the Lagrangian embedding $h_1^{v_f}|_{W_{\bar{p}_0}}$ is generic in the space of all mappings $h_1^v|_{W_{\bar{p}_0}}$ induced by general Hamiltonian vector fields $v \in \Gamma(g)$. (This extends the Theorem 1 in [11], p. 735).

REMARK 4.4. Let us choose a family of germs $W_{\bar{p}(q)}$ of fibers of T^*X defined by a section $X \to T^*X$, $X \ni q \to \bar{p}(q) \in T^*X$. Then we define

$$\mathrm{Exp}_q = \mathrm{Exp}_v|_{W_{\bar{p}(q)}}$$

and look on it as a family of maps parametrized by $q \in X$. C.T.C. Wall ([11], Conjecture 2, p. 735) conjectured that for a generic metric on X, i.e. $f = \Sigma_{ij} g^{ij} p_i p_j$, $\mathrm{Exp} = \{\mathrm{Exp}_q : q \in X\}$ is a generic n-parameter family of Lagrangian projections (cf. [1]). A straightforward calculation shows that the necessary condition (α) is not fulfilled. In this case we have $W = T^*X$. Thus the "Wall Conjecture" is not true (cf. [5, 3]). In fact there is an infinite number of constraints resulting from the obvious formula:

$$(h_t^v)_* v = v, \quad v = Jdf. \tag{23}$$

Since f is quadratic with respect to p, this implies that $(h_t^v)_* v$ must be linear with respect to v, and this is a strong constraint for h_t^v. To be more explicit, let $(p, Q) \to G_t(p, Q)$, $\det((\partial^2 G_t/\partial p \partial Q)(p, Q)) \neq 0$, be a generating function for h_t^v, (for a standard notation see [2]). By $(q, p) \to Q_t(q, p)$ we denote the solution of the equation $q - (\partial G_t/\partial p)(p, \bullet) = 0$. We define

$$\Phi^j(q, p) = \partial_{q_l} Q_t^j(q, p) \sum_i g^{il}(q) p_i - \partial_{p_s} Q_t^j(q, p) \sum_{ik} g^{ik}_{,s}(q) p_i p_k.$$

By a simple verification we find

$$\partial_p^\alpha (\Phi^j \circ (h_t^v)^{-1})(q, p) \equiv 0, \quad \text{for } |\alpha| > 1.$$

One can *conjecture* that all constraints satisfied by the family Exp arise from the one above simply by differentiation.

Now let us pass to the Exp-map in the sub-Riemannian geometry (cf. [9]). Let $M = T^*X$ be as above, and dim $X = n+1$. By $V \subset TX$ we denote a smooth distribution of hyperplanes on X, i.e. a subbundle of TX. All our arguments are valid for any dimension of V, however for simplicity of notation, we shall assume that codim $V = 1$. Locally V is annihilated by a 1-form,

$$\omega = dq^{n+1} + \sum_{i=1}^{n} A_i(q)\, \mathrm{d}q^i. \tag{24}$$

Let $H: T^*X \to R$ be a smooth function. We say that the Hamiltonian vector field v_H with hamiltonian H is *horizontal* if

$$(\pi_X)_* v_H|_{\bar{p}} \in V_{\pi_X(\bar{p})}, \quad \text{for all } \bar{p} \in T^*X.$$

By k we denote the sheaf of horizontal Hamiltonian vector fields. An easy check shows that if $v_f \in k$, then

$$f = \bar{f}(q, p'_1, \dots, p'_n),$$

for some smooth function $\bar{f}$ and $p'_i = p_i - A_i(q)p_{n+1}$.

Let V be equipped with a quadratic nondegenerate form $\langle .\,, .\rangle$ varying smoothly on $q \in X$. By g^{ij} we denote the inverse matrix to that defined by $\langle .\,, .\rangle$. Analogously to the usual Riemannian case we have the sheaf (subsheaf of k) h of horizontal geodesic vector fields defined by the quadratic Hamiltonians

$$f(q, p) = \sum_{ij=1}^{n} g^{ij}(q)(p_i - A_i(q)p_{n+1})(p_j - A_j(q)p_{n+1}).$$

Both these sheaves are subsheaves of the sheaf of Hamiltonian vector fields g. By $\bar{h}$ we denote the space of Hamiltonians quadratic in $p' = (p'_1, \dots, p'_n)$.

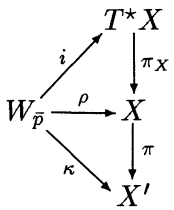
Let $X_i = (\partial/\partial q^i) - A_i(q)(\partial/\partial q^{n+1})$, $i = 1, \dots, n$; they give a basis of sections of V. Let $q \in X$. If the basic vector fields X_i, along with all their commutators span T_qX then the distribution V is said to satisfy the *Hörmander condition* at q. If this condition is fulfilled at every $q \in X$ then V is called also a *non-holonomic distribution* (cf. [9, 10]).

Let I be a submanifold of T^*X, dim $I \leqslant n+1$. If $\omega_X|_I = 0$, then we call I an isotropic submanifold. Let $\mathcal{I}_{\bar{p}}$, $\bar{p} \in T^*X$ denote the space of germs at $\bar{p}$, of isotropic submanifolds of dimension n in (T^*X, ω_X). We see that $\mathcal{I} = \cup_{\bar{p} \in T^*X}$ is preserved by the flow of symplectomorphisms h_t^v, $v \in \Gamma(g)$.

In what follows we assume that we are given a fibering π on X, $\pi: X \to X'$, such that ker $\pi_* \oplus V = TX$. Locally $\pi(q) = (q')$, $q = (q', q^{n+1})$, $q' = (q^1, \dots, q^n)$. Let $W_{\bar{p}} \in \mathcal{I}_{\bar{p}}$. The inclusion $i_{W_{\bar{p}}}: W_{\bar{p}} \to T^*X$ is called an isotropic immersion of $W_{\bar{p}}$, and $\rho_{W_{\bar{p}}} = \pi_X \circ i_{W_{\bar{p}}}$ an *isotropic projection.* Now we define

$$\kappa_{W_{\bar{p}}}: W_{\bar{p}} \to X', \quad \kappa_{W_{\bar{p}}} = \pi \circ \rho_{W_{\bar{p}}};$$

these maps we will call *subisotropic maps*. A smooth mapping $\kappa: W_{\bar{p}} \to X'$ is subisotropic if and only if there exists an isotropic immersion $i: W_{\bar{p}} \to T^*X$ such that the following diagram commutes



Let $f \in h$, and Exp_{v_f} be the corresponding Exp-map. We denote $\tilde{\mathrm{Exp}}_{v_f} = \pi \circ \mathrm{Exp}_{v_f}$ and we call $\tilde{\mathrm{Exp}}_{v_f}$ a *sub-Exp-map*. Adapting the proof of the Theorem 4.2 we obtain the following result.

THEOREM 4.5. *Let $\bar{p} \in T^*X$. Then there exists an open and dense set of quadratic Hamiltonians $h' \subset \bar{h}$, such that if $f \in h'$, then the sub-Exp-map $\tilde{\mathrm{Exp}}_{v_f}|_{W_{\bar{p}}}$, for any projection π, has only the singularities appearing in generic subisotropic maps.*

Now let $q_0 \in X$ and let H^{n-1} be an isotropic subspace of $T^*_{q_0}X$, $\bar{p}_0 \in H^{n-1}$, $\bar{p}_0 \neq 0$. What is interesting for us now is the description of all possible generic singularities of germs

$$(H, \bar{p}_0) \overset{\exp(g)}{\longrightarrow} X,$$

where g varies over the space of all Riemannian metrics on V. To get such a description we apply Theorem 4.5. Thus if $\pi: X \to X'$ is any projection (as in the statement of the Theorem) then all generic singularities of

$$\tilde{E} = \pi \circ \exp(g)$$

are of the form

$$p = \frac{\partial F}{\partial q}(\lambda, q)|_{\Sigma_F},$$

where

$$\Sigma_F = \left\{ (\lambda, q) \colon \left. \frac{\partial F}{\partial \lambda_i} \right|_{H^{n-1} \times X} = 0 \right\},$$

with the generating family F having only the generic singularities (see [4]).

To get $\exp(g)$ from $\pi \circ \exp(g)$ we remark that every line l in H^{n-1} passing through 0 is sent by $\exp(g)$ into a geodesic, and thus is a horizontal curve with a non-zero tangent vector, which is projected by π onto a non-zero vector. Let l_p be the line passing through 0 and p; let $\xi_0 = \bar{p}_0$ be considered as a vector in $T_{p_0} l_{\bar{p}_0}$, $x'_0 = \tilde{E}(\bar{p}_0)$, and

$$v'_0 = (\tilde{E}_*)_{\bar{p}_0}(\xi_0).$$

Let $N' \subset X'$ be any germ of a submanifold at x'_0 transversal to v'_0 and of codimension one. Let $N \subset X$ be any submanifold of codimension 2, such that $\pi|_N: N \to N'$ is a diffeomorphism and N is "generic" with respect to V. Now we lift $\tilde{E}$ to a map E into X such that N lies in the image of E and the curves $\tilde{E}(l_p)$ (p close to $\bar{p}_0$) are lifted into horizontal curves. We shall illustrate this procedure a little bit later, in the 2-dimensional case.

Now we investigate the generic subisotropic maps. We denote

$$D = \{p \in W_{\bar{p}}: \mathrm{Im}(\rho_*)_p \text{ is not transversal to } V_{\rho(p)}\}$$

and

$$\Delta = \kappa(D).$$

We will call Δ a horizontal set of ρ. Δ is the set of points of the image manifold $S = \rho(W_{\bar{p}})$, in which S is tangent to the distribution V. Let Γ denote the set of critical points of κ and $\Sigma = \kappa(\Gamma)$ denote the set of its critical values.

LEMMA 4.6. *Let V be a contact distribution (V satisfies the strong bracket generating hypothesis [9]), then for a generic subisotropic map κ, D is a curve.*

Proof. We show this fact for $n = 2$, $\dim X = 3$. $\pi: X \to X'$, $\pi(q^1, q^2, q^3) = (q^1, q^2)$. The general case is straightforward. Distribution V is annihilated by

$$\omega = \mathrm{d}q^3 + \sum_{i=1}^{2} A_i(q^1, q^2)\, \mathrm{d}q^i. \tag{25}$$

$\rho(W_{\bar{p}})$ is covered by geodesics. Without loss of generality we restrict our considerations to the vertical $W_{\bar{p}} \subset T^*_{\pi_X(\bar{p})}X$. We choose the parameterization $\{u_1, u_2\}$ of $W_{\bar{p}}$, such that u_1 parameterizes the geodesics (obviously horizontal with respect to V). Then

$$\frac{\partial \rho^3}{\partial u_1} + \sum_{i=1}^{2} A_i \circ \kappa \frac{\partial \rho^i}{\partial u_1} \equiv 0.$$

The second equation

$$F(u_1, u_2) = \left(\frac{\partial \rho^3}{\partial u_2} + \sum_{i=1}^{2} A_i \circ \kappa \frac{\partial \rho^i}{\partial u_2} \right)(u_1, u_2) = 0,$$

on maximal smooth strata of S defines a smooth curve. We see that $\nabla F \neq 0$. In fact

$$\frac{\partial F}{\partial u_1} = \left(\frac{\partial A_2}{\partial q^1} - \frac{\partial A_1}{\partial q^2}\right)\left(\frac{\partial \rho_1}{\partial u_2}\frac{\partial \rho_2}{\partial u_1} - \frac{\partial \rho_1}{\partial u_1}\frac{\partial \rho_2}{\partial u_2}\right) \neq 0$$

outside of the set of critical points of κ, because

$$[X_1, X_2] = \left(\frac{\partial A_2}{\partial q^1} - \frac{\partial A_1}{\partial q^2}\right)\frac{\partial}{\partial q^3} \neq 0.$$

Q.E.D.

Now we assume, $\dim X = 3$. For a contact distribution we have the following result.

THEOREM 4.7. *Let V be a contact distribution on R^3, annihilated by $\omega = dq^3 + \Sigma_{i=1}^2 A_i(q^1, q^2)\, dq^i$, $\pi: R^3 \to R^2: (q^1, q^2, q^3) \to (q^1, q^2)$, is the projection. Then for a generic subisotropic map;*

(1) *ρ is an immersion, $\Delta = \emptyset$ and κ is a diffeomorphism, fold or cusp-map.*
(2) *ρ is an immersion, $\Gamma = \emptyset$ and Δ is smooth curve.*
(3) *ρ is a singular map of corank 1, right-left equivalent ($\mathcal{A}$-equivalent, [8]) to Whitney's Cross-cap (S_0) with horizontal and critical sets tangent with the second order tangency.*

Proof. For generic isotropic map, the corresponding map κ is one of the Whitney's stable cases of smooth mappings $R^2, 0 \to R^2, 0$ provided S is a smooth hypersurface of R^3. If S is the remaining stable case – the Cross-cap (cf. [8]), then the subisotropic map κ is a fold-map. Distribution V, defined by ω is transversal to the fibers of π, so we easily see that, in the smooth case of S, the horizontal (Δ) and critical (Γ) sets are disjoint, which proves the first two cases.

For the lifting of κ we can write

$$z(u_1, u_2) = -\int_0^{u_1} x(s, u_2)\frac{\partial y}{\partial s}(s, u_2)\, ds + \phi(u_2),$$

where we use the notation $(q) = (x, y, z)$. Thus for x, y, z we write the following expansions

$$\begin{aligned} x &= \beta_1 u_1 + \beta_2 u_2 + \beta_{11} u_1^2 + \beta_{12} u_1 u_2 + \beta_{22} u_2^2 + \mathbf{m}^3, \\ y &= \alpha_1 u_1 + \alpha_2 u_2 + \alpha_{11} u_1^2 + \alpha_{12} u_1 u_2 + \alpha_{22} u_2^2 + \mathbf{m}^3, \\ z &= -\tfrac{2}{3}\alpha_{11}\beta_1 u_1^3 - (\tfrac{1}{2}\beta_1\alpha_{12} + \alpha_{11}\beta_2) u_1^2 u_2 - \alpha_{12}\beta_2 u_1 u_2^2 \\ &\quad + \gamma_2 u_2 + \gamma_{22} u_2^2 - \alpha_1 \int_0^{u_1} x(s, u_2)\, ds + \mathbf{m}^3, \end{aligned}$$

where $\mathbf{m}$ denotes the maximal ideal in the space of germs of smooth functions of variables u_1, u_2.

The cases $(\beta_1\alpha_2 - \alpha_1\beta_2) = 0$, and $\alpha_1 = 0$ may appear transversally, then generically $\gamma_2 \neq 0$ and we have the smooth case of S. So we have to assume $\alpha_1 = (\partial_y/\partial u_1) \neq 0$. In this case we define new coordinates of $W_{\bar{p}}$

$$(u_1,\ u_2) \to (y,\ u_2) = (y(u_1,\ u_2),\ u_2).$$

Now the equation $(\partial z/\partial u_1)+x(\partial y/\partial u_1) \equiv 0$ is transformed into $(\partial z/\partial y)(\partial y/\partial u_1)+x(\partial y/\partial u_1) \equiv 0$, which finally is equivalent to

$$\frac{\partial z}{\partial y}(y,\ u_2) = -x(y,\ u_2).$$

Thus

$$z(y,\ u_2) = -\int_0^y x(s,\ u_2)\,\mathrm{d}s + \phi(u_2).$$

By [4] $\kappa\colon (y,\ u_2) \to (x(y,\ u_2),\ y)$ is a fold-map, so $(\partial x/\partial u_2) = 0$ and $(\partial^2 x/\partial u_2^2) \neq 0$. Writing

$$x(y,\ u_2) = \beta_1 y + \beta_{11}y^2 + \beta_{12}yu_2 + \beta_{22}u_2^2 + \mathbf{m}^3$$

we have

$$\begin{aligned} z(y,\ u_2) = {} & \gamma_2 u_2 - \tfrac{1}{2}\beta_1 y^2 + \gamma_{22}u_2^2 - \tfrac{1}{3}\beta_{11}y^3 - \tfrac{1}{2}\beta_{12}y^2u_2 \\ & -\beta_{22}yu_2^2 + \gamma_{222}u_2^3 + \mathbf{m}^4. \end{aligned}$$

Finally

$$\begin{aligned} \rho\colon (y,\ u_2) \to {} & (\beta_1 y + \beta_{11}y^2 + \beta_{12}yu_2 + \beta_{22}u_2^2 + \mathbf{m}^3,\ y, \\ & \gamma_2 u_2 - \tfrac{1}{2}\beta_1 y^2 + \gamma_{22}u_2^2 + \mathbf{m}^3), \quad \beta_{22} \neq 0, \end{aligned}$$

and the singular case: $\gamma_2 = 0$ and $\gamma_{22} \neq 0$ may happen generically.

By a left coordinate change we obtain

$$j^2\rho(0) = (\beta_{12}yu_2 + \beta_{22}u_2^2,\ y,\ \gamma_{22}u_2^2).$$

The coordinate change

$$U_2 = u_2 + \frac{\beta_{12}}{2\beta_22}y$$

now transforms ρ to a map germ whose 2-jet is following

$$j^2\rho(0) = \left(\beta_{22}u_2^2 - \frac{\beta_{12}^2}{4\beta_{22}}y^2,\ y,\ \gamma_{22}\left(u_2 - \frac{\beta_{12}}{2\beta_{22}}y\right)^2\right).$$

The left coordinate change

$$X = \bar{X} - \frac{\beta_{12}^2}{4\beta_{22}} Y^2,$$

$$Z = \bar{Z} + \frac{\gamma_{22}\beta_{12}^2}{4\beta_{22}^2} Y^2 + \frac{\gamma_{22}}{\beta_{22}} \bar{X}$$

gives

$$j^2\rho(0) = \left(\beta_{22}u_2^2,\ y,\ -\frac{\gamma_{22}\beta_{12}}{\beta_{22}} u_2 y\right),$$

which is 2-determined and describes Whitney's Cross-cap.

Now we easily check that the equation of horizontal points D is transformed, in new coordinates, to the following one,

$$\frac{\partial z}{\partial u_2}(y,\ u_2) = 0,$$

i.e.

$$2\gamma_{22}u_2 - \tfrac{1}{2}\beta_{12}y^2 + 2\beta_{22}yu_2 + 3\gamma_{222}u_2^2 + \mathbf{m}^3 = 0.$$

Thus for D we obtain

$$u_2 = \frac{\beta_{12}}{4\gamma_{22}} y^2 + \mathbf{m}^3.$$

Analogously for the set of critical points of κ

$$\Gamma\colon u_2 = -\frac{\beta_{12}}{2\beta_{22}} y + \mathbf{m}^2.$$

So the order of the tangency of Δ and Σ is given by the formula

$$x = \frac{\beta_{12}^2}{4\beta_{22}} y^2 + \mathbf{m}^3.$$

Q.E.D.

REMARK 4.8. One can explicitly calculate the Exp-map in the case of Heisenberg group $\mathbf{H} = \mathbf{R}^3$ (cf. [9]), equipped with the distribution V annihilated by

$$\omega = \mathrm{d}z + \tfrac{1}{2}(y\,\mathrm{d}x - x\,\mathrm{d}y),$$

and Hamiltonian

$$H(p, q) = \tfrac{1}{2}(p_1 - \tfrac{1}{2}yp_3)^2 + \tfrac{1}{2}(p_2 + \tfrac{1}{2}xp_3)^2.$$

One computes

$$\begin{aligned} h_1^{vH}|_{\{q=0\}}(p) = \Bigg(& \tfrac{1}{2}(p_1(\cos p_3 + 1) - p_2 \sin p_3),\ \tfrac{1}{2}(p_2(\cos p_3 + 1) \\ & -p_1 \sin p_3),\ p_3, \frac{1}{p_3}(p_2(\cos p_3 - 1) + p_1 \sin p_3), \\ & \frac{1}{p_3}(-p_1(\cos p_3 - 1) \\ & +p_2 \sin p_3), (p_1^2 + p_2^2)\left(\frac{p_3 - \sin p_3}{2p_3^2}\right)\Bigg). \end{aligned}$$

We take $\pi(x, y, z) = (x, y)$. Then we have $\exp(g)\colon W_{\bar{p}} \to X$, and

$$\begin{aligned} \tilde{E}|_{W_{\bar{p}}} = \Bigg(& \frac{1}{p_3}(p_2(\cos p_3 - 1) + p_1 \sin p_3),\ \frac{1}{p_3}(-p_1(\cos p_3 - 1) \\ & +p_2 \sin p_3), (p_1^2 + p_2^2)\left(\frac{p_3 - \sin p_3}{2p_3^2}\right)\Bigg)\Bigg|_{W_{\bar{p}}}, \end{aligned}$$

where

$$W_{\bar{p}} = \{(p_1, p_2, p_3) \ :\ Ap_1 + Bp_2 + Cp_3 = 0\}.$$

By simple check we find that if

$$\bar{p} \in \{(p_1, p_2, p_3);\ p_3 = 2k\pi\},$$

and

$$p_3 = ap_1 + bp_2 + 2k\pi,$$

then $\exp(g)$ is not generic. In other cases it is an immersion.

The set of singular values of $\pi_X \circ h_1^{vH}$ (the usual caustic of Exp-map) is formed by the family of rotationally invariant paraboloids

$$z = (x^2 + y^2)\frac{2a - \sin 2a}{4(1 - \cos 2a)},$$

and the line $x = 0$, $y = 0$, where a is a solution of the equation tg $x = x$. Simplifying the system by isometry $(x, y, z) \to (x, y, z - \frac{1}{2}xy)$, we obtain the generating family for the isotropic map $h_1^{vH}|_{\{q=0\}}$, namely

$$F(x, y, z, \lambda) = x\lambda_1 \cos \lambda_3 + y\lambda_2 \cos \lambda_3 + z\lambda_3 - \frac{\sin 2\lambda_3}{4\lambda_3} + \frac{\lambda_1\lambda_2}{\lambda_3}(\cos \lambda_3 - 1) \cos \lambda_3.$$

Note added in proof. 1. We note that the integral formula (7) implies the accessibility criterion. 2. In the Hamiltonian case, for $p_0 \neq 0$, condition (β) is not satisfied. However the straightforward proof of Theorem 4.5 follows from the integral formula.

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