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IMRE Z. RUZSA

ANDRZEJ SCHINZEL

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## An application of Kloosterman sums

IMRE Z. RUZSA<sup>\*1</sup> and ANDRZEJ SCHINZEL<sup>2</sup>

<sup>1</sup> *Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Pf. 127, H-1364 Hungary*

<sup>2</sup> *Instytut Matematyczny PAN, ul. Śniadeckich 8, Skr. poczt. 137, 00-950 Warszawa, Poland*

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*To the memory of Professor A. V. Malyshev.*

Let  $n$  be a positive integer

$$A_n = \{a : 1 \leq a \leq n, (a, n) = 1\},$$

and for  $a \in A_n$  let  $\bar{a}$  denote the unique element of  $A_n$  satisfying  $a\bar{a} \equiv 1 \pmod{n}$ . For  $n$  odd,  $\varepsilon = 0$  or  $1$ ,  $\delta = \pm 1$  put

$$L_n^\varepsilon = \{a \in A_n : a - \bar{a} \equiv \varepsilon \pmod{2}\},$$

$$L_n^{\varepsilon, \delta} = \left\{ a \in A_n : a - \bar{a} \equiv \varepsilon \pmod{2}, \left( \frac{a}{n} \right) = \delta \right\}.$$

Zhang Wenpeng [5] recently conjectured that for every odd  $n$  and  $\eta > 0$

$$\#L_n^1 = \frac{1}{2}\phi(n) + O(n^{1/2+\eta})$$

and proved it, even in a somewhat stronger form for  $n$  being a prime power or a product of two primes.

On the other hand, G. Terjanian [4] conjectured that  $L_p^{\varepsilon, \delta} \neq \emptyset$  for every prime  $p > 29$  and every choice of  $\varepsilon$  and  $\delta$ . This conjecture has been proved by Chaładus [1] by applying Nagell's bound for the least quadratic nonresidue modulo  $p$ .

We prove the following theorem, which confirms Zhang's conjecture, improves his error term for  $n$  being a prime power, and improves Chaładus's theorem except for finitely many primes.

**THEOREM 1.** *For every choice of  $\varepsilon = 0, 1$  and  $\delta = \pm 1$  we have*

$$\#L_n^{\varepsilon, \delta} = \frac{\phi(n)}{4} c_{n, \delta} + O(2^{\nu(n)} \sqrt{n} (\log n)^2), \tag{1}$$

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where

$$c_{n,\delta} = \begin{cases} 1 + \delta & \text{if } n \text{ is a perfect square,} \\ 1 & \text{otherwise.} \end{cases}$$

and  $\nu(n)$  is the number of distinct prime factors of  $n$ .

Consider a positive integer  $n$ , not necessarily odd, a positive integer  $m$  coprime to  $n$ ,  $0 \leq j, k < m$ , an odd divisor  $r$  of  $n$ ,  $\delta = \pm 1$  and put

$$S_n^m(j, k, r, \delta) = \# \left\{ a \in A_n : a \equiv j \pmod{m}, \bar{a} \equiv k \pmod{m}, \left(\frac{a}{r}\right) = \delta \right\}.$$

We shall deduce Theorem 1 from the following estimate of this quantity.

**THEOREM 2.** *For any choice of  $m < n$ , coprime to  $n$ ,  $0 \leq j, k < m$ , odd  $r$  dividing  $n$  and  $\delta = \pm 1$  we have*

$$S_n^m(j, k, r, \delta) = \frac{\phi(n)}{2m^2} c_{r,\delta} + O(2^{\nu(n)} \sqrt{n} (\log n)^2),$$

where the constant in the  $O$  symbol is absolute and effective.

To obtain Theorem 1 from Theorem 2 we only need to observe that

$$\begin{aligned} \#L_n^{0,\delta} &= S_n^2(0, 0, n, \delta) + S_n^2(1, 1, n, \delta), \\ \#L_n^{1,\delta} &= S_n^2(0, 1, n, \delta) + S_n^2(1, 0, n, \delta). \end{aligned}$$

The proof of Theorem 2 is based on four lemmas.

**LEMMA 1.** *If  $r$  is an odd divisor of  $n$ , we have*

$$\left| \sum_{u \in A_n} \left(\frac{u}{r}\right) e\left(\frac{uv}{n}\right) \right| \leq \sqrt{n(v, n)},$$

where  $e(t) = \exp(2\pi it)$ .

*Proof.*  $\left(\frac{u}{r}\right)$  is a character mod  $n$ , whose conductor  $f$  is equal to the squarefree kernel of  $r$ . Hence by a known formula (see [2], Chapter IV, Sect. 20, assertion IV)

$$\left| \sum_{u \in A_n} \left(\frac{u}{r}\right) e\left(\frac{uv}{n}\right) \right| = \begin{cases} \frac{\phi(n)}{\phi(n/(v, n))} \left| \mu\left(\frac{n}{(v, n)}\right) \right| \sqrt{f} & \text{if } f \mid \frac{n}{(v, n)}, \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\phi\left(\frac{n}{(v, n)}\right) \geq \frac{\phi(n)}{(v, n)},$$

we obtain

$$\left| \sum_{u \in A_n} \left(\frac{u}{r}\right) e\left(\frac{uv}{n}\right) \right| \leq \begin{cases} \sqrt{f(v, n)} & \text{if } f(v, n) | n, \\ 0 & \text{otherwise,} \end{cases}$$

which gives the lemma. ■

LEMMA 2. For all integers  $v, w$  and an odd integer  $r$  dividing  $n$

$$\left| \sum_{u \in A_n} \left(\frac{u}{r}\right) e\left(\frac{uv + \bar{u}w}{n}\right) \right| \leq \sqrt{2n} 2^\nu \sqrt{(v, w, n)}, \tag{2}$$

where  $\nu$  is the number of distinct prime factors of  $n$ .

*Proof.* This is a slight improvement of a result of Malyshev [3], where instead of the last factor  $\min\{\sqrt{(v, n)}, \sqrt{(w, n)}\}$  is obtained. We indicate only the necessary changes to Malyshev’s proof to obtain (2).

We use  $K_r(v, w, n)$  to denote the sum in the left side of (2). For prime-powers Malyshev shows

$$|K_r(v, w, p^t)| \leq C_p p^{t/2} (v, p^t)^{1/2},$$

where  $C_p = 2$  for odd primes and  $C_2 = 2\sqrt{2}$ . By symmetry we also have

$$|K_r(v, w, p^t)| \leq C_p p^{t/2} (w, p^t)^{1/2},$$

and, taking into account that

$$\min\{(v, p^t), (w, p^t)\} = (v, w, p^t),$$

we conclude that

$$|K_r(v, w, p^t)| \leq C_p p^{t/2} (v, w, p^t)^{1/2}. \tag{3}$$

To treat the case of composite numbers Malyshev establishes the composition rule

$$K_r(v, w, n) = \pm \prod K_{p_i^{s_i}}(v, w_i, p_i^{t_i}), \tag{4}$$

where

$$n = \prod p_i^{t_i}, \quad r = \prod p_i^{s_i}$$

and the numbers  $w_i$  satisfy

$$w \equiv \sum w_i (n/p_i^{t_i})^2 \pmod{n}.$$

This implies that  $(v, w_i, p_i^{t_i}) = (v, w, p_i^{t_i})$  and hence

$$\prod (v, w_i, p_i^{t_i}) = (v, w, n).$$

Consequently on substituting (3) into (4) we obtain

$$|K_r(v, w, n)| \leq \sqrt{n} \sqrt{(v, w, n)} \prod_{p|n} C_p,$$

and (2) follows by noting that  $\prod C_p \leq \sqrt{2} \cdot 2^\nu$ . ■

LEMMA 3. *For any integer  $n \geq 2$  we have*

$$\sum_{v=1}^{n-1} \frac{\sqrt{(v, n)}}{v} \ll 2^\nu \log n,$$

where  $\nu$  is the number of distinct prime divisors of  $n$ .

*Proof.*

$$\sum_{v=1}^{n-1} \frac{\sqrt{(v, n)}}{v} \leq \sum_{d|n} \sum_{j=1}^{[n/d]} \frac{\sqrt{d}}{dj} \leq \sum_{d|n} d^{-1/2} \sum_{j=1}^n 1/j.$$

Here the second sum is  $O(\log n)$ . We estimate the first sum as follows:

$$\sum_{d|n} d^{-1/2} \leq \prod_{p|n} \sum_{i=0}^{\infty} p^{-i/2} = \prod_{p|n} \frac{1}{1 - p^{-1/2}} \ll 2^\nu,$$

since each term is at most 2, except possibly those corresponding to  $p = 2$  and  $p = 3$ . ■

LEMMA 4. *For any integer  $n \geq 2$  we have*

$$\sum_{v=1}^{n-1} \sum_{w=1}^{n-1} \frac{\sqrt{(v, w, n)}}{vw} \ll (\log n)^2.$$

*Proof.*

$$\sum_{v=1}^{n-1} \sum_{w=1}^{n-1} \frac{\sqrt{(v, w, n)}}{vw} \leq \sum_{d|n} \sum_{i=1}^{[n/d]} \sum_{j=1}^{[n/d]} \frac{\sqrt{d}}{(di)(dj)} \leq \sum_{d|n} d^{-3/2} \sum_{i=1}^n 1/i \sum_{j=1}^n 1/j.$$

Here the first sum is bounded from above by the convergent sum  $\sum_{k=1}^{\infty} k^{-3/2}$ , and the second and third sum is  $O(\log n)$ . ■

*Proof of Theorem 2.* For  $0 \leq j < m$ ,  $0 \leq u < n$  we define  $\phi_j(u)$  as

$$\phi_j(u) = \begin{cases} 1 & \text{if } u \equiv j \pmod{m}, \\ 0 & \text{otherwise} \end{cases}$$

and extend it periodically with period  $n$ . Clearly we have

$$S = S_n^m(j, k, r, \delta) = \frac{1}{2} \sum_{u \in A_n} \phi_j(u) \phi_k(\bar{u}) \left( 1 + \delta \left( \frac{u}{r} \right) \right). \tag{5}$$

We develop  $\phi_j$  into a trigonometric series:

$$\phi_j(u) = \sum_{v=0}^{n-1} \alpha_{jv} e \left( \frac{uv}{n} \right). \tag{6}$$

A substitution of expansion (6) into (5) yields

$$\begin{aligned} S &= \frac{1}{2} \sum_{v,w=0}^{n-1} \alpha_{jv} \alpha_{kw} \sum_{u \in A_n} e \left( \frac{uv + \bar{u}w}{n} \right) \left( 1 + \delta \left( \frac{u}{r} \right) \right) \\ &= \frac{1}{2} \sum_{v,w=0}^{n-1} \alpha_{jv} \alpha_{kw} T_{vw}. \end{aligned} \tag{7}$$

To estimate  $T_{vw}$  we distinguish four cases.

- (i) If  $v = w = 0$ , then clearly  $T_{vw} = \phi(n)c_{r,\delta}$ .
- (ii) If  $v \neq 0, w = 0$ , then we have

$$T_{vw} = \sum_{u=1}^{n-1} e \left( \frac{uv}{n} \right) \left( 1 + \delta \left( \frac{u}{r} \right) \right).$$

Applying Lemma 1 twice we obtain

$$|T_{vw}| \leq 2\sqrt{n(v, n)}. \tag{8}$$

- (iii) If  $v = 0, w \neq 0$ , then by symmetry

$$|T_{vw}| \leq 2\sqrt{n(w, n)}. \tag{9}$$

(iv) If  $v \neq 0$  and  $w \neq 0$ , then by Lemma 2

$$|T_{vw}| \leq 2^\nu (2n(v, w, n))^{1/2}. \tag{10}$$

Substituting these estimates into (7) we obtain

$$S = \frac{\phi(n)}{2} c_{r,\delta} \alpha_{j0} \alpha_{k0} + R, \tag{11}$$

where

$$|R| \leq 2\sqrt{n} \left( \sum_{v=1}^{n-1} |\alpha_{jv} \alpha_{k0}| \sqrt{(v, n)} + \sum_{w=1}^{n-1} |\alpha_{j0} \alpha_{kw}| \sqrt{(w, n)} + 2^\nu \sum_{v=1}^{n-1} \sum_{w=1}^{n-1} |\alpha_{jv} \alpha_{kw}| \sqrt{(v, w, n)} \right). \tag{12}$$

The coefficients can be determined from an inversion formula:

$$\begin{aligned} \alpha_{jv} &= \frac{1}{n} \sum_{u=0}^{n-1} \phi_j(u) e\left(-\frac{vu}{n}\right) \\ &= \frac{1}{n} \sum_{l=0}^L e\left(-\frac{v}{n}(j + lm)\right), \quad L = \left[ \frac{n-1-j}{m} \right]. \end{aligned} \tag{13}$$

In particular,

$$\alpha_{j0} = \frac{1}{n} \left[ \frac{n-1-j}{m} + 1 \right], \quad \frac{1}{m} - \frac{1}{n} \leq \alpha_{j0} \leq \frac{1}{m} + \frac{1}{n}.$$

Hence the main term of (11) satisfies

$$\frac{\phi(n)}{2} c_{r,\delta} \alpha_{j0} \alpha_{k0} = \frac{\phi(n)}{2m^2} c_{r,\delta} + R_1, \quad |R_1| \leq \frac{3}{m}. \tag{14}$$

On the other hand, the geometric series in (13) can be easily summed. With  $z = e(vm/n)$  we have

$$\alpha_{jv} = \frac{1}{n} e\left(-\frac{vj}{n}\right) \frac{1 - z^{L+1}}{1 - z},$$

thus

$$|\alpha_{jv}| = \frac{1}{n} \frac{|1 - z^{L+1}|}{|1 - z|} \leq \frac{1}{n} \frac{2}{|1 - z|} = \frac{1}{n |\sin \pi vm/n|}$$

(we used the fact that  $|1 - e(t)| = 2|\sin \pi t|$ ). As  $v$  runs from 1 to  $n - 1$ , the residue of  $vm$  modulo  $n$  assumes the values 1 to  $n - 1$ , since  $(m, n) = 1$ , and we have  $(vm, n) = (v, n)$ . Hence

$$\begin{aligned} \sum_{v=1}^{n-1} |\alpha_{jv}| \sqrt{(v, n)} &\leq \frac{1}{n} \sum_{v=1}^{n-1} \frac{\sqrt{(v, n)}}{|\sin \pi vm/n|} \\ &= \frac{1}{n} \sum_{v=1}^{n-1} \frac{\sqrt{(v, n)}}{\sin \pi v/n} \\ &\leq \frac{2}{n} \sum_{v=1}^{[n/2]} \frac{\sqrt{(v, n)}}{\sin \pi v/n}. \end{aligned}$$

Since  $\sin t \geq (2/\pi)t$  on  $[0, \pi/2]$ , this sum is

$$\leq \sum_{v=1}^{[n/2]} \frac{\sqrt{(v, n)}}{v} \ll 2^\nu \log n$$

by Lemma 3. Thus the first sum in estimate (12) of  $R$  is  $O(2^\nu \log n)$ , and by symmetry so is the second.

By the same arguments, the third sum can be estimated as follows:

$$\begin{aligned} \sum_{v=1}^{n-1} \sum_{w=1}^{n-1} |\alpha_{jv} \alpha_{kw}| \sqrt{(v, w, n)} &\leq \frac{1}{n^2} \sum_{v=1}^{n-1} \sum_{w=1}^{n-1} \frac{\sqrt{(v, w, n)}}{|\sin \pi vm/n \sin \pi wm/n|} \\ &= \frac{1}{n^2} \sum_{v=1}^{n-1} \sum_{w=1}^{n-1} \frac{\sqrt{(v, w, n)}}{\sin \pi v/n \sin \pi w/n} \\ &\leq \frac{4}{n^2} \sum_{v=1}^{[n/2]} \sum_{w=1}^{[n/2]} \frac{\sqrt{(v, w, n)}}{\sin \pi v/n \sin \pi w/n} \\ &\leq \sum_{v=1}^{[n/2]} \sum_{w=1}^{[n/2]} \frac{\sqrt{(v, w, n)}}{vw} \\ &\ll (\log n)^2 \end{aligned}$$

by Lemma 4.

Substituting these estimates into (12) we obtain

$$|R| \ll 2^\nu \sqrt{n} (\log n)^2. \tag{15}$$



Theorem 2 follows from (12), (14) and (15). ■

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