COMPOSITIO MATHEMATICA

FABIO BARDELLI
CIRO CILIBERTO
ALESSANDRO VERRA
Curves of minimal genus on a general abelian variety

Curves of minimal genus on a general abelian variety

FABIO BARDELLI[1], CIRO CILIBERTO[2] and ALESSANDRO VERRA[3] *
[1]*Università di Pisa, Dipartimento di Matematica, Via Buonarroti 2, 56100 Pisa (Italy)*
[2]*Università di Roma II, Dipartimento di Matematica, Via O. Raimondo, 00173 Roma (Italy)*
[3]*Università di Roma I, Dipartimento di Matematica, "G. Castelnuovo", P.le Aldo Moro, 00100 Roma Italy*

Contents

Introduction

As R. Torelli observed in 1914 [T], every abelian variety A can be realized as a quotient of a Jacobian of some smooth projective curve C. Indeed, it suffices to pick as C any curve admitting a morphism to A whose image *generates* A, in the sense that any translate of it passing through the origin of A generates A as a complex torus. Alternatively, one may embed A in a projective space with a very ample linear system and pick any smooth curve section C of A. Then the existence of an epimorphism of the Jacobian $J(C)$ of C onto A stems from an iterated application of the Lefschetz hyperplane section theorem.

In any event, it is clear that A can be represented as a quotient of a Jacobian in infinitely many ways. Actually, there are invariants for such a representation. A discrete invariant is provided by the dimension of the Jacobian of which A is a quotient, i.e. by the genus of the curve C, and another discrete invariant is given by the homology class of the image of C in A. Instead, continuous invariants are given by the moduli of all curves C of fixed genus admitting a morphism to A whose image generates A. Indeed, these curves can vary in several families, may be with different number of moduli. A natural question to ask in this context is to

describe all representations of A for which the first invariant mentioned above is the least possible, i.e. all representations of A as a quotient of a Jacobian of minimal dimension. In view of the previous remarks, this is essentially the same as looking at the family of curves of minimal geometric genus generating A. And since the question is clearly invariant by isogenies, one may, without loss of generality, restrict the attention to principally polarized abelian varieties.

In the present paper we mainly deal with this problem in the case A is a general principally polarized abelian variety of a given dimension $g \geqslant 4$ (the question being trivial for $g \leqslant 3$). Let us briefly describe our results.

In Section 1 we determine a lower bound for the minimal genus $\gamma(g)$ of a curve C such that a general principally polarized abelian variety A of dimension $g \geqslant 4$ is a quotient of $J(C)$. This lower bound, essentially proved by Alzati and Pirola in [AP], is quadratic in g and its proof uses a theorem from [CGT] which provides an upper bound for the dimension of families of Jacobians with non trivial endomorphisms.

In the case $g = 4$ the bound is sharp. Indeed, it says that $\gamma(4) \geqslant 7$ and we prove in Section 2 that $\gamma(4) = 7$. In fact we show, by computing the differential of a certain map between the appropriate moduli spaces, that the Prym varieties of genus 7 double coverings of genus 3 curves, branched at four points, are dense in the moduli space of abelian fourfolds with a polarization of type (1,2,2,2). This fact, by the way, yields, as we show, the unirationality of the latter moduli space: a result which reminds us, but, as far as we see, is neither a consequence of, nor implies, the unirationality of the moduli space of principally polarized abelian fourfolds, proved by Clemens in [Cl]. Furthermore, we prove in Section 3 that every curve of geometric genus 7 in a principally polarized abelian fourfold A is a double covering as above. The proof of this result is based on a rather delicate projective-geometrical analysis involving some linear series cut out on the canonical model of a genus 7 curve C lying on A by certain families of quadrics. These linear series arise in a natural way if one looks at the matter from the point of view of infinitesimal deformations, and observes that any such a deformation of A carries an infinitesimal deformation of C with it. The argument is inspired by a similar one contained in [CG].

In the case $g = 5$, the bound of Section 1 says that $\gamma(5) \geqslant 9$, but it is no more sharp since we prove in Section 4 that $\gamma(5) = 11$. It is well known that $\gamma(5) \leqslant 11$, since a general principally polarized abelian fivefold is the Prym variety of an unramified double covering of a genus 6 curve, and this happens in exactly 27 different ways, according to Donagi and Smith [DS]. In order to prove that $\gamma(5) \geqslant 11$ we use a projective-geometrical argument similar to the one exploited in Section 3.

In Section 5 we turn again to the study of genus 7 curves on a general principally polarized abelian fourfold A. We prove, by computing the degree of a certain map between the appropriate moduli spaces, that, up to translations, there are exactly $3 \cdot (2^8 - 1)$ genus 7 curves on A in twice the minimal class. Or, in other words, that

a general abelian fourfold with a polarization of type (1,2,2,2) can be expressed in exactly three ways as the Prym variety of a genus 7 double covering of a genus 3 curve.

In Section 6 we have collected a few observations, speculations and conjectures related to the subject of this paper and motivated by our results, as well as by those of other authors, especially Welters [W] and Debarre [D].

After this work had been completed and sent to the journal, we received a preprint from Pirola [P] in which, using infinitesimal deformation methods different from ours, he is able to improve the lower bound for $\gamma(g)$ given in Section 1, finding the better estimate $\gamma(g) > g(g-1)/2$. He is also able to prove that if the bound is attained, then either $g = 4$ or $g = 5$ and the general principally polarized abelian variety A of dimension g is isogenous to the Prym variety of a double covering. This gives a new proof of our results of Section 3 for $g = 4$ and completes our analysis of Section 4 for $g = 5$.

1. An estimate for the minimal genus of curves on a general abelian variety

Let A be a g-dimensional abelian variety over the complex field $\mathbf{C}$. We set

$$\gamma(A) = \min\{\gamma \in \mathbf{N}: \text{there is a non constant morphism } f\colon C \to A, \text{ where } C \text{ is a smooth irreducible projective curve of genus } \gamma\}.$$

We also define the *Jacobian dimension* of A to be the integer

$$
\begin{aligned}
j(A) &= \min\{\gamma \in \mathbf{N}: \text{there is a non constant morphism } f\colon C \to A, \text{ where } \\
&\qquad C \text{ is a smooth irreducible projective curve of genus } \gamma \text{ and } \\
&\qquad f(C) \text{ generates } A\} \\
&= \min\{\gamma \in \mathbf{N}: A \text{ can be represented as the quotient of a Jacobian } \\
&\qquad \text{of a smooth irreducible projective curve of genus } \gamma\}.
\end{aligned}
$$

We note that $\gamma(A) \leqslant j(A)$ and the equality holds if A is simple. Furthermore, if C is a smooth irreducible projective curve of geometric genus $g(C) = \gamma(A)$ and if $f\colon C \to A$ is a non constant morphism, then f is birational onto its image. Moreover, if $\approx$ denotes the isogeny relation between abelian varieties, then we clearly have

$$A \approx A' \Rightarrow \gamma(A) = \gamma(A') \quad \text{and} \quad j(A) = j(A').$$

We define $\gamma(g)$ (resp. $j(g)$) to be the maximum of $\gamma(A)$ (resp. of $j(A)$) as A varies in the set of all g-dimensional abelian varieties. Of course $\gamma(g) \leqslant j(g)$ and we will prove in a moment that $\gamma(g) = j(g)$.

Consider a commutative diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\;f\;} & \mathcal{B} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\
U & \xrightarrow{\;r\;} & B
\end{array} \qquad\qquad (1.1)$$

such that:

(i) $p: \mathcal{C} \to U$ is a smooth projective family of irreducible curves of genus γ over a smooth connected basis U;

(ii) $\pi: \mathcal{B} \to B$ is a smooth family of principally polarized abelian varieties of dimension g over a smooth connected basis B;

(iii) $\dim f(\mathcal{C}) = \dim r(U) + 1$.

If t is a point of U, we denote by C_t the fibre of $p: \mathcal{C} \to U$ over t, by A_t the fibre of $\pi: \mathcal{B} \to B$ over the point $r(t)$ and by $f_t: C_t \to A_t$ the restriction of f to C_t, which is never constant by our assumption (iii). Note that there is a natural morphism $\beta: B \to \mathcal{A}_g$, where $\mathcal{A}_g$ is the moduli space of principally polarized abelian varieties of dimension g. We denote by $\rho: U \to \mathcal{A}_g$ the map $\rho = \beta \circ r$, and we also assume that:

(iv) ρ is dominant.

We define $\gamma'(g)$ to be the minimum γ such that there is a diagram of type (1.1) enjoying the properties (i), ..., (iv). Again, if we have such a diagram with $\gamma = \gamma'(g)$, then for a general point t in U, $f_t: C_t \to A_t$ is birational onto its image. Note that by a *general point* of a variety we mean a point which varies in a suitable non empty Zariski open subset of the variety. The following proposition, although elementary, is quite useful.

PROPOSITION (1.2). *One has* $\gamma(g) = \gamma'(g) = j(g)$.

Proof. First we prove that $\gamma'(g) \leqslant \gamma(g)$. Fix any smooth family $\pi: \mathcal{B} \to B$ of principally polarized abelian varieties of dimension g over a smooth connected basis B, such that the natural morphism $\beta: B \to \mathcal{A}_g$ is surjective. Let S be the set of all commutative diagrams of type (1.1) with fixed second column $\pi: \mathcal{B} \to B$ and enjoying the properties (i), (ii) and (iii) above, and let B_γ be the subset of B consisting of the union of all the images of the corresponding maps r, as the diagram varies in S. Since the relative Hilbert scheme of curves of fixed genus γ of a flat family of projective varieties parametrized by an affine scheme is a countable union of projective schemes over the base, it is clear that B_γ can be also seen as a countable union of locally closed subsets of B, each one being the image of a map r for a suitable diagram in $S(\gamma)$. Taking into account the definition of $\gamma(g)$ we have

$$\bigcup_{\gamma \leqslant \gamma(g)} B_\gamma = B,$$

and then $\gamma'(g) \leqslant \gamma(g)$.

Secondly, we show that $j(g) \leqslant \gamma'(g)$, which, together with fact that $\gamma(g) \leqslant j(g)$, completes the proof of the proposition. Consider a commutative diagram of the type (1.1), enjoying properties (i), ..., (iv), and with $\gamma = \gamma'(g)$. Then $\rho(U)$ contains a non empty open subset $\mathcal{U}$ of $\mathcal{A}_g$. If t is a general point of U, then A_t is a general principally polarized abelian variety of dimension g. Thus A_t is simple, and therefore $f_t(C_t)$ generates A_t. Hence, after having may be shrinked $\mathcal{U}$, we may assume that for all abelian varieties A corresponding to points in $\mathcal{U}$, A is generated by a curve of genus $\gamma'(g)$. Consider any abelian variety A'. Since A' is isogenous to some abelian variety A represented by a point in $\mathcal{U}$ (see [Ba]), then we have

$$j(A') = j(A) \leqslant \gamma'(g),$$

hence $j(g) \leqslant \gamma'(g)$.

We observe that, as pointed out in the introduction, our main interest is basically concentrated on $j(g)$. The equality $j(g) = \gamma'(g)$ shows that $j(g)$ can be interpreted as the minimal genus of a curve on a general abelian variety of dimension g. As the proof of Proposition (1.2) shows, it is useful to introduce $\gamma(g)$ in order to compare $j(g)$ and $\gamma'(g)$.

There is a rather rough, but basic estimate for $\gamma(g)$ proved by Alzati and Pirola in [AP]. We briefly reproduce their proof here, adding a remark which enables us to slightly improve their bound. This will make it sharp, as we shall see in Section 2, at least for $g = 4$.

PROPOSITION (1.3). *If $g \geqslant 4$, then $\gamma(g) \geqslant [g(g+1)/4] + 3/2$.*

Proof. By Proposition (1.2) we may assume we have a diagram of the type (1.1) with $\gamma = \gamma(g)$ and enjoying properties (i), ..., (iv). As in the proof of Proposition (1.2), we see that, if t is a general point of U, then $f_t(C_t)$ generates A_t. Thus $\gamma = \gamma(g) \geqslant g$.

Let $\mathcal{M}_\gamma$ be the moduli space of curves of genus γ. Consider the natural morphism $\mu \colon U \to \mathcal{M}_\gamma$ which sends a point t in U to the isomorphism class of the fibre C_t of $p \colon C \to U$ over t. We denote by M the closure of its image in $\mathcal{M}_\gamma$. Then we consider the morphism $\rho \times \mu \colon U \to \mathcal{A}_g \times \mathcal{M}_\gamma$. Let V be the closure of its image in $\mathcal{A}_g \times \mathcal{M}_\gamma$. The general point of V corresponds to a pair $(A, C) \in \mathcal{A}_g \times \mathcal{M}_\gamma$ such that there is a non constant morphism $f \colon C \to A$. Of course ρ is dominant if and only if the first projection $V \to \mathcal{A}_g$ is such. Moreover the second projection $V \to M$ is finite. In fact, given a curve C corresponding to a point of M, the set of all morphisms of $J(C)$ to some abelian variety is discrete. Hence we have

$$\dim M = \dim V \geqslant \dim \mathcal{A}_g = g(g+1)/2. \tag{1.4}$$

This implies that $\gamma > g$, otherwise we would have $g(g+1)/2 \leqslant 3g - 3$, hence $g \leqslant 3$, a contradiction. But then, for t general in U, the map $J(C) \to A_t$ induced

by f_t has a positive-dimensional kernel, i.e. $J(C_t)$ is non simple. This means that M is contained in the set

$$\mathcal{S}_\gamma = \{C \in \mathcal{M}_\gamma \colon \operatorname{End}(J(C)) \neq \mathbf{Z}\}.$$

Then by [CGT] we have dim $M \leqslant 2\gamma - 2$. We claim that one has dim $M \leqslant 2\gamma - 3$, which, together with (1.4), yields the assertion. Indeed, dim $M = 2\gamma - 2$ implies that the curve C corresponding to a general point of M has a morphism onto some elliptic curve E and $J(C) \approx E \times X$ with X simple (see [CGT], remark (4.7)). Hence the kernel of $J(C) \to A$ should be isogenous to E, i.e. $\gamma = g + 1$. But then (1.4) would imply $g(g + 1)/2 \leqslant 2\gamma - 2 = 2g$, i.e. $g \leqslant 3$, a contradiction.

As pointed out in the introduction a recent preprint of Pirola [P] contains an improvement of the above estimate for $\gamma(g)$.

2. The minimal genus of curves on a general abelian fourfold

Proposition (1.3) yields $\gamma(4) \geqslant 7$. In this paragraph we will show that $\gamma(4) = 7$. In order to do this we will have to consider the family of genus 7 double coverings of genus 3 curves. We start by recalling a few general well known facts about double coverings, which will be often used in the sequel.

Let C be a smooth, irreducible, projective curve of genus g, let B be an effective divisor on C of even degree $2b$, let $\mathcal{L}$ be a line bundle on C such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(B)$. The datum of the triple $(C, B, \mathcal{L})$ is equivalent to the datum of a double covering $f \colon C' \to C$ branched at B. We will assume $\mathcal{L}$ to be non trivial if B is the zero divisor. Let $C^{(n)}$ be the n-fold symmetric product of C. Then, for B general in $C^{(2b)}$, C' is smooth, irreducible, of genus $g' = 2g - 1 + b$. The curve C' is endowed with a natural involution σ induced by f, which acts on its cohomology. In particular it acts on $H^0(C', \omega_{C'}^{\otimes i})$, for all $i \geqslant 1$, where $\omega_{C'}$ denotes, as usual, the dualizing sheaf of C'. One has the decomposition of $H^0(C', \omega_{C'}^{\otimes i})$ in $+1$ and -1 eigenspaces for the involution σ

$$H^0(C', \omega_{C'}^{\otimes i}) = H^0(C', \omega_{C'}^{\otimes i})^+ \oplus H^0(C', \omega_{C'}^{\otimes i})^-,$$

and there are natural isomorphisms

$$H^0(C', \omega_{C'}^{\otimes i})^+ \cong H^0(C, \omega_C^{\otimes i} \otimes \mathcal{L}^{\otimes i-1}),$$

$$H^0(C', \omega_{C'}^{\otimes i})^- \cong H^0(C, (\omega_C \otimes \mathcal{L})^{\otimes i}), \quad \text{if } i \text{ is odd,}$$

$$H^0(C', \omega_{C'}^{\otimes i})^+ \cong H^0(C, (\omega_C \otimes \mathcal{L})^{\otimes i}),$$

$$H^0(C', \omega_{C'}^{\otimes i})^- \cong H^0(C, \omega^{\otimes i} \otimes \mathcal{L}^{\otimes i-1}), \quad \text{if } i \text{ is even,}$$

for all $i \geqslant 1$. Consider now the canonical map for C'

$$\phi_{C'} \colon C' \to \mathbf{P} := \mathbf{P}(H^0(C', \omega_{C'})^*) \cong \mathbf{P}^{g'-1},$$

where we denote by * the dual vector space. Consider the two projection maps

$$\pi_+ : \mathbf{P}(H^0(C', \omega_{C'})^*) \to \mathbf{P}^+ := \mathbf{P}(H^0(C, \omega_C)^*) \cong \mathbf{P}^{g-1},$$

$$\pi_- : \mathbf{P}(H^0(C', \omega_{C'})^*) \to \mathbf{P}^- := \mathbf{P}(H^0(C, \omega_C \otimes \mathcal{L})^*) \cong \mathbf{P}^{g+b-2},$$

with centres $\mathbf{P}^-$ and $\mathbf{P}^+$ respectively. We set

$$p_+ = \pi_+ \circ \phi_{C'}, \quad p_- = \pi_- \circ \phi_{C'},$$

and consider the canonical map of C

$$\phi_C : C \to \mathbf{P}^+,$$

and the map

$$\alpha : C \to \mathbf{P}^-,$$

associated to the complete linear system $|\omega_C \otimes \mathcal{L}|$. We will denote by C^+ and C^- the images of these maps. One has $p_+ = \phi_C \circ f$, $p_- = \alpha \circ f$, both maps being two-to-one onto their images C^+ and C^-.

We will also consider the scroll F described by the lines in $\mathbf{P}$ joining all pairs of points of C conjugated under the involution σ. F intersects $\mathbf{P}^+$ along C^+ and $\mathbf{P}^-$ along C^-. Hence F can be regarded as joining pairs of corresponding points of C^+ and C^- and therefore

$$\begin{aligned} \deg(F) &= \deg(C^+) + \deg(C^-) = (2g - 2) + (2g - 2 + b) \\ &= 4(g - 1) + b. \end{aligned}$$

Next we consider the *norm morphism* $f_* : J(C') \to J(C)$ and we denote by X_f the connected component of zero of $\operatorname{Ker} f_*$. This is an abelian variety contained in $J(C')$ and the natural principal polarization of $J(C')$ induces on X_f a polarization of type $(1, 2, \ldots, 2)$ unless $b = 0$ or 1, in which case it induces the double of a principal polarization (see [BL], pg. 376). We shall call X_f, endowed with this polarization, the *Prym variety* of the covering $f : C' \to C$.

Let $\mathcal{R}(g, b)$ be the moduli space of all isomorphism classes of double coverings $f : C' \to C$ with C a smooth curve of genus g, C' irreducible and f branched at $2b$ distinct points of C. Note that we may as well interpret $\mathcal{R}(g, b)$ as the moduli space of all isomorphism classes of triples $(C, B, \mathcal{L})$, where C is a smooth genus g curve, B is an effective divisor on C formed by $2b$ distinct points, $\mathcal{L}$ is a line bundle on C such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(B)$, non trivial if B is zero. It is clear that

$$\dim \mathcal{R}(g, b) = 3g - 3 + 2b$$

if $g \geqslant 2$. Furthermore, there is a natural morphism

$$p_{(g,b)} \colon \mathcal{R}(g, b) \to \mathcal{A}_{g+b-1}(\delta),$$

where $\mathcal{A}_g(\delta)$ is the moduli space of abelian g-folds with a polarization of type $\delta = (\delta_1, \ldots, \delta_g)$, and δ is $(1, 2, \ldots, 2)$ unless $b = 0$ or 1, in which case it is $(1, \ldots, 1)$. The map sends the equivalence class of the covering $f \colon C' \to C$ to the equivalence class of the polarized abelian variety X_f. We shall call $p_{(g,b)}$ the *Prym map* of type (g, b). $\mathcal{R}(g, b)$ is a closed subscheme of $\bar{\mathcal{R}}(g, b)$, the moduli space of all *admissible* double coverings, introduced by Beauville in [B]. The Prym map can be extended to a *proper* map

$$\bar{p}_{(g,b)} \colon \bar{\mathcal{R}}(g, b) \to \mathcal{A}_{g+b-1}(\delta).$$

We will write $\mathcal{R}_g$, p_g etc. instead of $\mathcal{R}(g, 0)$, $p_{(g,0)}$ etc.

We now recall how to compute the differential of the Prym map $p_{(g,b)}$ at the equivalence class of a triple $\xi = (C, B, \mathcal{L})$ in $\mathcal{R}(g, b)$.

Let us denote by T_V the tangent sheaf to a variety V, and by $T_x(V)$ the Zariski tangent space to V at a point x of V. The versal deformation space of the double covering $f \colon C' \to C$ corresponding to the triple $\xi = (C, B, \mathcal{L})$ in $\mathcal{R}(g, b)$ is well known to be $H^1(C', T_{C'})^+$. The tangent space $T_0(X_f)$ to X_f at the origin is naturally isomorphic to $(H^{1,0}(C')^-)^*$. The polarization of X_f is given by an isogeny $\lambda \colon X_f \to X_f^\wedge$, where we denote by $\hat{A}$ the dual abelian variety $\mathrm{Pic}^0(A)$ of a given abelian variety A. The isogeny λ determines an isomorphism $T_0(X_f) \to T_0(X_f^\wedge)$, hence an involution ι on $T_0(X_f) \otimes T_0(X_f^\wedge)$ interchanging the factors. The deformation space of X_f as a complex manifold can be identified with

$$H^1(X_f, T_{X_f}) \cong T_0(X_f) \otimes T_0(X_f^\wedge).$$

The subspace of infinitesimal deformations of X_f which retain the polarization is the subspace of $T_0(X_f) \otimes T_0(X_f^\wedge)$ invariant under ι. This in turn can be identified with $\mathrm{Sym}^2(T_0(X_f))$. In conclusion, the Zariski tangent space to the Siegel space associated to $\mathcal{A}_{g+b-1}(\delta)$ at the class of X_f is naturally isomorphic to $\mathrm{Sym}^2(H^{1,0}(C')^-)^*$. The natural cup-product map

$$H^1(C', T_{C'})^+ \to \mathrm{Hom}(H^{1,0}(C')^-,$$

$$H^{0,1}(C')^-) \cong (H^{1,0}(C')^-)^* \otimes H^{0,1}(C')^-$$

has its image in the invariant space under the involution ι, hence it determines the map

$$H^1(C', T_{C'})^+ \to \mathrm{Sym}^2(H^{1,0}(C')^-)^*,$$

which computes the differential $dp_{(g,b)|\xi}$ at the equivalence class ξ of the double covering $f: C' \to C$. The codifferential of $p_{(g,b)}$ at ξ is then given by

$$dp_{(g,b)|\xi*}: \operatorname{Sym}^2 H^0(C', \omega_{C'})^- \to H^0(C', \omega_{C'}^{\otimes 2})^+,$$

hence, via the natural identifications mentioned before, by the map

$$w_C: \operatorname{Sym}^2 H^0(C, \omega_C \otimes \mathcal{L}) \to H^0(C, (\omega_C \otimes \mathcal{L})^{\otimes 2})$$

which is the natural product of sections.

Now we specialize to the case of genus 7 double coverings of genus 3 curves, i.e. to the case $g = 3$, $b = 2$. To ease notation we set

$$\mathcal{R}^+ = \mathcal{R}(3, 2), \quad \mathcal{A}_4^+ = \mathcal{A}_4(1, 2, 2, 2), \quad p^+ = p_{(3,2)}.$$

We want to study the map p^+. We need the:

LEMMA (2.1). *In the case $g = 3$, $b = 2$, the curve C^- lies on a quadric surface if and only if either C is hyperelliptic or $h^0(C, \mathcal{L}) > 0$.*

Proof. We have $0 \leqslant h^0(C, \mathcal{L}) \leqslant 2$. If $h^0(C, \mathcal{L}) = 0$, then the map α is an embedding and C^- is a smooth curve of degree 6 and genus 3 in $\mathbf{P}^3$. The curve C is hyperelliptic if and only if C^- lies on a quadric which is described by the lines joining pairs of points that are conjugated in the hyperelliptic involution. If $h^0(C, \mathcal{L}) = 1$ the map α is birational onto its image, C^- is a sextic with a node or a cusp, its arithmetic genus is 4 and it lies on a quadric. If $h^0(C, \mathcal{L}) = 2$ then C is hyperelliptic and α is composed with the hyperelliptic involution, hence C^- is a rational normal cubic in $\mathbf{P}^3$ and it lies on a quadric.

Now we define the following closed subsets of $\mathcal{R}^+$:

$$\mathcal{R}_{\text{eff}}^+ = \{\text{equivalence classes of triples } (C, B, \mathcal{L}) \text{ such that} \\ h^0(C, \mathcal{L}) \neq 0\},$$

$$\mathcal{R}_{\text{hyp}}^+ = \{\text{equivalence classes of triples } (C, B, \mathcal{L}) \text{ such that } C \\ \text{is hyperelliptic}\},$$

and we prove the:

THEOREM (2.2). *The map p^+ is dominant and generically finite and its ramification locus is $\mathcal{R}_{\text{eff}}^+ \cup \mathcal{R}_{\text{hyp}}^+$.*

Proof. We have $\dim \mathcal{R}^+ = \dim \mathcal{A}_4^+ = 10$. By Lemma (2.1), w_C is not injective neither surjective, and accordingly $dp_{(g,b)|\xi}$ is not an isomorphism at $\xi = (C, B, \mathcal{L})$, if and only if $(C, B, \mathcal{L})$ lies in $\mathcal{R}_{\text{eff}}^+ \cup \mathcal{R}_{\text{hyp}}^+$.

We will return to the study of the map p^+ in Section 5, and we will then compute its degree. Meanwhile, we are in position to prove the:

THEOREM (2.3). *One has $\gamma(4) = 7$.*

Proof. Since $\gamma(4) \geqslant 7$, it suffices to prove that for the general abelian variety A, endowed with a fixed polarization, there is an irreducible curve C of geometric genus 7 and a non constant map $C \to A$. By the Theorem (2.2), the general element A in $\mathcal{A}_4^+$ can be realized as X_f for some genus 7 double covering $f: C' \to C$ of a genus 3 curve. In particular we have a natural inclusion $A \to J(C')$. By dualizing and composing with the polarization map $\hat{A} \to A$, we get a surjective morphism $\mathrm{Pic}^0(C') \to A$, the composition of which with the Abel–Jacobi mapping $C' \to \mathrm{Pic}^0(C')$ is a non constant map $C' \to A$, as required.

We want to point out another consequence of Theorem (2.2):

THEOREM (2.4). *$\mathcal{A}_4^+$ is unirational.*

Proof. Fix four points $p_1, \ldots, p_4$ in general position in $\mathbf{P}^2$ and let us consider the 10-dimensional linear system L of all plane quartics passing through $p_1, \ldots, p_4$. We consider the open subset L' of L whose points correspond to smooth curves. Let C be a curve in L'. C is a curve of genus 3 endowed with the divisor of degree four $p_1 + p_2 + p_3 + p_4$. If $\mathcal{M}_{3,4}$ is the moduli space of 4-pointed curves of genus 3 we have a natural map $L' \to \mathcal{M}_{3,4}$ which is dominant. Indeed, if we give a general genus 3 curve C and 4 general points $x_1, \ldots, x_4$ on it we can realize C as a canonical plane quartic and then move the points $x_1, \ldots, x_4$ to $p_1, \ldots, p_4$ with a suitable projective transformation of $\mathbf{P}^2$.

We can now fix, on a general curve C of L', the line bundle of degree two $\mathcal{L} = \mathcal{O}_C(p_1 + p_2 + p_3 - p_4)$. Since C is general in moduli and $p_1 + p_2 + p_3 + p_4$ is general in $C^{(4)}$, there is only one divisor B of the complete linear series $|\mathcal{L}^{\otimes 2}|$ containing p_4. Then we may, after having suitably shrinked L', look at L' as a family of genus 3 curves C endowed with a line bundle $\mathcal{L}$ of degree two and with a divisor B in $|\mathcal{L}^{\otimes 2}|$. Accordingly we get a morphism

$$f: L' \to \mathcal{R}^+.$$

We claim that f is dominant. In fact let $(C, B, \mathcal{L})$ be a general point in $\mathcal{R}^+$, with $B = x_1 + \cdots + x_4$. Since B is a general divisor in the general g_4^1 given by $|\mathcal{L}^{\otimes 2}|$, then x_4 is a general point of C. Hence $|\mathcal{L} \otimes \mathcal{O}_C(x_4)|$ is an effective non special divisor $y_1 + y_2 + y_3$. In other words $\mathcal{L} \cong \mathcal{O}_C(y_1 + y_2 + y_3 - x_4)$ and, by the generality assumption, y_1, y_2, y_3, x_4 are general points on C. So we can realize C as a curve in L with $(y_1, y_2, y_3, x_4) = (p_1, p_2, p_3, p_4)$ and f sends C to the triple $(C, B, \mathcal{L})$. Note that this argument shows that $\deg(f) = 4$. The unirationality of $\mathcal{A}_4^+$ follows now from Theorem (2.2), since p^+ is dominant.

As proved by Clemens [Cl], $\mathcal{A}_4$ is also unirational, but we could not see any relationship between Clemens' result and Theorem (2.4) above.

3. Curves of genus seven on a general abelian fourfold

In this paragraph we will characterize the genus 7 curves on a general (principally polarized) abelian fourfold. Our main result, which somewhat inverts the results

of Section 2, is the following:

THEOREM (3.1). *Let A be a general abelian fourfold with a fixed polarization, let C be a smooth genus 7 curve with a non constant morphism ϕ: $C \to A$. Then C is a double covering of a genus 3 curve.*

The proof will be achieved in various steps, some of independent interest, which will be used also in Section 4 for a similar analysis concerning the case $g = 5$.

Let us start with a few general remarks related to an infinitesimal deformation approach to the problem of studying curves on a general principally polarized abelian variety of dimension $g \geqslant 4$. Suppose we have a diagram of the type (1.1) enjoying properties (i), ..., (iv) as in Section 1, from which we keep the notation. In particular we have the morphisms ρ: $U \to \mathcal{A}_g$, μ: $U \to M \subseteq \mathcal{M}_\gamma$, $\rho \times \mu$: $U \to V \subseteq \mathcal{A}_g \times \mathcal{M}_\gamma$. The proof of Proposition (1.3) shows that $\gamma > g$.

Let C be the curve corresponding to the general point in M. Then there is a general principally polarized abelian variety A of dimension g and a non constant morphism f: $C \to A$. Since A is simple, the induced morphism $J(C) \to A$ is surjective, hence $J(C) \approx A \times B$ with B a suitable principally polarized abelian variety of dimension $\gamma - g > 0$. Let a and b be positive integers such that $a + b = \gamma$ and let us set

$$\mathcal{S}(a, b) = \{C \in \mathcal{M}_\gamma : J(C) \approx A \times B \text{ with } A, B \text{ abelian varieties of}$$
$$\text{dimensions } \dim A = a \text{ and } \dim B = b\},$$

$\mathcal{S}(a, b)$ is countable union of closed Zariski subsets of $\mathcal{M}_\gamma$. Since $\mathcal{S}(g, \gamma - g) \supseteq M$, then M is contained in one of these closed Zariski subsets, which we denote by W. We accordingly have $T_C(W) \supseteq T_C(M)$. We set $T_A = T_0(A)$, $T_B = T_0(B)$ as subspaces of $T := T_0(J(C)) \cong H^1(C, \mathcal{O}_C) \cong H^0(C, \omega_C)^*$, and we set $\tau_A = \mathbf{P}(T_A)$, $\tau_B = \mathbf{P}(T_B)$ as projective subspaces of $\mathbf{P} = \mathbf{P}(T) \cong \mathbf{P}^{\gamma-1}$.

Consider the natural multiplication map

$$T_A^* \otimes T_B^* \to H^0(C, \omega_C^{\otimes 2}) \cong H^1(C, T_C)^* \cong T_C(\mathcal{M}_\gamma)^*,$$

and let H be its image. Then one has (see [CG], Proposition (2.2) and [CGT], Theorem (1.2)) that:

(i) either C is hyperelliptic and then $\dim M \leqslant \gamma$;

(ii) or C is not hyperelliptic and $T_C(W)^*$ is a quotient of $H^0(C, \omega_C^{\otimes 2})/H$, hence also $T_C(M)^*$ is such; in particular, by (1.4) we have

$$g(g + 1)/2 \leqslant \dim M \leqslant 3\gamma - 3 - \dim H.$$

In all the cases we will deal with we shall have $\gamma < g(g + 1)/2$, hence by (1.4) case (i) will never occur and we will always be in case (ii). Then we may give a concrete geometric interpretation to the various objects we encountered so far. First of all, we consider the canonical map for C

$$\phi_C \colon C \to \mathbf{P} = \mathbf{P}(T),$$

which embeds C in $\mathbf{P}$. By abusing notation, we will regard C as a curve in $\mathbf{P}$. T_A^*(resp. T_B^*), considered as a subspace of $T^* \cong H^0(C, \omega_C)$, corresponds to the linear series g_B(resp. g_A) cut out on C by the hyperplanes of $\mathbf{P}$ through τ_B(resp. τ_A). Then H, as a subspace of $H^0(C, \omega_C^{\otimes 2})$, corresponds to the linear series $g_A \oplus g_B$, the *minimal sum* of g_A and g_B, i.e. the minimal linear series containing all divisors on C which are sums of a divisor of g_A and of a divisor of g_B. In other words, this is the linear series cut out on C by the quadrics through τ_A and τ_B. Note that

$$\dim g_A = \gamma - g - 1, \quad \dim g_B = g - 1, \quad \dim H = \dim g_A \oplus g_B + 1.$$

We will denote by g_A' and g_B' the base point free linear series obtained from g_A and g_B by removing all their base points. Of course we have

$$\dim g_A' = \dim g_A, \quad \dim g_B' = \dim g_B, \quad \dim g_A' \oplus g_B' = \dim g_A \oplus g_B,$$

and, summing up, we find out first basic inequality

$$\begin{aligned} \dim g_A' \oplus g_B' = \dim g_A \oplus g_B = \dim H - 1 \leqslant 3\gamma - 4- \\ \dim M \leqslant 3\gamma - 4 - g(g+1)/2. \end{aligned} \tag{3.2}$$

The second basic information we are seeking is given by the following:

LEMMA (3.3). *In the above setting, C is not contained in any quadric cone in $\mathbf{P}$ with vertex τ_B.*

Proof. Remember that C corresponds to a general point of M. From the proof of Proposition (1.3) we know that the second projection $V \to M$ is étale at C, so we may identify $T_C(M)$ with $T_{(A,C)}(V)$, where (A, C) is any pair in V. Hence the differential of the first projection $V \to \mathcal{A}_g$ at (A, C) can be interpreted as a surjective map

$$d \colon T_C(M) \to T_A(\mathcal{A}_g).$$

Accordingly the codifferential d^* is injective. Now we have the injection

$$j_A \colon T_A(\mathcal{A}_g)^* \cong \mathrm{Sym}^2 T_A^* \to \mathrm{Sym}^2 T^* \cong \mathrm{Sym}^2 H^0(C, \omega_C)$$

whose geometrical interpretation is transparent:

$$\mathbf{P}(\mathrm{Im}(j_A)) = \{\text{quadric cones with vertex at } \tau_B\}.$$

Then the injectivity of d^* yields the assertion.

Our strategy for the proof of Theorem (3.1) will be, as in [CGT] and [CG] in a similar situation, to bound the dimension of H, i.e. of $g_A' \oplus g_B'$, by using a classical

theorem of Castelnuovo on the dimension of the minimal sum of two linear series. We recall the basic facts from Castelnuovo's theory we are going to use.

Let C be a smooth projective curve and let L be a linear series on C. If D is an effective divisor on C, we put

$$L(-D) = \{E: E \text{ is an effective divisor and } E + D \text{ is in } L\},$$

and we define

$$c(D, L) = \dim L - \dim L(-D),$$

i.e. $c(D, L)$ is the *number of conditions* imposed by D on L.

An effective reduced divisor D on C is said to be in *uniform position* with respect to L if any two effective divisors of the same degree contained in D impose the same number of conditions on L. The so called *uniform position theorem* (see [ACGH], pg. 112) says that, if L' is a base point free linear system on C, defining on C a morphism which is birational onto its image, then the general divisor D of L' is in uniform position with respect to all linear series L. Finally Castelnuovo's theorem (see [C]) says that:

THEOREM (3.4). *Let L and L' be linear series on C and let D be a divisor on C which is in uniform position with respect to both L and L'. Then*

$$c(D, L \oplus L') \geqslant \min\{\deg D, c(D, L) + c(D, L') - 1\}.$$

From now on, until the end of the paragraph, we specialize to the case $\gamma = 7$ and $g = 4$ and we keep the above notation. We have now

$$\dim \tau_A = 3, \quad \dim g'_A = 2, \quad \dim \tau_B = 2, \quad \dim g'_B = 3.$$

We will denote by p_A (resp. p_B) the projection of C from τ_B to τ_A (resp. from τ_A to τ_B) and by C_A (resp. C_B) its image. Note that p_A (resp. p_B) induces a map p'_A (resp. p'_B) from C to the normalization C'_A (resp. C'_B) of C_A (resp. C_B).

First of all, we prove a Lemma which we will often need later:

LEMMA (3.5). *Let C be the curve corresponding to the general point of M and assume that C has a non constant non birational map to a smooth curve Y of genus y. Then $y \leqslant 3$ and the equality holds if and only if the degree of the map is two, and it is branched at four points.*

Proof. The Hurwitz formula immediately yields $y \leqslant 4$. Assume $y = 4$. We have the splitting $J(C) \approx A \times B$. We also have an isogeny of $J(Y)$ onto an abelian subvariety of $J(C)$. Since A is a general abelian fourfold, A is simple and not isogenous to $J(Y)$. Hence $J(Y)$ should be isogenous to an abelian subvariety of B, a contradiction. If $y = 3$ the Hurwitz formula yields that the degree of the map

$f: C \to Y$ is either 2 or 3. But in the latter case f would be unramified and we would have dim $M \leqslant 6$, contradicting (1.4).

Next we prove the:

LEMMA (3.6). *Let C be the curve corresponding to the general point of M. Then:*

  (a) *either C is a double covering of a genus 3 curve,*
  (b) *or the map p_A is birational onto its image, i.e. the linear series g'_B defines a birational morphism.*

Proof. Assume p_A not birational. We know by Lemma (3.3) that C_A does not lie on any quadric surface in τ_A. Hence $\deg C_A \geqslant 5$. The only possibility is therefore $\deg C_A = 5$ or 6 and the degree of the map $p'_A: C \to C'_A$ to be two. Then either we are in case (a) or Lemma (3.5) yields $0 \leqslant g(C_A) \leqslant 2$. But $g(C_A) \neq 0$ since C is not hyperelliptic. So we have to exclude that $1 \leqslant g(C_A) \leqslant 2$. Suppose $g(C_A) = 1$. Since we have the splitting $J(C) \approx A \times B$ and A is general, we have

$$B \approx C'_A \times B',$$

where B' is a suitable principally polarized abelian surface. We have a morphism $\pi: C \to C'_A \times B'$ and, since C generates its Jacobian, $\pi(C)$ generates $C'_A \times B'$, hence its genus is at least 3. In view of Lemma (3.5), π has to be birational onto the image. Then, by applying Proposition (2.4) of [CGT], one easily finds dim $M \leqslant 9$, contradicting (3.2). The proof of the case $g(C_A) = 2$ is similar and therefore we omit it.

Next we consider again the splitting $J(C) \approx A \times B$ where C is the curve corresponding to the general point of M. Then we can find a suitable étale neighborhood D of C and a morphism

$$\eta: D \to \mathcal{A}_3$$

sending C to B. We will denote by δ its codifferential

$$d\eta^*_{|C}: \operatorname{Sym}^2 T^*_B \cong T_B(\mathcal{A}_3)^* \to T_C(M)^* \cong T_A(\mathcal{A}_4)^* \cong \operatorname{Sym}^2 T^*_A.$$

We have the following:

LEMMA (3.7). *Let C be the curve corresponding to the general point of M. Then:*

  (a) *either C is a double covering of a genus 3 curve,*
  (b) *or dim $\eta(D) \geqslant 5$ and therefore dim(coker δ) $\leqslant 5$.*
      Furthermore, one has dim $g'_A \oplus g'_B \leqslant 7$ and, if the equality holds, then:
  (i) *$|\omega_C^{\otimes 2}|$ is generated by $g_A \oplus g_B$ and by the linear system cut out on C by the quadric cones with vertex at τ_B;*

(ii) *the linear system of all quadrics in* $\mathbf{P}$ *containing* τ_B *cuts out the complete bicanonical series* $|\omega_C^{\otimes 2}|$ *on* C;

(iii) $\tau_B \cap C = \emptyset$, *hence* g_B *has no base points;*

(iv) *the linear system of all quadrics in* $\mathbf{P}$ *containing* τ_A *cuts out on* C *a linear series whose codimension in the complete bicanonical series is* $\dim(\operatorname{coker} \delta)$;

(v) *if* E *is the divisor cut out by* τ_A *on* C, *then* $\deg E \leqslant \dim(\operatorname{coker} \delta)$.

Proof. Let B be a general point in $\eta(D)$ and let C be a general point in an irreducible component F of the fibre of η over B. Then there is a non-constant map $C \to B$ whose image generates B. If this map is not birational onto its image, then Lemma (3.5) implies that we are in case (a). If the map $C \to B$ is birational onto its image then, according to Proposition (2.4) of [CGT], we have $\dim F \leqslant 5$, whence (b) holds.

The inequality $\dim g'_A \oplus g'_B \leqslant 7$ is nothing but (3.2) in the present case. Hence, if the equality holds, we find that $\dim M = 10$, the map $V \to \mathcal{A}_4$ is generically finite and the injection

$$\mathrm{d}^* \colon T_A(\mathcal{A}_4)^* \to T_C(M)^*$$

considered in the proof of Lemma (3.3) is an isomorphism. Furthermore the equality in (3.2) yields also the isomorphism

$$T_C(M)^* \cong H^0(C, \omega_C^{\otimes 2})/H.$$

All this easily implies (i). Assertion (ii) is a reformulation of (i), and (iii) is an easy consequence of it. Note now that we also have the injection

$$j_B \colon T_B(\mathcal{A}_3)^* \cong \operatorname{Sym}^2 T_B^* \to \operatorname{Sym}^2 T^* \cong \operatorname{Sym}^2 H^0(C, \omega_C),$$

and that

$$\mathbf{P}(\operatorname{Im}(j_B)) = \{\text{quadric cones with vertex at } \tau_A\}.$$

Part (iv) is then obvious and (v) is again a consequence of it.

Now we can prove the following:

LEMMA (3.8). *Let* C *be the curve corresponding to the general point of* M. *Then:*

(a) *either* C *is a double covering of a genus 3 curve,*

(b) *or* p_B *is not birational onto its image, i.e. the linear series* g'_A *does not define a birational morphism.*

Proof. Assume that C is not a double covering of a genus 3 curve and that p_B is birational onto its image C_B. Let D be the general divisor of the linear series g'_A. We claim that

$$c(D, g'_B) \geqslant 3. \tag{3.9}$$

Otherwise we would have $c(D, g'_B) \leqslant 2$ and indeed $c(D, g'_B) = 2$, since $\dim g'_A = 2$. Note that C_A is birational to C by Lemma (3.6). Hence the image of D on C_A via p_A should be formed by $\deg D$ distinct points lying on a line, because $c(D, g'_B) = 2$. Since D varies in a linear series of dimension 2, and therefore $\deg D > 2$, the non-degenerate curve C_A in $\mathbf{P}^3$ would possess a 2-dimensional family of trisecant lines, which, as well known, is a contradiction (see [CC], example (1.8)).

Now we apply Castelnuovo's Theorem (3.4):

$$c(D, g'_A \oplus g'_B) \geqslant \min\{\deg D, c(D, g'_A) + c(D, g'_B) - 1\} \geqslant 4,$$

since $c(D, g'_A) = 2$ and $\deg D \geqslant 6$ because C has genus 7. Then

$$\dim g'_A \oplus g'_B \geqslant 4 + \dim g'_B = 7, \tag{3.10}$$

and, by Lemma (3.7), the equality has to hold both in (3.10) and in (3.9). Then the image of D on the curve C_A spans a plane π_D. We denote by $D + D'$ the (pull-back on C of the) divisor cut out by π_D on C_A. As D varies in g'_A, D' also varies in a linear series. Let us consider the linear series $L = |D'|$.

We claim that $\dim L > 1$. First of all, we prove that $\dim L \geqslant 1$. If not, then the image of D' on C_A would be a point p of τ_A contained in each one of the planes π_D, as D varies in g'_A. In other words g'_A would be cut out on C_A, off p, by the net of all planes through p. But then $g'_A \oplus g'_B$ would be cut out on C_A by the quadrics through p. Since the equality holds in (3.10), this would imply the existence of a quadric in τ_A containing C_A, against Lemma (3.3).

Assume now, by contradiction, that $\dim L = 1$. Let B be the fixed divisor of L and let D'' be the general divisor of $L' = L(-B)$. Then D'' is formed by distinct points and $\deg D'' \geqslant 3$, since C is not hyperelliptic because $\gamma < g(g + 1)/2$. Furthermore D'' should be formed by points of C_A lying on a line. If E is the divisor cut out by τ_A on C, then by (iii) of Lemma (3.7), one has $E \equiv D'$, where $\equiv$ is the linear equivalence. Thus $E \in L$, hence it is easy to see that $E - B$, as a divisor on C, is formed $L' = L(-B)$. Then D'' is formed by distinct points and $\deg D'' \geqslant 3$, since C is not hyperelliptic because $\gamma < g(g + 1)/2$. Furthermore D'' should be formed by points of C_A lying on a line. If E is the divisor cut out by τ_A on C, then by (iii) of Lemma (3.7), one has $E \equiv D'$, where $\equiv$ is the linear equivalence. Thus $E \in L$, hence it is easy to see that $E - B$, as a divisor on C, is formed F_A, i.e. the planes of τ_A containing the line generators of F_A, cut out on C_A, off the divisors of L, the divisors of g'_A.

Take now two general points x and y of C_A. Since there is only one divisor of the linear series g'_A containing x and y, this means that the line xy has to meet F_A, off x and y, at a unique point. Thus the general chord of C_A is a flex-tangent to F_A, i.e. it has multiplicity of intersection 3 somewhere with F_A. Now the lines of the 2-dimensional family of the chords of C_A neither pass through the same point of τ_A nor meet one and the same curve $C' \neq C_A$ (see, for instance, [CC],

example (1.8)). This implies that the general chord of C_A is a flex-tangent to F_A at a general point of F_A.

Fix a general line generator r of F_A. For a general point x on r consider the unique flex tangent to F_A passing through x and distinct from r. This is a chord of C_A, hence it meets C_A at two points p_x, q_x. As x varies on r, the divisor $p_x + q_x$ varies in a rational, hence in a linear, series, and C would therefore be hyperelliptic, a contradiction.

We can now conclude the proof of the lemma. Since dim $L' \geqslant 2$, then deg $D'' \geqslant 6$, so in particular deg $D' \geqslant 6$. Otherwise L' would define a non birational morphism, and the only possibility would then be deg $D'' = 4$ and C hyperelliptic, a contradiction. Since we are assuming g'_A to be birational, we also have deg $D \geqslant 6$. Since $\deg(D + D') \leqslant 12$, we have $D' = D''$ and deg $D = $ deg $D' = 6$. But then, since $E \equiv D'$, we find deg $E = 6$, contradicting (v) of Lemma (3.7), since by part (b) of the same Lemma, we have deg $E \leqslant \dim(\operatorname{coker} \delta) \leqslant 5$.

We are now in a position to give the:

Proof of Theorem (3.1). Assume, by contradiction, that C, the curve corresponding to a general point of M, is not a double covering of a genus 3 curve. In view of the Lemmas (3.6) and (3.8), we have that p_B is not birational onto its image C_B, whereas p_A is birational onto its image C_A.

By the Lemma (3.5) we have $0 \leqslant g(C_B) \leqslant 2$. The cases $g(C_B) = 1, 2$ can be excluded as we did in the proof of the Lemma (3.6). So we are left with the case $g(C_B) = 0$. Then g'_A is composed with a rational involution L of degree $d > 2$. Let Z be a general divisor of this involution. Then

$$c(Z, g'_A) = 1,$$

whereas

$$c(Z, g'_B) \geqslant 2, \tag{3.12}$$

since p_A is birational onto its image, hence g'_B is not composed with L. Then

$$\dim g'_A \oplus g'_B - \dim g'_B \oplus L \geqslant c(Z, g'_A \oplus g'_B) \geqslant c(Z, g'_B) \geqslant 2$$

$$\dim g'_B \oplus L - \dim g'_B \geqslant c(Z, g'_B \oplus L) \geqslant c(Z, g'_B) \geqslant 2,$$

hence, by adding up, we get

$$\dim g'_A \oplus g'_B \geqslant 2c(Z, g'_B) + 3 \geqslant 7. \tag{3.13}$$

By Lemma (3.7) we see that equality has to hold in (3.13) and (3.12). Hence Z is formed by points on a line r on C_A in τ_A, whereas its image on τ_B is a point p on

C_B. Let us look at Z as a divisor on C in **P**. Z is contained in the following two projective spaces, both of dimension 4:

$$\pi_1 = \langle \tau_B, r \rangle, \quad \pi_2 = \langle \tau_A, p \rangle.$$

Since π_1 and π_2 span **P**, we have

$$\dim\langle Z \rangle \leqslant \dim(\pi_1 \cap \pi_2) = 2,$$

hence by Riemann–Roch we have $d \leqslant 4$. If $d = 3$, then C lies on a rational normal scroll F which projects from τ_B to a scroll F_A, which is easily seen not to be a cone (see (ii) of Lemma (3.7) and the proof of Lemma (3.8)). Let h be the degree of C_B and let D be the general divisor of g_A'. Then $D = Z_1 + \cdots + Z_h$, where $Z_1, \ldots, Z_h$ are divisors in L. We claim that $h = 2$. Otherwise one clearly has

$$c(D, g_A' \oplus g_B') \geqslant c(Z_1, g_A') + c(Z_2 + \cdots + Z_h, g_B')$$

$$= 1 + c(Z_2 + \cdots + Z_h, g_B') = 5,$$

hence $\dim g_A' \oplus g_B' \geqslant 8$, a contradiction. Then C_B should be conic, thus the degree of g_A' would be 6 and $\deg E = 6$, whereas by Lemma (3.7), (a) and (v) we have $\deg E \leqslant 5$. Hence we have $d = 4$. Then the planes spanned by the divisors of L describe a rational normal threefold Φ of degree 4 in **P**. These planes intersect τ_A in lines and τ_B in points. Hence Φ would intersect τ_A in a quadric containing C_A, against Lemma (3.3).

4. The minimal genus of curves on a general abelian fivefold

In the previous paragraphs we developed some tools which are also useful to determine $\gamma(5)$. The result is:

THEOREM (4.1). *One has $\gamma(5) = 11$.*

The strategy for the proof is similar to the one for the proof of Theorems (2.3) and (3.1). First we note that Proposition (1.3) yields $\gamma(5) \geqslant 9$. The case $\gamma(5) = 9$ was excluded in [AP], so we have to treat the case $\gamma(5) = 10$. Indeed, our argument could also be applied, with slight modifications, to the case $\gamma(5) = 9$. For the sake of brevity we will not dwell on this.

We keep the notation we introduced in Section 3, and assume, by contradiction, that there is a diagram of the type (1.1) enjoying properties (i), ..., (iv) as in Section 1 and with $\gamma = 10$ and $g = 5$. In particular we have the morphisms $\rho: U \to \mathcal{A}_5$, $\mu: U \to M \subseteq \mathcal{M}_{10}$, $\rho \times \mu: U \to V \subseteq \mathcal{A}_5 \times \mathcal{M}_{10}$. The general point of V corresponds to a pair $(A, C) \in \mathcal{A}_5 \times \mathcal{M}_{10}$ such that A is general in $\mathcal{A}_5$ and there is a non-constant morphism $C \to A$. Hence, if C is a general point of M,

then we have the splitting $J(C) \approx A \times B$ with B a suitable principally polarized abelian variety of dimension 5. In the present case we have

$$\dim \mathbf{P} = 9, \quad \dim \tau_A = \dim \tau_B = 4, \quad \dim g'_A = \dim g'_B = 4.$$

We will also keep the notation p_A, p_B, C_A, C_B etc. introduced in Section 3.

LEMMA (4.2). *The projections p_A and p_B are both not birational onto their images, i.e. both series g'_A and g'_B define non birational morphisms. Furthermore p_A determines a morphism of degree two of C onto C_A, and $0 < g(C_A) \leqslant 5$.*

   Proof. Suppose g'_A determines a birational morphism. Let D be the general divisor in g'_A and suppose that $g'_A \neq g'_B$. Then

$$c(D, g'_A) = 4, \quad c(D, g'_B) = 5.$$

By Castelnuovo's Theorem (3.5) and by Clifford's Theorem, we have

$$c(D, g'_A \oplus g'_B) \geqslant \min\{\deg D, 8\} = 8,$$

hence $\dim g'_A \oplus g'_B = 12$, contradicting (3.2). So we must have $g'_A = g'_B$ and $\dim g'_A \oplus g'_B = 11$, but this is easily seen to contradict Lemma (3.3).

   If the degree of p_A were bigger than two, then the degree of C_A would be 6 or less, and C_A would lie on some quadric of τ_A, against Lemma (3.3). Finally the Hurwitz formula says that $0 \leqslant g(C_A) \leqslant 5$, but $g(C_A) \neq 0$ since C is not hyperelliptic.

   The proof of Theorem (4.1) consists in discussing separately the various cases $1 \leqslant g(C_A) \leqslant 5$, excluding each of them. First we dispose of the case $g(C_A) = 5$ with the following lemma, whose easy proof we omit:

LEMMA (4.3). *Let C be as above. Then:*

   (i) *C has no non constant map to a smooth genus 5 curve Y;*
   (ii) *if C has a non constant map $f\colon C \to Y$ to a smooth genus 4 curve Y, then the degree of f is 2, Y is a general genus 4 curve and f is branched at 6 general points of Y;*
   (iii) *if C has a map $f\colon C \to Y$ to a smooth genus 3 curve Y, then the degree of the map is 2.*

   Now we will discuss the remaining cases. The cases $1 \leqslant g(C_A) \leqslant 3$ will be excluded by counting parameters. The case $g(C_A) = 4$ requires a more delicate projective-geometrical analysis.

Case $g(C_A) = 1$. We prove the:

LEMMA (4.4). *If $g(C_A) = 1$ then $g(C_B) \geqslant 1$ and the degree of the map p_B is 2 or 3.*

Proof. Suppose $g(C_B) = 0$. Then the degree of p_B is $d \geqslant 4$, since a bielliptic curve of genus $\gamma \geqslant 5$ is never trigonal. But then d should be 4, τ_A should intersect C at two points x, y, and C_B should be a rational normal quartic in τ_B. Hence g'_A should be complete, but this is easily seen to be a contradiction. Hence we have $g(C_B) \geqslant 1$ and $d \leqslant 3$.

Furthermore, we need a definition and a lemma:

DEFINITION (4.5). Let C, C', C'' be smooth, connected, projective curves, and let $f: C \to C'$ and $g: C \to C''$ be two surjective morphisms. We say that f and g are *independent* if the morphism $f \times g: C \to C' \times C''$ is birational onto its image.

LEMMA (4.6). *Let N be a locally closed, irreducible subvariety of $\mathcal{M}_\gamma$ whose general point corresponds to a curve C which admits two independent morphisms $f: C \to C'$ and $g: C \to C''$ to smooth curves of positive genera g' and g''. We have:*

(i) *if $g' = g'' = 1$, then $\dim N \leqslant \gamma$;*
(ii) *if $g' = 1$ and $g'' \geqslant 2$, then $\dim N \leqslant \gamma + 3g'' - 4$;*
(iii) *if $g' \geqslant g'' > 1$, then*

$$\dim N \leqslant 2\gamma - 2 - (2 \deg f - 3)(g' - 1) - (2 \deg g - 3)(g'' - 1).$$

Proof. Parts (i) and (ii) are an easy consequence of Proposition (2.4) of [CGT]. Part (iii) can be proved with the same technique used in the proof of that proposition. We omit the standard argument.

Now we go back to the analysis of the case $g(C_A) = 1$. First of all, it is easy to see that the two maps p'_A and p'_B are independent. Now we apply Lemma (4.6) and we have $g(C_B) \geqslant 3$. Moreover Lemma (4.3) implies $g(C_B) \leqslant 4$. Assume $g(C_B) = 3$. In view of the splitting $J(C) \approx A \times B$ we have

$$B \approx C'_A \times J(C'_B) \times E,$$

with E a suitable elliptic curve. Then there is a map $C \to E$ which is clearly independent of p'_A and Lemma (4.6), (i), applied to the two maps $C \to E$ and p'_A, leads to a contradiction.

Assume $g(C_B) = 4$. The existence of the two independent degree 2 maps p'_A, p'_B, leads to a non constant map $C'_A \to C'^{(2)}_B$, hence to a map $C'_A \to \text{Pic}^2(C'_B)$. Since C'_B is general in moduli (see Lemma (4.3), (ii)), the latter map should be constant and C'_B should be hyperelliptic, a contradiction.

Case $g(C_A) = 2$. We have the splitting

$$B \approx J(C'_A) \times B',$$

with B' a suitable principally polarized threefold. The projection map $C \to B'$ is not birational onto its image Y, otherwise we find a contradiction by Proposition (2.4) of [CGT]. Since Y has to generate B', we have $4 \geqslant g(Y) \geqslant 3$, but the

case $g(Y) = 4$ can be excluded as above. So we are left with the case $g(Y) = 3$, in which $C \to Y$ has degree 2 by Lemma (4.3), (iii). Let Y' be the normalization of Y and let $C \to Y'$ be the map induced by $C \to Y$. The two degree 2 maps p'_A and $C \to Y'$ are independent, hence they yield a map $C \to C'_A \times Y'$ which is birational onto its image X. Moreover we get two maps

$$Y' \to \mathrm{Pic}^2(C'_A), \quad C'_A \to \mathrm{Pic}^2(Y'),$$

which are either both constant or neither one of them is such (this can be easily deduced from [BL], Theorem (5.11), pg. 340; see also [S], Sections 6, 14).

Assume both maps are constant. Then Y' is hyperelliptic as well as C'_A. Let $\mathcal{L}$, $\mathcal{N}$ be the hyperelliptic line bundles on C'_A and Y' respectively and let p, q be the projections of $C'_A \times Y'$ on the two factors. Then $X \in |p^*\mathcal{L} \otimes q^*\mathcal{N}|$ (see again [BL], pg. 340). Note that dim $|p^*\mathcal{L} \otimes q^*\mathcal{N}| = 3$ and that, by the theorem of Bertini, the general curve in $|p^*\mathcal{L} \otimes q^*\mathcal{N}|$ is smooth of genus 11. Hence X is not the general curve in $|p^*\mathcal{L} \otimes q^*\mathcal{N}|$. But now a parameter count shows that dim $M \leqslant 10$, against (3.2). An analogous parameter count, which we leave to the reader, settles the case in which the map $C'_A \to \mathrm{Pic}^2(Y')$ is not constant.

The case $g(C_A) = 3$ can be treated in a similar way as the cases $f(C_A) = 1, 2$. Therefore we omit its discussion and turn to the:

Case $g(C_A) = 4$. Since C_A has genus 4 and is not contained in any quadric by Lemma (3.3), its degree is 9. Let us consider the scroll F described by the lines in **P** joining pairs of points of C which map to the same point of C_A. The degree of F is 15 (see Section 2). Since p_A has C_A as its image, F has to intersect τ_B along a curve Y, which is unisecant the line generators of F. As the degree of C_A is 9, the degree of Y is 6, hence it does not span τ_B.

We can be more precise about Y. In fact the covering $f := p'_A : C \to C'_A$ corresponds to a triple $(C'_A, B, \mathcal{L})$, where B is the branch divisor of degree 6, and $\mathcal{L}$ is such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_{C'_A}(B)$. Inside $\mathbf{P} \cong \mathbf{P}(H^0(C, \omega_C)^*)$ we have the two subspaces

$$\mathbf{P}^+ = \mathbf{P}(H^0(C'_A, \omega_{C'_A})^*) \cong \mathbf{P}^3, \quad \mathbf{P}^- = \mathbf{P}(H^0(C'_A, \omega_{C'_A} \otimes \mathcal{L})^*) \cong \mathbf{P}^5,$$

(see Section 2, of which we keep the notation). We have the canonical map $\phi_{C'_A}$ of C'_A in $\mathbf{P}^+$, which is an embedding (see Lemma (4.3), (ii)). Furthermore, we have the map $\alpha : C'_A \to \mathbf{P}^-$ determined by the complete linear system $|\omega_{C'_A} \otimes \mathcal{L}|$. The scroll F is described by the lines joining the corresponding points of $C'^+_A = \phi_{C'_A}(C'_A)$ and $C'^-_A = \alpha(C'_A)$. Since F spans **P**, it is clear that Y has to coincide with C'^+_A, hence $\mathbf{P}^+$ is contained in τ_B. Consider the pull back map $f^* : J(C'_A) \to J(C)$ which is injective (see [BL], pg. 376). We then look at $H(C'_A)$ as contained in $J(C)$. In view of the splitting $J(C) \approx A \times B$, we see that

$$B \approx J(C'_A) \times E,$$

with E a suitable elliptic curve. If η is the point of $\mathbf{P}$ corresponding to $\mathbf{P}(T_0(E))$, we have that τ_B is the join of η and $\mathbf{P}^+$, whereas $\mathbf{P}^-$ contains both τ_A and η.

Now we note that:

(i) p_B is not birational onto its image C_B (see Lemma (4.2));
(ii) by composing p_B with the projection of C_B from η to $\mathbf{P}^+$ we have nothing but the projection of C from $\mathbf{P}^-$ to $\mathbf{P}^+$, which in turn coincides with $f = p'_A$.

Hence p'_B should be also composed with the involution σ determined by p'_A. On the other hand τ_A is not unisecant the generators of F, otherwise F would be degenerate. Thus the projection p_B is not composed with σ, a contradiction.

Our argument proves so far that $\gamma(5) \geqslant 11$. Finally one has $\gamma(5) = 11$, since the general principally polarized abelian variety is the Prym variety associated to a genus 11 unbranched double covering of a general genus 6 curve (see [M]). This finishes the proof of Theorem (4.1).

We remark that our analysis still leaves the problem of classifying all genus 11 curves on a general principally polarized abelian fivefold (see Section 6). As pointed out in the introduction, the question has been solved by Pirola [P], after the present work was completed.

5. The restricted Prym map for principally polarized abelian fourfolds, its ramification and its degree

In this paragraph we will go deeper into the study of the map $p^+ \colon \mathcal{R}^+ \to \mathcal{A}_4^+$ introduced in Section 2. In particular we will determine its degree.

We consider the following moduli spaces:

$$\mathcal{R} = \{\text{isomorphism classes of admissible double coverings } f \colon C' \to C$$
$$\text{in } \mathcal{R}_5 \text{ such that both } C \text{ and } C' \text{ are irreducible with exactly}$$
$$\text{two nodes}\}$$

$$\mathcal{R}\mathcal{A}_4 = \{(A, e) \colon A \in \mathcal{A}_4, e \in A_2, e \neq 0\}$$

where, for a given complex torus A and for any positive integer n we denote by A_n the group of points of order n of A.

REMARK (5.1). $\mathcal{R}$ is a locally closed subscheme of $\bar{\mathcal{R}}_5$. If $f \colon C' \to C$ is a general element in $\bar{\mathcal{R}}_5$ then C' is smooth of genus 9 and C is smooth of genus 5, whereas if it is general in $\mathcal{R}$, then C' has geometric genus 7 and C has geometric genus 3. Note that, since f must be admissible, the tangent directions at the two nodes of C' are fixed under the involution determined by f.

REMARK (5.2). We have the extended Prym map (see [B])

$$\bar{p}_5 \colon \bar{\mathcal{R}}_5 \to \mathcal{A}_4.$$

To ease notation, we will denote it by $\bar{p}$. The fibre of $\bar{p}$ at a general point A of $\mathcal{A}_4$ is a compact surface S_A (described by Donagi in [D]). For each point $f: C' \to C$ in S_A there is the Abel–Prym map

$$\alpha: C' \to A$$

which is generically an embedding. Thus A contains a 2-dimensional family of genus 9 curves. The reason why we have $\gamma(4) = 7$ is that, as one may expect, only finitely many of these curves have two nodes, so that the corresponding double coverings sit in $\mathcal{R}$. As we will see, to determine the degree of p^+ is equivalent to determine the number of these double-nodal curves in A.

We will consider the following morphisms:

(i) $p: \mathcal{R} \to \mathcal{A}_4$, the *restricted Prym map*, is the restriction of the Prym map $\bar{p}: \bar{\mathcal{R}}_5 \to \mathcal{A}_4$ to $\mathcal{R}$;

(ii) $f: \mathcal{R}\mathcal{A}_4 \to \mathcal{A}_4$ is the obvious *forgetful map*;

(iii) $n: \mathcal{R} \to \mathcal{R}^+$, the *normalization map* which is defined as follows. A point $f: C' \to C$ in $\mathcal{R}$ determines a double covering $f^+: N' \to N$, where N'(resp. N) is the normalization of C'(resp. C). We will say that $f^+: N' \to N$ is the *normalization* of $f: C' \to C$. Since f^+ is ramified at the four points of N' sitting over the two nodes of N', the isomorphism class of $f^+: N' \to N$ is in $\mathcal{R}^+$. The map n sends the class of $f: C' \to C$ to the class of $f^+: N' \to N$;

(iv) $q: \mathcal{R}\mathcal{A}_4 \to \mathcal{A}_4^+$, the *quotient map* defined as follows. Let (A, e) be a point in $\mathcal{R}\mathcal{A}_4$. Consider the abelian fourfold A_e, the quotient of A by the translation t_e induced by e. We have the quotient morphism $\pi: A \to A_e$. If Θ is the principal polarization of A, then there is a polarization Θ_e on A_e such that

$$\Theta + t_e(\Theta) \sim \pi^*(\Theta_e),$$

where we denote by $\sim$ the algebraic equivalence. One easily computes $(\Theta_e)^4 = 8 \cdot 4!$ and $h^0(A_e, \mathcal{O}_{A_e}(\Theta_e)) = 8$, so that (A_e, Θ_e) sits in $\mathcal{A}_4^+$. The map q sends (A, e) to (A_e, Θ_e);

(v) $p': \mathcal{R} \to \mathcal{R}\mathcal{A}_4$, the *lifting map*, defined as follows. Let $f: C' \to C$ be a point in $\mathcal{R}$ and $f^+: N' \to N$ be its image in $\mathcal{R}^+$ under the normalization map. Consider the Prym varieties (A, Θ) of f and (A^+, Θ^+) of f^+. We have the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T' & \longrightarrow & \operatorname{Pic}^0(C') & \longrightarrow & \operatorname{Pic}^0(N') & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*^+} & & \\
0 & \longrightarrow & T & \longrightarrow & \operatorname{Pic}^0(C) & \longrightarrow & \operatorname{Pic}^0(N) & \longrightarrow & 0
\end{array}
$$

where f_*(resp. f_*^+) is the norm map related to the covering $f: C' \to C$(resp. $f^+: N' \to N$), and the horizontal rows are the standard extensions of generalized Jacobians. Taking the kernels of the norm maps, one gets, as in [B], pg. 159, the exact sequence

$$0 \to T_2' \to A \times \mathbf{Z}_2 \to A^+ \to 0,$$

where T_2' is the group of points of order two of T'. In particular we find an isogeny $g: A \to A^+$ and $\mathrm{Ker}(g) \cong (\mathbf{Z}_2)^{t-1}$, where $t = \dim T'$. In our situation we have $T \cong T' \cong (\mathbf{C}^*)^2$, so that $\mathrm{Ker}(g) \cong \mathbf{Z}_2$. This uniquely defines a non zero point $e \in A_2$. The map p' sends $f: C' \to C$ to the pair (A, e). Notice that

$$A^+ = A_e, \quad \Theta + t_e(\Theta) \sim g^*(\Theta^+). \tag{5.3}$$

LEMMA (5.4). *One has $p = f \circ p'$ and $q \circ p' = p^+ \circ n$.*
    Proof. The first assertion is trivial. The latter follows by (5.3).

Our objective will be the study of ramification and degree of the maps $p: \mathcal{R} \to \mathcal{A}_4$ and $p^+: \mathcal{R}^+ \to \mathcal{A}_4^+$, two things which are essentially equivalent in view of the previous lemma. So we will concentrate on the restricted Prym map $p: \mathcal{R} \to \mathcal{A}_4$. We let

$$\mathcal{R}_{\mathrm{eff}} = n^{-1}(\mathcal{R}_{\mathrm{eff}}^+), \ \mathcal{R}_{\mathrm{hyp}} = n^{-1}(\mathcal{R}_{\mathrm{hyp}}^+).$$

A first consequence of Lemma (5.4) is that:

PROPOSITION (5.5). *The ramification locus of $p: \mathcal{R} \to \mathcal{A}_4$ is $\mathcal{R}_{\mathrm{eff}} \cup \mathcal{R}_{\mathrm{hyp}}$.*
    Proof. Indeed, the maps n, f, q are clearly unramified and, according to Theorem (2.2), $\mathcal{R}_{\mathrm{eff}}^+ \cup \mathcal{R}_{\mathrm{hyp}}^+$ is the ramification locus of p^+.

Our strategy for determining the degree of p will be the following:

  (i) we will show that p is proper and unramified around a general point A of the Jacobian locus $\mathcal{J}_4$ of $\mathcal{A}_4$;
  (ii) we will compute the cardinality of $p^{-1}(A)$. So let $J(X) = (A, \Theta) \in \mathcal{J}_4$ be the Jacobian of a genus 4 curve X, and let $\mathcal{R}^*$ be the Zariski closure of $\mathcal{R}$ in $\bar{\mathcal{R}}_5$. We have the following:

LEMMA (5.6). *Let $f: C' \to C$ be an intersection point of $\mathcal{R}^*$ with the fibre S_A of $\bar{p}$ at $J(X) = (A, \Theta)$. Then, for a sufficiently general X:*
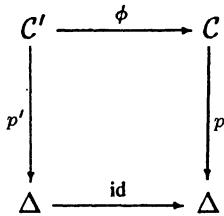
  (i) *there is a morphism $g: C \to \mathbf{P}^1$ of degree 3;*
  (ii) *$f: C' \to C$ is a point of $\mathcal{R}$, i.e. C is irreducible with exactly two nodes.*

    Proof. We check Beauville's list in [B] of families of admissible coverings $f: C' \to C$ whose Prym variety is a general fourfold Jacobian $J(X)$, in particular $J(X)$ has no vanishing thetanull. Assume there are no irreducible components of C' which are interchanged by the involution $\iota: C' \to C'$ determined by f.

Then, applying [B], 4.10, and excluding families whose Prym varieties are in the thetanull locus (see also [B], 7.3), we see that the only possibility is that there is a morphism $g: C \to \mathbf{P}^1$ of degree $d \leqslant 3$. Indeed we must have $d = 3$ by [M], Section 7. Assuming that some irreducible components of C' are interchanged by ι, and applying [B], 5.2 and 5.4, we find the so called *Beauville–Wirtinger coverings* (see [DS], IV, 1.4). In particular:

(a) either C is obtained from X by identifying two points,

(b) or $C = X \cup E$, with E of arithmetic genus $p_a(E) = 1$ and X intersecting E transversally at one point.

Let us check that a Beauville–Wirtinger covering $f: C' \to C$ is not in $\mathcal{R}^*$. Otherwise there would be a commutative diagram

$$
\begin{array}{ccc}
C' & \overset{\phi}{\longrightarrow} & C \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
\Delta & \overset{\mathrm{id}}{\longrightarrow} & \Delta
\end{array}
$$

where:

(1) Δ is the one-dimensional disc centered at 0;

(2) $p: C \to \Delta$ and $p': C' \to \Delta$ are flat families of curves;

(3) for each $t \in \Delta$, the induced map $\phi_t: C'_t \to C_t$ is an admissible covering which sits in $\mathcal{R}$ for $t \neq 0$, whereas $\phi_0: C'_0 \to C_0$ coincides with $\phi: C' \to C$.

Then the normalization $n: \mathcal{N} \to C$ of C is a flat family of connected genus 3 curves. In particular $n_0: N_0 \to C_0$ is a connected partial normalization of $C_0 = C$ with $p_a(N_0) = 3$. But for such a partial normalization the sum of the geometric genera of its irreducible components should be at most 3, which is impossible for curves in (a) or (b). This proves (i).

    To show (ii) note that, by the same argument, if $f: C' \to C$ is in $\mathcal{R}^*$ then C cannot be smooth or have only one node. Now we count dimensions. If the nodes are exactly two and C is irreducible, then $f: C' \to C$ has to sit in $\mathcal{R}$ since it is an admissible covering. Assume C is irreducible with at least three nodes. Then $g(N) \leqslant 2$ for the normalization N of C. Since there is a map $g: C \to \mathbf{P}^1$ of degree three, accordingly we have a g^1_3 on N, and the pairs of points of N corresponding to the nodes of C are *neutral* for this g^1_3 (see [Se]). This easily implies that C depends on at most 8 parameters, contrary to the fact that $f: C' \to C$ has to depend on 9 parameters at least, because its Prym variety has to be a general Jacobian fourfold. Finally assume C to be reducible. As above, the sum of the geometric genera of the irreducible components of C is at most 3. Then, applying again [B],

Lemma (4.11), and counting parameters one easily sees that the Prym varieties of all possible admissible coverings of C never define general Jacobian fourfolds.

We are now in position to conclude part (i) of our program:

THEOREM (5.7). *The restricted Prym map* $p: \mathcal{R} \to \mathcal{A}_4$ *is proper around the general point of* $\mathcal{J}_4$.

Proof. Indeed, the Prym map $\bar{p}: \bar{\mathcal{R}} \to \mathcal{A}_4$ is proper (see [B]), $\mathcal{R}^*$ is closed in $\bar{\mathcal{R}}$ by definition and, by the previous Lemma (5.6), for a general point $J(X) = (A, \Theta)$ of $\mathcal{J}_4$ we have $(\bar{p}_{|\mathcal{R}^*})^{-1}(J(X)) = p^{-1}(J(X))$.

THEOREM (5.8). *Let* $J(X) = (A, \Theta)$ *be a general point of* $\mathcal{J}_4$. *Then the restricted Prym map* $p: \mathcal{R} \to \mathcal{A}_4$ *is unramified at each point* $f: C' \to C$ *of* $p^{-1}(J(X))$.

Proof. Recall that, by Lemma (5.6), (ii), both C' and C are irreducible with exactly two nodes. Take the normalization $f^+: N' \to N$ of f. Then N' and N are both irreducible and the covering $f^+: N' \to N$ corresponds to the triple $(N, B, \mathcal{L})$, where $B = x_1 + x_2 + y_1 + y_2$ is the branch divisor of f^+, (x_1, x_2) and (y_1, y_2) are the pairs of points of N corresponding to the nodes of C, and $\mathcal{L}$ is a line bundle on N such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_N(B)$. By Proposition (5.5), p ramifies at $f: C' \to C$ if and only if: (i) either N is hyperelliptic, (ii) or $h^0(N, \mathcal{L}) \geqslant 1$. To exclude that this happens for a general Jacobian $J(X)$, we use Lemma (5.6), (i), which yields the existence of a morphism $g: C \to \mathbf{P}^1$ of degree 3. Of course this lifts to a morphism $g^+: N \to \mathbf{P}^1$ of degree 3 for which the pairs (x_1, x_2) and (y_1, y_2) are neutral. Since there is no base point free g_3^1 on a hyperelliptic curve of genus bigger than 2, then we are in case (ii) and $h^0(N, \mathcal{L}) = 1$. Let D be the unique divisor in $|\mathcal{L}|$ and let o be the unique point of N such that the g_3^1 determined by $g^+: N \to \mathbf{P}^1$ coincides with the linear series $|\omega_N(-o)|$. The pairs (x_1, x_2) and (y_1, y_2) being neutral for this series means that

$$x_1 + x_2 \in |\omega_N(-x_3 - o)|, \quad y_1 + y_2 \in |\omega_N(-y_3 - o)|,$$

with x_3, y_3 suitable points on N. Then

$$2D \equiv B = x_1 + x_2 + y_1 + y_2 \in |\omega_N^{\otimes 2}(-x_3 - y_3 - 2o)|,$$

i.e.

$$x_3 + y_3 + 2o \in |\omega_N^{\otimes 2}(-2D)|.$$

We claim that $\omega_N^{\otimes 2}(-2D) \neq \omega_N$. Otherwise $x_3 + y_3 + 2o \in |\omega_N|$, and then

$$x_1 + x_2 = y_3 + o \in |\omega_N(-x_3 - o)|, \quad y_1 + y_2 = x_3 + o \in |\omega_N(-y_3 - o)|.$$

Therefore we should have

$$B = x_1 + x_2 + y_1 + y_2 = x_3 + y_3 + 2o$$

a contradiction, since B is reduced. Now we claim that we have only finitely many choices for B inside $|\mathcal{L}^{\otimes 2}|$. Indeed, B is obtained from the divisor $x_3 + y_3 + 2o$ of $|\omega_N^{\otimes 2}(-2D)|$ by taking the two, uniquely defined, effective divisors $x_1 + x_2 \in |\omega_N(-x_3 - o)|$, $y_1 + y_2 \in |\omega_N(-y_3 - o)|$ and summing them up. Since $\dim|\omega_N^{\otimes 2}(-2D)| = 1$, this gives finitely many choices for B. In conclusion, the covering $f^+: N' \to N$ corresponds to a triple $(N, B, \mathcal{L})$ with $|\mathcal{L}|$ effective and B in a finite set of $|\mathcal{L}^{\otimes 2}|$. These triples depend on 8 parameters only, so the corresponding Prym varieties cannot dominate $\mathcal{J}_4$.

Now we turn to the second step of our proof, i.e. the computation of the degree of the fibre of p over the general point of $\mathcal{J}_4$. The main tool here is Recillas' trigonal construction ([R] and [DS], Part III, or [BL], chap. 12, Section 7) applied to our situation. Let X be a genus 4 curve, which for simplicity we assume to be non hyperelliptic. Let us consider the variety $W_4^1(X)$ of all the g_4^1's on X. This is isomorphic to the symmetric product $X^{(2)}$, the isomorphism being given by the map

$$D \in X^{(2)} \to |\omega_X(-D)| \in W_4^1(X).$$

Let us fix a point $\mathcal{L} \in W_4^1(X)$. This defines the following reduced curve of $X^{(2)}$

$$T'_{\mathcal{L}} = \{D \in X^{(2)}: h^0(X, \mathcal{L}(-D)) \neq 0\}.$$

The map

$$D \in T'_{\mathcal{L}} \to |\mathcal{L}(-D)| \in T'_{\mathcal{L}},$$

is an involution on $T'_{\mathcal{L}}$, so we have a quotient curve $T_{\mathcal{L}}$ and a double covering

$$f_{\mathcal{L}}: T'_{\mathcal{L}} \to T_{\mathcal{L}}.$$

The following facts are well known (see [DS], Part III):

(i) there is a morphism of degree 3 $g_{\mathcal{L}}: T_{\mathcal{L}} \to \mathbf{P}^1$,
(ii) if $\mathcal{L}$ is base point free, then $f_{\mathcal{L}}: T'_{\mathcal{L}} \to T_{\mathcal{L}}$ is admissible, determining a point of $\bar{\mathcal{R}}_5$ and the corresponding Prym variety is just $J(X)$,
(iii) hence there is a rational map

$$\tau_X: W_4^1(X) \to S_X,$$

where S_X is the fibre of the Prym map p over $J(X)$, sending $\mathcal{L}$ to $f_{\mathcal{L}}: T'_{\mathcal{L}} \to T_{\mathcal{L}}$, and this map is injective if the automorphism group of X is trivial. We have from [DS] a complete description of S_X: either a point $f: C' \to C$ is in the image of τ_X, or it is a Beauville–Wirtinger covering as in the proof of Lemma (5.6), cases (a) and (b).

We are now in a position to compute the degree of the fibre of p over $J(X)$, with X a general genus 4 curve. First we need the following:

LEMMA (5.9). *Let X be a general genus 4 curve. Then there is a bijection between the set of points of $p^{-1}(J(X))$ and the set of base points free g_4^1's $|\mathcal{L}|$ on X such that the corresponding curve $T_\mathcal{L}$ has two nodes.*

Proof. By the proof of Lemma (5.6), an element of $p^{-1}(J(X))$ cannot be a Beauville–Wirtinger covering. So $p^{-1}(J(X))$ sits in the image of τ_X, whence the assertion.

Let now $|\mathcal{L}|$ be a g_4^1 on X. We will say that $|\mathcal{L}|$ is k-*nodal* if k is the number of divisors in $|\mathcal{L}|$ of the type $2x + 2y$. From [DS], pgs. 47–48, we have that for a given $|\mathcal{L}|$ on a genus 4 curve X, the corresponding curve $T_\mathcal{L}$ is singular only in one of the following cases:

(i) $|\mathcal{L}|$ is not base point free, then $T'_\mathcal{L}$ is reducible and $f_\mathcal{L} \colon T'_\mathcal{L} \to T_\mathcal{L}$ is a Beauville–Wirtinger covering;

(ii) $|\mathcal{L}|$ is base point free and k-nodal, then $T'_\mathcal{L}$ is an irreducible curve with exactly k nodes.

Hence by Lemma (5.6), (ii), we have that the set $p^{-1}(J(X))$ is in one-to-one correspondence with the set of base point free 2-nodal g_4^1's on X. Note that Lemma (5.6), (ii), and the above remarks actually imply that the set of base point free 2-nodal g_4^1's on a general genus 4 curve X coincides with the set of base point free k-nodal g_4^1's on X, with $k \geqslant 2$.

LEMMA (5.10). *Let X be a general genus 4 curve. Then there are exactly $3 \cdot (2^8 - 1)$ distinct 2-nodal g_4^1's on X.*

Proof. Let $|\mathcal{L}|$ be a 2-nodal g_4^1 on X. To give $|\mathcal{L}|$ is equivalent to give a pair (D', D'') of effective divisors of degree 2 on X such that $D' - D''$ is a non trivial element in $\mathrm{Pic}^0(X)_2$. Poincaré's formulae compute all pairs of such divisors for a fixed non-trivial element of $\mathrm{Pic}^0(X)_2$ to be 3. Hence the required number is $3 \cdot (2^8 - 1)$.

We finally have the:

THEOREM (5.11). *The degree of the covering $p \colon \mathcal{R} \to \mathcal{A}_4$ is $3 \cdot (2^8 - 1)$, whereas the degree of the covering $p^+ \colon \mathcal{R}^+ \to \mathcal{A}_4^+$ is 3.*

Proof. The assertion about the degree of p follows by Lemmas (5.9) and (5.10). As for the degree of p^+, we go back to Lemma (5.4). It is quite easy to see that $\deg(n) = 3$. Furthermore $\deg(q) = 3$. Indeed, let (A^+, Θ^+) be an element in $\mathcal{A}_4^+$. In order to give an element in the fibre of q over (A^+, Θ^+) we have to give a non trivial element σ in $\mathrm{Pic}^0(A^+)_2$ such that the corresponding covering $\pi \colon A \to A^+$ has a principal polarization. Now A^+ has a polarization of type $(1,2,2,2)$ and, if we choose a non trivial element σ in $\mathrm{Pic}^0(A^+)_2$, the corresponding covering $\pi \colon A \to A^+$ has either an induced polarization of type $(2,2,2,2)$ or a polarization of type $(1,2,2,4)$. Those corresponding to the former type are clearly only three, they generate a group $G \cong \mathbf{Z}_2^2$ such that $\pi^*(G) \cong \mathbf{Z}_2$, whose non zero element we denote by e. Of course (A, e) is in $\mathcal{R}\mathcal{A}_4$ and $q(A, e) = (A^+, \Theta^+)$.

Finally $\deg(f) = 2^8 - 1$, hence $\deg(p') = 3$, and so $\deg(p^+) = 3$.

REMARK (5.12). Let $f: C' \to C$ be in $\mathcal{R}_{\text{hyp}}$. Since C is hyperelliptic, then C is tetragonal and so is its normalization N'. Let $f: C' \to C$ be in $\mathcal{R}_{\text{eff}}$. Let $f^+: N' \to N$ be its normalization. Then f^+ is branched at four points x_1, x_2, y_1, y_2 such that $x_1 + x_2 + y_1 + y_2 \equiv 2D$ with D effective of degree two. But then

$$(f^+)^*(D) \equiv (f^+)^{-1}(x_1 + x_2 + y_1 + y_2),$$

and N' is tetragonal. In both cases the tetragonal involution L on N' is fixed by the involution ι determined by f^+.

Conversely suppose that $f: C' \to C$ is in $\mathcal{R}$ and N' is tetragonal. Let L be the g_4^1 on N' and let us assume it is fixed by the involution ι determined by f^+ on N'. Then we have the following possibilities:

(i) every divisor of L is fixed by ι. Then the image of L on N is a g_2^1 and therefore $f: C' \to C$ is in $\mathcal{R}_{\text{hyp}}$;

(ii) L is fixed by ι but not every divisor of L is fixed by ι. In this case ι defines an involution on $L \cong \mathbf{P}^1$, which has two fixed points. A fixed point is here a divisor D of L fixed by ι. Then:

(ii$_1$) either there is an effective divisor E of degree 2 on N such that $D = (f^+)^*(E)$;

(ii$_2$) or there is a point x on N such that $D = (f^+)^*(x) + y_1 + y_2$, with y_1, y_2 ramification points of f^+;

(ii$_3$) or D is the ramification divisor of f^+.

The number of fibres of f^+ contained in some divisor of L is two (see [ACGH], chap. VIII). Hence one of the two fixed divisors for the action of ι on L is of type (ii$_1$) if and only if the other one is of type (ii$_3$) and then $f: C' \to C$ is in $\mathcal{R}_{\text{eff}}$. In case (ii$_2$), $f: C' \to C$ is neither in $\mathcal{R}_{\text{eff}}$ nor in $\mathcal{R}_{\text{hyp}}$.

REMARK (5.13). Part of the above analysis of the behaviour of p over $\mathcal{J}_4$ can be carried over to study the behaviour of p over the thetanull locus. For example it can be proved that the thetanull locus, as well as $\mathcal{J}_4$, does not lie in the branch locus of p.

6. A few concluding remarks, conjectures and speculations

We wish to conclude this paper by making some comments on the results of the previous paragraphs, and by discussing some related problems and conjectures.

To start with, let us point out that our results joined with Pirola's ones in [P] can be interpreted as saying that there are only countably many isomorphism classes of curves of minimal genus $\gamma(g)$ in a general principally polarized abelian variety A of dimension g, for $g = 4, 5$. This in turn yields that A can be represented in countably many ways as a quotient of a Jacobian of minimal dimension. The same result holds also for $g = 3$ (see [Ba]), and can be proved, with similar techniques

as in [Ba], for $g = 2$, while for $g = 1$ it is trivial. These results naturally suggest the following:

PROBLEMS (6.1). Is a general principally polarized abelian g-fold representable in countably many ways as the quotient of Jacobians of minimal dimension $\gamma(g)$? Or, equivalently: is a general principally polarized abelian g-fold A representable in finitely many ways as the quotient of Jacobians $J(C)$ of minimal dimension $\gamma(g)$ in such a way that the image of C in A belongs to a fixed homology class? And, more generally: for which integers $\gamma \geqslant j(A)$ is it possible to have only countably many representations of A as a quotient of a Jacobian of dimension γ?

Actually we tend to believe that the first question above should always have an affirmative answer. Indeed, let us suppose that on a general g-dimensional abelian variety A there is some positive dimensional family of curves of minimal geometric genus with varying moduli. By the completeness of the Hilbert scheme, we may assume that the family is compact, i.e. that it is parametrized by a non singular projective curve T. Thus we may suppose there is a smooth surface $\mathcal{C}$, a surjective morphism $\phi: \mathcal{C} \to T$ and a morphism $\psi: \mathcal{C} \to A$, such that for t general in T, the fibre C_t of ϕ over t is smooth irreducible of genus $\gamma(g)$ and ψ, restricted to C_t, is birational onto the image. The minimality assumption yields that all fibres of ϕ can be assumed to be smooth. Suppose that the global monodromy group of the family $\phi: \mathcal{C} \to T$ is finite, as it could very well be in such a situation. Then it could be made trivial after passing to a finite unramified covering of T. In this case there is no variation of the Hodge structure in the family (see [G]), in particular all curves C_t should be isomorphic, a contradiction.

In any event, a natural related question is to look for a characterization of those abelian g-folds containing continuous families of curves of geometric genus $\gamma(g)$. Note that for $g = 5$ all Jacobians enjoy this property. The question in particular makes sense for $g = 4$, where Theorem (2.2) should provide a starting point for this investigation. Of course one might also generalize the problems (6.1) asking the questions for any principally polarized abelian variety A, not just for the general one, and replacing $\gamma(g)$ with $j(A)$.

Next we recall that, given a principally polarized abelian g-fold (A, Θ_A), we have the so-called *minimal class*

$$\chi_A = \Theta_A^{g-1}/(g-1)!$$

in the homology ring of A. Then one can define

$$c(g) = \min\{c \in \mathbf{N}: \text{such that } c\chi_A \text{ is effective on a general principally} \\ \text{polarized abelian } g\text{-fold } A\}.$$

Now we want to address the question of what is the relation between $\gamma(g)$ and $c(g)$. More precisely, we formulate the following:

PROBLEMS (6.2). Is it true that curves of minimal genus among those realizing

the minimal multiple of the minimal class which becomes effective on a (general) principally polarized abelian g-fold are also curves of minimal genus? Conversely, is it true that all curves of minimal genus on a (general) principally polarized abelian g-fold can be realized as representatives of the minimal multiple of the minimal class which becomes effective?

Some comments and remarks are in order. Any time we have a covering $f: C' \to C$ of degree d, with C, C' smooth curves of genera γ, γ', we can look at the connected component of the origin of the kernel of the induced norm map $f_*: J(C') \to J(C)$. This is an abelian variety A of dimension $g = \gamma' - \gamma$, with a suitable polarization. One can ask whether, by varying the covering $f: C' \to C$, A varies in an open dense set of its moduli space. An easy count of parameters shows that, as soon as $g \geqslant 4$, this may happen only if $d = 2$ and either $\gamma = 3$, $\gamma' = 7, 8, 9$, $g = 4$, or $\gamma = 6$, $\gamma' = 11$, $g = 5$, which are essentially the cases we consider in this paper. This suggests that in order to produce geometrical constructions of general abelian g-folds, with $g \geqslant 6$, one needs to consider more complicated objects than coverings. Let us see what is the reason for this.

If $g = 4, 5$, the curves achieving the minimal genus $\gamma(g)$ in a general principally polarized abelian g-fold (A, Θ_A) can be realized as representatives of $2\chi_A$, in particular $c(4) = c(5) = 2$. This follows by the description of these curves as double coverings and by the results of Welters [W], which indicate that curves of twice the minimal class are *essentially* related to double coverings. Instead, higher multiples of the minimal class are related to singular correspondences on suitable curves, as discussed in [W]. On the other hand, recent results of Debarre [De] imply that

$$c(g) > (g/8)^{1/2} - 1/4.$$

Also, applying Welter's results in [W], it is quite easy to see that $c(g) \geqslant 3$ as soon as $g \geqslant 6$.

As observed by Kanev (see [K$_1$], [K$_2$], and [BL], chap. 12) if C is a smooth curve of genus g with a g^1_{c+2} with simple ramification points, then $J(C)$ contains a smooth curve whose class is $c\chi_{J(C)}$ and whose genus is $\gamma = cg + [c(c-1)/2]$. For $c = 2$ this is nothing but Recillas' construction [R] used in Section 5. In case $g = 4, 5$, if we take C general in moduli and, as we can, $c = 2$, then we have curves of twice the minimal class. For $g = 5$ they have genus 11 and can be deformed off the Jacobian locus to a general abelian fivefold, as we know. For $g = 4$ essentially the same happens. On a general Jacobian fourfold we have a 2-dimensional family of genus 9 curves. Finitely many of these acquire two nodes and these curves, of genus 7, can be deformed off the Jacobian locus.

Turning to the first unknown case $g = 6$, we can take $c = 3$ and C general of genus 6. Then $J(C)$ contains a two-dimensional family of genus 21 curves, parametrized by the family of g^1_5's on C, representing the triple of the minimal class. It is natural to conjecture that these curves can be deformed off the Jacobian locus to a general abelian sixfold. This would imply $c(6) = 3$ and $\gamma(6) \leqslant 21$.

In the circle of ideas of this paper, we also propose the:

PROBLEMS (6.3). For every integer $\gamma > 0$, study the locus $\mathcal{A}_g(\gamma)$ described by all A in $\mathcal{A}_g$ containing a generating curve of genus γ. In particular: is there a $\gamma_0 \geqslant \gamma(g)$ such that for all integers $\gamma \geqslant \gamma_0$ one has $\mathcal{A}_g(\gamma) = \mathcal{A}_g$? What is the maximal dimension $\underline{a}_g(\gamma)$ of a component of $\mathcal{A}_g(\gamma)$? Also, for every integer $n \geqslant 1$, study the locus $\mathcal{A}_{g,n}$ of $\mathcal{A}_g$ described by all A in $\mathcal{A}_g$ such that $n\chi_A$ becomes effective.

Notice that $\mathcal{A}_g(\gamma)$ is not just a locally closed subset in $\mathcal{A}_g$, but it is a countable union of locally closed subsets of $\mathcal{A}_g$. Nonetheless it could be interesting to describe all of its components, or at least those enjoying some suitable "minimality property" (e.g. of being contained in $\mathcal{A}_{g,n}$, with n minimal). We note that the first question in (6.3) has a different flavour according to whether $\gamma < \gamma(g)$ or $\gamma > \gamma(g)$. The second question has to do with the latter case. For instance it is clear that $\mathcal{A}_4(8) = \mathcal{A}_4(9) = \mathcal{A}_4$, but what about $\mathcal{A}_4(10)$? Does it coincide with $\mathcal{A}_4$, or does it at least contain $\mathcal{J}_4$? As for the third question, it is again interesting to look at the case $g = 4$. Here the sequence of the numbers $\underline{a}_4(\gamma)$, $\gamma = 1, \ldots, 6$, is 7, 6, 7, 9, 8, 9. It should be not difficult in this case to describe all components of $\mathcal{A}_4(\gamma)$, with $\gamma = 1, \ldots, 6$. On the other hand, we do not know much about $\mathcal{A}_{g,n}$.

Acknowledgement

References

[ACGH]   E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, Geometry of algebraic curves. I, *Grundlehren der Math.* 267, Springer-Verlag, 1984.

[AP]   A. Alzati and P. Pirola, On abelian subvarieties generated by symmetric correspondences, *Math. Z.* 205 (1990), 333–342.

[Ba]   F. Bardelli, Curves of genus three on a general abelian threefold and the non finite generation of the Griffiths group, in Arithmetic of Complex Manifolds, Proceedings, Erlangen, 1988, ed. W. Barth, H. Lange, *Springer Lecture Notes in Math.*, 1399 (1989), 10–26.

[B]   A. Beauville, Prym varieties and the Schottky problem, *Invent. Math.* 41 (1977), 149–196.

[BL]   Ch. Birkenhake and H. Lange, Complex abelian varieties, *Grundlehren der math. Wiss.*, Springer-Verlag, 1992.

[C]   C. Ciliberto, Alcune applicazioni di un classico procedimento di Castelnuovo, Sem. di variabili Complesse, Univ. di Bologna, 1982–83, 17–43.

[CC]   L. Chiantini and C. Ciliberto, A few remarks on the lifting problem, preprint (1992).

[CGT]   C. Ciliberto, G. van der Geer and M. Teixidor, On the number of parameters of curves whose Jacobians possess nontrivial endomorphisms, *J. Alg. Geom.* 1 (1992), 215–229.

[CG]   C. Ciliberto and G. van der Geer, Subvarieties of the moduli space of curves parametrizing Jacobians with non-trivial endomorphisms, *Am. J. Math.* 114 (1991), 551–570.

[Cl]   H. Clemens, Double solids, *Adv. in Math.* 47 (1983).

[D]   R. Donagi, The tetragonal construction, *Bull. Amer. Math. Soc.* 4 (1981), 181–185.

[De]   O. Debarre, Degree of curves in abelian varieties, preprint (1992).

[DS]   R. Donagi and R. Smith, The structure of the Prym map, *Acta Math.* 146 (1981), 25–102.

[G]   Ph. Griffiths, Periods of integrals over algebraic manifolds, III, *Publ. Math. IHES*, 38 (1970), 125–180.

[K₁]   V. Kanev, Theta divisors of generalized Prym varieties I, *Springer Lecture Notes in Math.* 1124 (1985), 166–215.

[K₂]   V. Kanev, Principal polarizations of Prym-Tjurin varieties, preprint (   ).

[M]   D. Mumford, Prym varieties I, *Contribution to Analysis*, Academic Press, 1974.

[P]   G.P. Pirola, Abel–Jacobi invariant and curves on generic abelian varieties, preprint (1992).

[R]   S. Recillas, Jacobian of curves with a g_4^1 are Prym varieties of trigonal curves, *Bol. Soc. Mat. Mexicana*, 19 (1974), 9–13.

[S]   F. Severi, Sulle corrispondenze fra i punti di una curva algebrica e sopra certe classi di superficie, *Mem. Accad. Sci.* Torino, (2) 54 (1903).

[Se]   E. Sernesi, Neutral linear series, preprint, Erlangen (1988).

[T]   R. Torelli, Sulle serie algebriche semplicemente infinite di gruppi di punti appartenenti a una curva algebrica, *Rend. Circ. Mat. Palermo*, 37 (1914).

[W]   G. Welters, Curves of twice the minimal class on principally polarized abelian varieties, *Indagationes Math.* 49 (1987), 87–109.