C. SOULÉ

Successive minima on arithmetic varieties

Compositio Mathematica, tome 96, nº 1 (1995), p. 85-98
<http://www.numdam.org/item?id=CM_1995__96_1_85_0>

Compositio Mathematica **96:** 85–98, 1995.

Successive minima on arithmetic varieties

C. SOULÉ
CNRS and IHES, 35 Route de Chartres 91440 Bures-sur-Yvette France

Received: 15 November 1993; accepted in final form 9 March 1994

Mumford's theory of stability, when applied to varieties over number fields, has interesting consequences, as was shown in recent years by several authors [12], [5], [13], [3], [16], [20]. In this paper, we use it to get informations on the successive minima of the lattice of sections of bundles on arithmetic varieties.

More precisely, let E be a projective module of rank N over the ring of integers in a number field K, and $E_K^\vee = \mathrm{Hom}(E, K)$. Consider a closed subvariety $X_K \subset \mathbb{P}(E_K^\vee)$ in the projective space of lines in $E_K^\vee$. Fix a hermitian metric on $E \otimes_{\mathbb{Z}} \mathbb{C}$. Bost proved in [3] that Chow semi-stability of X_K in $\mathbb{P}(E_K^\vee)$ implies a lower bound for the height of X_K (see 3.1 below). By a different method we show that the proof that X_K is semi-stable gives, in some cases, a stronger inequality (see however the remark in 3.1.2) which involves the successive minima of E. Our general result, Theorem 1, can be applied to surfaces of general type, Theorem 3, using the work of Gieseker [7], and to line bundles on smooth curves, Theorem 4, using the work of Morrison [14]. A variant of Theorem 1 gives results for rank two stable bundles on curves, Theorem 5, by using the work of Gieseker and Morrison [8]. Finally, we derive another inequality for successive minima on arithmetic surfaces, Theorem 6, from the vanishing theorem proved in [16].

I thank J.-B. Bost, J.-F. Burnol, I. Morrison, I. Reider, E. Ullmo and S. Zhang for helpful discussions, and the referee for pointing out several inaccuracies.

1. Preliminaries

1.1. Let M be a free $\mathbb{Z}$-module of finite rank and $\|\cdot\|$ a norm on the complex vector space $M \otimes_{\mathbb{Z}} \mathbb{C}$. We equip $M \otimes_{\mathbb{Z}} \mathbb{R}$ with the Haar measure for which the unit ball has volume equal to the volume of the standard euclidean ball of the same dimension and we let $\mathrm{covol}(M \otimes_{\mathbb{Z}} \mathbb{R}/M)$ be the covolume of M in $M \otimes_{\mathbb{Z}} \mathbb{R}$ for that measure. We then define the Euler characteristic of $(M, \|\cdot\|)$ to be the real number $\chi(M, \|\cdot\|) = -\log \mathrm{covol}(M \otimes_{\mathbb{Z}} \mathbb{R}/M)$.

Clearly, if $\|\cdot\|'$ is another norm on $M \otimes_{\mathbb{Z}} \mathbb{C}$ such that $\|x\| \leqslant \|x\|'$ for all $x \in M \otimes_{\mathbb{Z}} \mathbb{C}$, we have

$$\chi(M, \|\cdot\|') \leqslant \chi(M, \|\cdot\|).$$

1.2. Let K be a number field, of degree $[K : \mathbb{Q}]$, let $\mathcal{O}_K$ be its ring of integers, let $S = \mathrm{Spec}(\mathcal{O}_K)$ be the associated scheme, let Σ be the set of complex embeddings of K and let D_K be the discriminant of K over $\mathbb{Q}$. These notations will be valid throughout this paper.

If M is a torsion free $\mathcal{O}_K$-module of finite rank such that, for all $\sigma \in \Sigma$, the corresponding complex vector space $M_\sigma = M \otimes_{\mathcal{O}_K} \mathbb{C}$ is equipped with a norm $|\cdot|_\sigma$, we may think of M as a free $\mathbb{Z}$-module equipped with the norm $|\cdot|$ on $M \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_\sigma M_\sigma$ defined by

$$\left|\sum_\sigma x_\sigma\right| = \operatorname*{Sup}_\sigma |x_\sigma|_\sigma .$$

In particular, consider an hermitian vector bundle (E, h) over S, in the sense of [9]. In other words, E is a torsion free $\mathcal{O}_K$-module of finite rank and, for all $\sigma \in \Sigma$, E_σ is equipped with an hermitian scalar product h, compatible with the isomorphism $E_\sigma \simeq E_{\bar\sigma}$ induced by complex conjugation. We will then denote by $\|\cdot\|_\sigma$ the associated norm on E_σ and $\|\cdot\|$ the norm on $E \otimes_{\mathbb{Z}} \mathbb{C}$ defined as above. Also we let $\widehat{\deg}(E, h) \in \mathbb{R}$ be the *arithmetic degree* of (E, h), which can be computed as follows. Let N be the rank of E and $\Lambda^N E$ its top exterior power. We equip $\Lambda^N E$ with the metric induced by h : if $v_1, \dots, v_N$, $w_1, \dots, w_N$ lie in E_σ

$$\Lambda^N h(v_1 \wedge \cdots \wedge v_N, w_1 \wedge \cdots \wedge w_N) = \det(h(v_i, w_j), 1 \leqslant i, j \leqslant N).$$

We have then

$$\widehat{\deg}(E, h) = \widehat{\deg}(\Lambda^N E, \Lambda^N h).$$

On the other hand, if (L, h) is an hermitian line bundle on S, and if $s \in L$ is any nontrivial section, denote by $[L : \mathcal{O}_K s]$ the index in L of the submodule generated by s. We have

$$\widehat{\deg}(L, h) = \log[L : \mathcal{O}_K s] - \sum_{\sigma\in\Sigma} \log\|s\|_\sigma .$$

1.3.

LEMMA 1. *Let $\varphi : E \to M$ be a morphism of torsion free $\mathcal{O}_K$-modules of finite type, h a hermitian metric on E, with associated norm $\|\cdot\|_\sigma$ on E_σ, and $|\cdot|_\sigma$ a norm on each M_σ, $\sigma \in \Sigma$. We assume that, for all $x \in E_\sigma$, $|\varphi(x)|_\sigma \leqslant \|x\|_\sigma$. If $x_1, \dots, x_N$ in E are such that $\varphi(x_1), \dots, \varphi(x_N)$ is a basis of $M \otimes_{\mathcal{O}_K} K$, the following inequality holds*:

$$\chi(M, |\cdot|) \geqslant -[K : \mathbb{Q}] \sum_{i=1}^N \log\|x_i\| - \frac{N}{2} \log|D_K|.$$

Proof. Let $\| \cdot \|'_\sigma$ be the norm induced from $\| \cdot \|_\sigma$ by the projection map $E_\sigma \to M_\sigma$. Since $| \cdot |_\sigma \leqslant \| \cdot \|'_\sigma$ we deduce from 1.1 that $\chi(M, | \cdot |) \geqslant \chi(M, \| \cdot \|')$, so we may assume that $| \cdot |_\sigma = \| \cdot \|'_\sigma$. Furthermore, $\| \cdot \|'_\sigma$ is the norm coming from the hermitian metric h' induced by E_σ on M_σ, and $\|\varphi(x_i)\|' \leqslant \|x_i\|$, so we may assume that $(M, \| \cdot \|') = (E, \| \cdot \|)$.

We know that

$$\widehat{\deg}(E, h) = \chi(E, \| \cdot \|) + \frac{N}{2}\log|D_K|$$

(e.g. [4], (2.1.13)). The element $s = x_1 \wedge \cdots \wedge x_N$ of $\Lambda^N E$ is nonzero, so we get, by Hadamard inequality,

$$\widehat{\deg}(E, h) = \log[\Lambda^N E : \mathcal{O}_K s] - \sum_{\sigma \in \Sigma} \log\|x_1 \wedge \cdots \wedge x_N\|_\sigma \geqslant$$

$$-[K : \mathbb{Q}] \sum_{i=1}^{N} \log\|x_i\|.$$

The Lemma 1 follows from this.

1.4. Let (E, h) be a hermitian vector bundle of rank N over S. For any integer $i \leqslant N$ we let λ_i be the infimum of the set of real numbers λ such that there exist $v_1, \ldots, v_i$ in E, linearly independent over K and such that $\|v_\alpha\| \leqslant \lambda$, $1 \leqslant \alpha \leqslant i$. These are the *successive minima* of (E, h). We can choose $x_1, \ldots, x_N \in E$, linearly independent over K, such that $\|x_i\| = \lambda_{N-i+1}$.

If we let

$$\mu_i = \log \lambda_i, \quad 1 \leqslant i \leqslant N,$$

and

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \mu_i,$$

it follows from Bombieri–Vaaler's version of Minkowski's theorem on successive minima that

$$-N\mu[K : \mathbb{Q}] \leqslant \widehat{\deg}(E, h) \leqslant C(N, K) - N\mu[K : \mathbb{Q}], \tag{1}$$

where $C(N, K)$ is the following constant

$$\begin{aligned} C(N, K) &= N(r_1 + r_2)\log(2) \\ &\quad + N(\log|D_K|)/2 - r_1 \log V_N - r_2 \log V_{2N}, \end{aligned}$$

where r_1 and r_2 are the number of real and complex places of K, and V_n is the standard euclidean volume of the unit ball in $\mathbb{R}^n$ (see [4], 5.2.3).

1.5. The $\mathcal{O}_K$-module $\omega_S = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z})$ is locally free of rank one. We fix an hermitian metric on ω_S by deciding that the trace morphism $\mathrm{Tr} \in \omega_S$ has norm $|\mathrm{Tr}|_\sigma = 1$ (resp. $|\mathrm{Tr}|_\sigma = 2$), if $\sigma = \bar{\sigma}$ (resp. $\sigma \neq \bar{\sigma}$).

Let (E, h) be a hermitian vector bundle of rank N on S. We denote by $E^\vee = \mathrm{Hom}(E, \mathcal{O}_S)$ its dual and we equip $E' = E^\vee \otimes_{\mathcal{O}_S} \omega_S$ with the tensor product of the metric dual to h on $E^\vee$ with the chosen metric on ω_S. If $x = \Sigma_{\sigma\in\Sigma} x_\sigma$ lies in $E' \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_\sigma E'_\sigma$ we let

$$\|x\|' = \sum_{\sigma\in\Sigma} \|x_\sigma\|.$$

The $\mathbb{Z}$-modules underlying E and E', equipped with the norms $\|\cdot\|$ and $\|\cdot\|'$, are then dual to each other (see e.g. [10], 2.4.2). Let λ_i be the successive minima of $(E, \|\cdot\|)$ and λ'_i those of $(E', \|\cdot\|')$, $1 \leqslant i \leqslant n = \mathrm{rk}_{\mathbb{Z}}(E) = [K : \mathbb{Q}]N$. In other words, λ_i is the infimum of the real numbers $\lambda \geqslant 0$ such that there exist $v_1, \dots, v_i \in E$, linearly independent over $\mathbb{Z}$, with $\|v_\alpha\| \leqslant \lambda$ for all $\alpha \leqslant i$.

From [1] Theorem 2.1 and John theorem, as in op. cit. Section 3, we get the inequalities

$$\lambda_i \lambda'_{n+1-i} \leqslant n^{3/2}, \quad i = 1, \dots, n. \tag{2}$$

2. The main result

2.1. Let (E, h) and $x_1, \dots, x_N$ be as in 1.4 above, let $E^\vee = \mathrm{Hom}(E, \mathcal{O}_S)$ be the dual of E, and let $\mathbb{P}(E^\vee)$ be the associated projective space (representing lines in $E^\vee$).

Consider a closed subvariety $X_K \subset \mathbb{P}(E_K^\vee)$ of dimension d over K. We let $\deg(X_K) \in \mathbb{N}$ be its (algebraic) degree and $h(X_K) \in \mathbb{R}$ its Faltings height, denoted $h_F(X_K)$ in [4], (3.1.1) and (3.1.5). If $\overline{\mathcal{O}(1)}$ is the canonical line bundle on $\mathbb{P}(E^\vee)$ equipped with the metric induced from h, and if X is the Zariski closure of X_K in $\mathbb{P}(E^\vee)$, we have from [4], loc. cit.,

$$h(X_K) = \widehat{\deg}\left(\hat{c}_1(\overline{\mathcal{O}(1)})^{d+1} | X\right) \in \mathbb{R}.$$

When m is large enough, $m \geqslant m_0$ say, the cup-product map

$$\varphi : E_K^{\otimes m} \to H^0(X_K, \mathcal{O}(m))$$

is surjective, so that $H^0(X_K, \mathcal{O}(m))$ is generated by the monomials

$$x_1^{\alpha_1} \cdots x_N^{\alpha_N} = \varphi(x_1^{\otimes\alpha_1} \otimes \cdots \otimes x_N^{\otimes\alpha_N}),$$

$\alpha_1 + \cdots + \alpha_N = m$. A *special basis* is a basis of $H^0(X_K, \mathcal{O}(m))$ made of such elements.

Assume N real numbers $r_1, \ldots, r_N$ are given, and let $\mathbf{r} = (r_1, \ldots, r_N)$. We define the weight of x_i to be r_i, $1 \leqslant i \leqslant N$, the weight of a monomial in $E_K^{\otimes m}$ to be the sum of the weight of the x_i's occurring in it, and the weight of a monomial $u \in H^0(X_K, \mathcal{O}(m))$ to be the minimum $\mathrm{wt}_{\mathbf{r}}(u)$ of the weights of the monomials in the x_i's mapping to u by φ. The weight $\mathrm{wt}_{\mathbf{r}}(\mathcal{B})$ of a special basis $\mathcal{B}$ is the sum of the weights of its elements, and $w_{\mathbf{r}}(m)$ is the minimum weight of a special basis of $H^0(X_K, \mathcal{O}(m))$.

When $r_1, \ldots, r_N \in \mathbb{N}$, there is a natural integer $e_{\mathbf{r}}$ such that, as m goes to infinity,

$$w_{\mathbf{r}}(m) = e_{\mathbf{r}} \frac{m^{d+1}}{(d+1)!} + O(m^d)$$

([14], Corollary 3.3).

THEOREM 1. *Assume there exists a continuous function* $\psi : \mathbb{R}^N \to \mathbb{R}$ *such that* $\psi(tx) = t\psi(x)$ *for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^N$, *and such that* $e_{\mathbf{r}} \leqslant \psi(\mathbf{r})$ *when* $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_N = 0$ *are integers. Then the following inequality holds*:

$$\frac{h(X_K)}{[K:\mathbb{Q}]} + (d+1)\deg(X_K)\mu_1 + \psi(\mu_N - \mu_1, \mu_{N-1} - \mu_1, \ldots, \mu_2 - \mu_1, 0) \geqslant 0. \tag{3}$$

2.2. Our first step to prove Theorem 1 is the following. Fix real numbers $\varepsilon > 0$ and $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_N = 0$. Then there exists a constant C such that, for any positive integer $m \geqslant m_0$,

$$w_{\mathbf{r}}(m) \leqslant (\psi(\mathbf{r}) + \varepsilon) \frac{m^{d+1}}{(d+1)!} + Cm^d. \tag{4}$$

Indeed we may choose a positive real number $\eta > 0$ and rational numbers $s_i = p_i/q$ with $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_N = 0$, $|s_i - r_i| < \eta$, and $\psi(\mathbf{s}) \leqslant \psi(\mathbf{r}) + \varepsilon/2$.

If $m \geqslant m_0$ and if $\mathcal{B}$ is any special basis of $H^0(X_K, \mathcal{O}(m))$ we have

$$\sum_{u \in \mathcal{B}} \mathrm{wt}_{\mathbf{r}}(u) \leqslant \sum_{u \in \mathcal{B}} \mathrm{wt}_{\mathbf{s}}(u) + m\eta \,\mathrm{card}(\mathcal{B}).$$

By the usual theory of Hilbert polynomials,

$$h^0(X_K, \mathcal{O}(m)) = \deg(X_K) \frac{m^d}{d!} + O(m^{d-1}). \tag{5}$$

Therefore

$$w_{\mathbf{r}}(m) \leqslant w_{\mathbf{s}}(m) + \deg(X_K)\frac{m^{d+1}}{d!}\eta + O(m^d).$$

Since $w_{\mathbf{s}}(m) = w_{\mathbf{p}}(m)/q$ and $\psi(\mathbf{s}) = \psi(\mathbf{p})/q$, we get from our hypothesis on ψ the inequality

$$w_{\mathbf{s}}(m) \leqslant \psi(\mathbf{s})\frac{m^{d+1}}{(d+1)!} + O(m^d),$$

hence

$$w_{\mathbf{r}}(m) \leqslant \left(\psi(\mathbf{r}) + \frac{\varepsilon}{2} + (d+1)\deg(X_K)\eta\right)\frac{m^{d+1}}{(d+1)!} + O(m^d).$$

If η is small enough this means that

$$w_{\mathbf{r}}(m) \leqslant (\psi(\mathbf{r}) + \varepsilon)\frac{m^{d+1}}{(d+1)!} + O(m^d),$$

i.e. (4) holds.

2.3. Given $m \geqslant m_0$ we let $M = H^0(X, \mathcal{O}(m))$. If $\sigma \in \Sigma$, denote by X_σ the corresponding set of complex points of X_K. We equip $M_\sigma = H^0(X_\sigma, \mathcal{O}(m))$ with the sup norm on X_σ:

$$|s|_\sigma = \operatorname*{Sup}_{x \in X_\sigma} \|s(x)\|_\sigma,$$

where $\|\cdot\|_\sigma$ is the norm on $\mathcal{O}(1)^{\otimes m}$ induced by E. The morphism

$$\varphi : E^{\otimes m} \to M$$

is then norm decreasing. If $u = \varphi(x_1^{\otimes\alpha_1} \otimes \cdots \otimes x_N^{\otimes\alpha_N})$ is a monomial, we have

$$|u| \leqslant \|x_1^{\otimes\alpha_1} \otimes \cdots \otimes x_N^{\otimes\alpha_N}\| \leqslant \prod_{i=1}^{N} \|x_i\|^{\alpha_i}.$$

Let

$$r_i = \mu_{N-i+1} - \mu_1, \quad 1 \leqslant i \leqslant N.$$

Then $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_N = 0$ and the previous inequalities imply

$$\log|u| \leqslant \log\|x\| \leqslant \sum_{i=1}^{N} \alpha_i \log\|x_i\| = \sum_{i=1}^{N} \alpha_i r_i + m\mu_1,$$

where $x = x_1^{\otimes \alpha_1} \otimes \cdots \otimes x_N^{\otimes \alpha_N} \in E^{\otimes m}$.

By definition of $\mathrm{wt}_{\mathbf{r}}(u)$, for any $\varepsilon' > 0$ we may find x with $\varphi(x) = u$ and

$$\log\|x\| \leqslant \mathrm{wt}_{\mathbf{r}}(u) + \varepsilon' + m\mu_1.$$

Applying Lemma 1 we conclude from this that for any special basis $\mathcal{B}$ of M

$$\chi(M, |\cdot|) \geqslant -[K : \mathbb{Q}] \sum_{u \in \mathcal{B}} (\mathrm{wt}_{\mathbf{r}}(u) + \varepsilon' + m\mu_1) - h^0(X_K, \mathcal{O}(m)) \frac{\log|D_K|}{2},$$

hence

$$\chi(M, |\cdot|) \geqslant -[K : \mathbb{Q}](w_{\mathbf{r}}(m) + mh^0(X_K, \mathcal{O}(m))\mu_1) + O(m^d).$$

Using (4) and (5) we deduce that

$$\chi(M, |\cdot|) \geqslant -[K : \mathbb{Q}](\psi(\mathbf{r}) + \varepsilon + (d+1)\deg(X_K)\mu_1) \frac{m^{d+1}}{(d+1)!} + O(m^d). \quad (6)$$

On the other hand, by a result of Zhang, [19] Theorem 1.4, we have

$$\chi(M, |\cdot|) = h(X_K) \frac{m^{d+1}}{(d+1)!} + o(m^{d+1}).$$

Comparing with (6) for all $\varepsilon > 0$, we get the inequality (3).

3. Applications

3.1. Chow semi-stability

3.1.1. We keep the notations of Section 2.1 and denote by $X_{\bar{K}} = X_K \otimes_K \bar{K}$ the projective variety obtained from X_K by extending scalars from K to an algebraic closure $\bar{K}$. Let $E_{\bar{K}}^{\vee} = E^{\vee} \otimes_{\mathcal{O}_K} \bar{K}$.

THEOREM 2. *Assume that the projective variety $X_{\bar{K}} \subset \mathbb{P}(E_{\bar{K}}^{\vee})$ is Chow semi-stable. Then*

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + (d+1)\deg(X_K)\mu \geqslant 0. \quad (7)$$

Proof. Let V_i be the subspace of $E_{\bar{K}}$ generated by $x_1, \ldots, x_i$, $1 \leqslant i \leqslant N$. If $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_N = 0$ are integers, it follows from Mumford's criterion for semi-stability, [15] Theorem 2.9 applied to the weighted flag (V_i, r_i) and from [14] Corollary 3.3 that

$$e_{\mathbf{r}} \leqslant (d+1)\deg(X_K)\left(\sum_{i=1}^{N} r_i/N\right).$$

Therefore we may apply Theorem 1 with

$$\psi(\mathbf{r}) = (d+1)\deg(X_K)\left(\sum_{i=1}^{N} r_i/N\right).$$

We get

$$\frac{h(X_K)}{[K:\mathbb{Q}]} + (d+1)\deg(X_K)\left(\left(\sum_{i=1}^{N}(\mu_i-\mu_1)/N\right)+\mu_1\right) \geqslant 0,$$

i.e. (7) holds.

3.1.2. Using (1) we deduce from Theorem 2 the following

COROLLARY. Under the assumptions of Theorem 2,

$$\begin{aligned} & h(X_K) - (d+1)\deg(X_K)\widehat{\deg}(E, h)/N \\ & \quad \geqslant -(d+1)\deg(X_K)C(N, K)/N. \end{aligned}$$

This inequality is Bost's Theorem 1 in [3], except that the constant on the right hand side of this inequality is different from the one in loc. cit. (which is a constant multiple of $[K:\mathbb{Q}]$). In order to get a constant multiple of $[K:\mathbb{Q}]$ one could try to replace the successive minima μ_i, $1 \leqslant i \leqslant N$, by the slopes of the canonical polygon of Stuhler [17] and Grayson [11]. It is mentioned in [3] 4.3 that the inequality of loc. cit. can be applied to stable bundles on curves, surfaces of general type and abelian varieties.

3.2. Surfaces of general type

Let Y be a smooth surface of general type defined over K and $n \geqslant 5$ a fixed integer. The nth power L of the canonical line bundle on Y has then no base point [2]. With the notations of 2.1, we assume that $E_K = H^0(Y, L)$ and that X_K is the image of the morphism $Y \to \mathbb{P}(H^0(Y, L)^\vee)$.

THEOREM 3. *If n is big enough, the following inequality holds*:

$$\frac{h(X_K)}{[K:\mathbb{Q}]} + 3\deg(X_K)\mu \geqslant \deg(X_K)(\mu_N - \mu_1)/N.$$

Proof. This result follows from Theorem 1 and Gieseker's work [7]. Indeed, let $r(1) \leqslant r(2) \leqslant \cdots \leqslant r(N)$ be relative integers such that $\Sigma_{i=1}^{N} r(i) = 0$. Denote by $c_1(Y)^2$ the (algebraic) self intersection of the canonical line bundle on Y. Let $p \geqslant 1$ and $M \gg 0$ be integers and $m = M(p+1)$. Then, according to [7], Lemma 6.6, Lemma 5.15, Definition 5.3 and Section 2, the vector space $H^0(X_K, L^{\otimes m})$ has a distinguished basis of weight at most

$$\frac{M^3p^3}{2}\left(r(N)c_1(Y)^2n^2 - (Nr(N) + \tfrac{1}{3}(r(N) - r(1)))\frac{n^2}{N}c_1(Y)^2\right)$$

$$+o(M^3p^3)$$

$$= -\frac{M^3p^3}{6}\deg(X_K)(r(N) - r(1))/N + o(M^3p^3)$$

with respect to $(r(1), \ldots, r(N))$, as M goes to infinity. If $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_N = 0$ are integers, we let $r(i) = r_{N-i+1} - (\Sigma_{i=1}^{N} r_i/N)$. We get $e_{\mathbf{r}} \leqslant \psi(\mathbf{r})$ with

$$\psi(\mathbf{r}) = \left(-r_1 + 3\sum_{i=1}^{N} r_i\right)\deg(X_K)/N,$$

hence Theorem 3 follows from Theorem 1.

4. Smooth curves

4.1. We keep the notations of Section 2.1.

THEOREM 4. *Assume that $X_K \subset \mathbb{P}(E_K^\vee)$ is a smooth geometrically irreducible curve of genus g and degree $d_0 = \deg(X_K) \geqslant 2g+1$. Then the following inequality holds when $E_K = H^0(X_K, \mathcal{O}(1))$:*

$$\frac{h(X_K)}{[K:\mathbb{Q}]} + 2d_0\mu \geqslant \frac{2d_0 g(d_0 - 2g)}{d_0^2 + d_0 - 2g^2}(\mu - \mu_1). \tag{8}$$

Proof. By a result of Morrison, [14] Theorem 4.4, the hypotheses of Theorem 1 are satisfied with

$$\psi(\mathbf{r}) = \frac{2d_0^2}{d_0^2 + d_0 - 2g^2}\left(\sum_{i=1}^{N} r_i\right)$$

(as noticed by the referee, the computation in [14], loc. cit., is not correct; the constant above is what comes out instead). Since $N = h^0(X_K, \mathcal{O}(1)) = d_0 + 1 - g$, we get from (3) the inequality

$$\frac{h(X_K)}{[K:\mathbb{Q}]} + 2d_0\mu_1 + \frac{2d_0^2(d_0 + 1 - g)}{d_0^2 + d_0 - 2g^2}(\mu - \mu_1) \geqslant 0,$$

i.e. (8) holds.

REMARK. Another way to prove (8), which does not use Zhang's result [19] Theorem 1.4, consists in comparing the height of X_K with the height of its projections to $\mathbb{P}(V_i^\vee)$, where $V_i \subset E_K$ is the subspace generated by $x_1, \ldots, x_i$, $1 \leqslant i \leqslant N$. One may then combine [4] 3.3.2 with Morrison's combinatorial results, [14] Corollary 4.3 and Theorem 4.4, to obtain the inequality (8).

4.2. We shall now consider vector bundles of rank two on curves. Let X_K be a smooth geometrically irreducible curve of genus $g \geqslant 2$ over K, let F be a rank two vector bundle on X_K of degree d_0 (big enough with respect to g), let $L = \Lambda^2 F$ be the second exterior power of F, and let $F_{\bar{K}}$ be its restriction to $X_{\bar{K}}$. Assume (E, h) is an hermitian vector bundle on S such that $E_K = H^0(X_K, F)$. According to [8], Lemma 3.2, the map

$$\psi : \Lambda^2 H^0(X_K, F) \to H^0(X_K, L)$$

is surjective. Therefore the lattice $E' = \psi(\Lambda^2 E)$ is such that $E'_K = H^0(X_K, L)$, and we let h' be the metric induced by h on E'. We let $h(X_K)$ be the height of X_K for the projective embedding $X_K \subset \mathbb{P}(H^0(X_K, L)^\vee)$, with respect to (E', h'). Denote by $\lambda_1, \ldots, \lambda_N$ the successive minima of (E, h), $N = h^0(X_K, F) = d_0 + 2 - 2g$, $\mu_i = \log\lambda_i$, $1 \leqslant i \leqslant N$, and

$$\mu = \frac{1}{N}\sum_{i=1}^{N} \mu_i.$$

THEOREM 5. *There exists a positive constant $a(g, d_0)$ and an integer D such that if $d_0 > D$ and the bundle $F_{\bar{K}}$ is stable the following inequality holds*

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 4d_0\mu \geqslant a(g, d_0)(\mu - \mu_1);$$

furthermore, if $d_0 > D$ and $F_{\bar{K}}$ is semi-stable, then

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 4d_0\mu \geqslant 0.$$

Proof. Theorem 5 follows from [8] by a method similar to Theorem 1. Choose $x_1, \ldots, x_N \in E$, linearly independent over K, such that $\|x_i\| = \lambda_{N-i+1}$. Consider the morphism

$$\varphi : (\Lambda^2 E)^{\otimes m} \to M = H^0(X, \mathcal{O}(m)),$$

where X is the Zariski closure of X_K in $\mathbb{P}(E'^\vee)$, obtained by cup-product from the canonical morphism

$$\Lambda^2 E \to E' \to H^0(X, \mathcal{O}(1)).$$

When m is big enough, the image of φ has maximal rank over K. Given a set of N real numbers $\mathbf{r} = (r_1, \ldots, r_N)$, we define the weight of $y_{ij} = x_i \wedge x_j \in \Lambda^2 E$ to be $r_i + r_j$, $1 \leqslant i \neq j \leqslant N$. The weight of a monomial $y_{i_1 j_1} \otimes y_{i_2 j_2} \otimes \cdots \otimes y_{i_m j_m} \in (\Lambda^2 E)^{\otimes m}$ is the sum of the weights of its factors, a special basis $\mathcal{B}$ of $H^0(X_K, \mathcal{O}(m))$ is a basis made of the images by φ of some of these monomials. We define its weight $\mathrm{wt}_{\mathbf{r}}(\mathcal{B})$ as in 2.1, and $w_{\mathbf{r}}(m)$ is the minimum weight of a special basis of $H^0(X_K, \mathcal{O}(m))$. When m goes to infinity

$$w_{\mathbf{r}}(m) = e_{\mathbf{r}} \frac{m^2}{2} + O(m). \tag{9}$$

From the proof of [8], Theorem 5.1, it follows that, if $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_N = 0$ are rational numbers such that $r_1 + r_2 + \cdots + r_N = 1$ and if $F_{\bar{K}}$ is stable (resp. semi-stable) and d_0 is big enough, we have

$$e_{\mathbf{r}} \leqslant (4d_0 - a(g, d_0))/N$$

(resp. $e_{\mathbf{r}} \leqslant 4d_0/N$) for some positive constant $a(g, d_0)$. As in 2.2, we deduce from this that if $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_N = 0$ are real numbers, then

$$w_{\mathbf{r}}(m) \leqslant (\psi(\mathbf{r}) + \varepsilon) \frac{m^2}{2} + Cm,$$

with

$$\psi(\mathbf{r}) = \frac{4d_0 - a(g, d_0)}{N} \left(\sum_{i=1}^{N} r_i \right),$$

$$\left(\text{resp.} \quad \psi(\mathbf{r}) = \frac{4d_0}{N} \left(\sum_{i=1}^{N} r_i \right) \right).$$

If we equip $M = H^0(X, \mathcal{O}(m))$ with the sup-norm coming from the metric induced by E' on L, and if $u = \varphi(y_{i_1 j_1} \otimes y_{i_2 j_2} \otimes \cdots \otimes y_{i_m j_m})$ is a decomposable element, we have

$$\begin{aligned} |u| &\leqslant \|y_{i_1 j_1}\| \, \|y_{i_2 j_2}\| \cdots \|y_{i_m j_m}\| \\ &\leqslant \|x_{i_1}\| \, \|x_{j_1}\| \, \|x_{i_2}\| \cdots \|x_{i_m}\| \, \|x_{j_m}\|. \end{aligned}$$

If we let $r_i = \mu_{N-i+1} - \mu_1$, $1 \leqslant i \leqslant N$, it follows that

$$\log|u| \leqslant \mathrm{wt}_{\mathbf{r}}(u) + 2m\mu_1.$$

Therefore, using Lemma 1 as in 2.3, we get

$$\chi(M,\, |\cdot|) \geqslant -[K:\mathbb{Q}](w_{\mathbf{r}}(m) + 2m\, h^0(X_K,\, \mathcal{O}(m))\mu_1) + O(m). \tag{10}$$

Since

$$h^0(X_K,\, \mathcal{O}(m)) = d_0 m + O(1),$$

it follows from (9), (10) and [19] Theorem 1.4 as in 2.3, that

$$\frac{h(X_K)}{[K:\mathbb{Q}]} + 4d_0\mu_1 + (4d_0 - a(g,\, d_0))(\mu - \mu_1) \geqslant 0$$

if $F_{\bar{K}}$ is stable, and

$$\frac{h(X_K)}{[K:\mathbb{Q}]} + 4d_0\mu \geqslant 0$$

if $F_{\bar{K}}$ is semi-stable. This proves Theorem 5.

REMARK. From the proof of [8] Theorem 5.1, one can derive the following estimate:

$$a(g,\, d_0) \geqslant 0.8.$$

4.3. The vanishing theorem of [16] provides more information on the successive minima of sections of line bundles on curves. Namely, let $f : X \to S$ be a semi-stable curve over S, with geometrically irreducible generic fiber X_K. Consider a line bundle L on X of degree $m \geqslant 2$ on X_K. Choose an hermitian metric h on L with positive first Chern form $c_1(L,\, h)$.

We assume that the arithmetic degree of $\bar{L} = (L,\, h)$ on any irreducible divisor of X is nonnegative, and we let $\bar{L}^2 \in \mathbb{R}$ be the arithmetic self-intersection $\hat{c}_1(\bar{L})^2$ of the first Chern class of $\bar{L}$.

We equip the tangent space of $X(\mathbb{C})$ with the metric whose associated (normalized) Kähler form is $c_1(L,\, h)/m$, and the relative dualizing sheaf $\omega_{X/S}$ with the dual metric.

The $\mathcal{O}_K$-module $E' = H^0(X,\, L \otimes \omega_{X/S})$ is then equipped with the L^2-metric. If $x = \Sigma_{\sigma\in\Sigma} x_\sigma$ lies in $E' \otimes_{\mathbb{Z}} \mathbb{C}$ we let $\|x\|' = \Sigma_{\sigma\in\Sigma} \|x_\sigma\|_{L^2}$. Let $n = [K:\mathbb{Q}]h^0(X_K,\, L \otimes \omega_{X/S})$ be the rank of E' over $\mathbb{Z}$, λ'_n the top successive minimum of $(E',\, \|\cdot\|')$ and $\mu'_n = \log \lambda'_n$.

THEOREM 5.

(a) *Under the above assumptions, the following inequality holds*

$$\mu'_n \leqslant -\frac{\bar{L}^2}{m^2[K:\mathbb{Q}]} + \frac{\log|D_K|}{[K:\mathbb{Q}]} + 1 + \tfrac{3}{2}\log(n). \tag{11}$$

(b) *Assume furthermore that X_K has genus $g \geqslant 2$, that $\omega_{X/S}$ is equipped with the Arakelov metric, and that $\bar{L}$ is the k-th power of $\bar{\omega}_{X/S}$, $k \geqslant 1$. Then*

$$\mu_n' \leqslant -\frac{(k+1)\bar{\omega}_{X/S}^2}{4g(g-1)[K:\mathbb{Q}]} + \frac{\log|D_K|}{[K:\mathbb{Q}]} + 1 + \tfrac{3}{2}\log(n). \tag{12}$$

Proof. By Serre duality, if we let L^{-1} be the dual of L, the quotient of the $\mathcal{O}_K$-module $H^1(X, L^{-1})$ by its torsion subgroup, when equipped with the L^2-metric, is the dual of $H^0(X, L \otimes \omega_{X/S})$ over S. Let ω_S be as in 1.5 above, let λ_1 be the smallest norm $||v|| = \operatorname{Sup}_\sigma ||v_\sigma||_{L^2}$ of nonzero vectors v in

$$E = (H^1(X, L^{-1}) \otimes \omega_S)/\text{torsion} = H^1(X, L^{-1} \otimes f^*\omega_S)/\text{torsion},$$

and let $\mu_1 = \log \lambda_1$. From (2) we know that

$$\mu_n' \leqslant -\mu_1 + \tfrac{3}{2}\log(n). \tag{13}$$

Let $M = L \otimes f^*\omega_S^{-1}$ be equipped with the tensor product of the chosen metrics. Using [10], p. 355, we compute

$$\begin{aligned} \hat{c}_1(\bar{M})^2 &= \hat{c}_1(\bar{L})^2 - 2\hat{c}_1(\bar{L})\hat{c}_1(f^*\bar{\omega}_S) \\ &= \bar{L}^2 - 2m\,\widehat{\deg}(\bar{\omega}_S), \end{aligned}$$

where

$$\widehat{\deg}(\bar{\omega}_S) = \log|D_K| - 2r_2\log(2) \leqslant \log|D_K|$$

is the arithmetic degree of $\bar{\omega}_S$.

Similarly, let $P \in X(\bar{K})$ be an algebraic point on X_K, defined on a finite extension K' of K, and $u\colon \operatorname{Spec}(\mathcal{O}_{K'}) \to X$ the morphism defined by P. The normalized height of P with respect to $\bar{M}$ is then

$$\frac{\widehat{\deg}(u^*\bar{M})}{[K':K]} = \frac{\widehat{\deg}(u^*\bar{L})}{[K':K]} - \widehat{\deg}(\bar{\omega}_S).$$

From our hypotheses on $\bar{L}$ we get

$$\frac{\widehat{\deg}(u^*\bar{M})}{[K':K]} \geqslant -\widehat{\deg}(\bar{\omega}_S).$$

In case (a), we may then apply [16] Theorem 2 to get

$$\begin{aligned} [K:\mathbb{Q}]m^2(\mu_1+1) &\geqslant \hat{c}_1(\bar{M})^2 + (m^2-2m)e(\bar{M}) \\ &\geqslant \bar{L}^2 - m^2\,\widehat{\deg}(\bar{\omega}_S). \end{aligned} \tag{14}$$

The inequality (11) follows from (13) and (14). Similarly, in case (b), we get as in [16] Theorem 3 that

$$[K:\mathbb{Q}](\mu_1+1) \geqslant \frac{(k+1)\bar{\omega}^2_{X/S}}{4g(g-1)} - \widehat{\deg}(\bar{\omega}_S), \tag{15}$$

and (12) follows from (13) and (15).

REMARK. Since n is an affine function of k, Theorem 5(b) implies that λ'_n goes to zero as k goes to infinity. As was noticed by Ullmo, this proves that, if $k \geqslant k_0$, the lattice $H^0(X, \omega_{X/S}^{\otimes k+1})$ contains a set of sections of L^2-norm less than one which has maximal rank. This also follows from Zhang's result [18] Theorem 1.5, but this proof is effective in the sense that k_0 can be evaluated from (12).

References

1. W. Banaszczyk: New bounds in some transference theorems in the geometry of numbers, *Math. Annalen*, 296 (1993), 625–635.
2. E. Bombieri: Canonical models of surfaces of general type, *Publ. Math. IHES*, 42 (1973), 171–219.
3. J.-B. Bost: Semi-stability and heights of cycles, *Inventiones Math.*, 118 (1994), 223–253.
4. J.-B. Bost, H. Gillet, C. Soulé: Heights of projective varieties and positive Green forms, *Journal of the AMS*, 7 (1994), 903–1027.
5. J.-F. Burnol: Remarques sur la stabilité en arithmétique, International Mathematics Research Notices, *Duke Math. J.*, 6 (1992), 117–127.
6. G. Faltings: Calculus on arithmetic surfaces, *Annals of Math.*, 119 (1984), 387–424.
7. D. Gieseker: Global moduli for surfaces of general type, *Inventiones Math.*, 43 (1977), 233–282.
8. D. Gieseker and I. Morrison: Hilbert stability of rank-two bundles on curves, *J. Differential geometry*, 19 (1984), 1–29.
9. H. Gillet and C. Soulé: Characteristic classes for algebraic vector bundles with Hermitian metrics, I, II, *Annals of Math.*, 131 (1990), 163–203 and 205–238.
10. H. Gillet and C. Soulé: On the number of lattice points in convex symmetric bodies and their duals, *Israel Journal of Math.*, 74, no 2–3 (1991), 347–357.
11. D. Grayson: Reduction theory using semistability, *Comment. Math. Helvetici*, 59 (1984), 600–634.
12. Y. Miyaoka: Bogomolov inequality on arithmetic surfaces, talk at the Oberwolfach conference on "Arithmetical Algebraic Geometry", G. Harder and N. Katz org. (1988).
13. A. Moriwaki: Inequality of Bogomolov–Gieseker's type on arithmetic surfaces, *Duke Math. Journal*, 74 (1994), 713–761.
14. I. Morrison: Projective stability of ruled surfaces, *Inventiones Math.*, 56 (1980), 269–304.
15. D. Mumford: Stability of projective varieties, *l'Ens. Math.*, 23 (1977), 39–110.
16. C. Soulé: A vanishing theorem on arithmetic surfaces, *Inventiones Math.* 116 (1994), 577–599.
17. U. Stuhler: Eine Bemerkung zur Reduktionstheorie Quadratischen Formen, *Archiv. der Math.*, 27 (1976), 604–610.
18. S. Zhang: Positive line bundles on arithmetic surfaces, *Annals of Math.*, 136 (1992), 569–587.
19. S. Zhang: Positive line bundles on arithmetic varieties, preprint Princeton University (1992).
20. S. Zhang: Geometry reductivity at archimedean places, preprint Princeton University (1993).