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Whittaker models, nilpotent orbits and the asymptotics of Harish-Chandra modules

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Abstract. We study the existence of Whittaker models for Harish-Chandra modules. In a real rank two setting, we prove Matumoto's conjecture, establishing the equivalence of a nilpotent orbit condition, the existence of a Whittaker model and an asymptotic condition; the equivalence of these three conditions fails in higher rank.

1. Introduction

One of the most important theorems in the representation theory of a semisimple Lie group is the *Subrepresentation theorem.* Every irreducible admissible representation can be realized as an invariant subspace of some principal series representation. Using the theory of matrix coefficient asymptotics, one can give an elegant account that such embeddings must exist, but a complete determination of all embeddings is still mysterious and unknown. For certain problems, knowing all possible embeddings is not important. For example, in order to classify the irreducible admissible representations (i.e. *Langlands classification*), the embeddings one must understand are easily determined; in part, this is due to the fact that these embeddings are "maximal" among the set of all such embeddings. However, when studying embeddings into more general types of induced modules (e.g. the existence of Whittaker models), the non-maximal embeddings into principal series representations are of crucial importance. In this article, we locate embeddings of an opposite character from the maximal embeddings of Langlands classification; what one might refer to as "minimal embeddings". These are the most difficult embeddings to understand and, in general, there is no known procedure to compute them.

Our motivation is a conjecture of H. Matumoto [26] and his subsequent work [27], [28]. Simply put, the conjecture links three a priori different notions: the singularity theory of irreducible Harish-Chandra modules (as encoded in the associated variety of the annihilator), the theory of matrix

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coefficient asymptotics (as encoded by the Jacquet module), and the existence of embeddings into particular induced representations (referred to as Whittaker models). From one perspective, the conjecture implies the existence of very special "minimal embeddings" of representations into principal series representations; these minimal embeddings, when combined with prior work of Matumoto and Goodman–Wallach, yield Whittaker models. So, our ability to exhibit the right kind of minimal embeddings into principal series amounts to an existence theorem for Whittaker models; this is perhaps the most important consequence of this paper. However, in another light, one can view our results as an attempt to revisit and reinterpret the authors joint program with L. Casian in [8]–[10]. Whereas, the former program focused on the $\mathfrak{g}$-structure of Jacquet modules, the ideas in this paper advance the philosophy of describing "nice submodules" of Jacquet modules via a connection with the theory of nilpotent orbits. From this vantage point, adopting the Hecke module framework of [8]–[10], we are studying a delicate relationship between double cell Weyl group representations in the Harish-Chandra module setting and right cell Weyl group representations in a highest weight module setting. In the real rank two Hermitian symmetric case, we will prove Matumoto's conjecture is true. A detailed analysis in $Sp_6\mathbb{R}$ shows the conjecture fails, in general, for higher real rank. In addition, we will indicate the conjecture is "almost" true for the general real rank two case.

As usual, more precision requires much more notation and terminology. We fix G to be a connected semisimple real matrix group and $P_m = M_m A_m N_m \subset G$ a minimal parabolic subgroup compatible with an Iwasawa decomposition $G = KA_m N_m$. We denote real Lie algebras by the notation $\mathfrak{g}_o$, $\mathfrak{k}_o$, etc., their complexifications without the subscript "o". Fix an *Iwasawa Borel subalgebra* $\mathfrak{b} \subset \mathfrak{p}_m$, which induces a Bruhat ordering on the full Weyl group W; we choose the ordering so that e (resp. w_o) is the unique minimal (resp. maximal) element. We will be working primarily in one of two types of categories of representations; each setting requires some notation, all of which is standard and reviewed in Section 2. Specifically, we work within the category of Harish-Chandra modules $\mathscr{HC}_o$ with the same infinitesimal character as a fixed finite dimensional representation F of G. The irreducible and standard modules in this category are parametrized by a finite partially ordered set $\mathscr{D}$; if $\delta \in \mathscr{D}$, then $\bar{\pi}(\delta)$ and $\pi(\delta)$ denote the irreducible and standard modules, respectively. In addition, if $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, then we recall the category $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$ of highest weight modules. In this case, the set of minimal length right coset representatives W^P is a parameter set for the irreducible modules $L_\mathfrak{p}(w)$ and the generalized Verma modules $N_\mathfrak{p}(w)$; our conventions are setup so that $N_\mathfrak{p}(e) = L_\mathfrak{p}(e)$; see Section 2 for more details.

It is important to recall the assignment $V \rightsquigarrow \mathcal{O}_V$, which associates to each irreducible $\mathfrak{U}(\mathfrak{g})$-module V a nilpotent orbit $\mathcal{O}_V$ in $\mathfrak{g}^*$ (or $\mathfrak{g}$). This requires that we begin with the annihilator I_V of V in $\mathfrak{U}(\mathfrak{g})$; any such ideal is called a *primitive ideal*, by definition. The associated graded object $\operatorname{gr} I_V$ is a graded ideal in $\operatorname{gr}\mathfrak{U}(\mathfrak{g}) \cong S(\mathfrak{g})$. As such, it has an associated variety $\mathscr{V}(\operatorname{gr} I_V)$ of common zeros in $\mathfrak{g}^*$. Since I_V is graded (resp. G_{ad}-stable), this variety is a cone in $\mathfrak{g}^*$ (resp. is G_{ad}-stable). The ideal I_V meets the center $\mathfrak{z}(\mathfrak{g})$ in an ideal of codimension one and since the associated graded algebra of $\mathfrak{z}(\mathfrak{g})$ identifies with the space $S(\mathfrak{g})^{G_{ad}}$ of G_{ad}-invariant polynomials in $S(\mathfrak{g})$, it follows that $\operatorname{gr} I_V$ meets $\operatorname{gr}\mathfrak{z}(\mathfrak{g})$ in its augmentation ideal, consisting of all G_{ad}-invariant polynomials with zero constant term. Making appropriate identifications, this implies that $\mathscr{V}(\operatorname{gr} I_V)$ sits inside the nilcone

$$\mathscr{N} = \{X \in \mathfrak{g} \mid \operatorname{ad}(X) \text{ is nilpotent}\}.$$

From these remarks, using the finiteness theorem for nilpotent orbits [17, §3], we have that $\mathscr{V}(\operatorname{gr} I_V)$ is a finite union of nilpotent orbits. But, even more is true [6]:

$$\mathscr{V}(\operatorname{gr} I_V) = \overline{\mathcal{O}_V},$$

for some nilpotent orbit $\mathcal{O}_V$. These remarks describe the desired assignment

$$V \rightsquigarrow \mathcal{O}_V. \tag{1.1a}$$

We sometimes refer to $\mathcal{O}_V$ as the *nilpotent orbit associated to V.* Define the Gelfand–Kirillov dimension of V to be $\operatorname{Dim} V = \frac{1}{2}\dim_{\mathbb{C}}\mathcal{O}_V$; every coadjoint orbit carries a symplectic structure, which ensures its dimension is even [17, §1.4].

For our needs, one type of nilpotent orbit is of particular interest. The *Richardson Orbit* $\mathcal{O}_{\mathfrak{p}}$ associated to the parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the unique nilpotent orbit in $\mathfrak{g}$ which is dense in $\operatorname{Ad}(G_{ad}).\mathfrak{n}$; this orbit is denoted $\mathcal{O}_{\mathfrak{p}}$. For more details, see [17, §7].

Given a Harish-Chandra module V in $\mathscr{HC}_o$, define $J(V) = (\tilde{V})^*_{\mathfrak{b}\text{-locally finite}}$, where $\tilde{}$ (resp. $*$) refers to the admissible (resp. full) dual of V. This assignment defines a faithful exact covariant functor. We refer to $J(V)$ as the *Jacquet module of V.* The module $J(V)$ lies in the category $\mathcal{O}'(\mathfrak{g}, \mathfrak{p}_m)$, for all $V \in \mathscr{H}C_o$.

To make sense of one of our introductory remarks, it is important to recall

$$H^k(\mathfrak{n}_m, J(\bar{\pi}(\delta))) = H_k(\bar{\mathfrak{n}}_m, \bar{\pi}(\delta)), \tag{1.1b}$$

for all $k \in \mathbb{N}$. Information about submodules of $J(\bar{\pi}(\delta))$ will be encoded by highest weight vectors contributing to $H^o(\mathfrak{n}_m, J(\bar{\pi}(\delta)))$; combined with Frobenius reciprocity [20] we obtain embeddings of $\bar{\pi}(\delta)$ into principal series representations.

We seek to link the existence of "nice submodules" of $J(\bar{\pi}(\delta))$ with a condition on the nilpotent orbit $\mathcal{O}_{\bar{\pi}(\delta)}$. To carefully define these "nice submodules", define

$$W^P_{\text{soc}} = \{w \mid w \in W^P,\ L_{\mathfrak{p}}(w) \subset \operatorname{socle}(N_{\mathfrak{p}}(y)),\ \text{for some } y \in W^P\}, \tag{1.1c}$$

which is referred to as the *socular set for* $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$. This set is parametrized by those $L_{\mathfrak{p}}(w)$ with the proprty that $\operatorname{Dim} L_{\mathfrak{p}}(w) = \dim \mathfrak{n}$. For example, if $\mathfrak{p} = \mathfrak{b}$, then $W^P_{\text{soc}} = \{e\}$. Roughly speaking, as $\mathfrak{p}$ gets "bigger", the size of the socular set increases and the category $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$ gets "smaller". The importance of this set is clearly spelled out in Irving's work [21]. We now come to a central definition.

DEFINITION 1.2. Let $\bar{\pi}(\delta)$ be an irreducible Harish-Chandra module for G and $\mathfrak{p}$ a standard parabolic subalgebra of $\mathfrak{g}$. We say that $\bar{\pi}(\delta)$ has property $\mathfrak{p}$ if there exists an irreducible highest weight module L satisfying two conditions:

(a) L lies in the socle of $J(\bar{\pi}(\delta))$;
(b) $L = L_{\mathfrak{p}}(w)$ for some $w \in W^P_{\text{soc}}$.

Any such L satisfying (a) and (b) is called a $\mathfrak{p}$-*factor* for $J(\bar{\pi}(\delta))$.

Given $\mathfrak{p}$, an obvious problem is to classify the irreducible Harish-Chandra modules having property $\mathfrak{p}$. It is fairly easy to give a necessary condition; see Section 2 for a proof.

LEMMA 1.3. *If* $\bar{\pi}(\delta)$ *has property* $\mathfrak{p}$, *then* $\mathcal{O}_{\bar{\pi}(\delta)} = \mathcal{O}_{\mathfrak{p}}$.

Any hope of establishing the converse of (1.3) requires a more careful hypothesis on $\mathfrak{p}$. (As will become clear in the sequel, without additional hypothesis the converse of (1.3) fails.) A *Whittaker datum* Ψ is a triple $(P, \psi, \mathfrak{n})$, where $P = MAN$ is the Langlands decomposition of a parabolic subgroup of G and ψ is a character (one-dimensional representation) of $\mathfrak{n}$. We say that the Whittaker datum Ψ is *admissible* if the Richardson orbit associated to $\mathfrak{p}$ coincides with the orbit determined by ψ; i.e. $\mathcal{O}_{\mathfrak{p}} = G_{ad} \cdot \psi$. It is not true that all parabolic subalgebras admit admissible Whittaker datum. However, in Section 2 we establish the following well-known result; it ensures the main results of this paper are not vacuous.

LEMMA 1.4. *Let* $\mathfrak{p}$ *be an even Jacobson–Morozov parabolic subalgebra of* $\mathfrak{g}$ *arising as the complexification of a real parabolic subalgebra of* $\mathfrak{g}_o$. *Then* $\mathfrak{p}$ *admits admissible Whittaker datum.*

If $\mathscr{A}(G)$ is the space of real analytic functions on G, then under the left action we have the induced representation $\mathscr{A}(G;\Psi)$, which is just the space of real analytic sections of the line bundle over G/N determined by the one-dimensional representation $e^{-\psi}$. Given an arbitrary $\mathfrak{U}(\mathfrak{g})$-module V, if there exists an injective $\mathfrak{U}(\mathfrak{g})$-homomorphism $i: V \mapsto \mathscr{A}(G;\Psi)$, then we will say V has a Ψ*-global Whittaker model.*

Using our terminology, the next result was established by Matumoto, generalizing earlier work of Goodman–Wallach.

THEOREM 1.5 (Goodman–Wallach [19], Matumoto [26]). *Fix* Ψ *an admissible Whittaker datum for* G *and* $\bar{\pi}(\delta)$ *an irreducible Harish-Chandra module. If* $\bar{\pi}(\delta)$ *has property* $\mathfrak{p}$, *then* $\bar{\pi}(\delta)$ *has a* Ψ*-global Whittaker model.*

This leads us to our main problem of interest. Give a necessary and sufficient condition for the existence of a Ψ-global Whittaker model for $\bar{\pi}(\delta)$; or equivalently, necessary and sufficient conditions for Property $\mathfrak{p}$.

MATUMOTO'S CONJECTURE 1.6. *Let* $\mathfrak{p}$ *be an even Jacobson–Morozov parabolic subalgebra defined over* $\mathbb{R}$ *and* $\mathcal{O}_\mathfrak{p}$ *the corresponding Richardson orbit. Fix* Ψ *an admissible Whittaker datum for* G *and assume that* $\bar{\pi}(\delta)$ *is an irreducible Harish-Chandra module with* $\dim \bar{\pi}(\delta) = \dim \mathfrak{n}$. *The following are equivalent*:

(a) (*Singularity condition*) $\mathcal{O}_{\bar{\pi}(\delta)} = \mathcal{O}_\mathfrak{p}$;
(b) (*Whittaker condition*) $\bar{\pi}(\delta)$ *has a* Ψ*-global Whittaker model*;
(c) (*Asymptotic condition*) $\bar{\pi}(\delta)$ *has property* $\mathfrak{p}$.

Matumoto has made significant progress on this conjecture. First, in [25] he showed that (b) implies (a) and (1.5) is (c) implies (b). In case $P = P_m$, Casselman's Subrepresentation theorem shows that the Singularity condition implies the Asymptotic condition (and hence, the Whittaker condition). In addition, when G is a complex group, Matumoto [27] additionally established (a) implies (c), whence proving the conjecture. The implication "(a) $\Rightarrow$ (c)" is sometimes referred to as "the working hypothesis". We can now state the first main result of this paper.

THEOREM 1.7. *If* G *is of Hermitian symmetric type and of real rank two, then Matumoto's conjecture is true.*

In Section 9 we will give a detailed account of the validity of the working hypothesis in the case of $Sp_6\mathbb{R}$ and offer counterexamples to (1.6).

PROPOSITION 1.8. *In the case of* $Sp_6\mathbb{R}$, *the fundamental block of the finite dimensional representation F is a union of* 16 *double cells. Matumoto's conjecture is true on all but two of these double cells. On these two double cells the conjecture fails* (*i.e. the working hypothesis* (a) *implies* (c) *in* (1.6) *fails*).

In this sense, without further restricting the groups in question or representations of interest, (1.7) is the best general statement one can make. (We should point out that H. Matumoto has informed the author of counter-examples in $Sp_6\mathbb{R}$ using very different techniques.)

One might naturally ask to what extent one can remove the Hermitian symmetric hypothesis in (1.7). To comment on this, let us first recall the list of simple real rank two matrix groups, up to covering, amounts to 4 infinite families and 7 sporadic cases:

Hermitian symmetric	*Non-Hermitian symmetric*
$SU(2, q)$	$Sp(2, s)$
$SO_e(2, 2n-1)$	
$SO_e(2, 2n-2)$	
$SO^*(10)$	$Sl_3\mathbb{R}$
$Sp_4\mathbb{R}$	$Sl_3\mathbb{H}$
$E_{6(-14)}$	$E_{6(-26)}$
	$G_{2(2)}$

In Section 10, we address the non-Hermitian cases. We will see, in the case of $Sl_3\mathbb{R}$, $Sl_3\mathbb{H}$ and $E_{6(-26)}$, the only even Jacobson–Morozov parabolic defined over $\mathbb{R}$ is the minimal parabolic $\mathfrak{p}_m$ and in this setting (1.6) follows from Matumoto's work in [26]. The case of $G_{2(2)}$ is non-trivial, but still we are able to prove (1.6). This leaves the infinite family $Sp(2, s)$. We have verified (1.6) in the case of $s = 2$, but a general proof would require tools in the spirit of [5], which are currently unavailable. The ideas and techniques of proof we use for (1.7) will build upon the material in the Memoir [5], which was cast entirely in the Hermitian symmetric setting. Nevertheless, if (1.6) holds for the cases $s \geqslant 3$, we would then be able to remove the "Hermitian symmetric" assumption in (1.7).

Here is a brief outline of the content of each section of the paper. In Section 2, we introduce the necessary notation and terminology, most of which is standard. Section 3 will establish a useful reduction lemma; in effect, we are reduced to verifying (1.7) for one irreducible representation from each relevant double cell. This result is really a manifestation of the fact that the Jacquet functor "intertwines" double cell and right cell Weyl group representations. Section 4 outlines the basic strategy used in our proof of (1.7). The proof of the main result (1.7) is carried out in Sections 5 to 8 and $Sp_6\mathbb{R}$ is studied in Section 9. Non-Hermitian real rank two groups are discussed in Section 10.

2. Notation

In this section, we elaborate on some of the terminology used in the Introduction. The material is organized by topic for easier reference.

Module categories

Let $\mathscr{B}$ denote the *flag variety* of all Borel subalgebras in $\mathfrak{g}$. Recall, the complexification of K, denoted $\mathbf{K}$, acts on $\mathscr{B}$ with finitely many orbits, as does the complexification $\mathbf{P}$ of any parabolic subgroup of G.

In the setting of $\mathscr{H}C_o$, we introduce a parameter set $\mathscr{D}$ which will consist of pairs $\delta = (\mathscr{V}_{\delta_o}, \mathscr{L}_{\delta_o})$, where $\mathscr{V}_{\delta_o}$ is a $\mathbf{K}$-orbit in $\mathscr{B}$ and $\mathscr{L}_{\delta_o}$ is a $\mathbf{K}$-homogeneous line bundle on the orbit with flat connection. Equivalently, $\mathscr{D}$ can be described in terms of triples of Langlands data, as discussed in [33]. Continuing with the notation of [33], we have the basis of irreducible representations $\{\bar{\pi}(\delta) \mid \delta \in \mathscr{D}\}$ and the basis of *standard modules* $\{\pi(\delta) \mid \delta \in \mathscr{D}\}$ for the Grothendieck group $K(\mathscr{H}C_o) = \mathbb{Z}[\mathscr{D}]$. We remark that each standard module is a generalized principal series representation; i.e., an induced representation of the form $I_P(\sigma \otimes \nu)$, where $P = MAN$ is a standard cuspidal parabolic subgroup of G, σ is a relative discrete series representation of M and ν is a one dimensional (non-unitary) character of A.

If $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, we define the category $\mathscr{O}'(\mathfrak{g}, \mathfrak{p})$ of all finitely generated $\mathfrak{U}(\mathfrak{g})$-modules which are locally $\mathfrak{p}$-finite with the same infinitesimal character as F. This is a slight variant of the relative classical BGG categories $\mathscr{O}(\mathfrak{g}, \mathfrak{p})$, consisting of finitely generated $\mathfrak{U}(\mathfrak{g})$-modules which are locally $\mathfrak{p}$-finite, $\mathfrak{m} \oplus \mathfrak{a}$-semisimple and having the same generalized infinitesimal character as F; by a theorem of Soergel [30], these two categories are equivalent. As usual, let W_P be the parabolic subgroup determined by the Levi factor of $\mathfrak{p}$, with w_P the longest element (using the Bruhat order previously introduced on W). Let $\chi - \rho$ be the highest weight of the fixed finite dimensional representation F of G, where ρ is the half-sum of the positive roots determined by $\mathfrak{b}$. Let W^P denote the set of minimal length right coset representatives of $W_P \backslash W$. For each $w \in W^P$, define

$$N_{\mathfrak{p}}(w) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} E_P(w_P w w_o),$$

where $E_P(w_P w w_o)$ is the irreducible finite-dimensional representation of $\mathfrak{m} \oplus \mathfrak{a}$ of highest weight $w_P w w_o(\chi) - \rho$. The irreducible modules in $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$ are exhausted by taking the unique irreducible quotients $L_\mathfrak{p}(w)$ of $N_\mathfrak{p}(w)$, for $w \in W^P$. Our conventions are arranged so that $N_\mathfrak{p}(e) = L_\mathfrak{p}(e)$. In the obvious sense, the integral Grothendieck group of $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$ can be identified with $\mathbb{Z}[W^P]$.

We will often work simultaneously within two or more different relative categories $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$. However, in all cases, we can at least work within the category $\mathcal{O}'(\mathfrak{g}, \mathfrak{b})$, which contains all of the relative categories as subcategories. Thus, we institute the notational conventions

$$L_w = L_\mathfrak{b}(w), \quad N_w = N_\mathfrak{b}(w),$$

for all $w \in W$. In particular, this convention implies $L_\mathfrak{p}(w) = L_{w_P w}$, etc. Denote by I_w the annihilator of L_w in $\mathfrak{U}(\mathfrak{g})$.
For technical reasons (see (2.3) below), it is important to have on hand slight variants of the sets W^P and W^P_{soc}. Define

$$\mathcal{W}^P = \{w_P w \mid w \in W^P\}$$

$$\mathcal{W}^P_{\text{soc}} = \{w_P w \mid w \in W^P_{\text{soc}}\}.$$

More generally, suppose $\mathfrak{b} \subset \mathfrak{q} \subset \mathfrak{p}$ are parabolic subalgebras of $\mathfrak{g}$ with corresponding parameter sets W, W^Q and W^P, respectively. We can relate the longest elements w_P and w_Q of W_P and W_Q in the following useful way:

$$w_P = w_Q \cdot w_{P,Q}, \tag{2.1}$$

for some $w_{P,Q} \in W^Q$. Define

$$\mathcal{W}^{P,Q} = \{w_{P,Q} \cdot w \mid w \in W^P\},$$

then $\mathcal{W}^{P,B} = \mathcal{W}^P$.

Cells

Recall from [1] the definition of order relations $\leqslant_R$, $\leqslant_L$ and $\leqslant_{LR}$ on W, which lead to equivalence relations $\approx_R$, $\approx_L$ and $\approx_{LR}$ and equivalence classes of *right cells* $\mathcal{C}^R$, *left cells* $\mathcal{C}^L$ and *double cells* $\mathcal{C}^{LR}$ in W. We have partitions

$$W = \bigcup_{\text{right cells}} \mathcal{C}^R = \bigcup_{\text{left cells}} \mathcal{C}^L = \bigcup_{\text{double cells}} \mathcal{C}^{LR}. \tag{2.2}$$

So, we can use the notation $\mathscr{C}_w^R$ (resp. $\mathscr{C}_w^L, \mathscr{C}_w^{LR}$) to denote the unique right cell (resp. left cell, double cell) in W containing w. In turn, these cells give rise to representations of W.

Whereas the set W^P works nicely when parametrizing generalized Verma modules in $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$, it does *not* work so nicely when studying the right cell decomposition of W; this is where we need to use $\mathscr{W}^P$ instead of W^P. The useful observation is the following.

LEMMA 2.3. *The parameter set $\mathscr{W}^P$ will decompose into a union of right cells in W, sympathetic to* (2.2).

Along the same lines, we can define double cells $\mathscr{C}^G$ in $\mathscr{D}$ as in [34]; this definition can be viewed as paralleling the definition of right cells in W, in that they each can be realized via a relation defined in terms of the U_α construction. A precise definition is central to the paper. Suppose s is a simple reflection in W corresponding to the simple root α and assume that s is not in the tau invariant of $\delta \in \mathscr{D}$; see [34] for elaboration. We can consider the s-wall crossing $\theta_s(\bar{\pi}(\delta))$ (via a composition of translation functors). The module $\theta_s(\bar{\pi}(\delta))$ will have $\bar{\pi}(\delta)$ as a submodule and quotient and $\theta_s(\bar{\pi}(\delta))/\bar{\pi}(\delta)$ will have a semisimple module $U_s(\bar{\pi}(\delta))$ as a submodule; this is the Kazhdan–Lusztig conjecture for G, which is a theorem in the context of our paper (see [24], [33]). Thus, the module $\theta_s(\bar{\pi}(\delta))$ will have a Loewy filtration of the form

$\bar{\pi}(\delta)$
$U_s(\bar{\pi}(\delta))$
$\bar{\pi}(\delta)$

We define a relation $\leqslant_G$ on $\mathscr{D}$ as follows: $\delta \to_s \gamma$ if $\bar{\pi}(\gamma)$ occurs as a summand of $U_s(\bar{\pi}(\delta))$. The reflexive/transitive closure of this relation defines $\leqslant_G$. Finally, we say that $\delta \approx_G \gamma$ if $\delta \leqslant_G \gamma$ and $\gamma \leqslant_G \delta$. By definition, the double cells $\mathscr{C}^G$ in $\mathscr{D}$ are the equivalence classes under the relation $\approx_G$. Similar remarks apply to define the relations $\leqslant_R$ and $\approx_R$ on W, leading to the right cells $\mathscr{C}^R$ in W.

Proof of (1.3). If $\bar{\pi}(\delta)$ has property $\mathfrak{p}$, then $L_\mathfrak{p}(w) = L_{w_P w}$ is a submodule of $J(\bar{\pi}(\delta))$, for some $w \in W^P_{\text{soc}}$. First, we use the fact that any irreducible submodule of $J(\bar{\pi}(\delta))$ determines the same primitive ideal as $\bar{\pi}(\delta)$; in particular, $\bar{\pi}(\delta)$ and $L_\mathfrak{p}(w)$ have the same annihilator in $\mathfrak{U}(\mathfrak{g})$. Next, the socular set W^P_{soc} contains the minimal element e and $\mathscr{W}^P_{\text{soc}}$ is a right cell in W (by (2.2)) with minimal element w_P. Now, if $\mathscr{C}^R$ is *any* right cell in $\mathscr{W}^P$, then under the map

(1.1a), the nilpotent orbits $\mathcal{O}_{L_y}$ will coincide for all $y \in \mathscr{C}^R$. In particular, this shows that

$$\mathcal{O}_{L_{\mathfrak{p}(w)}} = \mathcal{O}_{L_{w_P w}} = \mathcal{O}_{L_{w_P}} = \mathcal{O}_{L_{\mathfrak{p}(e)}}.$$

But, an easy calculation shows that $\mathcal{O}_{L_{\mathfrak{p}(e)}} = \mathcal{O}_{N_{\mathfrak{p}(e)}} = \mathcal{O}_{\mathfrak{p}}$. □

Weight filtrations

It is useful to recall the program in [8]–[10] to describe a *weight filtration* for the Jacquet module $J(\bar{\pi}(\delta))$. This is a $\mathfrak{g}$-filtration with semisimple subquotients. A *weight filtration* for the Jacquet module $J(\bar{\pi}(\delta))$ arises as follows. Recalling the integral Grothendieck groups $\mathbb{Z}[\mathscr{D}]$ and $\mathbb{Z}[W^P]$, we extend scalars, obtaining

$$\mathscr{M}_G = \mathbb{Z}[u^{1/2}, u^{-1/2}] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathscr{D}] \quad \text{and} \quad \mathscr{M}_P = \mathbb{Z}[u^{1/2}, u^{-1/2}] \otimes_{\mathbb{Z}} \mathbb{Z}[W^P],$$

respectively. We refer to these extended Grothendieck groups as *Hecke modules*; this is justified since one can show that these objects are modules under an appropriately defined action of the Hecke algebra $\mathscr{H} = \mathscr{M}_B$ attached to the Weyl group; see [8], [9]. Once appropriate dictionaries are in place (via $\mathscr{D}$-modules, perverse sheaves and passage to positive characteristic) we may interpret weight filtrations of modules in $\mathscr{H}C_o$ or $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$ as elements of these Hecke modules. Under this dictionary, a typical standard irreducible Harish-Chandra module $\bar{\pi}(\delta)$ (resp. irreducible highest weight module $L_{\mathfrak{p}}(w)$) corresponds to a self-dual element $\hat{C}_\delta$ (resp. $\hat{C}_{w_P w}$) of $\mathscr{M}_G$ (resp. $\mathscr{M}_P$). The Jacquet functor J gives rise to a Hecke module map $\mathbb{J}: \mathscr{M}_G \mapsto \mathscr{M}_B = \mathscr{H}$ such that $\mathbb{J}(\hat{C}_\delta)$ determines a weight filtration of $J(\bar{\pi}(\delta))$; see [9]. We may write

$$\mathbb{J}(\hat{C}_\delta) = \sum_{w \in W, i \in \mathbb{Z}} a(w, i) u^{i/2} \hat{C}_w,$$

where $a(w, i)$ are integers. Define

$$r(\delta) = \min\{j \mid a(w, j) \neq 0, \text{ for some } w \in W\}.$$

A composition factor L_z of $J(\bar{\pi}(\delta))$ is said to lie in the *bottom weight layer* if $a(z, r(\delta)) \neq 0$. In particular, bottom weight layer factors are *among* the irreducible composition factors in the socle of $J(\bar{\pi}(\delta))$.

The theory of weight filtrations allows us to attach an invariant to each $\bar{\pi}(\delta)$. We define the *asymptotic length* of δ, denoted $ll_{asy}(\delta)$, to be the number of levels in the above weight filtration for $J(\bar{\pi}(\delta))$.

LEMMA 2.4. *ll_{asy} is constant along double cells.*

Proof. Suppose that $\delta \approx_G \gamma$, then we need to prove that $ll_{asy}(\delta) = ll_{asy}(\gamma)$. We know that there exist chains

$$\begin{aligned} \delta = \delta_1 &\xrightarrow[s_1]{} \delta_2 \xrightarrow[s_2]{} \cdots \xrightarrow[s_{c-1}]{} \delta_{c-1} = \gamma \\ \gamma = \gamma_1 &\xrightarrow[r_1]{} \gamma_2 \xrightarrow[r_2]{} \cdots \xrightarrow[r_{d-1}]{} \gamma_{d-1} = \delta. \end{aligned} \tag{2.5}$$

Let $L_{y_1}, \dots, L_{y_k}$ be the bottom weight layer factors of $J(\bar{\pi}(\delta))$; there may be multiplicities here. Since every irreducible submodule of $J(\bar{\pi}(\delta))$ determines the same primitive ideal as $\bar{\pi}(\delta)$, we see that $s_1 \notin \tau(L_{y_k})$, for $1 \leqslant i \leqslant k$. Let $L_{w_1}, \dots, L_{w_m}$ index all of the second from bottom weight layer factors of $J(\bar{\pi}(\delta))$ with $s_1 \notin \tau(L_{w_j})$. Let $L_{z_1}, \dots, L_{z_t}$ index all of the third from bottom weight layer factors of $J(\bar{\pi}(\delta))$ with $s_1 \notin \tau(L_{z_a})$, etc. Consider the following schematic layer filtration with semsimple subquotients.

$$\bigoplus_{1\leqslant j\leqslant m} L_{w_j} \oplus \bigoplus_{1\leqslant a\leqslant t} U_{s_1}(L_{z_a}) \oplus \cdots$$

$$\bigoplus_{1\leqslant i\leqslant k} L_{y_i} \oplus \bigoplus_{1\leqslant j\leqslant m} U_{s_1}(L_{w_j}) \bigoplus_{1\leqslant a\leqslant t} (L_{z_a})$$

$$\bigoplus_{1\leqslant i\leqslant k} U_{s_1}(L_{y_i}) \oplus \bigoplus_{1\leqslant j\leqslant m} L_{w_j}$$

$$\bigoplus_{1\leqslant i\leqslant k} L_{y_i}$$

By exactness of J, the algorithm of [8], the wall crossing 3-step filtration and tau invariant considerations, the bottom level (resp. top level) of this picture is exactly the bottom level (resp. top level) of $J(\bar{\pi}(\delta))$. In addition, by the algorithm in [8], given *any* irreducible summand $\bar{\pi}(\zeta)$ of $U_{s_1}(\bar{\pi}(\delta))$, the bottom weight layers factors of $J(\bar{\pi}(\zeta))$ will be among the second *or higher* level factors in this picture. By the self duality of this filtration, the top weight layer factors of $J(\bar{\pi}(\zeta))$ will be among the second from the top or *lower* level factors in the picture. We conclude that $ll_{asy}(\delta) \geqslant ll_{asy}(\delta_2)$. Iterating this argument and using (2.5), we have

$$ll_{asy}(\delta) = ll_{asy}(\delta_1) \geqslant ll_{asy}(\delta_2) \geqslant \cdots \geqslant ll_{asy}(\gamma) \geqslant ll_{asy}(\gamma_2) \geqslant \cdots \geqslant ll_{asy}(\delta). \qquad \square$$

Jacobson–Morozov parabolic subalgebras

Let $\{H, X, Y\}$ be a standard triple in $\mathfrak{g}$, consisting of neutral element H, nilpositive element X and nilnegative element Y. Decompose $\mathfrak{g}$ according to ad_H as

$$\mathfrak{g} = \mathfrak{g}^H \oplus \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i,$$

where $\mathfrak{g}_i = \{Z \in \mathfrak{g} \mid [H, Z] = iZ\}$. Then

$$\mathfrak{p} = \mathfrak{p}_X = \bigoplus_{i \geqslant 0} \mathfrak{g}_i = \mathfrak{g}^H \oplus \bigoplus_{i > 0} \mathfrak{g}_i = \mathfrak{l} \oplus \mathfrak{u} \tag{2.6}$$

is a parabolic subalgebra of $\mathfrak{g}$, called the *Jacobson–Morozov parabolic subalgebra* determined by X; see [17]. If $\mathfrak{g}_i$ is zero for all odd i, then we say the parabolic $\mathfrak{p}$ (or the nilpotent orbit $\mathcal{O}_X$, or the nilpotent element X) is even.

Proof of (1.4). Adopt the above notation. Recall that $X \in \mathfrak{g}_2$ and $Y \in \mathfrak{g}_{-2}$. Since X is even, $\mathfrak{u} = \mathfrak{g}_2 \oplus [\mathfrak{u}, \mathfrak{u}]$, so any Lie algebra homomorphism $\psi \in \mathfrak{u}^*$ is determined by $\psi|_{\mathfrak{g}_2} \in \mathfrak{g}_2^* = \mathfrak{g}_{-2}$, by Killing form duality. In particular, notice that $Y \in \mathfrak{g}_{-2}$, so Y determines a character $\psi_Y \in \mathbf{u}^*$.

We claim that $\mathcal{O}_{\psi_Y} = \mathcal{O}_Y = \mathcal{O}_{\mathfrak{p}}$, which will show that $\mathfrak{p}$ is admissible. To this end, recall $\dim \mathfrak{g}^Y = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$; see [17, (4.1.3)]. Since X is even, we then have $\dim \mathfrak{g}^Y = \dim \mathfrak{l}$, so $\dim \mathcal{O}_Y = 2 \dim \bar{\mathfrak{u}}$ and $Y \in \bar{\mathfrak{u}}$. But, by [17, (7.1.1)], there exists a unique nilpotent orbit in $\mathfrak{g}$ of dimension $2 \dim \bar{\mathfrak{u}}$ which meets $\bar{\mathfrak{u}}$. Thus, $\mathcal{O}_Y = \mathcal{O}_{\bar{\mathfrak{p}}}$. By a result of Lusztig–Spaltenstein [17], $\mathcal{O}_{\bar{\mathfrak{p}}} = \mathcal{O}_{\mathfrak{p}}$, showing that Y is a Richardson element for $\mathfrak{p}$ in $\mathfrak{g}_{-2}$. □

3. Property $\mathscr{C}^R$

Up front, it is important to realize that property $\mathfrak{p}$ is really a condition relating double cells in $\mathscr{D}$ and right cells in the Weyl group W. Namely, fix a standard parabolic subgroup $P = MAN$ of G and consider the socular set W_{soc}^P in W^P. Recall that $\mathscr{W}_{\text{soc}}^P$ is a right cell in W. In fact, one can characterize $\mathscr{W}_{\text{soc}}^{\mathfrak{p}}$ as the unique right cell in W with the property that the associated irreducible highest weight modules have GK-dimension equal to $\dim \mathfrak{n}$. These remarks point toward a variant of property $\mathfrak{p}$, which is more in tune with the philosophy of [8]–[10].

DEFINITION 3.1. Let $\bar{\pi}(\delta)$ be an irreducible Harish-Chandra module for G and $\mathscr{C}^R$ a right cell in W. We say that $\bar{\pi}(\delta)$ has property $\mathscr{C}^R$ if there exists an irreducible highest weight module L satisfying two conditions:

(a) L lies in the bottom weight layer of $J(\bar{\pi}(\delta))$;
(b) $L = L_w$ for some $w \in \mathscr{C}^R$.

Any such L satisfying (a) and (b) is called a *$\mathscr{C}^R$-factor* for $J(\bar{\pi}(\delta))$.

REMARKS 3.2. (i) Fix a parabolic subalgebra $\mathfrak{p}$. Suppose that $\mathscr{C}^R = \mathscr{C}^R_w$ for some $w \in \mathscr{W}^P_{\mathrm{soc}}$; i.e., suppose that $\mathscr{C}^R$ is the "bottom right cell" in $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$. Then property $\mathscr{C}^R_w$ implies property $\mathfrak{p}$, since any bottom weight layer factor lies in the socle of the Jacquet module.

(ii) It is entirely possible that a given irreducible Harish-Chandra module $\bar{\pi}(\delta)$ could satisfy property $\mathscr{C}^R$ and property $\mathscr{C}^{R'}$, for two different right cells $\mathscr{C}^R$ and $\mathscr{C}^{R'}$. However, any two such right cells lie in a common double cell $\mathscr{C}^{LR}$, since all irreducible submodules of $J(\bar{\pi}(\delta))$ determine the same primitive ideal.

LEMMA 3.3 (Reduction lemma). *Let $\mathscr{C}^G$ be a double cell in $\mathscr{D}$ and $\delta \in \mathscr{C}^G$. If $\bar{\pi}(\delta)$ has property $\mathscr{C}^R$, then $\bar{\pi}(\gamma)$ has property $\mathscr{C}^R$, for every $\gamma \in \mathscr{C}^G$.*

Proof. Let $\delta \approx_G \gamma$, which implies that $\gamma \leqslant_G \delta$ and $\delta \leqslant_G \gamma$. First, suppose that $\gamma \rightarrow_s \delta$. Let L_y be a bottom weight layer factor for $J(\bar{\pi}(\delta))$. By (2.3) and its proof, there exists a bottom weight layer factor L_w for $J(\bar{\pi}(\gamma))$ such that L_y is a summand of $U_s(L_w)$, whence $w \leqslant_R y$. Iterating this argument and using the second string of relations in (2.5), we obtain the same conclusion: if $\gamma \leqslant_G \delta$ and L_y is a bottom weight layer factor of $J(\bar{\pi}(\delta))$, then there exists a bottom weight layer factor L_w for $J(\bar{\pi}(\gamma))$ such that $w \leqslant_R y$. Reversing the roles of δ and γ and starting with the bottom layer factor L_w for $J(\bar{\pi}(\gamma))$, we can use the first string in (2.5) to find a bottom weight layer factor L_z for $J(\bar{\pi}(\delta))$ such that $z \leqslant_R w$. Since any two irreducible submodules of $J(\bar{\pi}(\delta))$ must determine the same primitive ideal, we have shown: $z \leqslant_R w \leqslant_R y$ and $z \approx_L y$. Equivalently, we have $I_{z^{-1}} \subset I_{y^{-1}}$ and $\mathcal{O}_{L_{z^{-1}}} = \mathcal{O}_{L_{y^{-1}}}$ By an old result of Borho and Kraft, there cannot be a proper inclusion between these two primitive ideals. Thus, $y \approx_R w \approx_R z$, as desired. □

The following corollary will not be needed, but helps to reinterpret our results.

COROLLARY 3.4. *If $x \in \mathscr{C}^R$, then there exists some $\zeta \in \mathscr{C}^G$ such that L_x is in the bottom weight layer of $J(\bar{\pi}(\zeta))$.*

We can define a map $\hat{\mathbb{J}}: \mathscr{C}^G \mapsto \mathbb{C}[\mathscr{C}^R]$, by $\hat{\mathbb{J}}(\gamma) =$ the complex span of those y such that L_y is a $\mathscr{C}^R$ factor for $J(\bar{\pi}(\gamma))$. Lemma 3.3 implies that $\hat{\mathbb{J}}(\gamma)$ is non-zero for every $\gamma \in \mathscr{C}^G$; a faithfullness result. The corollary tells us that given any $x \in \mathscr{C}^R$, we can find *some* $\zeta \in \mathscr{C}^G$ such that $\mathbb{C}x$ is a subspace of $\hat{\mathbb{J}}(\zeta)$; a surjectivity type result.

REMARK 3.5. A slight generalization of (3.3) will be useful. First, we can modify Definition (3.1)(a) and require that L lies in the bottom weight layer of an indecomposable summand of $J(\bar{\pi}(\delta))$; lets call the resulting definition "generalized property $\mathscr{C}^R$". The same proof now allows one to prove a generalization of (3.3): if $\bar{\pi}(\delta)$ has generalized property $\mathscr{C}^R$, then $\bar{\pi}(\gamma)$ has generalized property $\mathscr{C}^R$, for every $\gamma \in \mathscr{C}^G$. This leads to a generalization $\hat{\mathbb{J}}'$ of the map above; the difference is that the image of $\hat{\mathbb{J}}'$ may properly contain that of $\hat{\mathbb{J}}$. These generalizations are picking out a potentially bigger piece of the socle of $J(\bar{\pi}(\delta))$.

4. Strategy

Our proof of (1.7) will proceed in a case by case manner, but the general approach is the same for all of the groups we consider. Indeed, this approach applies for any group G, in principle. For this reason, we outline below the steps followed and the general techniques required to carry out each step. The specifics for each group are handled in the individul sections that follow.

Step 0. A reduction

We begin by laying out the strategy one follows to prove (1.6) for a particular group G. To begin with, we are reduced to considering those irreducible representations $\bar{\pi}(\delta)$, where δ lies in a double cell $\mathscr{C}^G \subset \mathscr{D}$ and the associated nilpotent orbit of $\mathscr{C}^G$ is the closure of a Richardson orbit $\mathscr{O}_{\mathfrak{p}}$ attached to an even Jacobson–Morozov parabolic subalgebra $\mathfrak{p}$, which is defined over $\mathbb{R}$. This really just amounts to the imposition of the hypothesis of (1.6). From the Introduction, the proof of (1.6) on such a double cell $\mathscr{C}^G$ is reduced to verifying property $\mathfrak{p}$ for every irreducible representation $\bar{\pi}(\delta)$, $\delta \in \mathscr{C}^G$. The Reduction lemma (3.3) reduces us to verifying property $\mathscr{W}^P_{\mathrm{soc}}$ for *just one* $\delta \in \mathscr{C}^G$. In effect, this reduction gives us sufficient freedom to pick a $\bar{\pi}(\delta)$ whose Jacquet module contains a predictable composition factor in the socle.

Step 1. Admissible parabolic subalgebras

The first order of business is to determine all of the even Jacobson–Morozov parabolic subalgebras defined over $\mathbb{R}$. To carry this out, we might as well assume we are working with standard parabolic subalgebras. The possible parabolic subalgebras defined over $\mathbb{R}$ are easily read off from the Satake diagram attached to the pair $(\mathfrak{g}, K)$. We now describe an assignment

$$\mathfrak{p} \rightsquigarrow \Delta(\mathfrak{p})$$

from standard parabolic subalgebras defined over $\mathbb{R}$ to weighted Dynkin diagrams (Dynkin diagrams with labels 0, 1 or 2 attached to the nodes). Todo so, let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be a parabolic subalgebra of $\mathfrak{g}$ defined over $\mathbb{R}$. Define a weighted Dynkin diagram $\Delta(\mathfrak{p})$ as follows. Attach a label 0 (resp. 2) to each node of the Dynkin diagram representing a simple root which lies in (resp. does not lie in) the Levi factor $\mathfrak{l}$.

It remains to determine if $\Delta(\mathfrak{p})$ is the weighted diagram of *some* nilpotent orbit. A priori this is not an easy problem, since the map from nilpotent orbits to weighted Dynkin diagrams is not onto; see [17]. However, in all of the cases of interest, we will explicitly exhibit a nilpotent orbit $\mathcal{O}_X$ with weighted Dynkin diagram $\Delta(\mathcal{O}_X) = \Delta(\mathfrak{p})$. Once this is accomplished, we can draw two conclusions: first, $\mathfrak{p} = \mathfrak{p}_X$ is the Jacobson–Morozov parabolic subalgebra attached to X (or to any standard triple containing X). This is immediate from the construction reviewed in (2.5) and the interpretation of the weighted Dynkin diagrams. Secondly, the nilpotent orbit $\mathcal{O}_X$ is even, since the only labels involved in the diagram $\Delta(\mathfrak{p})$ are even.

It is worth noting that the complexified minimal parabolic subalgebra $\mathfrak{p}_m$ is always an even Jacobson–Morozov parabolic subalgebra. Also, the dimension of any nilpotent orbit attached to an even Jacobson–Morozov parabolic subalgebra is easily computed using [17, (4.1.3)].

Step 2. Relative category structure

The Jacquet module $J(\bar{\pi}(\delta))$ lies in the category $\mathcal{O}'(\mathfrak{g}, \mathfrak{p}_m)$. However, the study of modules with property $\mathfrak{p}$ (or property $\mathscr{C}^R$) requires that we work with the categories $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$, for various $\mathfrak{p}$ containing $\mathfrak{p}_m$. In turn, a description of the socular sets W^P_{soc} will become important, as well as the fuller right cell decomposition of $\mathscr{W}^P$. It is at this stage we will make the strongest use of our small real rank assumption. If $\mathfrak{p}$ is an even Jacobson–Morozov parabolic subalgebra defined over $\mathbb{R}$ properly containing $\mathfrak{p}_m$, then combining work in [2], [4], [13] and [16] describe most of the parametrizing posets W^P that arise and (at least) the socular right cells. Thus, for our purposes, we additionally need some structure theoretic information about W^{P_m}. This is done on a case-by-case basis, but [29] describes a general model.

Step 3. Double cells

We need to have on hand a useful description of the parameter set $\mathscr{D}$ and its decomposition into double cells. The required descriptions are mostly contained in [5]; other cases (e.g. $G_{2(2)}$ or $Sp_6\mathbb{R}$) will be described as needed. The decomposition into double cells is straightforward, given the definition of the relation $\approx_G$ in Section 2.

Step 4. Implementation of the Reduction lemma

In order to prove (1.7), we are reduced to checking a technical condition on the structure of the Jacquet module $J(\bar{\pi}(\delta))$ for one δ in a given double cell $\mathscr{C}^G$. This will require knowing something about the socular factors of $J(\bar{\pi}(\delta))$. It is important to realize [10] outlines several procedures for computing such factors. Three points deserve special mention, but we generally assume familiarity with the program in [8]–[10].

(i) We can define $l_g(\delta)$ to be the dimension of the **K**-orbit $\mathscr{V}_{\delta_o}$. Let r_G be the dimension of a closed **K**-orbit in $\mathscr{B}$. Then

$$r_G \leqslant l_g(\delta) \leqslant \dim \mathscr{B},$$

for all $\delta \in \mathscr{D}$. From the remarks in [10], we know that if L_w is a composition factor of $J(\bar{\pi}(\delta))$, then $l(w) \leqslant l_g(\delta)$, with equality holding only for the leading composition factors (i.e. those which correspond to leading asymptotic exponents).

(ii) The weight filtrations of Jacquet modules considered in this paper are all self-dual, so there is an obvious notion of the "middle layer"; this is where all the leading composition factors lie. We can assign a integral weight i to each weight level above and below the middle, with the middle having weight 0, by convention. So, the first level above (resp. below) the middle has weight 1 (resp. -1), etc. (actually for technical reasons, the framework of [8]–[10] requires one to work with half-integral weights $i/2$, but the terminology remains the same). Given this notion of weight, a parity condition is established in [10]. If L_w occurs in the ith and jth weight layers of $J(\bar{\pi}(\delta))$, then $i \equiv j \pmod 2$.

(iii) A composition factor L_w of $J(\bar{\pi}(\delta))$ is called *smooth* if it occurs in the $\pm[l_g(\delta) - l(w)]$ levels of the weight filtration. For our purposes, the theory of *smooth exponents* in [10] will be most useful and is assumed throughout. The point is that smooth composition factors of $J(\bar{\pi}(\delta))$ can be computed non-recursively, without requiring full blown knowledge of Kazhdan–Lusztig and Lusztig–Vogan polynomials. Having carried out this calculation, we arrive at what we sometimes will refer to as the "smooth skeleton" of $J(\bar{\pi}(\delta))$. The tricky point comes in showing certain smooth composition factors actually lie in the socle or bottom weight layer of the Jacquet module. This is precisely the point where we will be using the freedom afforded by (3.3) to choose *any* δ we wish in the double cell $\mathscr{C}^G$; making a *wise* choice of $\delta \in \mathscr{C}^G$ makes study of the smooth factors of $J(\bar{\pi}(\delta))$ much easier.

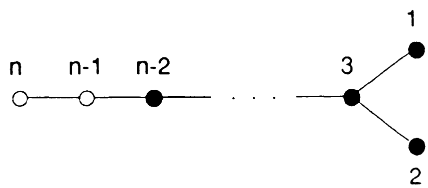
Step 5. The quasi-large case

The largest possible nilpotent orbit associated to $\bar{\pi}(\delta)$ will be $\mathcal{O}_{\mathfrak{p}_m}$; in such cases we say that $\bar{\pi}(\delta)$ is a *quasi-large* representation. In the case when G is a quasi-split group, $\mathfrak{b} = \mathfrak{p}_m$ and this collapses to the condition that $\mathcal{O}_{\bar{\pi}(\delta)} = \mathcal{O}_{\mathfrak{b}}$ is the principal nilpotent orbit; such representations are called *large* in [31]. It follows directly from [26] that (1.6) holds for any quasi-large representation.

Though not needed in the sequel, it is worth noting that G always does admit a quasi-large representation. One way to see this is as follows. First, by the Osborne conjecture and the character formula for an induced representation, the character of the Jacquet module of any principal series representation $I_{P_m}(E)$ is a sum of characters of category $\mathcal{O}'(\mathfrak{g}, \mathfrak{p}_m)$ generalized Verma modules; at least one of these generalized Verma modules is an actual submodule of $J(I_{P_m}(E))$. It follows that $\dim \mathcal{O}_{\bar{\pi}(\delta)} \leqslant \dim \mathcal{O}_{\mathfrak{p}_m}$, for all $\bar{\pi}(\delta)$, by the Subrepresentation theorem. By the exactness of J, the Jacquet module of some composition factor $\bar{\pi}(\gamma)$ of $I_{P_m}(E)$ contains a socular composition factor of $N_{\mathfrak{p}_m}(w)$. But, any such $\bar{\pi}(\gamma)$ satisfies $\mathcal{O}_{\bar{\pi}(\gamma)} = \mathcal{O}_{\mathfrak{p}_m}$; i.e., $\bar{\pi}(\gamma)$ is quasi-large.

5. $SO_e(2, 2n-2)$

Throughout this section, we assume $G = SO_e(2, 2n-2)$ and label the corresponding Satake diagram as follows.



Step 1. The minimal parabolic subalgebra $\mathfrak{p}_m$ is determined by the set of simple roots $S_m = \{\alpha_1, \ldots, \alpha_{n-2}\}$. There are two other proper standard parabolic subalgebras defined over $\mathbb{R}$. Let $\mathfrak{p} = \mathfrak{l}_P \oplus \mathfrak{n}_P$ denote the parabolic subalgebra with Levi factor simple roots determined by the set

$$S_P = \{\alpha_1, \ldots, \alpha_{n-2}, \alpha_n\} \quad \text{and} \quad \mathfrak{q} = \mathfrak{l}_Q \oplus \mathfrak{n}_Q$$

will be the parabolic subalgebra with Levi factor simple roots determined by the set $S_Q = \{\alpha_1, \ldots, \alpha_{n-1}\}$.

Recall the nilpotent orbits in $\mathfrak{so}_{2n}$ are parametrized by partitions $\mathbf{d} = [d_1, \ldots, d_{2n}]$ of $2n$ in which even parts occur with even multiplicity, except that very even partitions (those with only even parts) correspond to two orbits; see

[17, §5.1]. Consider the partitions

$$d_m = 51^{2n-5},$$
$$d_P = 3^2 1^{2n-6},$$
$$d_Q = 31^{2n-3}.$$

If X_m, X_P and X_Q are corresponding nilpotent elements, then by [17, §5.3] the weighted Dynkin diagrams attached to these nilpotents will be

$$\mathcal{O}_{d_m} = 2 \quad 2 \quad 0 \quad \cdots \quad 0 \begin{matrix} 0 \\ 0 \end{matrix}$$

$$\mathcal{O}_{d_P} = 0 \quad 2 \quad 0 \quad \cdots \quad 0 \begin{matrix} 0 \\ 0 \end{matrix}$$

$$\mathcal{O}_{d_Q} = 2 \quad 0 \quad 0 \quad \cdots \quad 0 \begin{matrix} 0 \\ 0 \end{matrix}$$

LEMMA 5.1. *The subalgebras* $\mathfrak{p}_m$, $\mathfrak{p}$ *and* $\mathfrak{q}$ *are even Jacobson–Morozov parabolic subalgebras of* $\mathfrak{so}_{2n}$. *In fact,* $\mathfrak{p}_{X_m} = \mathfrak{p}_m$, $\mathfrak{p}_{X_P} = \mathfrak{p}$ *and* $\mathfrak{p}_{X_Q} = \mathfrak{q}$. *The Richardson orbits attached to these three parabolic subalgebras have dimensions* $\dim \mathcal{O}_{\mathfrak{p}_m} = 8n - 12$, $\dim \mathcal{O}_{\mathfrak{p}} = 8n - 14$, *and* $\dim \mathcal{O}_{\mathfrak{q}} = 4n - 4$. *In particular, all three subalgebras admit admissible Whittaker datum for G.*

Step 2. It is useful to have on hand a solid understanding of the categories $\mathcal{O}'(\mathfrak{g}, \mathfrak{p}_m)$, $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$ and $\mathcal{O}'(\mathfrak{g}, \mathfrak{q})$. The parametrizing posets W^P and W^Q for the categories $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$ and $\mathcal{O}'(\mathfrak{g}, \mathfrak{q})$ are discussed thoroughly in [3, §3] and [13, Fig. 8.4], respectively. The reader needs to make two observations when converting [3] and [13] information to this paper. First, take note of our initial labeling of the Satake diagram, which is opposite the convention in the referenced papers. Secondly, the diagram in [13, Fig. 8.4] is for $\mathfrak{g} = \mathfrak{so}_{2n+2}$. The poset W^{P_m} will consist of two copies of W^P pasted together

$$W^{P_m} = W^P_{\text{top}} \cup W^P_{\text{bott}},$$

where $W^P = W^P_{\text{top}} = W^P_{\text{bott}}$ as labeled posets. The manner in which these two posets are attached to one another is best illustrated by consulting and using the labels in [4, Fig. 2.1]. Along the right-hand edge of W^P, consider the $2n$ parameters

$$(n-1, n-2), (n-2, n-3), (n-3, n-4), \ldots, (2, 1), (1, 0), (1, 0^*);$$
$$(0, 1^*), (0^*, 1^*), (1^*, 2^*), \ldots, (n-4^*, n-3^*), (n-3^*, n-2^*), (n-2^*, n-1^*).$$

Label these parameters as $w_1, \ldots, w_n, y_1, \ldots, y_n$ in $W^P = W^P_{\text{top}}$ and $\bar{w}_1, \ldots, \bar{w}_n$, $\bar{y}_1, \ldots, \bar{y}_n$ in $W^P = W^P_{\text{bott}}$. The additional weak order relations in W^{P_m}, beyond those in W^P_{top} and W^P_{bott}, are as follows

$$w_i \xrightarrow[n+1-i]{} \bar{w}_i \quad \text{and} \quad y_i \xrightarrow[i]{} \bar{y}_i, \quad 1 \leqslant i \leqslant n.$$

The socular sets in W^P and W^Q are given in [4, §2] and [13, Fig. 8.8], respectively. The bottom right cell in W^{P_m} (the socular set) is described as follows. If we index $W^P = W^P_{\text{bott}}$ as in [3, Fig. 4.3], then $W^{P_m}_{\text{soc}} = \{y \mid y \leqslant (1^*, 2^*)\}$.

Step 3. On the Harish-Chandra module level, recall the parametrization of $\mathscr{D}$ in [5, (2.10)]. Using the notation of this reference, recall that $\mathscr{D} = \mathscr{D}_0 \cup \mathscr{D}_1$, where $\mathscr{D}_0$ is the block corresponding to the finite dimensional module F and $\mathscr{D}_1$ is its complement. Following the notation of [5, (2.10)], we write $\mathscr{C}^G_{(i,j)}, \mathscr{C}^G_{\hat{i}}$, etc. to denote the double cell in $\mathscr{D}$ containing $\delta_{(i,j)}, \delta_{\hat{i}}$, etc.

LEMMA 5.2. (i) *The block $\mathscr{D}_0$ is a union of nine double cells as follows*

$$\mathscr{C}^G_{\hat{1}}, \quad \mathscr{C}^G_{\widehat{2n}}$$
$$\mathscr{C}^G_{\hat{2}}, \quad \mathscr{C}^G_{\widehat{2n-1}}$$
$$\mathscr{C}^G_{(n-1,0^*)}, \quad \mathscr{C}^G_{\overline{(n-1,0^*)}}$$
$$\mathscr{C}^G_{(1,n-1^\#)}$$
$$\mathscr{C}^G_{\hat{n}}$$
$$\mathscr{C}^G_{(2,n-2^\#)}.$$

(ii) *The block $\mathscr{D}_1$ is a union of two double cells as follows*

$$\mathscr{D}_1 = \mathscr{C}^G_{(1,1)^\flat} \cup \mathscr{C}^G_{(n-1,1)^\flat}.$$

EXAMPLE. In the case of $SO_e(2,6)$, we refer the reader to [5, Fig. 2.6] for the diagram of $\mathscr{D}_o$. Using these labels, the double cell decomposition is as follows

$$\mathscr{C}^G_{\hat{1}} = \{\hat{1}, 32^*, 31^*\}, \quad \mathscr{C}^G_{\hat{8}} = \{\hat{8}, \overline{32^*}, \overline{31^*}\}$$
$$\mathscr{C}^G_{\hat{2}} = \{\hat{2}, 21^*, \hat{3}\}, \quad \mathscr{C}^G_{\hat{7}} = \{\hat{7}, \overline{21^*}, \hat{6}\}$$
$$\mathscr{G}^G_{30^*} = \{30^*, 30, 31, 32\}, \quad \mathscr{C}^G_{\overline{30^*}} = \{\overline{30^*}, \overline{30}, \overline{31}, \overline{32}\}$$
$$\mathscr{C}^G_{13^\#} = \{13^\#\}$$
$$\mathscr{C}^G_{\hat{4}} = \{\hat{4}, \hat{5}, 10^*, 10, \overline{10}, \overline{10^*}, 20^*, 20, \overline{20^*}, \overline{20}, 31^\#, 21, \overline{21}, 21^\#\}$$
$$\mathscr{C}^G_{22^\#} = \{22^\#, 12^\#, 11^\#\}.$$

Referring to [5, Fig. 2.4] with $n = 4$, the double cell decomposition of $\mathscr{D}_1$ is as follows

$$\mathscr{C}^G_{31^b} = \{31^b, 21^b, 22^b\}, \quad \mathscr{C}^G_{11^b} = \{11^b, 12^b, 13^b\}.$$

Step 4. We now proceed to verify (1.7) by considering a number of subcases.

Proof of (1.7). First, suppose that $\bar{\pi}(\delta)$ is a highest weight module. In our notation, this means that δ lies in one of the following sets:

(a) $\mathscr{C}^G_{\hat{1}} \cup \mathscr{C}^G_{\widehat{2n}}$,
(b) $\mathscr{C}^G_{(n-1,0^*)} \cup \mathscr{C}^G_{\overline{(n-1,0^*)}}$
(c) $\mathscr{C}^G_{(1,n-1\#)}$.

As a parameter set, we can identify

$$W^Q = \mathscr{C}^G_{\hat{1}} \cup \mathscr{C}^G_{(n-1,0^*)} \cup \mathscr{C}^G_{(1,n-1\#)} \quad \text{and} \quad W^Q = \mathscr{C}^G_{\widehat{2n}} \cup \mathscr{C}^G_{\overline{(n-1,0^*)}} \cup \mathscr{C}^G_{(1,n-1\#)}.$$

In so doing, [12] tells us that the Jacquet module $J(\bar{\pi}(\delta))$ of any δ in (a), (b) or (c) is just the corresponding irreducible module in $\mathcal{O}'(\mathfrak{g}, \mathfrak{q})$. Using [13, Fig. 8.8], it follows that the GK-dimension of $\bar{\pi}(\delta)$ with δ of type (a) is $2n - 2$, while case (b) gives GK-dimension $2n - 3$ and case (c) has GK-dimension 0. By (5.1), only the two double cells in (a) are of interest. But, as just noted, [12] shows that $J(\bar{\pi}_{\hat{1}}) = J(\bar{\pi}_{\widehat{2n}}) = L_{\mathfrak{q}}(e)$. Since e is the bottom element in W^Q_{soc}, this proves $\bar{\pi}_{\hat{1}}$ and $\bar{\pi}_{\widehat{2n}}$ have property $\mathscr{W}^Q_{\text{soc}}$ and so (3.3) verifies (1.7) in this case.

The second case to consider is when δ lies in $\mathscr{C}^G_{\hat{2}} \cup \mathscr{C}^G_{\widehat{2n-1}}$. By the Hecht–Schmid character identity, there is a generalized principal series representation $I_P(\sigma \otimes \nu)$ of G having the structure

$\bar{\pi}_{(n-1,n-2^*)}$
$\bar{\pi}_{\hat{1}} \oplus \bar{\pi}_{\hat{2}}$

and a similar remark holds for a generalized principal series representation involving $\bar{\pi}_{\widehat{2n}}$ and $\bar{\pi}_{\widehat{2n-1}}$. By [31],

$$\operatorname{Dim} I_P(\sigma \otimes \nu) = \dim \mathfrak{n}_P + \dim \sigma = 4n - 7 + 1 = 4n - 6.$$

So, $\mathscr{C}^G_{\hat{2}} \cup \mathscr{C}^G_{\widehat{2n-1}}$ indexes quasi-large representations of G; i.e., those of maximal possible GK dimension. By Step 5 of Section 4, (1.7) holds in this case.

Thirdly, consider the big double cell $\mathscr{C}^G_{\hat{n}}$. Notice that $\tau(\bar{\pi}_{\hat{n}}) = \{s_3, s_4, \ldots, s_n\}$. By tau invariant and length considerations (as in (i) of Step 4 in Section 4), the remarks in Step 2 above lead one to conclude that there are only three *possible* irreducible composition factors of $J(\bar{\pi}_{\hat{n}})$; label these as L_y, L_w, L_z. The theory outlined in Step 4 of Section 4 shows

$$l_g(\delta_{\hat{n}}) = l(y) = l(w) = l(z) + 1$$

$$z = w_{P_m} w' s_3,$$

where w' is the maximal element in $W^{P_m}_{\text{soc}}$. In addition, L_y and L_w both occur as composition factors exactly once (they are the leading terms) and the factor L_z occurs once as a smooth composition factor in the bottom weight layer. Remarks in Step 2 above show that $z \in \mathscr{W}^P_{\text{soc}}$. This shows that $J(\bar{\pi}_{\hat{n}})$ has property $\mathscr{W}^P_{\text{soc}}$, so by (3.3), this verifies (1.7) in this case.

Next, consider the double cell $\mathscr{C}^G_{(2,n-2^*)}$. Using Step 2, one checks

$$\tau(\bar{\pi}_{(2,n-2^*)}) = \{s_1, s_2, \ldots, s_{n-1}\},$$

whence there are only three *possible* composition factors of $J(\bar{\pi}_{(2,n-2^*)})$, denoted L_w, L_y and L_z, satisfying

$$w = w_{P_m} w^{P_m} s_{n-1} s_n,$$

$$y = w_{P_m} w^{P_m} s_{n-1} s_{n-2} \cdots s_2 s_1 s_3 s_4 \cdots s_{n-1} s_n,$$

$$z = w_Q,$$

$$l(w) = l_g(\delta_{(2,n-2^*)}),$$

$$l(y) = l(z) + 1.$$

Further, applying the theory reviewed in Section 4 shows that $J(\bar{\pi}_{(2,n-2^*)})$ has a weight filtration of the form $k \cdot L_y < t \cdot L_z \oplus L_w < k \cdot L_y$. From this information, we cannot quite conclude that $\bar{\pi}_{(2,n-2^*)}$ has property $\mathfrak{q}$, even though z indexes the bottom socular element in $\mathscr{W}^Q_{\text{soc}}$. To do this, we need to show that at least one copy of L_z drops into the socle; equivalently, that one copy of L_z does not extend L_y inside this Jacquet module. (It is true that the two modules extend in $\mathscr{O}'(\mathfrak{g}, \mathfrak{q})$.) We first show that $\bar{\pi}_{(2,n-2^*)}$ occurs as a quotient of the principal series representation $I_{P_m}(w_{Q,P_m})$ of G determined by w_{Q,P_m}; recall (2.1). To verify this, we use the generalized principal series filtrations in [5], combined with standard wall crossing arguments and the theory of [11] to compute a weight filtration on $I_{P_m}(w_{Q,P_m})$ and draw the desired conclusion. Then by (1.1b), there exists a non-zero map

$$N_z \mapsto J(\bar{\pi}_{(2,n-2)^*}).$$

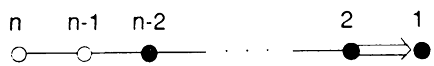
This shows that one copy of L_z splits off as an indecomposable summand of

$J(\bar{\pi}_{(2,n-2^{\#})})$, since $l(y) > l(z)$. By (3.5) we conclude (1.7). (Although not needed for our proof, one can additionally check that $t = 2$. Furthermore, $\hat{\mathbb{J}}$ maps onto a right cell $\mathscr{C}_y^R \neq \mathscr{W}_{\rm soc}^Q$, whereas $\hat{\mathbb{J}}'$ would map onto $\mathscr{C}_y^R \cup \mathscr{W}_{\rm soc}^Q$.)

It remains to consider the block $\mathscr{D}_1$. By (5.2)(ii), we have two cases to consider. The module $\bar{\pi}_{(n-1,1)^\flat}$ is an irreducible standard module, which in this case is an irreducible principal series representation. The GK-dimension of such a module is $\dim \mathfrak{n}_m$, which shows the double cell $\mathscr{C}^G_{(n-1,1)^\flat}$ indexes quasi-large modules. We can apply Step 5 of Section 4 and verify (1.7) in this case. Finally, consider the module $\bar{\pi}_{(1,1)^\flat}$. Using the theory discussed in Section 4, one can compute the smooth skeleton of the Jacquet module and verify that the filtration has five levels with the maximal element z of $\mathscr{W}^Q_{\rm soc}$ indexing a bottom smooth weight layer factor of $J(\bar{\pi}_{11^\flat})$. We claim L_z is in the bottom weight layer of $J(\bar{\pi}_{11^\flat})$; a priori there can be many non-smooth weight layers below the bottom smooth layer. It is enough to show $ll_{asy}(\delta_{11^\flat}) = 5$. By (2.4), it is then enough to show $ll_{asy}(\delta_{(1,n-1)^\flat}) \leqslant 5$; note that $\bar{\pi}_{(1,n-1)^\flat}$ is a *largest growth representation*. Since $\tau(\bar{\pi}_{(1,n-1)^\flat}) = \{s_1, \ldots, s_{n-1}\}$, one can check there are at most five irreducible composition factors of $J(\bar{\pi}_{(1,n-1)^\flat})$. Two of these factors fit into a three layer smooth skeleton. By parity considerations, either the remaining factors fit into a 5-layer filtration with L_{w_Q} a bottom weight layer factor, or, the weight filtration on $J(\bar{\pi}_{(1,n-1)^\flat})$ has length 3. We now apply (3.3) to finish (1.7). □

6. $SO_e(2, 2n-1)$

Throughout this section, we assume $G = SO_e(2, 2n-1)$ and label the corresponding Satake diagram as follows



The arguments in this section are very similar to those in Section 5.

Step 1. There are three parabolic subalgebras defined over $\mathbb{R}$ of interest: the minimal parabolic subalgebra $\mathfrak{p}_m$ (determined by the set of simple roots $S_m = \{\alpha_1, \ldots, \alpha_{n-2}\}$), the parabolic subalgebra $\mathfrak{p} = \mathfrak{l}_P \oplus \mathfrak{n}_P$ (with Levi factor simple roots determined by the set

$$S_P = \{\alpha_1, \ldots, \alpha_{n-2}, \alpha_n\}) \quad \text{and} \quad \mathfrak{q} = \mathfrak{l}_Q \oplus \mathfrak{n}_Q$$

(with Levi factor simple roots determined by the set $S_Q = \{\alpha_1, \ldots, \alpha_{n-1}\}$).

The nilpotent orbits in $\mathfrak{so}_{2n+1}$ are in one-to-one correspondence with the set of partitions $\mathbf{d} = [d_1, \ldots, d_{2n+1}]$ of $2n+1$ in which even parts occur with even multiplicity; see [17, §5.1]. Consider the partitions

$$d_m = 51^{2n-4},$$
$$d_P = 3^2 1^{2n-5},$$
$$d_Q = 31^{2n-2}.$$

If X_m, X_P and X_Q are corresponding nilpotent elements, then as in Section 5, we arrive at

LEMMA 6.1. *The subalgebras $\mathfrak{p}_m$, $\mathfrak{p}$ and $\mathfrak{q}$ are even Jacobson–Morozov parabolic subalgebras of $\mathfrak{so}_{2n+1}$. In fact, $\mathfrak{p}_{X_m} = \mathfrak{p}_m$, $\mathfrak{p}_{X_P} = \mathfrak{p}$ and $\mathfrak{p}_{X_Q} = \mathfrak{q}$. The Richardson orbits attached to these three parabolic subalgebras have dimensions $\dim \mathcal{O}_{\mathfrak{p}_m} = 8n - 8$, $\dim \mathcal{O}_{\mathfrak{p}} = 8n - 10$, and $\dim \mathcal{O}_{\mathfrak{q}} = 4n - 2$. In particular, all three subalgebras admit admissible Whittaker datum for G.*

Step 2. The parametrizing posets W^P and W^Q for the categories $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$ and $\mathcal{O}'(\mathfrak{g}, \mathfrak{q})$ are discussed thoroughly in [13, Fig. 8.6] and [13, Fig. 8.4], respectively. The reader needs to make two observations when converting [13] information to this paper: first, take note of our initial labeling of the Satake diagram, which is opposite the convention in the referenced paper. Secondly, the diagram in [13, Fig. 8.6] describes the poset of type $(B_{n+1}, A_1 \times B_{n-1})$, so a shift in complex rank must be taken into account. The poset W^{P_m} will consist of two copies of W^P pasted together

$$W^{P_m} = W^P_{\text{top}} \cup W^P_{\text{bott}},$$

where $W^P = W^P_{\text{top}} = W^P_{\text{bott}}$ as labeled posets. The manner in which these two posets are attached to one another is best illustrated by consulting and using the labels for W^P in [13, Fig. 8.6] (where "n" is replaced by "$n-1$", to account for the shift in complex dimension). Along the right-hand edge of W^P, consider the parameters

$$01, 12, \dots, (n-2, n-1);$$
$$(n-2, n-1)^\theta, \dots, 12^\theta, 01^\theta.$$

Next, we label these parameters $w_1, \dots, w_{n-1}$, $y_1, \dots, y_{n-1}$ in $W^P = W^P_{\text{top}}$ and $\bar{w}_1, \dots, \bar{w}_{n-1}$, $\bar{y}_1, \dots, \bar{y}_{n-1}$ in $W^P = W^P_{\text{bott}}$. The additional weak order relations in W^{P_m}, beyond those in W^P_{top} and W^P_{bott}, are as follows

$$w_i \xrightarrow[n+1-i]{} \bar{w}_i \quad \text{and} \quad y_i \xrightarrow[i+1]{} \bar{y}_i, \quad 1 \leqslant i \leqslant n-1.$$

The socular sets in W^P and W^Q are given in [13, Fig. 8.11] and [13, Fig. 8.8], respectively. The socular set in W^{P_m} is described as follows: if we identify $W^P = W^P_{\text{bott}}$, then

$$W^{P_m}_{\text{soc}} = \{w \in W^P_{\text{bott}} \mid l(w) \leqslant 2n - 4\}.$$

Step 3. On the Harish-Chandra module level, recall the parametrization of $\mathscr{D}$ in [5, (2.8)]. We use the notation $\mathscr{C}^G_{(i,j)}, \mathscr{C}^G_{i*}$, etc. to denote the double cell in $\mathscr{D}$ containing $\delta_{(i,j)}, \delta_{i*}$, etc.

LEMMA 6.2. *We have $\mathscr{D}$ is a union of seven double cells as follows*

$$\mathscr{C}^G_{1*}, \quad \mathscr{C}^G_{2n*}$$

$$\mathscr{C}^G_{2*}, \quad \mathscr{C}^G_{2n-1*}$$

$$\mathscr{C}^G_{\hat{1}}$$

$$\mathscr{C}^G_{(1,n-1)^\#},$$

$$\mathscr{C}^G_{(1,n-1^\flat)}.$$

EXAMPLE. Consider $G = SO_e(2, 7)$ and recall the diagram in [5, Fig. 2.3]. Using these labels, the double cell decomposition of (6.2) is as follows

$$\mathscr{C}^G_{1*} = \{1*, 06, 05, 04, 03, 02, 01\}, \quad \mathscr{C}^G_{8*} = \{8*, \overline{06}, \overline{05}, \overline{04}, \overline{03}, \overline{02}, \overline{01}\}$$

$$\mathscr{C}^G_{2*} = \{2*, 3*, 4*, 15, 24, 14, 23, 13, 12\}$$

$$\mathscr{C}^G_{7*} = \{5*, 6*, 7*, \overline{15}, \overline{24}, \overline{14}, \overline{23}, \overline{13}, \overline{12}\}$$

$$\mathscr{C}^G_{\hat{1}} = \{\hat{1}, \hat{2}, \hat{3}, 31^\#, 31^\flat, 21^\#, 21^\flat, 22^\flat\}$$

$$\mathscr{C}^G_{13^\#} = \{13^\#\}$$

$$\mathscr{C}^G_{13^\flat} = \{13^\flat, 12^\flat, 11^\flat, \hat{4}, 11^\#, 12^\#, 22^\#\}.$$

Step 4. We now proceed to verify (1.7) by considering a number of subcases.

Proof of (1.7). First, suppose that $\bar{\pi}(\delta)$ is a highest weight module. This means that δ lies in one of the following sets

(a) $\mathscr{C}^G_{1*} \cup \mathscr{C}^G_{2n*}$,
(b) $\mathscr{C}^G_{(1,n-1^\#)}$.

As a poset, we can identify $W^Q = \mathscr{C}^G_{1*} \cup \mathscr{C}^G_{(1,n-1^\#)}$ and $W^Q = \mathscr{C}^G_{2n*} \cup \mathscr{C}^G_{(1,n-1^\#)}$. Now, argue just as in the highest weight module case in Section 5; we omit the details. Conclude that $\bar{\pi}(\delta)$ of type (a) have property $\mathscr{W}^Q_{\text{soc}}$ and (1.7) holds.

The second case to consider is when δ lies in $\mathscr{C}^G(2^*) \cup \mathscr{C}^G(2n-1^*)$. As in Section 5, use a Hecht–Schmid character identity to see these correspond to quasi-large representations of G; i.e., those of maximal possible GK dimension. By Step 5 of Section 4, (1.7) holds in this case.

Thirdly, consider the double cell $\mathscr{C}^G_{\hat{1}}$. Notice that $\tau(\bar{\pi}_{\hat{1}}) = \{s_1, s_3, s_4, \ldots, s_n\}$. By tau invariant and length considerations (as in (i) of Step 4 in Section 4), the remarks in Step 2 above lead one to conclude that there are only three *possible* irreducible composition factors of $J(\bar{\pi}_{\hat{1}})$; label these as L_y, L_w, L_z. The theory outlined in Step 4 of Section 4 shows

$$l_g(\delta_{\hat{1}}) = l(w) = l(y) + 1 = l(z) + 3$$
$$w = w_{P_m} s_n s_{n-1} \cdots s_2 s_1 s_n s_{n-1} \cdots s_3$$
$$y = w_{P_m} s_{n-1} \cdots s_2 s_1 s_n s_{n-1} \cdots s_3$$
$$z = w s_1 s_3 s_2.$$

It follows from Step 2 that $z \in \mathscr{W}^P_{\text{soc}}$. Also, the theory of smooth exponents shows that L_w is a leading term with multiplicity two in the middle layer, L_y is a smooth exponent with multiplicity one in weight layers ± 1 and L_z is a non-smooth composition factor of multiplicity $t \geqslant 0$ in weight layers ± 1. In summary, this shows that $J(\bar{\pi}_{\hat{1}})$ has a weight filtration of the form

$t \cdot L_z \oplus L_y$
$L_w \oplus L_w$
$t \cdot L_z \oplus L_y$

By (3.3), we will have verified (1.7) on the double cell $\mathscr{C}^G_{\hat{1}}$ if we can show $t > 0$. To do this, as we reasoned at the end of Section 5, it is enough to show that $\bar{\pi}_{\hat{1}}$ occurs as a quotient of the principal series representation $I_{P_m}(w_{P_m} z)$ indexed by $w_{P_m} z$; equivalently, that $\bar{\pi}_{\hat{1}}$ occurs as a submodule of $I_{P_m}(d(w_{P_m} z))$, where $d(\cdot)$ is the duality on the poset W^{P_m}. This is easily checked using the generalized principal series filtrations in [5] and standard wall crossing arguments.

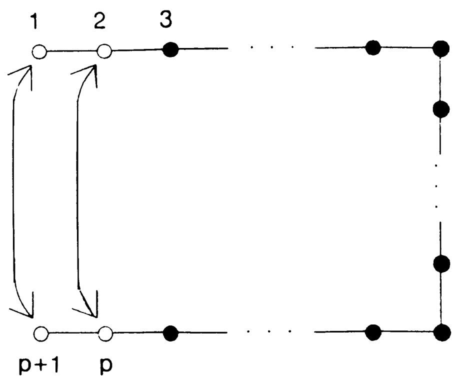
Finally, consider the double cell $\mathscr{C}^G_{(1,n-1^\flat)}$. Argue just as in the previous case to obtain a three level smooth skeleton for $J(\bar{\pi}_{1,n-1^\flat})$ with L_z in the bottom smooth weight layer and z the maximal element in $\mathscr{W}^Q_{\text{soc}}$. By the proof of (2.4) and the previous case,

$$ll_{asy}(\delta_{1,n-1^\flat}) \leqslant ll_{asy}(\delta_{\hat{1}}) = 3.$$

Thus, the smooth factor L_z actually lies in the bottom weight layer and we are done by (3.3).

7. $SU(2, p)$

Throughout this section, we assume $G = SU(2, p)$, $p \geqslant 2$ and proceed to verify (1.7). The essential ideas are of an inductive nature, using ideas from [5]. Begin by recalling the Satake diagram



There are only three proper standard parabolic subalgebras of $\mathfrak{g}$ defined over $\mathbb{R}$: $\mathfrak{p}_m$ with Levi factor simple roots $\{\alpha_3, \ldots, \alpha_{p-1}\}$; $\mathfrak{p}$ with Levi simple roots $\{\alpha_2, \ldots, \alpha_p\}$; $\mathfrak{q}$ with Levi factor omitting the simple roots α_2 and α_p. Notice, if $p = 2, 3$, then $\mathfrak{p}_m = \mathfrak{b}$ and G is quasi-split.

Recall, the nilpotent orbits in $\mathfrak{sl}_{p+2}$ are in one-to-one correspondence with partitions of $p + 2$ [17]. Define the following partitions of $p + 2$

$$d_m = \begin{cases} 4 & \text{for } p = 2 \\ 51^{p-3} & \text{otherwise,} \end{cases}$$

$$d_P = 31^{p-1},$$

$$d_Q = \begin{cases} 2^2 & \text{if } p = 2 \\ 32 & \text{if } p = 3 \\ 3^2 1^{p-4} & \text{if } p \geqslant 4. \end{cases}$$

By an old result of Kraft [17, §7.2], it is easy to see that

$$\mathcal{O}_{d_m} = \mathcal{O}_{\mathfrak{p}_m}, \quad \mathcal{O}_{d_P} = \mathcal{O}_{\mathfrak{p}} \quad \text{and} \quad \mathcal{O}_{d_Q} = \mathcal{O}_{\mathfrak{q}}.$$

In turn, by [17, §3.6], we can compute the weighted Dynkin diagrams of these three nilpotent orbits. We easily arrive at the following conclusion.

LEMMA 7.1. *The parabolic subalgebras* $\mathfrak{p}_m$ *and* $\mathfrak{p}$ *are even Jacobson–Morozov parabolic subalgebras. The subalgebra* $\mathfrak{q}$ *is an even Jacobson–Morozov parabolic subalgebra if and only if* $p \neq 3$.

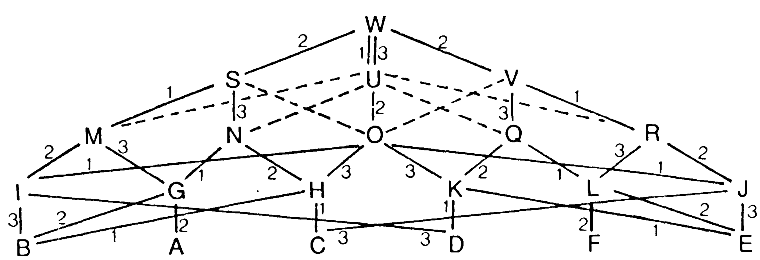
Let $\mathscr{D}_o$ denote the block of the finite-dimensional representation F. A combinatorial parametrization of the set $\mathscr{D}_o$ is described in [18] as follows: to each element of $\mathscr{D}_o$ is associated an arrangement δ of the numbers $1, 2, \ldots, p+2$ in singles or pairs. Each single number has a $+$ sign or a $-$ sign attached to it. Let r be the number of pairs in δ; then we require that $0 \leqslant r \leqslant 2$, and that the number of $+$ signs in δ equal $p - r$. Although the ordering of the single numbers, pairs, and of elements within the pairs in δ is immaterial, we shall always write parameters in the following "standard ordering": the sequence of numbers obtained by deleting the second coordinate of each pair in δ, should increase from left to right; and the second coordinate of each pair in δ should be greater than the corresponding first coordinate. For example, a typical δ for $SU(2,4)$ is $\delta = 1^+2^-(36)4^+5^+$. We refer the reader to [5, (2.7)] for a description of the action of simple root reflections on $\mathscr{D}_o$, the notions of *real, imaginary,* or *complex* reflections for δ, etc. The following result is discussed in [18].

LEMMA 7.2. *If* $p > 2$, *then* $\mathscr{D} = \mathscr{D}_o$. *In case* $p = 2$, *then* $\mathscr{D} = \mathscr{D}_o \cup \mathscr{D}_1$, *where* $\mathscr{D}_1$ *is the Vogan IC4-dual (in the sense of* [34]) *of the fundamental block for* Spin(5, 1) *as parametrized in* [13, *Fig.* 4.3].

This lemma points to special considerations for $SU(2,2)$; we dispense with this case next.

LEMMA 7.3. *Matumoto's conjecture is true for* $SU(2,2)$.

Proof. A parametrization of the fundamental block $\mathscr{D}_o$ is given below. To eliminate confussion, the parameter labeled F in this diagram is *not* the fixed finite-dimensional module of G; in this picture W is the fixed finite



Block of the trivial representation of SU(2,2) dimensional module. The fundamental block decomposes into a union of 10 double cells; these are itemized below, together with their associated nilpotent orbits. This nilpotent orbit can be read off from our Jacquet module remarks.

By Step 5 of Section 4, (1.7) holds for $\mathscr{C}_B^G \cup \mathscr{C}_E^G$. From [12], $J(\bar{\pi}_A) = J(\bar{\pi}_F) = L_{s_1s_3}$ and $s_1s_3 \in \mathscr{W}_{\text{soc}}^Q$; by (3.3) we conclude that (1.7) holds on the double cells $\mathscr{C}_A^G \cup \mathscr{C}_F^G$. A direct calculation produces a weight filtration on $J(\bar{\pi}_O)$

$L_{s_2s_1s_3}$
$L_{w_os_1s_2} \oplus L_{w_os_3s_2} \oplus L_{s_1s_3} \oplus L_{s_1s_3}$
$L_{s_2s_1s_3}$

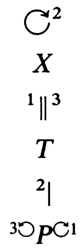
Since the dimension of the space of extensions between any two irreducible modules in $\mathcal{O}'(\mathfrak{g}, \mathfrak{p}_m)$ is at most one, at least one copy of $L_{s_1s_3}$ splits off as an indecomposable summand of $J(\bar{\pi}_O)$. Now apply (3.5) to verify (1.7) on the double cell $\mathscr{C}_O^G$. It only remains to consider $\mathscr{C}_C^G \cup \mathscr{C}_D^G$. We directly compute a weight filtration on $J(\bar{\pi}_C)$ as

L_{s_2}
$L_{s_1s_2} \oplus L_{s_3s_2}$
L_{s_2}

Since s_2 is the bottom element of $\mathscr{W}_{\text{soc}}^P$, we have shown that $\bar{\pi}_C$ has property $\mathscr{W}_{\text{soc}}^P$. The same argument applies to $\bar{\pi}_D$, which has a Jacquet module with an identical weight filtration. Apply (3.3), verifying (1.7) on $\mathscr{C}_C^G \cup \mathscr{C}_D^G$. This completes verification of Matumoto's conjecture on $\mathscr{D}_o$.

Double cell	Elements	Associated nilpotent
$\mathscr{C}_B^G$	$\{B\}$	$\mathcal{O}_4$
$\mathscr{C}_E^G$	$\{E\}$	$\mathcal{O}_4$
$\mathscr{C}_C^G$	$\{C, H, J\}$	$\mathcal{O}_{31}$
$\mathscr{C}_D^G$	$\{D, K, I\}$	$\mathcal{O}_{31}$
$\mathscr{C}_A^G$	$\{A, G\}$	$\mathcal{O}_{2^2}$
$\mathscr{C}_F^G$	$\{F, L\}$	$\mathcal{O}_{2^2}$
$\mathscr{C}_O^G$	$\{U, O\}$	$\mathcal{O}_{2^2}$
$\mathscr{C}_M^G$	$\{M, S, N\}$	$\mathcal{O}_{21^2}$
$\mathscr{C}_R^G$	$\{R, V, Q\}$	$\mathcal{O}_{21^2}$
$\mathscr{C}_W^G$	$\{W\}$	$\mathcal{O}_{1^4}$

By (7.2), parametrize $\mathscr{D}_1$ as follows



In this case, there are two double cells

$$\mathscr{C}_P^G = \{P\} \quad \text{and} \quad \mathscr{C}_T^G = \{T, X\}.$$

Tau invariant considerations tell us that $\bar{\pi}_P$ is a large representation, so Step 5 of Section 4 applies. A direct application of the theory in [8]–[10] shows that $J(\bar{\pi}_X)$ has a weight filtration

$L_{s_1s_3}$
$L_{w_os_2} \oplus L_{s_2s_1s_3}$
$L_{w_o} \oplus L_{w_os_1s_2} \oplus L_{w_os_3s_2} \oplus L_{s_1s_3} \oplus L_{s_1s_3}$
$L_{w_os_2} \oplus L_{s_2s_1s_3}$
$L_{s_1s_3}$

Since $s_1s_3 = w_Q \in \mathscr{W}_{\text{soc}}^Q$, we conclude that $\bar{\pi}_X$ has property $\mathscr{W}_{\text{soc}}^Q$. By (3.3), this verifies (1.7) on the double cell $\mathscr{C}_T^G$. □

In view of the above, we will set

$$\mathscr{D} = \mathscr{D}_o$$

for the remainder of this section and assume we are always working within the block of the finite dimensional module F. If $\bar{\pi}(\delta)$ is quasi-large we apply Step 5 of Section 4. The remaining cases arise when the associated nilpotent orbit in (1.1a) is either of the Richardson orbits $\mathcal{O}_{\mathrm{p}}$ or $\mathcal{O}_{\mathrm{q}}$.

The first step is to see what possible nilpotent orbits arise as $\mathcal{O}_{\bar{\pi}(\delta)}$ for some $\delta \in \mathscr{D}$. To do this, recall the notion of a tableau from [17]. In [18], it is shown how to associate a tableau $T(\delta)$ involving the numbers $1, \ldots, p+2$ to each $\delta \in \mathscr{D}$. Actually, Garfinkle associates a pair of tableaux to each δ; this is what is required to determine the wave front set of $\bar{\pi}(\delta)$, which is the closure of a real nilpotent orbit [17, §9]. However, the associated variety of the annihilator of $\bar{\pi}(\delta)$, being the closure of a complex nilpotent orbit, is determined simply by the tableau of numbers; in fact, really just by the shape of the tableau. Using the model [5, (2.7)] and the really just by the shape of the tableau. Using the model [5,(2.7)] and the algorithm of [18] it is an elementary exercise to verify the following. Recall, using our parameter set for $\mathscr{D}$, δ determines a discrete series representation if and only if it involves no paired entries.

LEMMA 7.4. (i) *Assume $G = SU(2,3)$ and $\delta \in \mathscr{D}$, then the associated nilpotent orbit $\mathcal{O}_{\bar{\pi}(\delta)}$ is among the following (and each case arises)*

$$\mathcal{O}_5, \mathcal{O}_{41}, \mathcal{O}_{32}, \mathcal{O}_{31^2}, \mathcal{O}_{2^21}, \mathcal{O}_{21^3}, \mathcal{O}_{1^5}.$$

In fact, all but the last two orbits arise from discrete series parameters.

(ii) *Assume $G = SU(2,p)$, $p \geqslant 4$ and $\delta \in \mathscr{D}$, then the associated nilpotent orbit $\mathcal{O}_{\bar{\pi}(\delta)}$ is among the following (and each case arises)*

$$\mathcal{O}_{51^{p-3}}, \mathcal{O}_{3^21^{p-4}}, \mathcal{O}_{41^{p-2}}, \mathcal{O}_{321^{p-3}}, \mathcal{O}_{31^{p-1}}, \mathcal{O}_{2^21^{p-2}}, \mathcal{O}_{21^p}, \mathcal{O}_{1^{p+2}}.$$

In fact, all but the last two orbits arise from discrete series parameters.

REMARKS. Recalling the proof of (7.3), in the cases of $SU(2,2)$ and $SU(2,3)$ every nilpotent orbit actually arises as the associated nilpotent orbit of some $\bar{\pi}(\delta)$. For $p > 3$ this is no longer true.

The case of $\mathcal{O}_{\bar{\pi}(\delta)} = \mathcal{O}_{\mathrm{p}}$

The result in (7.4) does not quite determine the double cells with a given associated nilpotent orbit. In the current case, we seek to determine all parameters $\delta \in \mathscr{D}$ with $T(\delta)$ having shape 31^{p-1}. Applying the algorithm of [18] we obtain the following result.

LEMMA 7.5. *Assume $p \geqslant 3$. There are exactly two double cells $\mathscr{C}^G_{\bar{1}234\cdots p+1\overline{p+2}}$ and $\mathscr{C}^G_{1\bar{2}(3,p+2)45\cdots p+1}$ in $\mathscr{D}$ with associated nilpotent orbit $\mathcal{O}_{\mathrm{p}}$.*

LEMMA 7.6. *If $\mathcal{O}_{\bar{\pi}(\delta)} = \mathcal{O}_{\mathrm{p}}$, then Matumoto's conjecture is true.*

Proof. Begin with $\mathscr{C}^G_{\bar{1}234\cdots p+1\overline{p+2}}$. The representative element $\delta_{\bar{1}234\cdots p+1\overline{p+2}}$ of this double cell indexes a discrete series parameter of G. It is easy to compute the complement of the tau invariant to be

$$\tau(\delta_{\bar{1}234\cdots p+1\overline{p+2}})^c = \{s_1, s_{p+1}\}.$$

Next, applying the considerations in Step 4 of Section 4, we find that $J(\bar{\pi}_{\bar{1}234\cdots p+1\overline{p+2}})$ has the smooth skeleton of the weight filtration of the form

$$L_z < L_w \oplus L_y < L_z,$$

where

$$z = w_{P_m} w_{P,P_m} = w_P,$$

$$w = w_{P_m} s_1 w_{P,P_m}$$

$$y = w_{P_m} s_{p+1} w_{P,P_m}$$

$$w_{P,P_m} = s_2 s_3 s_4 \cdots s_p s_{p-1} \cdots s_3 s_2.$$

Now, the claim is that L_z is actually contained in the socle of

$$J(\bar{\pi}_{\bar{1}234\cdots p+1\overline{p+2}}).$$

To see this, note that any composition factor in the socle must have the same tau invariant as $\bar{\pi}_{\bar{1}234\cdots p+1\overline{p+2}}$. Thus, one needs to determine the elements $x \in \mathscr{W}^{P_m}$ which satisfy both this tau invariant condition and $l(x) \leqslant r_G = l_g(\delta_{\bar{1}234\cdots p+1\overline{p+2}})$. One checks that $x = z = w_P$, y or w. By [10], comparing the lengths of w, y and z, the precise arrangement of all occurences of L_y, L_w and L_z in a weight filtration of the Jacquet module will be given by the smooth skeleton. In turn, the bottom weight layer (which is contained in the socle) must then be a multiple of L_z. Thus, in view of (3.3), this proves that property $\mathscr{W}^P_{\text{soc}}$ holds on $\mathscr{C}^G_{\bar{1}234\cdots p+1\overline{p+2}}$.

Next, consider the double cell $\mathscr{C}^G_{1\bar{2}(3,p+2)45\cdots p+1}$ and the representative element $\delta_{1\bar{2}(3,p+2)45\cdots p+1}$. We compute that the complement of the tau invariant is

$$\tau(\delta_{1\bar{2}(3,p+2)45\cdots p+1})^c = \{s_1, s_2\}.$$

Applying the theory of Step 4 in Section 4, we find that $J(\bar{\pi}_{1\bar{2}(3,p+2)45\cdots p+1})$ has the smooth skeleton of the weight filtration of the form

$$L_z \oplus L_x < L_w \oplus L_y \oplus L_u < L_z \oplus L_x,$$

where

$$y = s_1 w_P s_{p+1} s_p \cdots s_3,$$

$$z = w_P s_{p+1} s_p \cdots s_3,$$

$$x = s_1 (w_P s_2) s_{p+1} s_p \cdots s_3,$$

$$w = w_P s_{p+1} s_p \cdots s_3 s_2$$

$$u = s_1 (w_P s_2) s_{p+1} s_p \cdots s_3 s_1.$$

Note that $l(w_P s_2) = l(w_P) - 1$, so that the middle weight layer factors in the above smooth skeleton are indeed leading terms (as they must be, by the definition of smoothness). Also, observe that $z \in \mathscr{W}^P_{\text{soc}}$. By (3.3), Matumoto's conjecture will hold on the double cell $\mathscr{C}^G_{1\bar{2}(3,p+2)45\cdots p+1}$ if we can verify that L_z is in the bottom weight layer of the Jacquet module. To see this, we need to study the intersection $\mathscr{C}^L_z \cap \mathscr{W}^{P_m}$. The tableau of the left cell of z is

$$\begin{matrix} 1 & 2 & 3 \\ 4 & & \\ 5 & & \\ \vdots & & \\ p+2, & & \end{matrix}$$

which coincides with the tableau of the double cell $\mathscr{C}^G_{1\bar{2}(3,p+2)45\cdots p+1}$, since every irreducible submodule of the Jacquet module determines the same primitive ideal as $\bar{\pi}_{1\bar{2}(3,p+2)45\cdots p+1}$. Using *Knuth equivalences* from the elementary theory of combinatorics, write out all posssible Weyl group elements with the above tableau and then check that the element z is a minimal element in $\mathscr{C}^L_z \cap \mathscr{W}^{P_m}$. So, the bottom weight layer of the smooth skeleton is in fact the bottom weight layer of the full weight filtration: if there were another (non-smooth) weight layer below the layer containing L_z, there would exist a bottom layer factor L_v with $l(v) < l(z)$ and tableaux $T(v) = T(z)$. □

The case of $\mathcal{O}_{\bar{\pi}(\delta)} = \mathcal{O}_{\mathfrak{q}}$

This is the nasty case to consider. Roughly speaking, we want to argue in an inductive fashion, using some of the ideas in [5]. However, the induction cannot begin until the case of $SU(2, p)$, for $p \geqslant 4$. Though potentially disturbing at first glance, this fits perfectly with the fact (see (7.1)) that $\mathfrak{q}$ is an even Jacobson–Morozov parabolic subalgebra if and only if $p \neq 4$. So, for the remainder of this subsection, we will work with this assumption on p. The first bost is the next result which at least ensures we can get away with considering double cells represented by discrete series parameters; this was *not* true in the previous case. The proof proceeds as (7.5) and is left to the reader.

LEMMA 7.7. *Assume* $p \geqslant 4$, *then there are precisely* $p - 3$ *double cells in* $\mathscr{D}$ *with associated nilpotent orbit* $\mathcal{O}_{\mathfrak{q}}$. *These double cells may be represented as* *follows*

$$\mathscr{C}^{G}_{12\bar{3}\bar{4}5\cdots p+2}$$

$$\mathscr{C}^{G}_{123\bar{4}\bar{5}6\cdots p+2}$$

$$\vdots$$

$$\mathscr{C}^{G}_{12\cdots \overline{p-1}\bar{p}p+1p+2}.$$

Our proof will depend upon inductively establishing a structure theoretic fact about the Jacquet module of each discrete series double cell representative in (7.7). Since the Jacquet functor intertwines the Hecke modules $\mathscr{M}_{P_m}$ and $\mathscr{M}_G$, this will require that we first setup an inductive machine to understand the structure of each of the underlying parameter sets $\mathscr{D}$ and W^{P_m}. To avoid confusion, we introduce the notations $\mathscr{D}(i)$ and $W^{P_m}(i)$ to represent the **K**-orbit and $\mathbf{P}_m$-orbit parametrizing posets for $SU(2, i)$. First, in the Harish-Chandra module setting, we will introduce two maps Ψ_- and Ψ_+ relating parameters in $\mathscr{D}(p-1)$ with those in $\mathscr{D}(p)$. Secondly, in a similar way we will introduce two maps ψ_- and ψ_+ relating parameters in $W^{P_m}(p-1)$ with those in $W^{P_m}(p)$.

We begin on the Harish-Chandra module level. Define an injective map

$$\Psi_-: \mathscr{D}(p-1) \mapsto \mathscr{D}(p).$$

as follows: given a tuple δ^{p-1} of signed entries and pairs of the numbers $\{1, \ldots, p+1\}$, $\Psi_-(\delta^{p-1})$ is obtained by replacing each entry i_k of δ^{p-1} by $i_k + 1$ (but do not alter signs or pairings) then tacking a "1" on at the beginning. This defines an injective mapping as desired. In addition, define $\tilde{\mathscr{D}}(p-1)$ to consist of the parameters $\psi_-(\delta^{p-1})$, $\delta^{p-1} \in \mathscr{D}(p-1)$ with a typical edge label j in the diagram replaced by $j+1$. Then $\tilde{\mathscr{D}}(p-1)$ is poset isomorphic to a subdiagram of $\mathscr{D}(p)$. Similarly, define

$$\Psi_+: \mathscr{D}(p-1) \mapsto \mathscr{D}(p)$$

as follows: given a tuple δ^{p-1} of signed entries and pairs of the numbers $\{1, \ldots, p+1\}$, $\Psi_+(\delta^{p-1})$ is obtained by tacking a "$p+2$" on at the end of δ^{p-1}. This defines an injective mapping as desired. In addition, define $\tilde{\tilde{\mathscr{D}}}(p-1)$ to consist of the parameters $\Psi_+(\delta^{p-1})$, $\delta^{p-1} \in \mathscr{D}(p-1)$ with a typical edge label j unchanged, then $\tilde{\tilde{\mathscr{D}}}(p-1)$ is poset isomorphic to a subdiagram of $\mathscr{D}(p)$. As a consequence of relating the Harish-Chandra parameter sets for two groups, we automatically know that the Lusztig–Vogan data attached to $SU(2, p-1)$ is contained in the Lusztig–Vogan data for $SU(2, p)$. (Taking this reasoning one step further, when working with a general $SU(q, p)$, the hardest case is usually $SU(p, p)$.)

For example, consider the discrete series parameters $\delta_{12\bar{3}\bar{4}567}$ and $\delta_{123\bar{4}\bar{5}67}$ in $\mathscr{D}(5)$ for $SU(2, 5)$, then

$$\Psi_+(\delta_{12\bar{3}\bar{4}56}) = \delta_{12\bar{3}\bar{4}567},$$

$$\Psi_-(\delta_{12\bar{3}\bar{4}56}) = \delta_{123\bar{4}\bar{5}67}.$$

More generally, and of most interest to our current problem, we easily obtain the following result which tells us that the double cell representatives of interest for $SU(2, p)$ are represented, inductively, by analogous double cell representatives for $SU(2, p-1)$.

LEMMA 7.8. *Assume $p \geqslant 5$ and label the double cell representatives in $SU(2, p)$ of (7.7) as*

$$\delta(1) = (12\bar{3}\bar{4}5 \cdots p+2)$$

$$\delta(2) = (123\bar{4}\bar{5}6 \cdots p+2)$$

$$\vdots$$

$$\delta(p-3) = (12 \cdots \overline{p-1}\bar{p}p+1p+2).$$

For each $\delta(i)$, there exists a discrete series parameter $\delta^{p-1}(i)$ for $SU(2, p-1)$ such that the associated nilpotent orbit of $\bar{\pi}(\delta^{p-1}(i))$ is $\mathcal{O}_{3^2 1^{p-5}}$ and either $\Psi_-(\delta^{p-1}(i)) = \delta(i)$ or $\Psi_+(\delta^{p-1}(i)) = \delta(i)$.

The next step is to set up the inductive machine on the $W^{P_m}(p)$ level. Define an injective map

$$\psi_- : W^{P_m}(p-1) \mapsto W^{P_m}(p)$$

as follows: given a minimal length coset representative $w \in W^{P_m}(p-1)$, write $w = s_{i_1} \cdots s_{i_k}$ and set $\tilde{w} = s_{i_1+1} \cdots s_{i_k+1}$. Define $\psi_-(w) = s_2 s_1 \tilde{w}$. Then ψ_- maps $W^{P_m}(p-1)$ bijectively onto an interior subset of $W^{P_m}(p)$; this is *not* one of the obvious parabolic subposets. In the case of ψ_+, we define

$$\psi_+ : W^{P_m}(p-1) \mapsto W^{P_m}(p)$$

as follows: given a minimal length coset representative $w \in W^{P_m}(p-1)$, write $w = s_{i_1} \cdots s_{i_k}$ and set $\psi_+(w) = s_p s_{p+1} w$. Then ψ_+ maps $W^{P_m}(p-1)$ bijectively onto an interior subset of $W^{P_m}(p)$.

Linking the two above inductive pictures that we first recall from Section 2 the Hecke module map

$$\mathbb{J}(p) : \mathscr{M}_{SU(2,p)} \mapsto \mathscr{M}_{P_m}(p),$$

where $\mathscr{M}_{P_m}(p)$ is the Hecke module associated to the category $\mathscr{O}'(\mathfrak{g}, \mathfrak{p}_m)$ of $SU(2, p)$. By [10, §3], the calculation of a weight filtration for $J(\bar{\pi}(\delta))$ amounts to the calculation of a family of intertwining polynomials $\mathbf{d}_{\bar{\pi}(\delta),w}$ which relate (in a sophisticated way) the Kazhdan–Lusztig theory of $W^{P_m}(p)$ with the Lusztig–Vogan theory of $\mathscr{D}(p)$. Define

$$\tilde{\mathbb{J}}_{\pm} : \mathscr{M}_{SU(2,p-1)} \mapsto \mathscr{M}_{P_m}(p);\ \tilde{\mathbb{J}}_{\pm} = \psi_{\pm} \circ \mathbb{J}(p-1).$$

The content of the above discussion distills to a crucial observation.

OBSERVATION 7.9. *Assume $p \geqslant 3$ and that $\delta \in \mathscr{D}(p)$ arises as $\Psi_{\pm}(\delta^{p-1})$, for some $\delta^{p-1} \in \mathscr{D}(p-1)$. Then a portion of a weight filtration of $J(\bar{\pi}(\delta))$ is determined by a weight filtration for $J(\bar{\pi}(\delta^{p-1}))$ via the map $\tilde{\mathbb{J}}_{\pm}$.*

We refer to the portion of the weight filtration in (7.9) as the *inductive weight skeleton* of $J(\bar{\pi}(\delta))$ and denote it by writing

$$\tilde{\mathbb{J}}_{\pm}(\hat{C}_{\delta^{p-1}}) \subset \mathbb{J}(\hat{C}_{\delta}). \tag{7.10}$$

The utility of (7.9) hinges upon a priori knowledge of the structure theoretic aspect of a weight filtration on $J(\bar{\pi}(\delta))$ we wish to inductively detect. For the purpose at hand, this feature is isolated next.

LEMMA 7.11. *In the case of $SU(2, 4)$, $J(\bar{\pi}_{12\bar{3}\bar{4}56})$ has a five layer weight filtration with the bottom weight layer equal to $L_{s_3s_5s_1}$.*

Proof. In this case, a tedious but direct calculation will show that a weight filtration on $J(\bar{\pi}_{12\bar{3}\bar{4}56})$ has the form

$tL_{s_3s_5s_1}$
$L_{x_1} \oplus L_{x_2} \oplus L_{x_3} \oplus$ stuff
$L_{u_1} \oplus L_{u_2} \oplus L_{u_3} \oplus L_{u_4} \oplus$ stuff
$L_{x_1} \oplus L_{x_2} \oplus L_{x_3} \oplus$ stuff
$tL_{s_3s_5s_1}$

where "stuff" involves no smooth or socular factors, $t \geqslant 0$ and L_{x_i}, L_{u_i} are all smooth composition factors. The tricky issue is to show $t \geqslant 1$. This is best shown by first calculating the tableaux $T(\bar{\pi}_{12\overline{34}56}) = T(L_{s_3 s_5 s_1})$ is of the form

$$\begin{matrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{matrix}$$

In addition, one can check that one of the next-to-bottom smooth factors has a tableau of shape 321. Since every irreducible submodule of $J(\bar{\pi}_{12\overline{34}56})$ must define the same primitive ideal as $\bar{\pi}_{12\overline{34}56}$, this will force $t \geqslant 1$. Finally, extending the results in [5], via standard wall-crossing arguments, one can check that none of the standard modules attached to Langlands data involving P_m has the composition factor $\bar{\pi}_{12\overline{34}56}$ with muliplicity greater than one. Since every principal series representation has the same character as some standard module induced from P_m, and since the occurrence of $tL_{s_3 s_5 s_1}$ as submodule of the socle of $J(\bar{\pi}_{12\overline{34}56})$ ensures at least t embeddings of $\bar{\pi}_{12\overline{34}56}$ into some principal series representation, we conclude that $t = 1$. □

Finally, we are in position to prove the main result in this subcase, finishing the proof of (1.7) for $SU(2, p)$.

LEMMA 7.12. *If $\mathcal{O}_{\bar{\pi}(\delta)} = \mathcal{O}_{\mathfrak{q}}$, then Matumoto's conjecture holds for $\bar{\pi}(\delta)$.*

Proof. By (7.1) and (7.3), we may assume that $p \geqslant 4$. By (7.7) and (7.8), we are reduced to considering $\bar{\pi}(\delta(i))$, $1 \leqslant i \leqslant p - 3$, $p \geqslant 4$ and we can assume that there exist a discrete series parameter $\delta^{p-1}(i)$ for $SU(2, p - 1)$ such that either $\Psi_-(\delta^{p-1}(i)) = \delta(i)$ or $\Psi_+(\delta^{p-1}(i)) = \delta(i)$; for definiteness, say $\Psi_+(\delta^{p-1}(i)) = \delta(i)$. We argue by induction on p, using (7.9)–(7.11) to produce a 5-layer inductive weight skeleton $\tilde{\mathbb{J}}_{\pm}(\hat{C}_{\delta^{p-1}})$ with bottom layer factor L_z the unique minimal element of $\mathcal{W}^{Q}_{\mathrm{soc}}$ having $T(z) = T(\bar{\pi}(\delta(i)))$. □

8. Other Hermitian rank two cases

As noted at the end of Section 1, the proof of (1.7) is now reduced to three groups: $Sp_4\mathbb{R}$, SO^*10 and $E_{6(-14)}$. We handle each case individually.

$Sp_4\mathbb{R}$

Assume that G is the real rank two symplectic group and label the Satake diagram as

$$\begin{matrix} \circ & \Leftarrow & \circ \\ 2 & & 1 \end{matrix}$$

This is a split group with three proper standard parabolic subgroups defined over $\mathbb{R}: \mathfrak{b} = \mathfrak{p}_m, \mathfrak{p}_2$ (the parabolic subalgebra with the short simple root α_2 a root of the Levi factor) and $\mathfrak{p}_1$ (the parabolic subalgebra with the long simple root α_1 a root of the Levi factor). The nilpotent orbits in $\mathfrak{sp}_4$ are in one-to-one correspondence with partitions of 4 in which odd parts occur with even multiplicity; [17, §5.1]. Thus, there are four nilpotent orbits

$$\mathcal{O}_{1^4}, \mathcal{O}_{21^2}, \mathcal{O}_{2^2}, \mathcal{O}_4.$$

The orbit $\mathcal{O}_4$ is the regular orbit of dimension 8 which coincides with $\mathcal{O}_\mathfrak{b}$ and the orbit $\mathcal{O}_{2^2}$ is the subregular orbit of dimension 6 which coincides with $\mathcal{O}_{\mathfrak{p}_2} = \mathcal{O}_{\mathfrak{p}_1}$. Both of these nilpotent orbits are even, however, a consideration of weighted Dynkin diagrams shows that only $\mathfrak{b}$ and $\mathfrak{p}_2$ are Jacobson–Morozov parabolic subalgebras; there is no even nilpotent orbit with weighted Dynkin diagram: $2 \Leftarrow 0$.

In Fig. 2 of [8, §4], we parametrized the set $\mathcal{D}$ (referred to as $\mathcal{D}_K$) and W (referred to as $\mathcal{D}_N$). Two comments are in order: first, the label F in $\mathcal{D}$ should not be confused with the fixed finite-dimensional representation of G in Section 1; in this example, L is actually the label for the finite-dimensional module. Secondly, in the picture for $\mathcal{D}$, one should note that the simple reflection s_1 is *not* in the tau invariant of K; this is typically denoted by a circle labeled "1" above the parameter K. Using these notations, we arrive at the right cell decomposition of W

$$W = \mathscr{C}_e^R \cup \mathscr{C}_2^R \cup \mathscr{C}_1^R \cup \mathscr{C}_{2121}^R,$$

where $\mathscr{C}_e^R = \{e\}$, $\mathscr{C}_1^R = \{1, 12, 121\}$, $\mathscr{C}_2^R = \{2, 21, 212\}$ and $\mathscr{C}_{2121}^R = \{2121\}$. Also, using these parameters,

$$\mathscr{W}^{P_1} = \{1, 12, 121, 2121\}, \quad \mathscr{W}_{\text{soc}}^{P_1} = \{1, 12, 121\}$$
$$\mathscr{W}^{P_2} = \{2, 21, 212, 2121\}, \quad \mathscr{W}_{\text{soc}}^{P_2} = \{2, 21, 212\}.$$

The double cell decomposition of $\mathcal{D}$ is given by

$$\mathcal{D} = \mathscr{C}_A^G \cup \mathscr{C}_D^G \cup \mathscr{C}_L^G \cup \mathscr{C}_B^G \cup \mathscr{C}_C^G \cup \mathscr{C}_F^G,$$

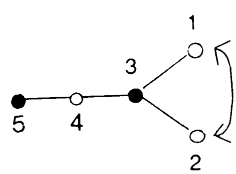
where $\mathscr{C}_A^G = \{A, E, H\}$, $\mathscr{C}_D^G = \{D, G, J\}$, $\mathscr{C}_L^G = \{L\}$, $\mathscr{C}_B^G = \{B\}$, $\mathscr{C}_C^G = \{C\}$ and $\mathscr{C}_F^G = \{F, I, K\}$.

The Jacquet module weight filtrations are given in [8, 4.12]. (We incorrectly asserted that these were the same as the socle filtrations. As will become clear below, this cannot be true.) From this data, it is clear that $\bar{\pi}(\delta_B)$ and $\bar{\pi}(\delta_C)$ are large representations, since they have property b. In this case, (1.7) would follow from Kostant's classical work in [22]. On the double cells $\mathscr{C}_A^G$ and $\mathscr{C}_D^G$ we see that the irreducibles have property $\mathscr{W}_{\mathrm{soc}}^{P_2}$, so (1.7) holds. Since $\bar{\pi}(\delta)_L$ is the finite dimensional module, it remains to consider the double cell $\mathscr{C}_F^G$. But, again, the weight filtrations allow one to read off the property $\mathscr{W}_{\mathrm{soc}}^{P_2}$ holds. This completes the proof of (1.7) for this group.

(8.1) REMARK. It is also true that the irreducibles attached to the double cell $\mathscr{C}_F^G$ satisfy property $\mathscr{W}_{\mathrm{soc}}^{P_1}$; this follows since all Ext groups in $\mathcal{O}'(\mathfrak{g}, \mathfrak{p}_m)$ have dimension at most one, hence a middle level weight factor in $\mathscr{W}_{\mathrm{soc}}^{P_1}$ will drop to the socle. For example, $J(K)$ has $L_{212} \oplus L_{12}$ in its socle. This is why our assertion in [8, 4.12] that the weight and socle filtrations coincide was incorrect. (In addition, from our perspective, this indicates why the bottom weight level aspect of (3.2) is more natural than the socle aspect of (1.2).) But, again, the parabolic subalgebra $\mathfrak{p}_1$ is not of Jacobson–Morozov type, so this is of no importance in studying (1.7). In fact, note that the irreducibles attached to the double cells $\mathscr{C}_A^G$ and $\mathscr{C}_D^G$ do not have property $\mathfrak{p}_1$, even though the associated variety of these cells is $\mathcal{O}_{\mathfrak{p}_1}$. This shows that some condition on the parabolic subalgebra is needed to establish the converse of (1.3).

*SO**10

In this subsection, $G = SO^*10$ and we label the Satake diagram



There are three proper standard parabolic subalgebras over $\mathbb{R}$ to consider. The minimal parabolic subalgebra $\mathfrak{p}_m$ (having Levi factor simple roots $S_m = \{\alpha_3, \alpha_5\}$), $\mathfrak{p}$ (having Levi factor simple roots $S_P = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5\}$) and $\mathfrak{q}$ (having Levi factor simple roots $S_Q = \{\alpha_3, \alpha_4, \alpha_5\}$). The nilpotent orbits in $\mathfrak{so}_{10}$ are in one-to-one correspondence with partitions of 10 in which even parts occur with even multiplicity; see [17, §5.1] for the general theory required to compute the Hasse diagram of the 16 nilpotent orbits in this example. Consider the partitions

$$d_m = 5^2$$
$$d_P = 3^2 1^4$$

leading to the nilpotent elements X_m and X_P, respectively. Using [17, §5.3], the corresponding nilpotent orbits have weighted Dynkin diagrams

$$\mathcal{O}_m = \mathcal{O}_{X_m} = 0 \quad 2 \quad 0 \begin{matrix} 2 \\ 2 \end{matrix}$$

$$\mathcal{O}_P = \mathcal{O}_{X_P} = 0 \quad 2 \quad 0 \begin{matrix} 0 \\ 0 \end{matrix}$$

It is immediate that $\mathfrak{p}_m$ and $\mathfrak{p}$ are both even Jacobson–Morozov parabolic subalgebras with $\mathcal{O}_m = \mathcal{O}_{\mathfrak{p}_m}$, $\dim \mathcal{O}_m = 36$, $\mathcal{O}_P = \mathcal{O}_{\mathfrak{p}}$ and $\dim \mathcal{O}_P = 26$. One can directly compute all 16 weighted Dynkin diagrams and see that

$$0 \quad 0 \quad 0 \begin{matrix} 2 \\ 2 \end{matrix}$$

never arises. From this we conclude that $\mathfrak{q}$ is not an even Jacobson–Morozov parabolic subalgebra.

The poset W^P is described in [4, Fig. 2.1], taking into account the reversed labeling of simple roots for D_5. Further, since [4] computed the socles of the generalized Verma modules in $\mathcal{O}'(\mathfrak{g}, \mathfrak{p})$, Irving's work [21] tells us that we can read off the socular set. Using the cited labels and denoting by $\leqslant_w$ the weak order on W^P given by simple reflections only, we see that

$$W^P_{\text{soc}} = \{w \in W^P \mid w \leqslant_w (3, 1^*)\}.$$

The poset W^{P_m} will not be reproduced here, but the theory in [29] applies to fairly easily arrive at the information we will need below; this poset contains 480 elements.

The poset $\mathcal{D}$ is given in [5, (2.11)]. We concretely reproduce this parameter set in Table 8.1, labeling the elements from 1 to 156 and follow the interpretation conventions of [5, Appendix]. In particular, we use the notation $\bar{\pi}_i$ to denote the irreducible Harish-Chandra module indexed by i and $\mathcal{C}_i^G$ to denote the double cell containing δ_i, etc. We have included this tabular data, since the reader can then (tediously) compute the double cell decomposition of $\mathcal{D}$, arriving at

$$\mathcal{D}=\mathcal{C}^G_1\cup\mathcal{C}^G_2\cup\mathcal{C}^G_3\cup\mathcal{C}^G_4\cup\mathcal{C}^G_6\cup\mathcal{C}^G_{12}\cup\mathcal{C}^G_{16}\cup\mathcal{C}^G_{92}\cup\mathcal{C}^G_{122}\cup\mathcal{C}^G_{124}\cup\mathcal{C}^G_{127}\cup\mathcal{C}^G_{156},$$

where

$\mathcal{C}^G_1 = \{1, 17, 37, 58, 59, 81, 82, 103, 104, 123\}$

$\mathcal{C}^G_2 = \{2, 13, 14, 15, 18, 19, 24, 34, 35, 38, 39, 40, 41, 44, 56, 61, 62, 64, 66, 89\}$

$\mathcal{C}^G_3 = \{3, 5, 9, 11, 21, 22, 25, 31, 45, 71\}$

$\mathcal{C}^G_4 = \{4, 20, 23, 42, 43, 47, 68, 69, 73, 93, 94, 98, 99, 100, 116, 117, 118, 120, 133, 134\}$

$\mathcal{C}^G_6 = \{6, 7, 8, 10, 26, 27, 28, 29, 30, 46, 49, 50, 51, 52, 53, 72, 74, 75, 77, 97\}$

$\mathcal{C}^G_{12} = \{12, 33, 55, 60, 79, 83, 102, 105, 106, 125\}$

$\mathcal{C}^G_{16} = \{16, 32, 36, 48, 54, 57, 63, 70, 80, 84, 85, 86, 87, 95, 107, 108, 109, 110, 126, 128\}$

$\mathcal{C}^G_{92} = \{65, 67, 76, 78, 88, 90, 91, 92, 96, 101, 111, 112, 113, 114, 115, 119, 121, 129, 130, 132, 135, 142, 143, 146, 151\}$

$\mathcal{C}^G_{122} = \{122, 136, 137, 147, 153\}$

$\mathcal{C}^G_{124} = \{124, 138, 139, 148, 154\}$

$\mathcal{C}^G_{127} = \{127, 131, 140, 141, 144, 145, 149, 150, 152, 155\}$

$\mathcal{C}^G_{156} = \{156\}$

The associated varieties of these twelve double cells can be computed by generalizing the program in [18]; this calculation was provided by D. Garfinkle:

Double cell	Associated variety
$\mathcal{C}^G_3$	$\mathcal{O}_{5^2}$
$\mathcal{C}^G_2\cup\mathcal{C}^G_6$	$\mathcal{O}_{4^21^2}$
$\mathcal{C}^G_4\cup\mathcal{C}^G_{16}$	$\mathcal{O}_{3^22^2}$
$\mathcal{C}^G_{92}$	$\mathcal{O}_{3^21^4}$
$\mathcal{C}^G_1\cup\mathcal{C}^G_{12}\cup\mathcal{C}^G_{127}$	$\mathcal{O}_{2^41^2}$
$\mathcal{G}^G_{122}\cup\mathcal{C}^G_{124}$	$\mathcal{O}_{2^21^6}$
$\mathcal{C}^G_{156}$	$\mathcal{O}_{1^{10}}$

Thus, only the double cells $\mathcal{C}^G_3$ and $\mathcal{C}^G_{92}$ are of interest. Since $\dim \mathcal{O}_{5^2} = 2\dim \mathfrak{n}_m$, the representations associated to $\mathcal{C}^G_3$ are quasi-large and Step 5 of Section 4 verifies (1.7) in this case. It remains to consider $\mathcal{C}\mathcal{G}_2$. We will focus on the irreducible representation $\bar{\pi}_{92}$ and show it satisfies property $\mathcal{W}^P_{\text{soc}}$, from which (3.3) finishes our proof of (1.7) for this group. Using the information in Table 8.1, it follows that $\tau(\bar{\pi}_{92}) = \{s_1, s_2, s_3, s_5\}$. This fact, together with the length considerations in Step 4 of Section 4 cut down calculation, using the techniques outlined in Section 4, we arrive at the following smooth skeleton of the weight filtration of $J(\bar{\pi}_{92})$

L_z
$L_w \oplus L_y,$
L_z

where

$$l_g(\bar{\pi}_{92}) = l(w) = l(y)$$

$$w = w_o s_4 s_3 s_2 s_1 s_3 s_4$$

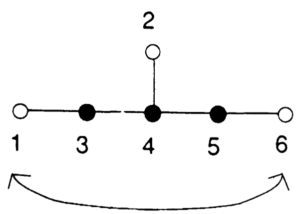
$$y = w_o s_2 s_1 s_4 s_3 s_5 s_4$$

$$z = w_P w_{(3,1^*)} s_3 s_4.$$

There are five other possible non-smooth composition factors. Arguments as in Sections 5 and 6 suffice to show that property $\mathcal{W}^P_{\text{soc}}$ holds; in fact, $L_z \subset \text{socle}(J(\bar{\pi}_{92}))$.

$E_{6(-14)}$

Finally, we come to the exceptional real rank two Hermitian symmetric form of E_6. (The author wishes to thank and acknowledge D. Garfinkle for helpful comments on the double cell structure for this group.) We label the Satake diagram



There are three proper standard parabolic subalgebras over $\mathbb{R}$ to consider. The minimal parabolic subalgebra $\mathfrak{p}_m$ (having Levi factor simple roots $S_m = \{\alpha_3, \alpha_4, \alpha_5\}$), $\mathfrak{p}$ (having Levi factor simple roots $S_P = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$) and $\mathfrak{q}$

TABLE 1
The $\mathscr{D}$ set for SO^*10

Parameter	l_g	dim A	Simples 1	2	3	4	5	Other
1	0	0	17	−1	−1	−1	−1	
2	0	0	18	−1	−1	19	−1	
3	0	0	−1	20	21	−1	22	
4	0	0	−1	20	−1	−1	23	
5	0	0	24	−1	21	19	25	
6	0	0	−1	26	−1	27	−1	
7	0	0	−1	26	28	29	−1	
8	0	0	17	−1	28	−1	−1	
9	0	0	−1	30	31	27	22	
10	0	0	−1	30	−1	29	23	
11	0	0	32	−1	31	−1	25	
12	0	0	−1	33	−1	−1	−1	
13	0	0	−1	33	34	−1	−1	
14	0	0	18	−1	34	35	−1	
15	0	0	24	−1	−1	35	36	
16	0	0	32	−1	−1	−1	36	
17	1	1	1	−1	37	−1	−1	8(1)
18	1	1	2	−1	38	39	−1	14(1)
19	1	1	39	−1	40	2	41	5(4)
20	1	1	−1	3	42	−1	43	4(2)
21	1	1	44	42	3	40	45	5(3)
22	1	1	−1	43	45	46	3	9(5)
23	1	1	−1	43	−1	47	4	10(5)
24	1	1	5	−1	44	39	48	15(1)
25	1	1	48	−1	45	41	5	11(5)
26	1	1	−1	6	49	50	−1	7(2)
27	1	1	−1	50	51	6	46	9(4)
28	1	1	37	49	7	52	−1	8(3)
29	1	1	−1	50	52	7	47	10(4)
30	1	1	−1	9	53	50	43	10(2)
31	1	1	54	53	9	51	45	11(3)
32	1	1	11	−1	54	−1	48	16(1)
33	1	1	−1	12	55	−1	−1	13(2)
34	1	1	38	55	13	56	−1	14(3)
35	1	1	39	−1	56	14	57	15(4)
36	1	1	48	−1	−1	57	15	16(5)
37	2	1	28	58	17	59	−1	
38	2	1	34	60	18	61	−1	
39	2	2	19	−1	62	18	63	35(1) 24(4)
40	2	1	64	65	19	21	66	
41	2	1	63	−1	66	25	19	
42	2	1	67	21	20	65	68	
43	2	2	−1	22	68	69	20	23(2) 30(5)
44	2	1	21	67	24	64	70	
45	2	2	70	68	22	71	21	25(3) 31(5)

TABLE 1 (*cont.*)

			Simples					
Parameter	l_g	dim A	1	2	3	4	5	Other
46	2	1	−1	69	72	22	27	
47	2	1	−1	69	73	23	29	
48	2	2	25	−1	70	63	24	36(1) 32(5)
49	2	1	58	28	26	74	−1	
50	2	2	−1	27	75	26	69	29(2) 30(4)
51	2	1	76	77	27	31	72	
52	2	1	59	74	29	28	73	
53	2	1	78	31	30	77	68	
54	2	1	31	78	32	76	70	
55	2	1	60	34	33	79	−1	
56	2	1	61	79	35	34	80	
57	2	1	63	−1	80	36	35	
58	3	1	49	37	−1	81	−1	
59	3	1	52	81	−1	37	82	
60	3	1	55	38	−1	83	−1	
61	3	1	56	83	84	38	85	39
62	3	2	84	86	39	84	87	56 44 40 38
63	3	2	41	−1	87	48	39	57(1)
64	3	1	40	88	84	44	89	39
65	3	1	88	40	−1	42	90	
66	3	1	89	90	41	91	40	45
67	3	1	42	44	−1	88	92	
68	3	2	92	45	43	93	42	53(5)
69	3	2	−1	46	94	43	50	47(2)
70	3	2	45	92	48	95	44	54(5)
71	3	2	95	93	91	45	91	51 46 41 40
72	3	1	96	97	46	91	51	45
73	3	1	82	98	47	−1	52	
74	3	1	81	52	99	49	98	50
75	3	2	100	99	50	99	94	53 52 51 49
76	3	1	51	101	−1	54	96	
77	3	1	101	51	99	53	97	50
78	3	1	53	54	−1	101	92	
79	3	1	83	56	−1	55	102	
80	3	1	85	102	57	−1	56	
81	4	1	74	59	103	58	104	
82	4	1	73	104	−1	−1	59	
83	4	1	79	61	105	60	106	
84	4	2	62	107	61	62	108	64(3)
85	4	1	80	106	108	−1	61	63
86	4	2	107	62	−1	107	109	79 67 65 60
87	4	2	108	109	63	110	62	80 70 66
88	4	1	65	64	111	67	112	
89	4	1	66	112	108	113	64	70 63
90	4	1	112	66	−1	114	65	68

TABLE 1 (*cont.*)

Parameter	l_g	dim A	Simples 1	2	3	4	5	Other
91	4	2	113	114	71	66	71	72(4)
92	4	2	68	70	−1	115	67	78(5)
93	4	2	115	71	116	68	114	77 69 65
94	4	2	117	118	69	116	75	73 72 68
95	4	2	71	115	110	70	113	76 64 63
96	4	1	72	119	−1	113	76	70
97	4	1	119	72	118	114	77	69 68
98	4	1	104	73	118	−1	74	69
99	4	2	120	75	74	75	118	77(3)
100	4	2	75	120	−1	120	117	78 76 59 58
101	4	1	77	76	121	78	119	
102	4	1	106	80	−1	−1	79	
103	5	1	122	−1	81	−1	123	99
104	5	1	98	82	123	−1	81	
105	5	1	−1	124	83	−1	125	84
106	5	1	102	85	125	−1	83	
107	5	2	86	84	124	86	126	88 83
108	5	2	87	126	85	127	84	89(3)
109	5	2	126	87	−1	128	86	102 92 90
110	5	2	127	128	95	87	127	91 84
111	5	1	−1	124	88	−1	129	84
112	5	1	90	89	129	130	88	92
113	5	2	91	130	127	89	95	96(4)
114	5	2	130	91	131	90	93	97(4)
115	5	2	93	95	132	92	130	101 88
116	5	2	133	131	93	94	131	99 91
117	5	2	94	134	−1	133	100	96 92 82
118	5	2	134	94	97	131	99	98(3)
119	5	1	97	96	135	130	101	92
120	5	2	99	100	122	100	134	101 81
121	5	1	122	−1	101	−1	135	99
122	6	2	103	−1	120	−1	136	121(1)
123	6	1	136	−1	104	137	103	118
124	6	2	−1	105	107	−1	138	111(2)
125	6	1	−1	138	106	139	105	108
126	6	2	109	108	138	140	107	112 106
127	6	2	110	140	113	108	110	
128	6	2	140	110	141	109	140	115 114 107
129	6	1	−1	138	112	142	111	108
130	6	2	114	113	143	112	115	119(4)
131	6	2	144	116	114	118	116	
132	6	2	145	141	115	−1	143	121 116 111 110
133	6	2	116	144	145	117	144	120 115 113
134	6	2	118	117	136	144	120	119 104
135	6	1	136	−1	119	146	121	118

TABLE 1 (*cont.*)

Parameter	l_g	dim A	Simples 1	2	3	4	5	Other
136	7	2	123	−1	134	147	122	135(1)
137	7	1	147	−1	−1	123	−1	131
138	7	2	−1	125	126	148	124	129(2)
139	7	1	−1	148	−1	125	−1	127
140	7	2	128	127	149	126	128	130
141	7	2	150	132	128	−1	149	131 124
142	7	1	−1	148	151	129	−1	130 127
143	7	2	152	149	130	151	132	135 131 129 127
144	7	2	131	133	152	134	133	130
145	7	2	132	150	133	−1	152	127 122
146	7	1	147	−1	151	135	−1	131 130
147	8	2	137	−1	153	136	−1	146(1) 144
148	8	2	−1	139	154	138	−1	142(2) 140
149	8	2	155	143	140	154	141	138
150	8	2	141	145	155	−1	155	144 140
151	8	2	153	154	142	143	−1	146(3)
152	8	2	143	155	144	153	145	136
153	9	2	151	156	147	152	−1	
154	9	2	156	151	148	149	−1	
155	9	2	149	152	150	156	150	
156	10	2	154	153	−1	155	−1	

(having Levi factor simple roots $S_Q = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$). The 21 nilpotent orbits in E_6 are listed according to weighted Dynkin diagram in [17, §8.4]. Using the labels in the cited reference, we have

$$\mathcal{O}_{\mathfrak{p}_m} = \mathcal{O}_{A_4} = \begin{matrix} & & 2 & & \\ 2 & 0 & 0 & 0 & 2 \end{matrix}$$

$$\mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{A_2} = \begin{matrix} & & 2 & & \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$\mathcal{O}_{\mathfrak{q}} = \mathcal{O}_{2A_2} = \begin{matrix} & & 0 & & \\ 2 & 0 & 0 & 0 & 2 \end{matrix}$$

It is immediate that $\mathfrak{p}_m$, $\mathfrak{p}$ and $\mathfrak{q}$ are all even Jacobson–Morozov parabolic subalgebras.

The poset W^P is fairly easily described: $|W^P| = 72$ and the diagram is given in [3, Fig. 5.1]. From [4] we find $W^P_{\text{soc}} = \{w \mid w \leqslant_w 55\}$. In the cases of W^{P_m} and

TABLE 2
Nilpotent orbits for $\mathfrak{sp}_6$

Nilpotent orbit	Weighted Dynkin diagram	Even ?	Richardson ?
$\mathcal{O}_6$	2—2⇐2	Yes	$\mathcal{O}_{\mathfrak{b}}$
$\mathcal{O}_{42}$	2—0⇐2	Yes	$\mathcal{O}_{\mathfrak{p}_i}$, $1 \leqslant i \leqslant 3$
$\mathcal{O}_{41^2}$	2—1⇐0	No	—
$\mathcal{O}_{3^2}$	0—2⇐0	Yes	$\mathcal{O}_{\mathfrak{p}_{13}}$
$\mathcal{O}_{2^3}$	0—0⇐2	Yes	$\mathcal{O}_{\mathfrak{p}_{23}}$
$\mathcal{O}_{2^21^2}$	0—1⇐0	No	$\mathcal{O}_{\mathfrak{p}_{12}}$
$\mathcal{O}_{21^4}$	1—0⇐0	No	Rigid
$\mathcal{O}_{1^6}$	0—0⇐0	Yes	$\mathcal{O}_{\mathfrak{g}}$

W^Q, $|W^{P_m}| = 2160$ and $|W^Q| = 270$, respectively; a computer is our approach to handling these cases.

The poset $\mathscr{D}$ contains 513 elements and is given in [5, §9]. Use the notation $\bar{\pi}_i$ to denote the irreducible Harish-Chandra module indexed by i and $\mathscr{C}_i^G$ to denote the double cell containing δ_i, etc. The reader can (tediously) compute the double cell decomposition of $\mathscr{D}$.

$$\mathscr{D} = \mathscr{C}_{364}^G \cup \mathscr{C}_{13}^G \cup \mathscr{C}_{513}^G \cup \mathscr{C}_{510}^G \cup \mathscr{C}_{512}^G \cup \mathscr{C}_{511}^G \cup \mathscr{C}_{1}^G \cup \mathscr{C}_{27}^G \cup \mathscr{C}_{180}^G \cup \mathscr{C}_{162}^G \cup \mathscr{C}_{2}^G \cup \mathscr{C}_{21}^G \cup \mathscr{C}_{7}^G.$$

In order to know which double cells must be studied, we need to know their associated varieties. This can be read off from the table below, which computes the dimension of the associated W representation and Lusztig's "a function"; see [23].

Double cell	Lusztig a	Dimension of W representation
$\mathscr{C}_{364}^G$	15	45
$\mathscr{C}_{13}^G$	12	24
$\mathscr{C}_{513}^G$	36	1
$\mathscr{C}_{510}^G$	25	6
$\mathscr{C}_{512}^G$	25	6
$\mathscr{C}_{511}^G$	20	20
$\mathscr{C}_{1}^G$	20	20
$\mathscr{C}_{27}^G$	20	20
$\mathscr{C}_{162}^G$	13	64
$\mathscr{C}_{180}^G$	13	64
$\mathscr{C}_{2}^G$	10	81
$\mathscr{C}_{7}^G$	6	81
$\mathscr{C}_{21}^G$	10	81

We need only consider the 45 element double cell $\mathscr{C}^G_{364}$ with associated variety $\overline{\mathcal{O}_p}$ and the 24 element double cell $\mathscr{C}^G_{13}$ with associated variety $\overline{\mathcal{O}_q}$; the double cells with associated variety $\mathcal{O}_{p_m}$ can be handled by Step 5 of Section 4. To handle each of these cases we investigate the structure of certain degenerate series representations.

Begin with the case of $\mathscr{C}^G_{364}$ and the representative irreducible module $\bar{\pi}_{364}$. Tau invariant considerations and the remarks in Step 4 of Section 4 allow one to show that $J(\bar{\pi}_{364})$ has a weight filtration with $t \cdot L_z$ occurring in a bottom weight layer with $t \geqslant 0$ and $z = w_P$; i.e., z is the bottom element of $\mathscr{W}^P_{\text{soc}}$. Thus, the real crux is to argue that the composition factor L_z actually occurs in the Jacquet module. The parameter z is the minimal *possible* corresponding to a composition factor of $J(\bar{\pi}^{364})$.

To show L_z is a composition factor of $J(\bar{\pi}_{364})$, we will study the structure of certain degenerate series representations. Recalling the notation in Section 2, given $w \in W^P$, we define

$$I_P(w) = I_P(E_P(w_P w w_o)),$$

which is a P-degenerate series (induced from finite dimensional data) representation of G. The first order of business is to describe how one computes a weight filtration for every P-degenerate series. The Levi factor factor L_P of P has a semisimple part isomorphic to $SU(5, 1)$, so its irreducibles are parametrized by the set $\mathscr{D}^P$ in [13, Fig. 4.4], with $n = 5$. In $\mathscr{D}^P$, only the top node, labeled "01" indexes a finite dimensional L-module $\bar{\pi}_P(01)$, so we first describe a weight filtration on $I_P(\bar{\pi}_P(01))$. This proceeds via a bootstrap argument on the levels in $\mathscr{D}^P$. Begin with the induced modules

$$I_P(\bar{\pi}_P(0)), \ldots, I_P(\bar{\pi}_P(5)) \tag{8.2}$$

corresponding to inducing the irreducibles along the bottom row of $\mathscr{D}^P$ up to G. The irreducibles $\bar{\pi}_P(i)$, $0 \leqslant i \leqslant 5$ are relative discrete series representations of L. Thus, the induced modules in (8.2) are really generalized principal series for G, induced up from P; i.e. standard modules. In the parameters of [5], we find

$$I_P(\bar{\pi}_P(0)) = \pi_{455}, \quad I_P(\bar{\pi}_P(1)) = \pi_{460},$$
$$I_P(\bar{\pi}_P(2)) = \pi_{470}, \quad I_P(\bar{\pi}_P(3)) = \pi_{474},$$
$$I_P(\bar{\pi}_P(4)) = \pi_{475}, \quad I_P(\bar{\pi}_P(5)) = \pi_{477}.$$

From [13], we know the weight filtrations of the L-standard modules

$\pi_P(i, 5-i)$, $0 \leqslant i \leqslant 4$. An induction in stages argument, the main theorem in [11] and the previous information computes a weight filtration on $I_P(\bar{\pi}_P(i, 5-i))$, $0 \leqslant i \leqslant 5$. For example, we find $I_P(\bar{\pi}_P(05))$ has a weight filtration

$\bar{\pi}_{478}$
$\bar{\pi}_{456} \oplus \bar{\pi}_{454}$
$\bar{\pi}_{428}$
$\bar{\pi}_{108}$

Continuing on in this way, we eventually arrive at a weight filtration for $I_P(\bar{\pi}_P(01)) = I_P(w^P)$

$\bar{\pi}_{513}$
$\bar{\pi}_{511}$
$\bar{\pi}_{364}$

We now have the exact sequences

$$I_{P_m}(w^{P_m}) \mapsto I_P(w^P) \mapsto 0,$$
$$0 \mapsto \bar{\pi}_{364} \mapsto I_P(w^P).$$

Hence, by duality, the Subrepresentation theorem on L_P and induction in stages, we obtain the exact sequence

$$I_{P_m}(\tilde{E}) \mapsto I_P(e) \mapsto \bar{\pi}_{364} \mapsto 0,$$

for some finite dimensional P_m-module $\tilde{E}$. Recalling (2.1), we have that $I_{P_m}(\tilde{E}) = I_{P_m}(w_{P,P_m})$. By (1.1b), this shows that there must be a non-zero map from L_{w_P} into $J(\bar{\pi}_{364})$, as desired. Apply (3.3) to verify (1.7) in this case.

Next, consider the case of $\mathscr{C}^G_{13}$ and the representative irreducible module $\bar{\pi}_{13}$. We argue identically as in the above case. Namely, one first shows that the bottom weight layer of $J(\bar{\pi}_{13})$ possibly contains L_{w_Q}, where w_Q is the minimal element of $\mathscr{W}^Q_{\mathrm{soc}}$. To show it actually does occur, one needs to study the degenerate series for the Levi factor L_Q of Q. In this setting, the semisimple part of L_Q is isomorphic to $SO_e(7, 1)$ and its irreducible representations are parametrized in [13, Fig. 4.3]. Using the notation of this reference, we need to study weight filtrations of

$$I_Q(\bar{\pi}_Q(04)), \quad I_Q(\bar{\pi}_Q(03)), \quad I_Q(\bar{\pi}_Q(02)), \quad I_Q(\bar{\pi}_Q(01));$$

only the last module $I_Q(\bar{\pi}_Q(01))$ is a Q-degenerate series for G. Take note of the fact that $I_Q(\bar{\pi}_Q(04)) = \pi_{497}$, in the notation of [5]. Bootstrapping as in the previous case, we eventually arrive at the weight filtration for $I_Q(\bar{\pi}_Q(01)) = I_Q(w^Q)$

$\bar{\pi}_{513}$
$\bar{\pi}_{512} \oplus \bar{\pi}_{510}$
$\bar{\pi}_{507} \oplus \bar{\pi}_{383}$
$\bar{\pi}_{336} \oplus \bar{\pi}_{310}$
$\bar{\pi}_{13}$

We now have the exact sequences

$$I_{P_m}(w^{P_m}) \mapsto I_Q(w^Q) \mapsto 0,$$
$$0 \mapsto \bar{\pi}_{13} \mapsto I_Q(w^Q).$$

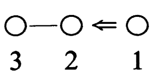
Hence, by duality, the Subrepresentation theorem on L_Q and induction in stages, the exact sequence

$$I_{P_m}(\tilde{E}) \to I_Q(e) \to \bar{\pi}_{13} \to 0$$

for some finite dimensional P_m-module $\tilde{E}$. Recalling (2.1), we have that $I_{P_m}(\tilde{E}) = I_{P_m}(w_{Q,P_m})$. In summary, this shows that there must be a non-zero map from L_{w_Q} into $J(\bar{\pi}_{13})$, as desired. Apply (3.3) to verify (1.7) in this case.

9. $Sp_6\mathbb{R}$

Throughout this section, G will denote the real rank three symplectic group. We give a full account of the validity of (1.6) for the fundamental block; similar ideas apply to the other blocks, but no new insights appear. Label the simple roots of the Satake diagram as



This is a split group, so every standard parabolic subalgebra is defined over $\mathbb{R}$; we label the seven proper parabolic subalgebras follows: $\mathfrak{b} = \mathfrak{p}_m$ is the minimal parabolic subalgebra, $\mathfrak{p}_i = \mathfrak{l}_i \oplus \mathfrak{n}_i$, $1 \leqslant i \leqslant 3$ is the parabolic subalgebra whose Levi factor $\mathfrak{l}_i$ has simple root α_i, and $\mathfrak{p}_{ij} = \mathfrak{l}_{ij} \oplus \mathfrak{n}_{ij}$, $1 \leqslant i < j \leqslant 3$ is the parabolic subalgebra whose Levi factor has simple roots $\{\alpha_i, \alpha_j\}$.

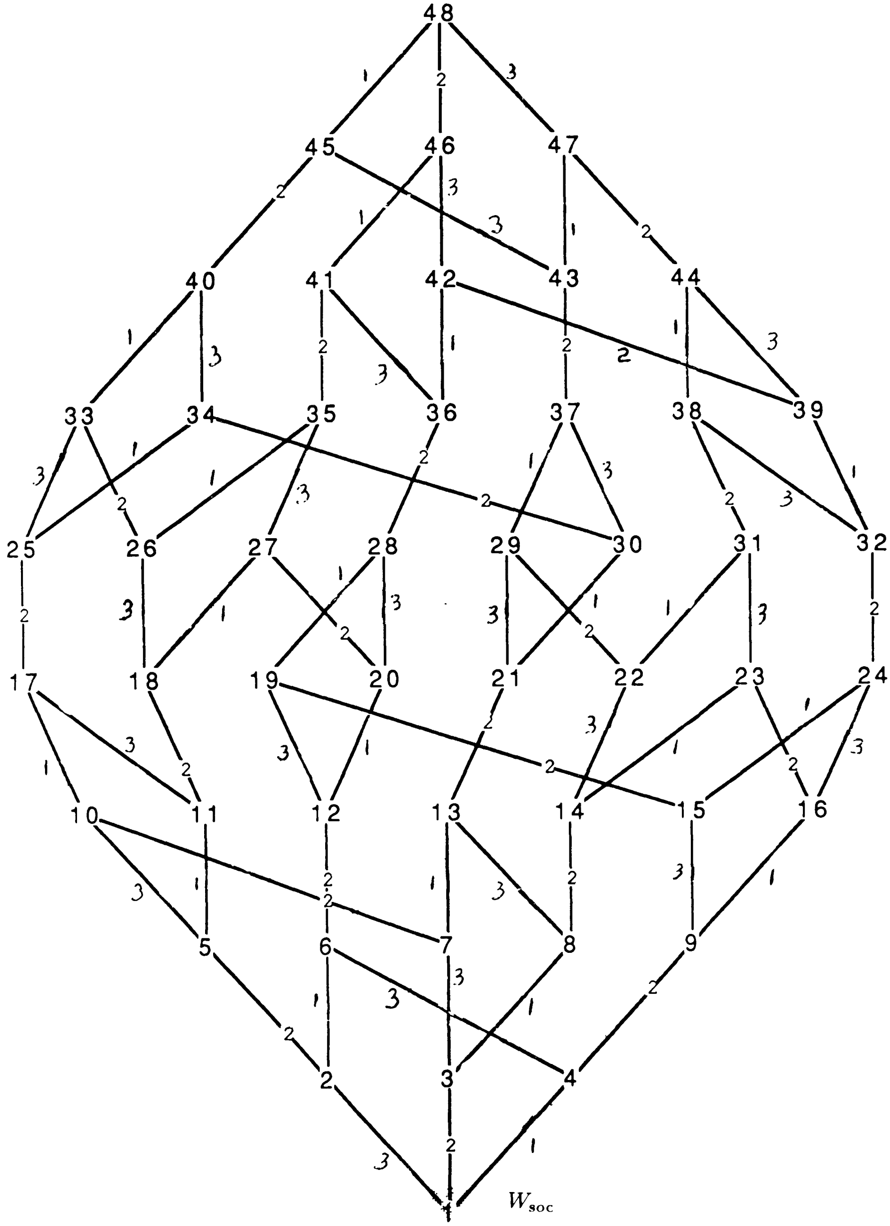


Fig. 1. W of type C_3.

The nilpotent orbits in $\mathfrak{sp}_6$ are in one-to-one corespondence with the partitions of 6 in which odd parts occur with even multiplicity. One can easily compute the weighted Dynkin diagram of each orbit, determine the even orbits and identify the Richardson orbits $\mathcal{O}_{\mathfrak{p}_I}$, $\mathcal{O}_{\mathfrak{p}_J}$; see Table 2.

From this table, it is immediate that $\mathfrak{b}$, $\mathfrak{p}_2$, $\mathfrak{p}_{13}$ and $\mathfrak{p}_{23}$ are the even Jacobson–Morozov standard parabolic subalgebras.

We will need to work with the full category $\mathcal{O}'(\mathfrak{g}, \mathfrak{b})$, as well as the subcategories correspondeing to the four even Jacobson–Morozov parabolic subalgebras. In Figure 1 we indicate a parametrization of $W = \mathscr{W}^B$ and in Fig. 2 we give the parametrizations of $\mathscr{W}^{P_2}$, $\mathscr{W}^{P_{13}}$ and $\mathscr{W}^{P_{23}}$. We have further included the socular right cells for W and the other posets.

The set $\mathscr{D}$ will decompose as a union of four blocks:

$$\mathscr{D} = \mathscr{D}_o \cup \mathscr{D}_b \cup \mathscr{D}_c \cup \mathscr{D}_d,$$

where $\mathscr{D}_o$ is labeled in Figure 3, $\mathscr{D}_b$ is given in [6, Fig. 3.2], $\mathscr{D}_c$ is given in [15, §5] and $\mathscr{D}_d$ is a singleton block. Following the notation in Fig. 3, $\mathscr{D}_o$ will decompose into a union of 16 double cells

$$\mathscr{D}_o = \mathscr{C}_1^G \cup \mathscr{C}_2^G \cup \mathscr{C}_{27}^G \cup \mathscr{C}_{51}^G \cup \mathscr{C}_5^G \cup \mathscr{C}_7^G \cup \mathscr{C}_3^G \cup \mathscr{C}_6^G \cup \mathscr{C}_4^G \cup \mathscr{C}_8^G \cup \mathscr{C}_{21}^G \cup \mathscr{C}_{22}^G \cup \mathscr{C}_{24}^G \cup \mathscr{C}_{39}^G \cup \mathscr{C}_{29}^G \cup \mathscr{C}_{48}^G.$$

Begin by tabulating the elements in each double cell and the nilpotent determining their associated varieties; these can be computed using a generalization of Garfinkle's work in [18] (as used in our discussion of $SU(p, q)$ in Section 6) or one can read this off from the Jacquet module calculations given below (since any socular factor of the Jacquet module determines the same annihilator as the corresponding irreducible Harish-Chandra module).

Double cell	Elements	Associated nilpotent
$\mathscr{C}_4^G$	{4}	$\mathcal{O}_6$
$\mathscr{C}_8^G$	{8}	$\mathcal{O}_6$
$\mathscr{C}_5^G$	{5, 14, 23, 34, 42}	$\mathcal{O}_{42}$
$\mathscr{C}_7^G$	{7, 16, 25, 36, 44}	$\mathcal{O}_{42}$
$\mathscr{C}_3^G$	{3, 11, 13, 19, 30}	$\mathcal{O}_{42}$
$\mathscr{C}_6^G$	{6, 12, 15, 20, 31}	$\mathcal{O}_{42}$
$\mathscr{C}_{27}^G$	{27, 26, 37, 45}	$\mathcal{O}_{2^21^2}$
$\mathscr{C}_{29}^G$	{28, 29, 38, 46}	$\mathcal{O}_{2^21^2}$
$\mathscr{C}_{48}^G$	{48, 49, 50, 53}	$\mathcal{O}_{2^21^2}$
$\mathscr{C}_{21}^G$	{21, 32, 40}	$\mathcal{O}_{3^2}$
$\mathscr{C}_{22}^G$	{22, 33, 41}	$\mathcal{O}_{2^3}$
$\mathscr{C}_{24}^G$	{24, 35, 43}	$\mathcal{O}_{2^3}$
$\mathscr{C}_{39}^G$	{39, 47, 52}	$\mathcal{O}_{2^3}$
$\mathscr{C}_1^G$	{1, 9, 17}	$\mathcal{O}_{2^3}$
$\mathscr{C}_2^G$	{2, 10, 18}	$\mathcal{O}_{2^3}$
$\mathscr{C}_{51}^G$	{51}	$\mathcal{O}_{1^6}$

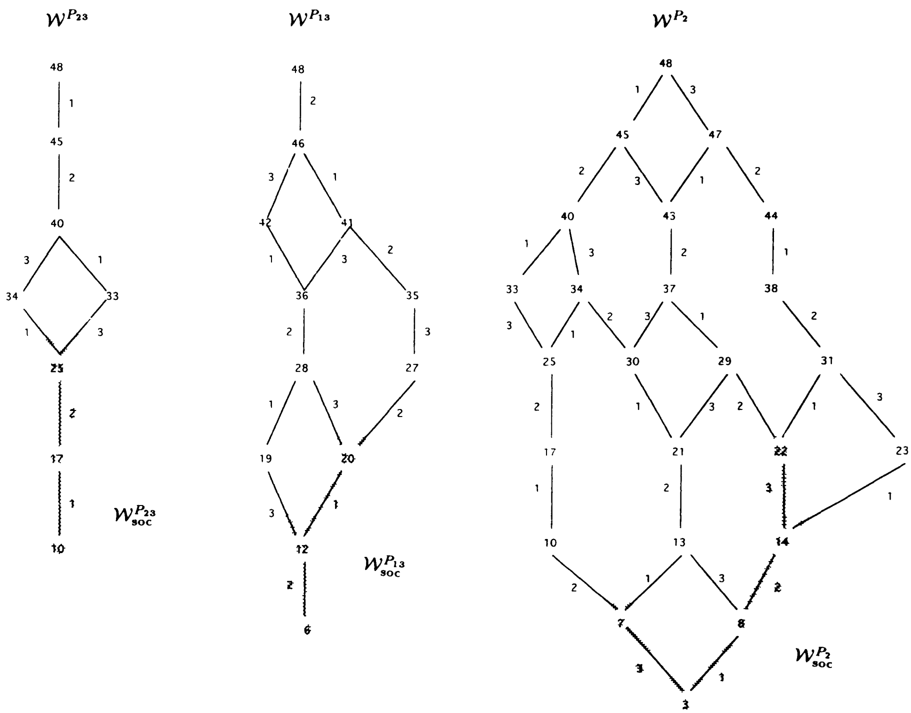


Fig. 2. The posets $\mathscr{W}^{P_2}$, $\mathscr{W}^{P_{23}}$ and $\mathscr{W}^{P_{13}}$.

The two double cells with associated nilpotent orbit $\mathcal{O}_6$ correspond to large representations and [22] establishes (1.6). Also, $\bar{\pi}_{51} = F$ is the finite dimensional module, so there is nothing to check for (1.6). This handles the easy cases.

The theory in Section 4 leads to the same five layer weight filtration for $J(\bar{\pi}_5)$ and $J(\bar{\pi}_7)$ (using labels in Fig. 1)

L_2
$L_7 \oplus L_7 \oplus L_6$
$L_{10} \oplus L_{13} \oplus L_{13} \oplus L_2 \oplus L_2$
$L_7 \oplus L_7 \oplus L_6$
L_2

Since $\dim \mathrm{Ext}^1(L_2, L_7) = 1$ in $\mathcal{O}'(\mathfrak{g}, \mathfrak{b})$, we see that $\bar{\pi}_5$ and $\bar{\pi}_7$ have L_7 in the socle of the Jacquet modules. But, from Fig. 2, L_7 is in $\mathscr{W}^{P_2}_{\mathrm{soc}}$ for $\mathcal{O}'(\mathfrak{g}, \mathfrak{p}_2)$; conclude that the modules have property $\mathfrak{p}_2$. Similarly, we conclude that property $\mathscr{W}^{P_2}_{\mathrm{soc}}$ holds on the double cells $\mathscr{C}^G_5 \cup \mathscr{C}^G_7$. This proves (1.6) in these cases.

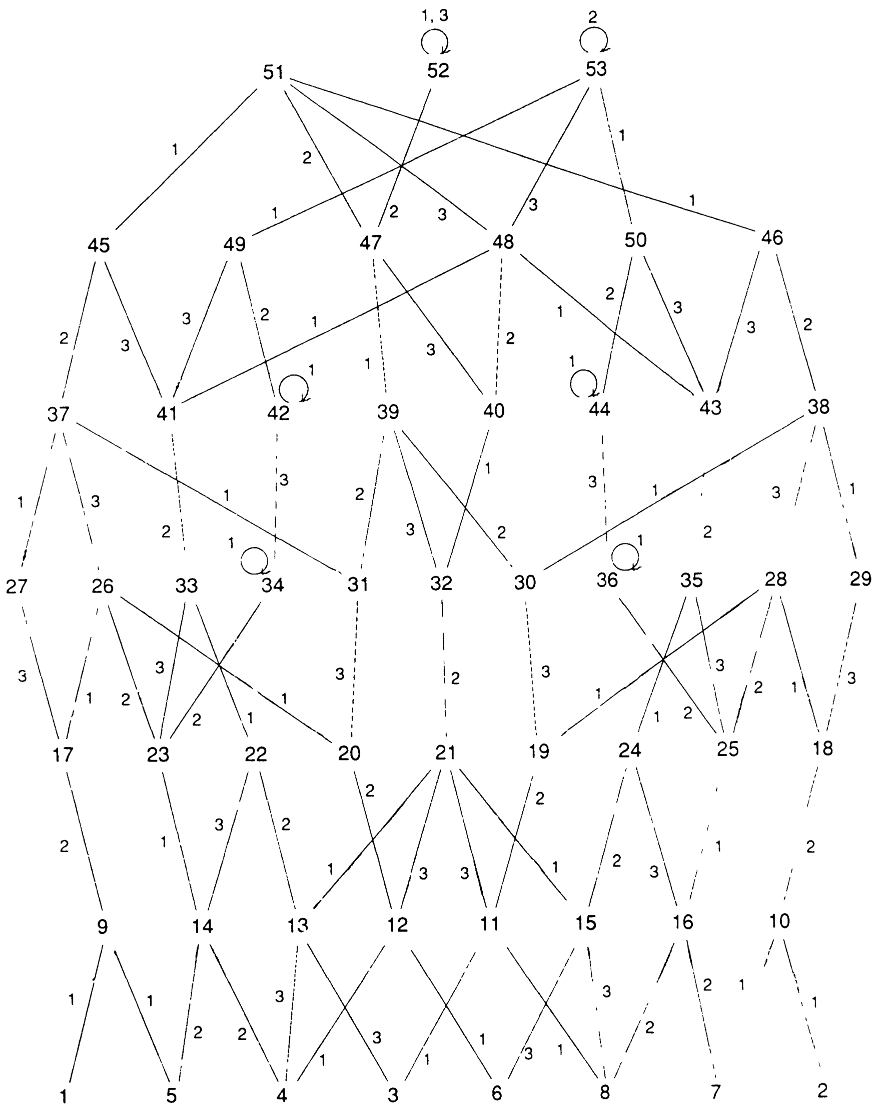


Fig. 3. $\mathscr{D}_o$ for $Sp_6\mathbb{R}$.

For the double cells $\mathscr{C}_3^G \cup \mathscr{C}_6^G$, the associated nilpotent orbit is again $\mathcal{O}_{42}$. However, in this case, we easily check that L_3 lies in the bottom weight layer of both $J(\bar{\pi}_3)$ and $J(\bar{\pi}_6)$. By Fig. 2 and (3.3), we have verified (1.6) on these two double cells.

The double cells $\mathscr{C}_1^G \cup \mathscr{C}_2^G$ index highest weight modules. From [12] we easily read off the Jacquet modules

$$J(\bar{\pi}_1) = J(\bar{\pi}_2) = L_{10}.$$

From Fig. 2, the cells $\mathscr{C}_1^G \cup \mathscr{C}_2^G$ have property $\mathscr{W}^{P_{23}}$; so, by (3.3), (1.6) holds in these cases.

In the case of $\mathscr{C}_{21}^G$ we argue as with $\mathscr{C}_5^G$. Although the Jacquet modules do not have bottom weight layer factors of the desired type (i.e. socular for $\mathcal{O}'(\mathfrak{g}, \mathfrak{p}_{13})$) one can argue that a second from the bottom layer factor attached to $\mathscr{W}_{\mathrm{soc}}^{P_{13}}$ lies in the socle. Thus, property $\mathfrak{p}_{13}$ holds and (1.6) holds in this case.

The above remarks have verified (1.6) for all but three double cells, $\mathscr{C}_{22}^G$, $\mathscr{C}_{24}^G$ and $\mathscr{C}_{39}^G$. Here is the main result of this section.

PROPOSITION 9.1. *In the case of $Sp_6\mathbb{R}$, Matumoto's conjecture is true for all double cells except $\mathscr{C}_{22}^G$ and $\mathscr{C}_{24}^G$; on these two double cells the conjecture fails*

Proof. Unfortunately, the techniques used thus far (namely, computation of a weight filtration for the Jacquet module) do not quite suffice. To see the delicate nature of what is at stake, consider the module $J(\bar{\pi}_{22})$, which is easily shown to have weight filtration

L_{19}
$L_{29} \oplus L_{29} \oplus L_{10}$
L_{19}

Obviously, L_{19} is in the bottom weight layer and so all the irredicibles $\bar{\pi}(\delta)$ attached to $\delta \in \mathscr{C}_{22}^G$ have property $\mathscr{C}_{19}^R$. The whole ball game comes down to deciding if L_{10} does nor does not drop into the socle of this Jacquet module. To settle the question we will study the structure of the P_{23}-degenerate series representations.

The semisimple part of the Levi factor L_{23} of P_{23} is isomorphic to $SL_3\mathbb{R}$. The irreducible representations of $SL_3\mathbb{R}$ corresponding to the block of the finite dimensional representation are labeled in [34, Fig. 16.2]. In that picture, $\pi_{P_{23}}(\gamma^3)$ is an irreducible standard module and $\bar{\pi}_{P_{23}}(\gamma_1^0)$ is the finite dimensional module of the Levi factor of P_{23}. By an induction in stages argument, one checks that

$$I_{P_{23}}(\pi_{P_{23}}(\gamma^3)) = I_B(w_{40}) = \pi_{40},$$

where we use the convention that $I_B(w_i)$ is the principal series representation of G indexed by the parameter labeled i in Fig. 1. A bootstrap argument (just like we employed in the $E_{6(-14)}$ case in Section 8), using the knowledge of the structure of standard modules of $SL_3\mathbb{R}$ (which is in [34, §16]) and the theory of [11] eventually shows that the P_{23}-degenerate series representation

$I_{P_{23}}(w^{P_{23}}) = I_{P_{23}}(\bar{\pi}(\gamma_1^o))$ has weight filtration

$\bar{\pi}_{51}$
$\bar{\pi}_{46} \oplus \bar{\pi}_{45}$
$\bar{\pi}_{39} \oplus \bar{\pi}_2 \oplus \bar{\pi}_1$

(9.3)

Next, apply the theory of [11] and arrive at Fig. 4 which describes the weight filtration of all P_{23}-degenerate series representations in $\mathscr{H}\mathscr{C}_o$. By inspection, the reader will note. If $\delta \in \mathscr{C}_{22}^G \cup \mathscr{C}_{24}^G$, then $\bar{\pi}(\delta)$ is *not* a composition factor of any P_{23}-degenerate series. On the other hand, suppose (1.6) were true for (say) $\bar{\pi}_{22}$. Then by the remarks at the beginning of this proof, we know that L_{10} is in the socle of $J(\bar{\pi}_{22})$. This implies a non-degenerate pairing $L_{10} \otimes \bar{\pi}_{22}$, leading to an injection

$$0 \mapsto \bar{\pi}_{22} \mapsto L_{10}^*.$$

But, L_{10} is the irreducible quotient of $N_{p_{23}}(e)$, so that $\bar{\pi}_{22}$ embedds as a submodule of

$$N_{p_{23}}(e)_{(K)}^* = I_{P_{23}}(w^{P_{23}});$$

a contradiction. Thus, (1.6) fails for $\bar{\pi}_{22}$. The same argument works for all $\bar{\pi}(\delta)$, with $\delta \in \mathscr{C}_{22}^G \cup \mathscr{C}_{24}^G$.

Finally, consider $\bar{\pi}_{39}$ and the associated cell $\mathscr{C}_{39}^G$. From (9.3) and duality, we see that there exists an exact sequence

$$I_{P_{23}}(e) \mapsto \bar{\pi}_{39} \mapsto 0.$$

In addition, the Subrepresentation theorem applied to the Levi factor of P_{23} and induction and stages shows that we have an exact sequence

$$I_B(\tilde{E}) \mapsto I_{P_{23}}(e) \mapsto 0,$$

for some finite-dimensional B-module $\tilde{E}$. But, again, as outlined in the $E_{6(-14)}$ cased of Section 8, we see that $I_B(\tilde{E}) = I_B(w_{B,P_m})$, where $w_{10} = w_{P_{23}} = w_B w_{B,P_m} = e w_{B,P_m} = w_{B,P_m}$, in the parametrization of Fig. 1. In other words, we have shown that $\bar{\pi}_{39}$ is a quotient of $I_B(w_{10})$. By (1.1b), this implies the existence of a non-zero map

$$N_{w_{10}} \mapsto J(\bar{\pi}_{39}).$$

$$\begin{array}{c} \overline{\pi}_{51} \\ \hline \overline{\pi}_{46} + \overline{\pi}_{45} \\ \hline \overline{\pi}_{39} + \overline{\pi}_{2} + \overline{\pi}_{1} \end{array}$$

$$\begin{array}{c} \overline{\pi}_{46} + \overline{\pi}_{45} \\ \hline \overline{\pi}_{38} + \overline{\pi}_{51} + \overline{\pi}_{37} + \overline{\pi}_{39} + \overline{\pi}_{2} + \overline{\pi}_{1} \\ \hline \overline{\pi}_{47} + \overline{\pi}_{9} + \overline{\pi}_{10} \end{array}$$

$$\begin{array}{c} \overline{\pi}_{38} + \overline{\pi}_{37} \\ \hline \overline{\pi}_{29} + \overline{\pi}_{28} + \overline{\pi}_{27} + \overline{\pi}_{26} + \overline{\pi}_{47} + \overline{\pi}_{9} + \overline{\pi}_{10} \\ \hline \overline{\pi}_{52} + \overline{\pi}_{17} + \overline{\pi}_{18} \end{array}$$

$$\begin{array}{c} \overline{\pi}_{29} + \overline{\pi}_{27} \\ \hline \overline{\pi}_{52} + \overline{\pi}_{17} + \overline{\pi}_{18} \\ \hline \overline{\pi}_{26} + \overline{\pi}_{28} \end{array} \qquad \begin{array}{c} \overline{\pi}_{26} + \overline{\pi}_{28} \\ \hline \overline{\pi}_{52} + \overline{\pi}_{17} + \overline{\pi}_{18} \\ \hline \overline{\pi}_{29} + \overline{\pi}_{27} \end{array}$$

$$\begin{array}{c} \overline{\pi}_{52} + \overline{\pi}_{17} + \overline{\pi}_{18} \\ \hline \overline{\pi}_{29} + \overline{\pi}_{28} + \overline{\pi}_{27} + \overline{\pi}_{26} + \overline{\pi}_{47} + \overline{\pi}_{9} + \overline{\pi}_{10} \\ \hline \overline{\pi}_{38} + \overline{\pi}_{37} \end{array}$$

$$\begin{array}{c} \overline{\pi}_{47} + \overline{\pi}_{9} + \overline{\pi}_{10} \\ \hline \overline{\pi}_{38} + \overline{\pi}_{51} + \overline{\pi}_{37} + \overline{\pi}_{39} + \overline{\pi}_{2} + \overline{\pi}_{1} \\ \hline \overline{\pi}_{46} + \overline{\pi}_{45} \end{array}$$

$$\begin{array}{c} \overline{\pi}_{39} + \overline{\pi}_{2} + \overline{\pi}_{1} \\ \hline \overline{\pi}_{46} + \overline{\pi}_{45} \\ \hline \overline{\pi}_{51} \end{array}$$

Fig. 4. P_{23}-degenerate series.

Now, $J(\bar{\pi}_{39})$ has a weight filtration of the form

L_{29}
$L_{33} \oplus L_{38} \oplus L_{19} \oplus L_{19}$
$L_{41} \oplus L_{41} \oplus L_{29} \oplus L_{29} \oplus L_{10}$
$L_{33} \oplus L_{38} \oplus L_{19} \oplus L_{19}$
L_{29}

This can only happen if L_{10} splits off as an indecomposable summand of $J(\bar{\pi}_{39})$. But, then property $\mathfrak{p}_{23}$ is immediate for all $\bar{\pi}(\delta)$, $\delta \in \mathscr{C}^G_{39}$, by (3.5).

10. Non-Hermitian rank two

If one tries to extend our consideration of (1.6) from the Hermitian to the general real rank two case, five additional groups (up to covering) require consideration

$SL_3\mathbb{R}$, $SL_3\mathbb{H}$, $E_{6(-26)}$, $G_{2(2)}$, $Sp(2, s)$.

We discuss the first four cases individually and verify that (1.6) holds. This is also true for the remaining group when $s = 2$, though we will omit the details. Combined with (1.7), this leads to an obvious conjecture; generalizing the program in [5] to $Sp(2, s)$ should provide sufficient machinery to verify the conjecture in these cases.

CONJECTURE 10.1. *Matumoto's conjecture* (1.6) *is true for real rank two groups*

$SL_3\mathbb{R}$.

As noted in the proof of (9.1), the irreducible Harish-Chandra modules in this case are parametrized in [34, Fig. 16.2]. If α, β are the simple roots, then there are three parabolic subalgebras defined over $\mathbb{R}$ of interest. The Borel subalgebra $\mathfrak{b}$ (since G is split) and the parabolic subalgebras $\mathfrak{p}_\alpha$ and $\mathfrak{p}_\beta$, where α and β are the simple roots in the Levi factor, respectively.

Since the nilpotent orbits in $\mathfrak{sl}_3$ are in one to one correspondence with partitions of 3, there are three nilpotent orbits

$\mathcal{O}_3$, $\mathcal{O}_{21}\mathcal{O}_{1^3}$.

A computation of the weighted Dynkin diagrams quickly shows that only

$\mathcal{O}_3$ and $\mathcal{O}_{1^3}$ are even; thus, (1.6) will only concern $\bar{\pi}(\delta)$ with $\mathcal{O}_{\bar{\pi}(\delta)} = \mathcal{O}_3$. Since these would be large representations, [22] proves (1.6) in this case.

10.2 REMARK. It is worth commenting that the converse of (1.3) does hold for this group. Namely, following the notation of [34, Fig. 16.2] we find

Double cell	Elements	Associated nilpotent
$\mathscr{C}^G_{\gamma^3}$	$\{\gamma^3\}$	$\mathcal{O}_3$
$\mathscr{C}^G_{\gamma^1}$	$\{\gamma^1, \gamma^o_{1,-1,-1}\}$	$\mathcal{O}_{21}$
$\mathscr{C}^G_{\gamma^2}$	$\{\gamma^2, \gamma^o_{-1,-1,1}\}$	$\mathcal{O}_{21}$
$\mathscr{C}^G_{\gamma^o_1}$	$\{\gamma^o_1\}$	$\mathcal{O}_{1^3}$

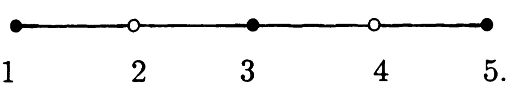
It is easy to compute a weight filtration for $J(\bar{\pi}(\gamma^2))$

L_β
$L_{\alpha\beta} \oplus L_{\alpha\beta}$
L_β

It is obvious that $\bar{\pi}(\gamma^2)$ has property $\mathfrak{p}_\beta$, since the Levi factor has simple root β and $W^{P_\beta} = \{e, \alpha, \alpha\beta\}$. Since all Ext^1 groups have dimension at most one in $\mathcal{O}'(\mathfrak{g}, \mathfrak{b})$, at least one copy of $L_{\alpha\beta}$ drops to the socle of $J(\bar{\pi}(\gamma^2))$; since $W^{P_\alpha} = \{e, \beta, \beta\alpha\}$, we conclude that $\bar{\pi}(\gamma^2)$ has property $\mathfrak{p}_\alpha$. Similar remarks apply to $\mathscr{C}^G_{\gamma^1}$

$SL_3\mathbb{H}$ and $E_{6(-26)}$

These two groups each have only one conjugacy class of Cartan subgroups. In the case of $SL_3\mathbb{H}$, the Satake diagram is

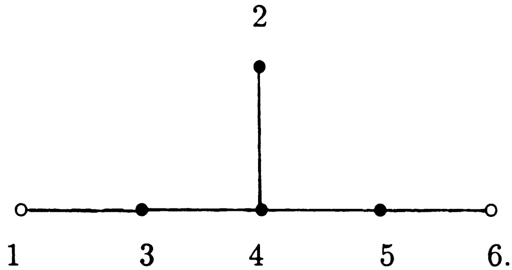


The minimal parabolic subalgebra $\mathfrak{p}_m$ has Levi factor simple roots α_1, α_3 and α_5. Thus, the only proper standard parabolic subalgebras defined over $\mathbb{R}$ are $\mathfrak{p}_2$ and $\mathfrak{p}_4$ which have Levi factor omitting the simple root α_2 and α_4, respectively. If either of these two maximal parabolic subalgebras was an even Jacobson–Morozov parabolic subalgebra, then either

$$0\ 2\ 0\ 0\ 0 \quad \text{or} \quad 0\ 0\ 0\ 2\ 0$$

would be the weighted Dynkin diagram of some nilpotent orbit in $\mathfrak{sl}_6$; this is a contradiction to [17, (3.6.5)]. Thus, $\mathfrak{p}_m$ is the only parabolic subalgebra fitting into the hypothesis of (1.6) and Step 5 of Section 4 verifies the conjecture in this setting.

In the case of $E_{6(-26)}$, the Satake diagram is



Arguing exactly as in the $SL_3\mathbb{H}$ case, we see that $\mathfrak{p}_m$ is the only even Jacobson–Morozov parabolic subalgebra defined over $\mathbb{R}$. Again, Step 5 of Section 4 verifies (1.6) in this case

$G_{2(2)}$.

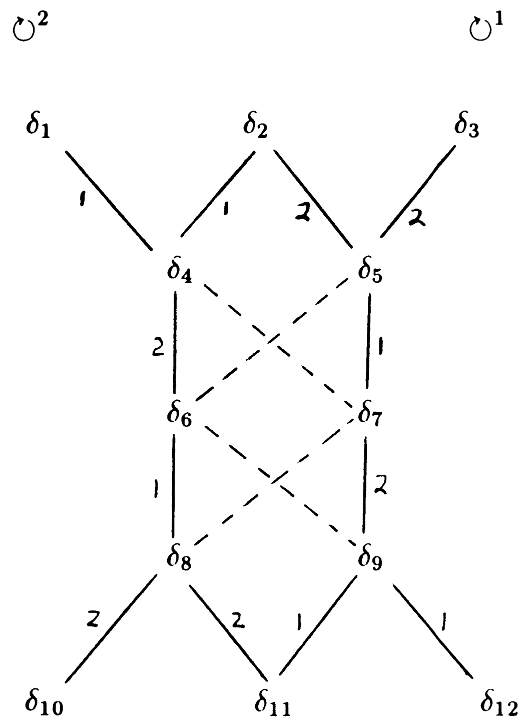


Fig. 5. $\mathscr{D}_o$ for $G_{2(2)}$.

For this split group, let α_1 (resp. α_2) be the simple short (resp. long) root. Denote by $\mathfrak{p}_i$ the standard maximal parabolic subalgebra with α_i a simple root of the Levi factor. The nilpotent orbits of $\mathfrak{g}_2$ are given in [17, §8.4] together with their weighted Dynkin diagrams. Conclude that only $\mathfrak{b} = \mathfrak{p}_m$ and $\mathfrak{p}_1$ are even Jacobson–Morozov parabolic subalgebras. In the obvious sense, parametrize W and W^{P_1} as follows

$$W = \{e, 1, 2, 12, 21, 121, 212, 1212, 2121, 12121, 21212, 121212\}$$

$$W^{P_1} = \{e, 2, 21, 212, 2121, 21212\}.$$

The set $\mathscr{D} = \mathscr{D}_o \cup \mathscr{D}_1$, where $\mathscr{D}_o$ is the block of the finite dimensional module and $\mathscr{D}_1$ is a singleton block corresponding to an irreducible principal series representation. An irreducible principal series is large and [22] handles (1.6) in this case. We are reduced to studying the block $\mathscr{D}_o$, which is given in Fig. 5. The corresponding double cells and their associated nilpotent orbits are tabulated below

Double cell	Elements	Associated nilpotent
$\mathscr{C}^G_{\delta_{11}}$	$\{\delta_{11}\}$	$\mathcal{O}_\mathfrak{b}$
$\mathscr{C}^G_{\delta_{10}}$	$\{\delta_{10}, \delta_8, \delta_6, \delta_4, \delta_1\}$	$\mathcal{O}_{\mathfrak{p}_1} = \mathcal{O}_{\mathfrak{p}_2}$
$\mathscr{C}^G_{\delta_{12}}$	$\{\delta_{12}, \delta_9, \delta_7, \delta_5, \delta_3\}$	$\mathcal{O}_{\mathfrak{p}_1} = \mathcal{O}_{\mathfrak{p}_2}$
$\mathscr{C}^G_{\delta_2}$	$\{\delta_2\}$	$\{0\}$

The cell $\mathscr{C}^G_{11}$ corresponds to a large representation and [22] handles (1.6) in this case. The two double cells $\mathscr{C}^G_{10} \cup \mathscr{C}^G_{12}$ have associated nilpotent orbit equal to the subregular orbit $\mathcal{O}_{\mathfrak{p}_1}$. In the case of $\mathscr{C}^G_{10}$, one computes that the bottom weight layer contains an element of $\mathscr{W}^{P_1}$, so (1.6) holds. For the double cell $\mathscr{C}^G_{12}$, one computes a weight filtration on $J(\bar{\pi}(\delta_{12}))$ to be

L_2
$L_{12} \oplus L_{12}$
L_2

Knowing that the dimension of the Ext^1 groups in $\mathcal{O}'(\mathfrak{g}, \mathfrak{b})$ is at most one, we see that one copy of L_{12} splits off as an indecomposable summand of $J(\bar{\pi}(\delta_{12}))$. This shows that $\bar{\pi}_{12}$ has property $\mathscr{W}^{P_1}$ and by (3.5), we see (1.6) holds on this double cell.

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