COMPOSITIO MATHEMATICA

S. M. BHATWADEKAR

AMARTYA K. DUTTA

On A^1-fibrations of subalgebras of polynomial algebras

Compositio Mathematica, tome 95, nº 3 (1995), p. 263-285

<http://www.numdam.org/item?id=CM_1995__95_3_263_0>

Compositio Mathematica **95**: 263–285, 1995.

On $\mathbf{A}^1$-fibrations of subalgebras of polynomial algebras

S. M. BHATWADEKAR & AMARTYA K. DUTTA

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay-400 005, India

Received 2 August 1993; accepted in final form 9 February 1994

1. Introduction

Let R be a commutative ring with unity. Let $R^{[n]}$ denote a polynomial ring in n variables over R. For a prime ideal P of R, $k(P)$ denotes the field R_P/PR_P. An R-algebra A is said to be an $\mathbf{A}^r$-fibration over R if

(i) A is finitely generated over R.
(ii) A is flat over R.
(iii) $A \otimes_R k(P) = k(P)^{[r]}$ $\forall P \in \operatorname{Spec}(R)$.

In ([A], 3.4), T. Asanuma has given a structure theorem for an $\mathbf{A}^r$-fibration A over a commutative noetherian ring R (see (2.6) of our paper). The statement of the theorem shows that *a necessary condition for an R-algebra A to be an $\mathbf{A}^r$-fibration over R is that A must be R-isomorphic to an R-subalgebra of $R^{[n]}$ for some n.* Therefore it is natural to ask: *What fibre conditions would be sufficient for an R-subalgebra of $R^{[n]}$ to be an $\mathbf{A}^r$-fibration over R?* In this paper we investigate the above question for $r = 1$. (See Theorems A and B below putting $m = 0$, i.e., $S = R$.)

THEOREM A. *Let R be a commutative noetherian ring containing a field of characteristic zero, $S = R^{[m]}$ for some integer $m \geqslant 0$ and A an S-subalgebra of $S^{[n]}$ such that*

(i) *A is R-flat.*
(ii) *$A \otimes_R k(P)$ are $(m+1)$-dimensional factorial domains for all minimal prime ideals P of R.*
(iii) *$A \otimes_R k(P)$ are integral domains for all height one prime ideals P of R.*

Then A is an $\mathbf{A}^1$-fibration over S.

THEOREM B. *Let R be a commutative noetherian ring, $S = R^{[m]}$ for some integer $m \geqslant 0$ and A an S-subalgebra of $S^{[n]}$ such that*

(i) *A is R-flat.*
(ii) *$A \otimes_R k(P)$ are factorial domains for all*

$$P \in \operatorname{Spec}(R) \quad \text{and} \quad \dim(A \otimes_R k(P)) = m + 1$$

for all minimal prime ideals P of R.

Then A is an $\mathbf{A}^1$-fibration over S.

The motivation for proving our theorems in such generality came from the following result ([B-D], 3.4): *if $S = R^{[m]}$ and $S \hookrightarrow A \hookrightarrow S^{[n]}$ where $A = R^{[m+1]}$ then A is an $\mathbf{A}^1$-fibration over S.* We investigated this phenomenon more closely and examined what conditions on A over R are actually needed to conclude that A is an $\mathbf{A}^1$-fibration over S. These considerations led to our Theorems A and B (for proofs see (3.12) and (3.14)).

As a consequence of our Theorems A and B, it follows using well-known results (see (3.16)) that if R_{red} is seminormal, then under the hypotheses of either Theorems A or B, we have $A \cong \operatorname{Sym}_S(P \otimes_R S)$ for a finitely generated projective R-module P of rank one. We shall make further discussions about our theorems in Section 4.

In Section 2 we set up notations and quote some results for later use. In Section 3 we prove our main theorems. We give examples in Section 4 to illustrate that the conditions in Theorems A and B are the best possible.

2. Preliminaries

In this section we set up notations and state some results for later use. Throughout this paper all rings will be commutative with unity.

For a commutative ring R,

$R^{[n]}$: polynomial ring in n variables over R,
R^* : the multiplicative group of invertible elements of R.

For a prime ideal P of R,

$k(P) : R_P/PR_P$.

For a finitely generated R-algebra A,

$\Omega_{A/R}$: universal module of R-differentials of A.

(2.1) DEFINITION. An R-algebra A is said to be an $\mathbf{A}^r$-*fibration* over R if the following hold:

(i) A is finitely generated over R.
(ii) A is R-flat.
(iii) $A \otimes_R k(P) = k(P)^{[r]}$ for all $P \in \mathrm{Spec}(R)$.

(2.2) DEFINITION. Let k be a field, $\bar{k}$ the algebraic closure of k and let D be a k-algebra. A D-algebra B is said to be an $\mathbf{A}^1$-*form* over D (with respect to k) if $B \otimes_k \bar{k} = (D \otimes_k \bar{k})^{[1]}$.

(2.3) DEFINITION. Let k be a field and $\bar{k}$ denote the algebraic closure of k. A k-algebra B is said to be *geometrically integral over* k if $B \otimes_k \bar{k}$ is an integral domain.

(2.4) DEFINITION. A reduced ring R is said to be *seminormal* if it satisfies the condition: for $b, c \in R$ with $b^3 = c^2$, there is an $a \in R$ such that $a^2 = b$, $a^3 = c$.

(2.5) DEFINITION. Let A be a ring and S a subring of A. An S-algebra homomorphism $\alpha: A \to S$ is called a *retract* from A to S.

We will now quote some results which will be needed in this paper. We begin with a result of T. Asanuma ([A], 3.4):

THEOREM 2.6. *If A is an $\mathbf{A}^r$-fibration over a commutative noetherian ring R, then $\Omega_{A/R}$ is a projective A-module and A is (upto an R-isomorphism) an R-subalgebra of $R^{[n]}$ for some n such that $A^{[n]} \cong \mathrm{Sym}_{R^{[n]}}(\Omega_{A/R} \otimes_A R^{[n]})$ as R-algebras.*

Next we state a result which appears in ([A-E-H], 4.1) and ([R-S], 3.4):

THEOREM 2.7. *If $S \hookrightarrow A \hookrightarrow S^{[n]}$ are factorial domains such that the transcendence degree of A over S is one, then $A = S^{[1]}$.*

We now state a theorem of Bass-Connell-Wright ([B-C-W], 4.4):

THEOREM 2.8. *Let A be a finitely presented S-algebra. Suppose that for all maximal ideals P of S, the S_P-algebra A_P is S_P-isomorphic to the symmetric algebra of some S_P-module. Then A is S-isomorphic to the symmetric algebra $\mathrm{Sym}_S(M)$ of a finitely presented S-module M.*

We now quote a theorem on finite generation due to N. Onoda ([O], 2.20).

THEOREM 2.9. *Let R be a noetherian domain and let A be an overdomain of R such that*

(i) *There exists a non-zero element $f \in A$ for which A_f is a finitely generated R-algebra.*
(ii) *A_M is a finitely generated R_M-algebra for all maximal ideals M of R.*

Then A is a finitely generated R-algebra.

The following result is due to E. Hamann ([H], 2.6):

THEOREM 2.10. *Let R be a noetherian ring such that R_{red} is seminormal. Then $R^{[1]}$ is R-invariant, i.e., if A is an R-algebra such that $A^{[m]} = R^{[m+1]}$ as R-algebras, then $A = R^{[1]}$.*

The next theorem is due to R.G. Swan ([S], 6.1):

THEOREM 2.11. *Let R be a ring such that R_{red} is seminormal. Then* $\text{Pic}(R^{[n]}) = \text{Pic}(R)\ \forall n \geqslant 1$.

The following result occurs in ([B-D], 3.4).

THEOREM 2.12. *Let R be a commutative noetherian ring such that either R_{red} is seminormal or R contains a field of characteristic zero. Let S and A be R-algebras such that $S = R^{[m]}$, $A = R^{[m+1]}$ and $S \hookrightarrow A \hookrightarrow S^{[n]}$. Then $A = S^{[1]}$.*

3. Main theorems

In this section we shall prove our main theorems (Theorems 3.12 and 3.14). For the proofs of these theorems we need some lemmas. We begin with:

LEMMA 3.1. *Let $R \hookrightarrow S \hookrightarrow A \hookrightarrow C$ be commutative rings. Suppose that P is a prime ideal of R of* ht 0 *such that PS, PA and PC are all prime ideals of S, A and C respectively. Then $PC \cap A = PA$, $PA \cap S = PS$ and $PS \cap R = P$ so that*

$$R/P \hookrightarrow (R/P) \otimes_R S \hookrightarrow (R/P) \otimes_R A \hookrightarrow (R/P) \otimes_R C.$$

Proof. Since $\text{ht}\, P = 0$, R_P is a zero-dimensional local ring with unique prime ideal PR_P and hence $\text{nil}(R_P) = PR_P$. Therefore it is easy to see that $PS \cap R = P$, $PA \cap R = P$ and $PC \cap R = P$. Hence localising at P, it is enough to assume that R is a zero-dimensional local ring with unique prime ideal P. Then $PA \cap S$ is a prime ideal of S all of whose elements are nilpotent. But $PA \cap S \supseteq PS$ and by hypothesis PS is a prime ideal of S. Hence $PA \cap S = PS$. Similarly $PC \cap A = PA$. □

LEMMA 3.2. *Let $R \hookrightarrow S \hookrightarrow A \hookrightarrow C$ be flat R-algebras over a noetherian ring R such that for all minimal prime ideals P of R, PS, PA and PC are prime ideals of S, A and C respectively. Then*

$$(R/\text{nil } R) \hookrightarrow (R/\text{nil } R) \otimes_R S \hookrightarrow (R/\text{nil } R) \otimes_R A \hookrightarrow (R/\text{nil } R) \otimes_R C.$$

Proof. It is enough to prove that $(\text{nil } R)S = \text{nil } S$, $(\text{nil } R)A = \text{nil } A$ and $(\text{nil } R)C = \text{nil } C$. We show that $(\text{nil } R)C = \text{nil } C$; the other equalities follow similarly. Now $(\text{nil } R)C \subseteq \text{nil } C$. Let $P_1, \ldots, P_t$ be the minimal prime ideals of R. Then $\text{nil } C \subseteq P_i C \; \forall i$, and by flatness of C over R ([M], 3.H),

$$\text{nil } C \subseteq \bigcap_{1 \leqslant i \leqslant t} P_i C = \left(\bigcap_{1 \leqslant i \leqslant t} P_i \right) C = (\text{nil } R)C.$$

Thus $(\text{nil } R)C = \text{nil } C$. □

LEMMA 3.3. *Let R be a commutative ring, I a nilpotent ideal of R, A an R-algebra such that A/IA is finitely generated over R/I. Then A is finitely generated over R.*

Proof. Let $x_1, \ldots, x_n \in A$ be such that their images generate A/IA over R/I. Then the map $R[x_1, \ldots, x_n] \to A/IA$ is a surjection and hence $A = R[x_1, \ldots, x_n] + IA$. Since I is nilpotent it follows that $A = R[x_1, \ldots, x_n]$. □

LEMMA 3.4. *Let R be a noetherian ring and R_1 an overring of R which is finitely generated as an R-module. If A is a flat R-algebra such that $A \otimes_R R_1$ is a finitely generated R_1-algebra, then A is a finitely generated R-algebra.*

Proof. Since A is R-flat, $A \hookrightarrow A \otimes_R R_1$ and we identify A with its image in $A \otimes_R R_1$. By hypothesis, $A \otimes_R R_1$ is clearly a finitely generated R-algebra. Let $x_1, \ldots, x_m$ be the generators of $A \otimes_R R_1$ over R. Since $A \otimes_R R_1$ is a finite A-module, each x_i satisfies an integral equation over A. Let $B = R[a_{ij}]_{1 \leq j \leqslant n_i, 1 \leqslant i \leqslant m}$ be the R-subalgebra of A generated by the coefficients a_{ij} of such equations. Then clearly $A \otimes_R R_1$ is a finite module over the noetherian ring B. Hence A, being a B-submodule of $A \otimes_R R_1$, is a finite module over B. Therefore, A is a finitely generated R-algebra. □

As a consequence of Lemma 3.4 we have the following:

COROLLARY 3.5. *Let R be a reduced noetherian ring and A a flat R-algebra such that A/PA is a finitely generated overdomain of R/P for all minimal prime ideals P of R. Then A is finitely generated over R.*

LEMMA 3.6. *Let k be a field and B be a normal overdomain of k. Let Q be a finitely generated prime ideal of B of height at most one. If B/Q is geometrically integral over k, then B is also geometrically integral over k.*

Proof. If $Q = 0$, it is obvious. If $Q \neq 0$, then B_Q is a one-dimensional normal local domain. Since all its prime ideals are finitely generated, it is also noetherian ([N], 3.4). Thus B_Q is a discrete valuation ring with maximal ideal QB_Q and residue field L, where L is the quotient field of B/Q. Let $\bar{k}$ be the algebraic closure of k. To check that $B \otimes_k \bar{k}$ is a domain it is enough to prove that $B_Q \otimes_k \bar{k}$ is a domain. By faithful flatness, it is enough to prove that $B_Q \otimes_k k_1$ is a domain for any finite extension k_1 of k inside $\bar{k}$. Now $\dim(B_Q \otimes_k k_1) = \dim(B_Q) = 1$. Since any maximal ideal of $B_Q \otimes_k k_1$ contracts to the unique maximal ideal QB_Q of B_Q, it contains $QB_Q \otimes_k k_1$. But $(B_Q \otimes_k k_1)/(QB_Q \otimes_k k_1) = L \otimes_k k_1$ which is a field (since B/Q is geometrically integral and L is the quotient field of B/Q). Therefore $QB_Q \otimes_k k_1$ is the unique maximal ideal of the noetherian one-dimensional ring $B_Q \otimes_k k_1$ and it is principal (since QB_Q is principal). Thus $B_Q \otimes_k k_1$ is a discrete valuation ring and hence a domain. Hence the result. □

LEMMA 3.7. *Let k be a perfect field, D a noetherian normal k-domain and B an $\mathbf{A}^1$-form over D (with respect to k). Then B is D-isomorphic to* $\mathrm{Sym}_D(M)$ *for some finitely generated projective D-module M of rank one. In particular if $D = k^{[m]}$ for some integer m, then $B = D^{[1]}$.*

Proof. Since B is an $\mathbf{A}^1$-form over D (with respect to k) and k is perfect, it is easy to see that B is finitely generated and flat over D and that there exists a finite separable extension $\tilde{k}$ of k such that $B \otimes_k \tilde{k} = (D \otimes_k \tilde{k})^{[1]}$. By (2.8) it is enough to assume that D is local and prove that $B = D^{[1]}$. We now show that $B \otimes_D k(P) = k(P)^{[1]}$ $\forall P \in \mathrm{Spec}(D)$. Let $B_1 = B \otimes_D k(P)$. We have $B_1 \otimes_{k(P)} (k(P) \otimes_k \tilde{k}) = (k(P) \otimes_k \tilde{k})^{[1]}$. It is easy to see that $k(P) \otimes_k \tilde{k}$ is a finite direct product of finite separable extension L_i of $k(P)$ and hence $B_1 \otimes_{k(P)} L_i = L_i^{[1]}$ for each i. Since L_i is a finite separable extension of $k(P)$, it follows by a well-known result that $B_1 (= B \otimes_D k(P)) = k(P)^{[1]}$.

Thus B is an $\mathbf{A}^1$-fibration over D and hence by (2.6), $B^{[n]}$ is D-isomorphic to $\mathrm{Sym}_{D^{[n]}}(\Omega_{B/D} \otimes_B D^{[n]})$ for some n and $\Omega_{B/D}$ is a projective B-module of rank 1. Since D is a local normal domain, it follows that $\Omega_{B/D} \otimes_B D^{[n]}$ is actually a free $D^{[n]}$-module. Thus $B^{[n]} = D^{[n+1]}$. Now by (2.10), $B = D^{[1]}$. Hence the result. □

LEMMA 3.8. *Let $R \hookrightarrow S \hookrightarrow A$ be commutative noetherian rings such that*

(i) *S and A are R-flat.*
(ii) $A \otimes_R k(P) = (S \otimes_R k(P))^{[1]}$ $\forall P \in \mathrm{Spec}(R)$.

Then A is flat over S and $A \otimes_S k(Q) = k(Q)^{[1]}$ $\forall Q \in \mathrm{Spec}(S)$.

Proof. Let $Q' \in \mathrm{Spec}(A)$. Let $Q = Q' \cap S$ and $P = Q \cap R$. Then we get the following local homomorphisms:

$$R_P \to S_Q \to A_{Q'}.$$

Now $A \otimes_R k(P)$ being flat over $S \otimes_R k(P)$,

$$A_{Q'} \otimes_{R_P} k(P) \text{ is flat over } S_Q \otimes_{R_P} k(P).$$

Also A being R-flat, $A_{Q'}$ is flat over R_P and S being R-flat, S_Q is flat over R_P. Therefore, by a standard result ([M], 20.G), $A_{Q'}$ is flat over S_Q. This shows ([M], 3.J) that A is flat over S.

We now prove the fibre condition. Let $Q \in \operatorname{Spec}(S)$. Let $P = Q \cap R$, $A_1 = A \otimes_R k(P)$ and $S_1 = S \otimes_R k(P)$. Then

$$A \otimes_S k(Q) = A \otimes_S S_1 \otimes_{S_1} k(Q) = A_1 \otimes_{S_1} k(Q) = k(Q)^{[1]}. \qquad \square$$

LEMMA 3.9. *Let R be a noetherian local ring and A a flat R-algebra. Suppose that there exists a regular sequence x, y in R and an element $\sigma \in GL_n(R_{xy})$ such that*

(i) $A_x = R_x[F_1, \dots, F_n] = R_x^{[n]}$.
(ii) $A_y = R_y[G_1, \dots, G_n] = R_y^{[n]}$.
(iii) $[F_1, \dots, F_n]\sigma = [G_1, \dots, G_n]$.

Then A is a graded R-algebra. Moreover if A is an R-subalgebra of a finitely generated flat graded R-algebra $C = \oplus_{l \geq 0} C^{(l)}$ with $C^{(0)} = R$ (the gradation of A need not be compatible with the gradation of C), then $A = R^{[n]}$.

Proof. We have

$$A_x = \bigoplus_{m \geq 0} B^{(m)} \quad \text{and} \quad A_y = \bigoplus_{m \geq 0} D^{(m)}$$

where

$$B^{(m)} = \bigoplus_{i_1 + \cdots + i_n = m} R_x F_1^{i_1} F_2^{i_2} \cdots F_n^{i_n}$$

and

$$D^{(m)} = \bigoplus_{i_1 + \cdots + i_n = m} R_y G_1^{i_1} G_2^{i_2} \cdots G_n^{i_n}.$$

As R is local, x and y form an R-sequence and A is R-flat, it follows easily that

$$R = R_x \cap R_y \quad \text{and} \quad A = A_x \cap A_y$$

(considered as subrings of R_{xy} and A_{xy} respectively). Since

$$A_{xy} = \bigoplus_{m\geq 0} (B^{(m)})_y = \bigoplus_{m\geq 0} (D^{(m)})_x$$

and $(B^{(m)})_y = (D^{(m)})_x$ for all m (as $\sigma \in GL_n(R_{xy})$), we have

$$A(=A_x \cap A_y) = \bigoplus_{m\geq 0} A^{(m)} \quad \text{where} \quad A^{(m)} = B^{(m)} \cap D^{(m)}.$$

Thus A is a graded R-algebra.

Now assume that A is contained in a finitely generated graded flat R-algebra $C = \oplus_{l\geq 0} C^{(l)}$ with $C^{(0)} = R$. We first show that $A^{(1)}$ is a finitely generated R-module. Now

$$(A^{(1)})_x = B^{(1)} = \bigoplus_{1\leqslant i\leqslant n} R_x F_i$$

is clearly an R_x-submodule of $\oplus_{0\leqslant l\leqslant d} C_x^{(l)}$ for sufficiently large d. It easily follows that $A^{(1)}$ is an R-submodule of $\oplus_{0\leqslant l\leqslant d} C^{(l)}$ so that $A^{(1)}$ is a finitely generated R-module. Moreover A and hence $A^{(1)}$ being flat over R, it follows that $A^{(1)}$ is actually free of rank n over R. Let $A^{(1)} = \oplus_{1\leqslant i\leqslant n} RH_i$. Now it is easy to see that $A = R[H_1, \ldots, H_n]$ $(=R^{[n]})$. □

THEOREM 3.10. *Let R be a noetherian normal domain with quotient field K. Let $S = R^{[m]}$ for some integer $m \geqslant 0$. Let A be an S-subalgebra of $S[X_1, \ldots, X_n]$ $(=S^{[n]})$ such that the following conditions hold:*

(i) *A is R-flat.*
(ii) *$A \otimes_R K$ is a factorial domain and* $\dim(A \otimes_R K) = m + 1$.
(iii) *$A \otimes_R k(P)$ are integral domains for all height one prime ideals P of R.*

Then A is S-isomorphic to $\mathrm{Sym}_S(E \otimes_R S)$ *where E is a finitely generated projective R-module of rank one.*

Proof. Since $S[X_1, \ldots, X_n]$ is a finitely generated overdomain of R and $R \hookrightarrow A \hookrightarrow S[X_1, \ldots, X_n]$, by ([G], (2.1)), there exists an element $f(\neq 0) \in A$ such that A_f is a finitely generated R-algebra. Therefore by (2.9), A would be finitely generated over R (and hence over S) if A is locally finitely generated over R. Hence by (2.8) and the fact that $\mathrm{Pic}(R^{[m]}) = \mathrm{Pic}(R)$ for a normal domain R, it is enough to assume that R is a local noetherian normal domain with maximal ideal M and show that $A = S^{[1]}$.

Note that since $\dim(A \otimes_R K) = m + 1$, by ([$G$], 2.3(b)) it follows that the transcendence degree of $A \otimes_R K$ over $S \otimes_R K$ is one. Therefore as

$$S \otimes_R K(=K^{[m]}) \hookrightarrow A \otimes_R K \hookrightarrow (S \otimes_R K)^{[n]}(=K^{[m+n]})$$

are factorial domains, by (2.7), it follows that

$$A \otimes_R K = (S \otimes_R K)^{[1]}(=K^{[m+1]}). \tag{3.10.1}$$

We now prove that $A = S^{[1]}$ by induction on $\dim R(=\operatorname{ht} M)$.

If $\dim R = 0$, then $R = K$ and we are through by (3.10.1).

Let $\dim R = 1$. Then R is a discrete valuation ring with maximal ideal $M = (\pi)$. Now $A_\pi(=A \otimes_R K)$ is a factorial domain; and by condition (iii), $A/\pi A$ is an integral domain. Also since $A \hookrightarrow S[X_1, \ldots, X_n]$ and S is noetherian, $\bigcap_{l \geq 0} \pi^l A = (0)$. Hence A is a factorial domain. Therefore, as $\operatorname{tr deg}_S A$ is one, by (2.7), $A = S^{[1]}$.

Now let $\dim R \geq 2$. Let $c(\neq 0) \in M$. Then $\dim R_c < \dim R$. By induction hypothesis, A_c is locally finitely generated over R_c and hence by ([G], 2.1) and (2.9), A_c is finitely generated over R_c. Also, by (3.10.1) $A_c \otimes_{R_c} K = (S_c \otimes_{R_c} K)^{[1]}$. Therefore it is easy to see that there exists $x \in M$ such that $A_x = S_x[F]$ for some $F \in A \cap (X_1, \ldots, X_n)S[X_1, \ldots, X_n]$.

Let $P_1, \ldots, P_t$ be the associated prime ideals of xR. Since R is a normal domain, $\operatorname{ht}(P_i) = 1$ for all i, $1 \leq i \leq t$. Let

$$T = R \Big\backslash \left(\bigcup_{i=1}^{t} P_i \right).$$

Then $T^{-1}R$ being a semi-local Dedekind domain is a P.I.D. Now

$$T^{-1}R \hookrightarrow T^{-1}S \hookrightarrow T^{-1}A \hookrightarrow (T^{-1}S)[X_1, \ldots, X_n].$$

Let $T^{-1}P_i = (u_i)T^{-1}R$, $1 \leq i \leq t$. By conditions (i) and (iii), A/P_iA are domains and hence $T^{-1}A/(u_i)$ $(=T^{-1}(A/P_iA))$ are domains. Thus $u_1, \ldots, u_t$ are prime elements of $T^{-1}A$ such that $T^{-1}A[1/u_1, \ldots, 1/u_t]$ $(=A \otimes_R K)$ is a factorial domain. Also as $T^{-1}A \hookrightarrow (T^{-1}S)[X_1, \ldots, X_n]$, $\bigcap_{l \geq 0} u_i^l T^{-1}A = (0)$, for all i. Therefore $T^{-1}A$ is a factorial domain and hence as $\operatorname{tr deg}_S A = 1$, by (2.7), $T^{-1}A = (T^{-1}S)^{[1]}$.

We now show that there exists $y \in T$ such that $A_y = S_y^{[1]}$. Choose $s \in T \cap M$. Then $\dim(R_s) < \dim R$. By induction hypothesis A_s is locally finitely generated over R_s and hence by ([G], 2.1) and (2.9), A_s is finitely generated over R_s. Now since $T^{-1}A_s = T^{-1}S_s^{[1]}$ it follows that there exists $y \in T \cap M$ such that $A_y = S_y[G]$ where G can be chosen to be in

$$A \cap (X_1, \ldots, X_n)S[X_1, \ldots, X_n].$$

By construction the pair (x, y) form a sequence in R. Also since

$$F, G \in A \cap (X_1, \ldots, X_n)S[X_1, \ldots, X_n] \quad \text{with} \quad A_{xy} = S_{xy}[F] = S_{xy}[G]$$

it follows that $G = uF$ for some $u \in (S_{xy})^*$ $(= R^*_{xy})$. Let $S = R[W_1, \ldots, W_m]$. Now

$$A_x = S_x[F] = R_x[W_1, \ldots, W_m, F]$$

and

$$A_y = S_y[G] = R_y[W_1, \ldots, W_m, G]$$

with $G = uF$, $u \in R^*_{xy}$. Hence by Lemma (3.9), it follows that $A = R^{[m+1]}$. Now using (2.12) we conclude that $A = S^{[1]}$. □

PROPOSITION 3.11. *Let R be a commutative noetherian ring, $S = R^{[m]}$ for some integer $m \geqslant 0$, A an S-subalgebra of $S^{[n]}$ such that*

(i) *A is R-flat.*
(ii) *$A \otimes_R k(P)$ are factorial domains of Krull dimension $m + 1$ for all minimal prime ideals P of R.*
(iii) *$A \otimes_R k(P)$ are geometrically integral over $k(P)$ for all height one prime ideals P of R.*

Then the following hold:

(a) *$A \otimes_R k(P)$ are $\mathbf{A}^1$-forms over $S \otimes_R k(P)$ (with respect to $k(P)$) for all prime ideals P of R.*
(b) *A is finitely generated over R.*

Proof. We first prove (a). Since it is a fibre condition, replacing R by R_P, we can assume that R is a local noetherian ring with maximal ideal P. We prove (a) by induction on $\text{ht}(P)(= \dim R)$.

Let $\text{ht}\, P = 0$. By (3.1), we can replace R by R/P and assume that R is the field $k(P)$. Hence using (3.10), we in fact deduce that

$$A \otimes_R k(P) = (S \otimes_R k(P))^{[1]}$$

Now let $\text{ht}\, P \geqslant 1$. By inductive hypothesis we assume that $A \otimes_R k(Q)$ are $\mathbf{A}^1$-forms over $S \otimes_R k(Q)$ $\forall Q \in \text{Spec}(R)$ with $\text{ht}\, Q < \text{ht}\, P$. Further by the proof of ht 0 case we also assume that

$$A \otimes_R k(Q) = (S \otimes_R k(Q))^{[1]} \text{ for all minimal prime ideals } Q \text{ of } R. \quad (3.11.1)$$

Let $\hat{R}$ denote the completion of R. Then $\hat{R}$ is a complete local ring with maximal ideal $\hat{P}$ such that $R/P \cong \hat{R}/\hat{P}$. Now $\hat{R}$ being R-flat, $A \otimes_R \hat{R}$ is $\hat{R}$-flat and

$$\hat{R} \hookrightarrow S \otimes_R \hat{R} \hookrightarrow A \otimes_R \hat{R} \hookrightarrow (S \otimes_R \hat{R})^{[n]}.$$

For any prime ideal $\bar{Q}$ of $\hat{R}$ with $\operatorname{ht}(\bar{Q}) < \operatorname{ht}(\hat{P})$, let $Q = \bar{Q} \cap R$. Then we know that $\operatorname{ht}(Q) < \operatorname{ht}(P)$ so that $A \otimes_R k(Q)$ is an $\mathbf{A}^1$-form over $S \otimes_R k(Q)$ with respect to $k(Q)$ and hence it is easy to see that

$$(A \otimes_R \hat{R}) \otimes_{\hat{R}} k(\bar{Q}) \text{ is an } \mathbf{A}^1\text{-form over } (S \otimes_R \hat{R}) \otimes_{\hat{R}} k(\bar{Q}) \text{ w.r.t. } k(\bar{Q}).$$

Moreover if $\bar{Q}$ is a minimal prime ideal of $\hat{R}$ then by the R-flatness of $\hat{R}$, Q is a minimal prime ideal of R (by the "going-down theorem" ([M], 5.D)) so that by (3.11.1) it follows that

$$(A \otimes_R \hat{R}) \otimes_{\hat{R}} k(\bar{Q}) = ((S \otimes_R \hat{R}) \otimes_{\hat{R}} k(\bar{Q}))^{[1]}.$$

Thus to prove (a) it is enough to assume that R is a *complete* local ring. Using (3.1) we can replace R by R/Q_0 where Q_0 is a minimal prime ideal of R such that $\dim R = \dim(R/Q_0)$ and further assume R to be a *complete local noetherian domain* with maximal ideal P. Thus to prove (a) it is enough to prove the following:

Let R be a complete noetherian local domain (of dimension $\geqslant 1$) with maximal ideal P and residue field $k(=R/P)$, $S = R^{[m]}$ and A an S-subalgebra of $S^{[n]}$ such that

(i) *A is R*-flat.
(ii) $A \otimes_R K = (S \otimes_R K)^{[1]}$ where K is the quotient field of R.
(iii) $A \otimes_R k(Q)$ are $\mathbf{A}^1$-forms over $S \otimes_R k(Q)$ (with respect to $k(Q)$) for all non-maximal prime ideals Q of R.
(iv) Moreover if $\dim R(=\operatorname{ht} P) = 1$, then $A \otimes_R k$ is geometrically integral over k.

Then $A \otimes_R k$ is an $\mathbf{A}^1$-form over $S \otimes_R k$ (with respect to k).

Let $\tilde{R}$ denote the normalisation of R. It is known that $\tilde{R}$ is a local domain which is finitely generated as an R-module ([M], p. 234). Let $\tilde{P}$ denote the maximal ideal of $\tilde{R}$ and let $\tilde{k} = \tilde{R}/\tilde{P}$. Let $\tilde{S} = S \otimes_R \tilde{R}$ and $\tilde{A} = A \otimes_R \tilde{R}$. To prove that $A \otimes_R k$ is an $\mathbf{A}^1$-form over $S \otimes_R k$ (with respect to k), it is obviously enough to prove that $A \otimes_R \tilde{k} = (S \otimes_R \tilde{k})^{[1]}$, i.e., to prove that $\tilde{A} \otimes_{\tilde{R}} \tilde{k} = (\tilde{S} \otimes_{\tilde{R}} \tilde{k})^{[1]}$. We in fact prove the following:

Claim. $\tilde{A} = \tilde{S}^{[1]}$.

Proof of the claim. By R-flatness of A and K, we have

$$\tilde{R} \hookrightarrow \tilde{S} \hookrightarrow \tilde{A} \hookrightarrow \tilde{S}^{[n]}$$

and that $\tilde{A}$ is $\tilde{R}$-flat. Condition (ii) implies that $\tilde{A} \otimes_{\tilde{R}} K = (\tilde{S} \otimes_{\tilde{R}} K)^{[1]}$.

We now show that for any prime ideal $\tilde{Q}$ of $\tilde{R}$ of height one, $\tilde{A} \otimes_{\tilde{R}} k(\tilde{Q})$ $(= A \otimes_R k(\tilde{Q}))$ is a domain. If dim $R(=\text{ht } P) = 1$, then clearly $\tilde{Q} = \tilde{P}$ and by condition (iv) it follows that $\tilde{A} \otimes_{\tilde{R}} k(\tilde{Q})$ $(= A \otimes_R \tilde{k})$ is a domain. On the other hand if $\dim R (= \text{ht } P) \geqslant 2$, then by condition (iii), it follows that $A \otimes_R k(\tilde{Q})$ is an $\mathbf{A}^1$-form over $S \otimes_R k(\tilde{Q})$ $(= k(\tilde{Q})^{[m]})$ with respect to $k(\tilde{Q})$ and hence is a domain.

Now by applying Theorem 3.10 we deduce that $\tilde{A} = \tilde{S}^{[1]}$ proving the claim and hence (a).

We now prove (b). Note that by (a) (which we have already proved) all the fibres $A \otimes_R k(Q)$ are $\mathbf{A}^1$-forms over $S \otimes_R k(Q)$. Thus to prove (b) we can make the following reductions: first by (3.2) and (3.3) we can replace R by $R/\text{nil } R$ and assume that R is a noetherian reduced ring. Next by (3.1) and (3.5) we can replace R by R/Q where Q is a minimal prime ideal of R and assume R to be a noetherian domain. Hence by ([G], 2.1) and (2.9) A would be finitely generated over R if A is locally finitely generated over R; so we may further assume R to be a local noetherian domain with quotient field K. Since $\dim(A \otimes_R K) = m + 1$, by ([G], 2.3(b)) the transcendence degree of $A \otimes_R K$ over $S \otimes_R K$ is one. Therefore as $S \otimes_R K \hookrightarrow A \otimes_R K \hookrightarrow (S \otimes_R K)^{[n]}$ are factorial domains, by (2.7), $A \otimes_R K = (S \otimes_R K)^{[1]}$.

Now let $\hat{R}$ denote the completion of R. Since $\hat{R}$ is faithfully flat over R, to prove that A is finitely generated over R, it is enough to prove that $A \otimes_R \hat{R}$ is finitely generated over $\hat{R}$. Note that

$$\hat{R} \hookrightarrow S \otimes_R \hat{R} \hookrightarrow A \otimes_R \hat{R} \hookrightarrow (S \otimes_R \hat{R})^{[n]}.$$

Also from the fibre conditions on A it is easy to see (using (a)) that $(A \otimes_R \hat{R}) \otimes_{\hat{R}} k(\bar{Q})$ are $\mathbf{A}^1$-forms over $(S \otimes_R \hat{R}) \otimes_{\hat{R}} k(\bar{Q})$ (w.r.t. $k(\bar{Q})$) for all prime ideals $\bar{Q}$ of $\hat{R}$. Moreover since $A \otimes_R K = (S \otimes_R K)^{[1]}$, and since by flatness of $\hat{R}$ over R, the minimal prime ideals of $\hat{R}$ contract to (0) in R (by the "going-down theorem"), we have

$$(A \otimes_R \hat{R}) \otimes_{\hat{R}} k(\bar{Q}) = ((S \otimes_R \hat{R}) \otimes_{\hat{R}} k(\bar{Q}))^{[1]}$$

for all minimal prime ideals $\bar{Q}$ of $\hat{R}$. Thus to prove (b) we may assume R to be a complete local noetherian ring and as before (using (3.2), (3.3) and then (3.1), (3.5)) we may further assume R to be a *complete local noetherian domain with quotient field* K. Note that now we have the additional data:

(1) $A \otimes_R K = (S \otimes_R K)^{[1]}$ and
(2) $A \otimes_R k(Q)$ *are* $\mathbf{A}^1$-forms over $S \otimes_R k(Q)$ (with respect to $k(Q)$) for all prime ideals Q of R.

Now the normalisation $\tilde{R}$ of R is a finite R-module ([M], p. 234) and hence is a *noetherian* normal local domain. Also $A \otimes_R \tilde{R}$ is $\tilde{R}$-flat and by condition (1) $(A \otimes_R \tilde{R}) \otimes_{\tilde{R}} K = ((S \otimes_R \tilde{R}) \otimes_{\tilde{R}} K)^{[1]}$. Moreover from condition (2) it follows that $(A \otimes_R \tilde{R}) \otimes_{\tilde{R}} k(\tilde{Q})$ are domains for all prime ideals $\tilde{Q}$ of $\tilde{R}$. Hence by (3.10) we have $A \otimes_R \tilde{R} = (S \otimes_R \tilde{R})^{[1]} = \tilde{R}^{[m+1]}$. Therefore, by (3.4), it follows that A is a finitely generated R-algebra proving (b).

This completes the proof of Proposition 3.11. □

As a consequence of Proposition 3.11 we now prove Theorem A mentioned in the introduction.

THEOREM 3.12. *Let R be a commutative noetherian ring containing a field of characteristic zero,* $S = R^{[m]}$ *for some integer* $m \geqslant 0$, *A an S-subalgebra of* $S^{[n]}$ *such that*

(i) *A is R-flat.*
(ii) $A \otimes_R k(P)$ *are factorial domains of Krull dimension* $m + 1$ *for all minimal prime ideals P of R.*
(iii) $A \otimes_R k(P)$ *are integral domains for all height one prime ideals P of R.*

Then A is an $\mathbf{A}^1$*-fibration over S.*

Proof. Since R contains a field of characteristic zero, in view of Proposition 3.11, Lemma 3.7 and Lemma 3.8, it is enough to show that $A \otimes_R k(P)$ are geometrically integral over $k(P)$ for all height one prime ideals P of R.

Let P be a prime ideal of R of height one. We shall prove the geometric integrality of $A \otimes_R k(P)$ over $k(P)$ by proving the stronger statement

$$A \otimes_R k(P) = (S \otimes_R k(P))^{[1]}.$$

Replacing R by R_P we may now assume that R is a one-dimensional local noetherian ring with maximal ideal P and residue field $k\,(=k(P))$. Further applying Lemma 3.1 we can replace R by R/Q for a minimal prime ideal Q of R and assume that R is a domain to start with. Thus the proof of Theorem 3.12 reduces to providing the following statement:

Let R be a one-dimensional noetherian local domain with quotient field K, maximal ideal P and residue field k of characteristic zero. Let $S = R^{[m]}$ for some integer $m \geqslant 0$. Let A be an S-subalgebra of $S[X_1, \dots, X_n]\,(=S^{[n]})$ such that

(i) A is R-flat.
(ii) $A \otimes_R K$ is a factorial domain and $\dim(A \otimes_R K) = m + 1$.
(iii) $A \otimes_R k$ is an integral domain.

Then

$$A \otimes_R k = (S \otimes_R k)^{[1]}\,(=k^{[m+1]}). \tag{3.12.1}$$

The rest of the proof would thus be devoted to proving (3.12.1).

Since R is a one-dimensional noetherian local domain, it is easy to see by the Krull-Akizuki theorem ([N], p. 115) that there exists a discrete valuation ring $(\tilde{R}, \pi)$ such that $R \hookrightarrow \tilde{R} \hookrightarrow K$ and the residue field $\tilde{k}\,(=\tilde{R}/\pi)$ is finite over k. Moreover by the condition on the characteristic of k, $\tilde{k}$ is separable over k. Let $\tilde{S} = S \otimes_R \tilde{R}$ and $\tilde{A} = A \otimes_R \tilde{R}$. By R-flatness of A and K, it follows that $\tilde{S} \hookrightarrow \tilde{A} \hookrightarrow \tilde{S}[X_1, \ldots, X_n]$.

We first show that there exists $U \in \tilde{S}[X_1, \ldots, X_n]$ such that

$$\tilde{A} \hookrightarrow \tilde{S}[U] \quad \text{and} \quad \tilde{A}[1/\pi] = \tilde{S}[1/\pi][U].$$

Let $C = \tilde{A}[1/\pi] \cap \tilde{S}[X_1, \ldots, X_n]$. Then clearly

$$\tilde{R} \hookrightarrow \tilde{S} \hookrightarrow \tilde{A} \hookrightarrow C \hookrightarrow \tilde{S}[X_1, \ldots, X_n] \quad \text{and} \quad \tilde{A}[1/\pi] = C[1/\pi].$$

Therefore $C \otimes_{\tilde{R}} K\,(=C[1/\pi] = \tilde{A}[1/\pi] = A \otimes_R K)$ is a factorial domain and $\dim(C \otimes_{\tilde{R}} K) = m + 1$. From the construction of C it is easy to check that $\pi C = \pi \tilde{S}[X_1, \ldots, X_n] \cap C$. Hence $C/\pi C$ is a domain. Therefore, $\tilde{R}$ being a d.v.r. and C being an overdomain of $\tilde{S}$, by Theorem 3.10, it follows that $C = \tilde{S}^{[1]}$, say, $C = \tilde{S}[U]$. Then

$$\tilde{A} \hookrightarrow \tilde{S}[U] \hookrightarrow \tilde{S}[X_1, \ldots, X_n] \quad \text{and} \quad \tilde{A}[1/\pi] = \tilde{S}[1/\pi][U].$$

Now let $Z = \pi^l U$ where l is the least integer such that $\pi^l U \in \tilde{A}$.

Claim. $\tilde{A} = \tilde{S}[Z]$.

Assume the claim for the time being. In view of the claim,

$$(A \otimes_R k) \otimes_k \tilde{k}(=\tilde{A} \otimes_{\tilde{R}} \tilde{k} = \tilde{S}[Z] \otimes_{\tilde{R}} \tilde{k}) = ((S \otimes_R k) \otimes_k \tilde{k})^{[1]}.$$

Since $S \otimes_R k \hookrightarrow A \otimes_R k$ (as there is retract from $A \otimes_R k$ to $S \otimes_R k$) and $S \otimes_R k = k^{[m]}$, by (3.7), we would have $A \otimes_R k = (S \otimes_R k)^{[1]}$, proving (3.12.1).

Thus the proof of the theorem will be complete if we prove the claim.

Proof of the claim. The claim is obvious if $l = 0$. So assume that $l \geqslant 1$.

Since $\tilde{S}[Z][1/\pi] = \tilde{A}[1/\pi]$, in order to prove the claim it is enough to prove that the canonical map $\tilde{S}[Z]/(\pi) \to \tilde{A}/\pi\tilde{A}$ is injective.

Let $B = A \otimes_R k$. Since $\tilde{k}$ is a separable extension of k and B is a domain (by condition (iii)) it follows that $B \otimes_k \tilde{k} = \tilde{A}/\pi\tilde{A}$ is a reduced ring. Since $\tilde{k}$ is finite over k, $\tilde{A}/\pi\tilde{A}$ is a finitely generated flat module over the domain B

and hence it has only finitely many minimal prime ideals. Let $P_1, \ldots, P_s$ be those minimal prime ideals of $\tilde{A}$ which contain $\pi\tilde{A}$. Let $Q_i = P_i \cap \tilde{S}[Z]$.

The proof of the claim will be complete if we show that $Q_i = \pi\tilde{S}[Z]$ for some i. For in that case the canonical map $\tilde{S}[Z]/(\pi) \to \tilde{A}/\pi\tilde{A}$ will be injective, since the composite map $\tilde{S}[Z]/(\pi)\,(=\tilde{S}[Z]/Q_i) \to \tilde{A}/\pi\tilde{A} \to \tilde{A}/P_i$ will be injective. (Note that this will also show that π is actually a prime element of $\tilde{A}$.)

We now show that $Q_i = \pi\tilde{S}[Z]$ for some i. Suppose not. Then $\forall i$, $1 \leqslant i \leqslant s$, $\mathrm{ht}(Q_i) \geqslant 2$ (since $\pi \in Q_i$). By reindexing if necessary assume that $Q_1, \ldots, Q_t$ are all the distinct members of the family $\{Q_i \mid 1 \leqslant i \leqslant s\}$. We show separately that $t = 1$ and $t > 1$ are not possible (by arriving at contradictions). That would complete the proof of the claim.

$t = 1$

If $t = 1$, there is only one (distinct) Q_i, call it Q. In this case $P_1, \ldots, P_s$ are the minimal prime ideals of $\tilde{A}$ (containing $\pi\tilde{A}$) which all contract to Q in $\tilde{S}[Z]$. Now

$$\tilde{A}/\pi\tilde{A} = \tilde{A}/(P_1 \cap \cdots \cap P_s). \text{ As } \tilde{S} \hookrightarrow \tilde{A} \hookrightarrow \tilde{S}[U],$$

there is a retract from $\tilde{A}$ to $\tilde{S}$ which induces a retract from $\tilde{A}/\pi\tilde{A}\,(=\tilde{A} \otimes_{\tilde{R}} \tilde{k})$ to $\tilde{S}/\pi\tilde{S}\,(=\tilde{S} \otimes_{\tilde{R}} \tilde{k})$. Since $\pi\tilde{A} \cap \tilde{S}[Z] = Q$, the above retract induces a retract

$$\tilde{S}[Z]/Q \to \tilde{S}/\pi\tilde{S}.$$

Now $\dim(\tilde{S}/\pi\tilde{S}) = \dim(\tilde{k}^{[m]}) = m$ and $\dim(\tilde{S}[Z]) = \dim \tilde{S} + 1 = \dim \tilde{R} + m + 1 = m + 2$. As $\mathrm{ht}\, Q \geqslant 2$, from dimension considerations the *retract* $\tilde{S}[Z]/Q \to \tilde{S}/\pi\tilde{S}$ is actually an isomorphism. That clearly shows that

$$Z - g \in Q(\subseteq \pi\tilde{A}) \quad \text{for some } g \in \tilde{S}.$$

Since $Z = \pi^l U$ with $l \geqslant 1$, clearly $g \in \pi\tilde{S}[U] \cap \tilde{S} = \pi\tilde{S} \subseteq \pi\tilde{A}$. But then $\pi^{l-1}U\,(=Z/\pi) \in \tilde{A}$ contradicting the choice of l. Thus $t \neq 1$.

We now show that $t \geqslant 2$ is not possible.

$t = 2$

If possible let $t \geqslant 2$. Let P_1, P_2 be minimal prime ideals of $\tilde{A}$ (containing $\pi\tilde{A}$) such that $Q_1\,(=P_1 \cap \tilde{S}[Z]) \neq Q_2\,(=P_2 \cap \tilde{S}[Z])$. Let $\bar{P}_i$ be the image of P_i in $\tilde{A}/\pi\tilde{A}$ and $\bar{Q}_i$ be the image of Q_i in $(\tilde{S}/\pi\tilde{S})[Z]$ $(i = 1, 2)$. Let $I = \bar{Q}_1 + \bar{Q}_2$, $\tilde{J} = I \cap (\tilde{S}/\pi\tilde{S})$ and $J = \tilde{J} \cap (S \otimes_R k)$.

Recall that Q_1 and Q_2 are distinct prime ideals each of height $\geqslant 2$ containing π so that $\operatorname{ht} \bar{Q}_i \geqslant 1$ $(i = 1, 2)$ and hence $\operatorname{ht}(I) \geqslant 2$. Therefore $\tilde{J} \neq 0$. As $\tilde{S}/\pi\tilde{S} (= S \otimes_R \tilde{k} = \tilde{k}^{[m]})$ is a domain which is integral over its subring $S \otimes_R k$ it follows that $J \neq 0$. Let h be a non-zero element of J.

Since k is infinite (being a field of characteristic zero) and there is a retract from $B (= A \otimes_R k)$ to $S \otimes_R k (= k^{[m]})$, it is easy to see that there exists a maximal ideal N of B such that

$$h \notin N \quad \text{and} \quad B/N = k.$$

Now B being a domain, $B \hookrightarrow B_N$ and hence by flatness

$$\tilde{A}/\pi\tilde{A} (= B \otimes_k \tilde{k}) \hookrightarrow B_N \otimes_k \tilde{k}. \tag{3.12.2}$$

Since B_N is a ring with the unique maximal ideal NB_N and residue field k and since $\tilde{k}$ is a finite extension of k, clearly $B_N \otimes_k \tilde{k}$ is a ring with unique maximal ideal $NB_N \otimes_k \tilde{k}$ and residue field $\tilde{k}$. Note that h being a unit in B_N is also a unit in $B_N \otimes_k \tilde{k}$.

Now since $B_N \otimes_k \tilde{k}$ is a localisation of $\tilde{A}/\pi\tilde{A} (= B \otimes_k \tilde{k})$ (and since $B_N \otimes_k \tilde{k}$ has a unique maximal ideal) it is easy to see that there exists a prime ideal M of $\tilde{A}/\pi\tilde{A}$ such that

$$B_N \otimes_k \tilde{k} = (\tilde{A}/\pi\tilde{A})_M. \tag{3.12.3}$$

By (3.12.2) and (3.12.3) the map $\tilde{A}/\pi\tilde{A} \to (\tilde{A}/\pi\tilde{A})_M$ is injective so that the zero-divisors of $\tilde{A}/\pi\tilde{A}$ are contained in M. Hence the minimal prime ideals $\bar{P}_1$ and $\bar{P}_2$ are contained in M. Therefore $h \in \bar{P}_1 + \bar{P}_2 \subseteq M$. But vide (3.12.3) this contradicts the earlier observation that h is a unit in $(\tilde{A}/\pi\tilde{A})_M$. The contradiction shows that $t \ngeqslant 2$.

This proves the claim and hence Theorem 3.12. □

REMARK 3.13. Theorem 3.12 would not be valid without the assumption that R contains a field of characteristic zero as Examples 4.1 and 4.2 of Section 4 illustrate. The problem is two-fold: first of all the assumption that the *fibres at height one prime ideals are integral* would not in general ensure that they are *geometrically integral* (see Example 4.1). Secondly even if the fibres at the height one prime ideals are geometrically integral, that would merely imply (by Proposition 3.11) that the fibres are $\mathbf{A}^1$-*forms*—but as Example 4.2 shows, the $\mathbf{A}^1$-forms need not be $\mathbf{A}^1$ because of the existence of non-trivial inseparable $\mathbf{A}^1$-forms.

We now prove Theorem B mentioned in the introduction. In view of Remark 3.13 this theorem seems to be the best possible in the general case.

THEOREM 3.14. *Let R be a commutative noetherian ring, $S = R^{[m]}$ for some integer $m \geqslant 0$ and A an S-subalgebra of $S^{[n]}$ such that the following hold:*

(i) *A is R-flat.*
(ii) *$A \otimes_R k(P)$ are factorial domains for all prime ideals P of R and $\dim(A \otimes_R k(P)) = m + 1$ for all minimal prime ideals P of R.*

Then A is an $\mathbf{A}^1$-fibration over S.

Proof. We first show that $A \otimes_R k(P)$ are geometrically integral over $k(P)$ for all prime ideals P of R. Fix $P \in \operatorname{Spec}(R)$. We can make the following reductions: replacing R by R_P we assume that R is a noetherian local ring with maximal ideal P and residue field $k(=k(P))$. Next by applying (3.1) we may assume (by replacing R by R/Q where Q is a minimal prime ideal of R) that R is a noetherian local domain with quotient field K. Since $K \hookrightarrow A \otimes_R K \hookrightarrow (S \otimes_R K)^{[n]} (= K^{[m+n]})$ and $\dim A \otimes_R K = m + 1$, by ([G], 2.3(b)), $\operatorname{tr deg}_R A = m + 1$. Hence by ([G], 1.2), $\dim A \leqslant \dim R + m + 1$. Moreover as $A \otimes_R k(Q)$ are domains for all $Q \in \operatorname{Spec}(R)$ and A is R-flat, it follows that $QA \in \operatorname{Spec}(A)$ for all $Q \in \operatorname{Spec}(R)$. Therefore it is easy to see that $\operatorname{ht} PA \geqslant \operatorname{ht} P (= \dim R)$ and hence

$$\dim(A \otimes_R k) (= \dim A/PA) \leqslant \dim A - \dim R \leqslant m + 1.$$

Let $B = A \otimes_R k$. As $S \hookrightarrow A \hookrightarrow S^{[n]}$, there is a retract from A to S which induces a retract from B to $S \otimes_R k$ with kernel I. Since $\dim B \leqslant m + 1$ and $\dim(S \otimes_R k) (= \dim k^{[m]}) = m$, it follows that $\operatorname{ht} I \leqslant 1$ and therefore I is principal (as B is a factorial domain). Now since $B/I (= k^{[m]})$ is geometrically integral over k, by Lemma (3.6), it follows that $B = A \otimes_R k$ is geometrically integral over k.

From Proposition 3.11 it now follows that A is finitely generated over R and that $A \otimes_R k(P)$ are $\mathbf{A}^1$-forms over $S \otimes_R k(P)$ (with respect to $k(P)$) for all prime ideals P of R. By virtue of (3.8) the proof of the theorem would now be complete if we show that

$$A \otimes_R k(P) = (S \otimes_R k(P))^{[1]} \quad \text{for all } P \in \operatorname{Spec}(R).$$

Again fix a prime ideal P and let $k = k(P)$ and $\bar{k}$ the algebraic closure of k. As $A \otimes_R k$ is an $\mathbf{A}^1$-form over $S \otimes_R k (= k^{[m]})$, $\dim(A \otimes_R k) = m + 1$. Recall that there is a retract from $A \otimes_R k$ to $S \otimes_R k (= k^{[m]})$. Since $A \otimes_R k$ is a factorial domain of dimension $m + 1$, the kernel of this retract is a non-zero principal prime deal, say generated by F. Then F also generates the kernel of the induced retract from $A \otimes_R \bar{k} (= (S \otimes_R \bar{k})^{[1]})$ to $S \otimes_R \bar{k}$. Hence $A \otimes_R \bar{k} = (S \otimes_R \bar{k})[F]$. Now by faithful flatness of $\bar{k}$ over k,

$$A \otimes_R k(P) (= A \otimes_R k) = (S \otimes_R k)[F] = (S \otimes_R k(P))^{[1]}.$$

This completes the proof of the theorem. □

REMARK 3.15. The proof of the theorem (along with Lemma (3.6)) actually shows the following:

Let R be a noetherian ring, $S = R^{[m]}$ for some integer $m \geqslant 0$ and A an S-subalgebra of $S^{[n]}$ such that

(i) A is R-flat.
(ii) $A \otimes_R k(P)$ are factorial domains of Krull dimension $m + 1$ for all minimal prime ideals P of R.
(iii) For every height one prime ideals P of R, $A \otimes_R k(P)$ is either a factorial domain or a noetherian normal domain.

Then the following hold:

(a) $A \otimes_R k(P)$ are $\mathbf{A}^1$-forms over $S \otimes_R k(P)$ with respect to $k(P)$ for all prime ideals P of R.
(b) A is finitely generated over R.

Moreover for any prime ideal P for which $A \otimes_R k(P)$ is a factorial domain, it would in fact follow that $A \otimes_R k(P) = (S \otimes_R k(P))^{[1]}$.

COROLLARY 3.16. *Let R be a noetherian ring such that R_{red} is seminormal. Then under the hypotheses of either Theorem* 3.12 *or Theorem* 3.14, *A is S-isomorphic to* $\text{Sym}_S(P \otimes_R S)$ *where P is a finitely generated projective R-module of rank one.*

Proof. By our results A is an $\mathbf{A}^1$-fibration over S and hence by (2.6), there exists an integer l such that A is an S-subalgebra of $S^{[l]}$ and that $A^{[l]}$ is S-isomorphic to $\text{Sym}_{S^{[l]}}(Q)$ where Q is a finitely generated projective $S^{[l]}$-module of rank one. Now as R_{red} and hence S_{red} is seminormal, by (2.11), there exists a finitely generated projective S-module Q' of rank one such that $Q' \otimes_S S^{[l]} \cong Q$. Hence for every maximal ideal M of S, $A_M^{[l]} = S_M^{[l+1]}$ (as S_M-algebras). Therefore S_{red} being seminormal, by (2.10), $A_M = S_M^{[1]}$ for all maximal ideals M of S. Thus applying (2.8) we get that A is S-isomorphic to $\text{Sym}_S(P')$ where P' is a finitely generated projective S-module of rank one. Again by (2.11), $P' \cong P \otimes_R S$ for some finitely generated projective R-module P of rank one. Hence the result. □

4. Examples

In this section we discuss the necessity of the hypotheses in our theorems with the help of examples. For simplicity we shall take $m = 0$, i.e., $S = R$ in our theorems.

In Theorems 3.10 and 3.12 we have deduced (under the assumption that either R is normal or R contains a field of characteristic zero) that all fibres in the given set-up are $\mathbf{A}^1$ by merely imposing the condition that the fibres at minimal prime ideals are factorial and the fibres at height one prime ideals are integral. The first two examples which we are giving below (i.e., Examples 4.1 and 4.2) show that in general the fibres at all the non-closed points of Spec(R) being factorial (or even $\mathbf{A}^1$) need not imply that an integral fibre at a closed point is factorial even when R is a complete local domain. (Note that it is easy to construct examples of R-subalgebras of $R[X]$ even over a discrete valuation ring R such that the generic fibre is $\mathbf{A}^1$ but the special fibre is not an integral domain. For instance, let R be a d.v.r. with parameter π and let $A = R[U, V]/(\pi U - V^2) \cong R[\pi X^2, \pi X] \hookrightarrow R[X]$. Also note that the condition on fibres at minimal primes cannot be deduced from other conditions. To see this consider any discrete valuation ring (R, π) with quotient field K and residue field k. Let $A = R[X^2] + (1 - \pi X^2)R[X] \hookrightarrow R[X]$. Then A is a finitely generated flat R-algebra and the closed fibre $A \otimes_R k = k[X]$. But clearly $A \otimes_R K$ is not even normal as $X^2 \in A \otimes_R K$, $X\,(=X(1 - \pi X^2)/(1 - \pi X^2))$ is an element of the quotient field of $A \otimes_R K$ but $X \notin A \otimes_R K$).

First we give an example of a one-dimensional noetherian local domain R and a finitely generated flat R-subalgebra A of $R[X]$ whose generic fibre is $\mathbf{A}^1$ but whose closed fibre is integral but not normal. In the second example R is a noetherian complete local domain of dimension n and A a finitely generated flat R-subalgebra of $R[X]$ whose fibres at all non-closed points of Spec(R) are $\mathbf{A}^1$ while the fibre at the closed point is a purely inseparable $\mathbf{A}^1$-form. (See also Remarks 3.13 and 3.15 in this connection.)

EXAMPLE 4.1. Let $\mathbf{Z}$ denote the ring of integers, p a prime integer and $\mathbf{F}_p = \mathbf{Z}/(p)$. Let

$$R = \mathbf{Z}[T^p]_{(p[T^p])} + (p[T])\mathbf{Z}[T]_{(p[T])}.$$

Then R is a finite module over $\mathbf{Z}[T^p]_{(p[T^p])}$ and hence is noetherian. It is local with residue field $k = \mathbf{F}_p(T^p)$. Its normalisation $\tilde{R} = \mathbf{Z}[T]_{(p[T])}$ is local with residue field $\tilde{k} = \mathbf{F}_p(T)$. The quotient field of R (and $\tilde{R}$) is $K = \mathbf{Q}(T)$ where $\mathbf{Q}$ denotes the field of rational numbers.

Note that the element $pV - (TU + V)^p \in R[U, V]$ (using binomial expansion and the fact that $p\tilde{R} \subset R$). Now let

$$A = R[U, V]/(pV - (TU + V)^p).$$

Since the defining equation is monic in V, A is a flat (in fact a free) $R[U]$-module. Also $R[1/p] = K$ and hence

$$A[1/p] = A \otimes_R K = K[U, V]/(pV - (TU + V)^p) = K^{[1]}$$

as $K[pV, TU + V] = K[U, V]$. This also shows that A is a domain (since by R-flatness of A, $A \hookrightarrow A \otimes_R K$). Hence the map $\phi: R[U, V] \to R[X]$ given by

$$\phi(U) = pX - (pT)^{p-1}X^p,$$
$$\phi(V) = p^{p-1}T^pX^p$$

clearly induces an injective R-homomorphism $A \hookrightarrow R[X]$. As observed before $A \otimes_R K = K^{[1]}$. But

$$A \otimes_R k = k[U, V]/(T^pU^p + V^p)$$

which is an integral domain (since $k = \mathbf{F}_p(T^p)$) but is not even geometrically integral as $A \otimes_R \tilde{k} = \tilde{k}[U, V]/(TU + V)^p$ (note that $\tilde{k} = \mathbf{F}_p(T)$). □

EXAMPLE 4.2. Let k be a non-perfect field of characteristic p and let $\beta \in k$ be such that $T^p - \beta$ is irreducible in $k[T]$. Let $L = k[T]/(T^p - \beta) = k(\alpha)$, say, where $\alpha^p = \beta$. Now let

$$R = k + (Y_1, \ldots, Y_n)L[[Y_1, \ldots, Y_n]]$$

(considered as a subring of $L[[Y_1, \ldots, Y_n]]$) and let

$$A = R[U, V]/(V - \beta V^p - U^p) = R[u, v]$$

where u, v are the images of U and V respectively in A. R is a finite module over $k[[Y_1, \ldots, Y_n]]$ and hence is noetherian. It is local with residue field k. A is a finitely generated R-algebra and being a free module over $R[V]$ it is also R-flat.

Now let $\tilde{R}$ denote the normalisation of R. Then $\tilde{R} = L[[Y_1, \ldots, Y_n]]$ and the ideal $(Y_1, \ldots, Y_n)L[[Y_1, \ldots, Y_n]]$ is the conductor of $\tilde{R}$ in R. Also since

$$V - \beta V^p - U^p = V - (U + \alpha V)^p,$$

we have $A \otimes_R \tilde{R} = \tilde{R}^{[1]}$. By R-flatness of A, $A \hookrightarrow A \otimes_R \tilde{R} (= \tilde{R}^{[1]})$ and is therefore a domain. Note that $\alpha Y_1^p \in R$, so that we can define the map $\phi: R[U, V] \to R[X]$ given by

$$\phi(U) = Y_1 X - \alpha Y_1^p X^p,$$

$$\phi(V) = Y_1^p X^p.$$

Since A is a domain, using the map ϕ it is easy to see that $R \hookrightarrow A \hookrightarrow R[X]$.
Now since $A \otimes_R \tilde{R} = \tilde{R}^{[1]}$, it follows that

$$A \otimes_R k(Q) = k(Q)^{[1]} \quad \text{for all } Q \in \operatorname{Spec}(\tilde{R}).$$

Now if P is any non-maximal prime ideal of R, then clearly $R_P = \tilde{R}_P$ so that $A \otimes_R k(P) = k(P)^{[1]}$. On the other hand if P is the maximal ideal of R, then

$$A \otimes_R k(P)\ (= A \otimes_R k) = k[U, V]/(V - \beta V^p - U^p)$$

which is an $\mathbf{A}^1$-form over $k(P)$, since

$$(A \otimes_R k(P)) \otimes_{k(P)} L = L[U, V]/(V - (\alpha V + U)^p) = L^{[1]}$$

as $L[U, \alpha V + U] = L[U, V]$. But $A \otimes_R k(P)\ (= A \otimes_R k) \neq k^{[1]}$ since the valuation ring $k[1/u]_{(1/u)}[v/u]$ of a place of $A \otimes_R k$ at infinity has residue field $L\,(\neq k)$. □

We now discuss the "Flatness" condition with two examples. A priori there is no reason why even nice fibre conditions would imply flatness. For instance consider the following example of a two-dimensional regular local ring R and a finitely generated R-subalgebra of $R[X]$ all of whose non-closed fibres are $\mathbf{A}^1$ but whose closed fibre is $\mathbf{A}^2$:

EXAMPLE 4.3. Let (R, M) be a regular local ring of dimension 2 with quotient field K and (π_1, π_2) be a regular system of parameters for R. Let $A = R[\pi_1 X, \pi_2 X] \hookrightarrow R[X]$. Then $A \cong R[U, V]/(\pi_2 U - \pi_1 V)$. If P is a prime ideal of height at most one in R, then either π_1 or $\pi_2 \notin P$ and hence $A_P = R_P^{[1]}$. Thus $A \otimes_R k(P) = k(P)^{[1]}$ for all non-maximal prime ideals P of R. But the closed fibre $A \otimes_R (R/M) = (R/M)^{[2]}$. Since $A/\pi_1 A$ is not a domain, obviously A is not R-flat. □

However the motivation for considering the flatness question comes from Theorem (2.7) which we state again:

If $R \hookrightarrow A \hookrightarrow R[X]\ (= R^{[1]})$ are factorial domains and $A \neq R$, then $A = R^{[1]}$.

Note that when R is a noetherian factorial domain with quotient field K and $R \hookrightarrow A \hookrightarrow R[X]$ (with $A \neq R$), the factoriality of A is easily seen to be equivalent to the conditions:

(a) $A \otimes_R K$ is a factorial domain of dimension one.
(b) $PA \in \mathrm{Spec}(A)$ for all height one prime ideals P of R.

Thus for a noetherian factorial domain and an R-subalgebra A of $R[X]$, even with such mild assumptions as (a) and (b), Theorem (2.7) says that $A = R^{[1]}$, in particular, A is R-flat. Note that in Example 4.3 the condition (b) is violated.

Therefore one might be tempted to ask, as a generalisation of Theorem (2.7), the question: For a noetherian domain R (say, containing a field of characteristic zero), and an R-subalgebra A of $R[X]$ satisfying the conditions (a) and (b) above, is A an $\mathbf{A}^1$-fibration over R? Note that in view of Theorem 3.12 this is equivalent to asking: is A flat over R? The following example shows that this need not be true in general. Thus in Theorem 3.12 we cannot remove the "flatness" assumption even if in place of condition (iii) we impose the stronger condition that $PA \in \mathrm{Spec}(A)$ for all height one prime ideals P of R.

EXAMPLE 4.4. Let $\mathbf{C}$ and $\mathbf{R}$ denote the field of complex numbers and real numbers respectively. Let

$$R = \mathbf{R} \oplus (T)C[[T]] = \mathbf{R} \oplus (T)\mathbf{R}[[T]] \oplus (iT)\mathbf{R}[[T]].$$

Then $R \cong \mathbf{R}[[T]][Y]/(Y^2 + T^2)$. Thus R is a local one-dimensional noetherian domain with maximal ideal $M = (T, iT)$, quotient field $K = \mathbf{C}((T))$ and residue field $k = \mathbf{R}$. The normalisation $\tilde{R}$ of R is $\mathbf{C}[[T]]$ with residue field $\tilde{k} = \mathbf{C}$. Let

$$Q = (U + iV)\tilde{R}[U, V] \cap R[U, V].$$

It is easy to see that

$$Q = (U^2 + V^2, TU + iTV, iTU - TV)R[U, V].$$

Let $A = R[U, V]/Q$. The map $\phi: R[U, V] \to R[X]$ defined by

$$\phi(U) = TX \quad \text{and} \quad \phi(V) = iTX$$

induces an injective map $A \hookrightarrow R[X]$. Now $A \otimes_R K = K[U, V]/(U + iV) = K^{[1]}$. But

$$A \otimes_R k = \mathbf{R}[U, V]/(U^2 + V^2)$$

which is a domain but is not regular. □

In view of Example 4.4 and the preceding discussions we ask the following question:

QUESTION 4.5. Let R be a noetherian *normal* local domain with quotient field K and A an R-subalgebra of $R[X]$ such that $A \otimes_R K = K^{[1]}$ and $PA \in \operatorname{Spec}(A)$ for all prime ideals P of R of height one. Then is A flat over R, or equivalently (by Theorem 3.10), is $A = R^{[1]}$?

Note that if $\dim R = 1$, then by the above hypotheses both R and A are factorial so that by (2.7) $A = R^{[1]}$.

Acknowledgement

The authors sincerely thank N. Mohan Kumar for his help regarding Lemma 3.9.

References

[A] T. Asanuma, Polynomial fibre rings of algebras over noetherian rings, *Invent. Math.* 87 (1987) 101–127.

[A-E-H] S. S. Abhyankar, P. Eakin and W. Heinzer, On the uniqueness of the coefficient ring in a polynomial ring, *J. Algebra* 23 (1972) 310–342.

[B-C-W] H. Bass, E. H. Connell and D. L. Wright, Locally polynomial algebras are symmetric algebras, *Invent. Math.* 38 (1977) 279–299.

[B-D] S. M. Bhatwadekar and A. K. Dutta, On residual variables and stably polynomial algebras, *Comm. Algebra* 21(2) (1993) 635–645.

[G] J. M. Giral, Krull dimension, transcendence degree and subalgebras of finitely generated algebras, *Arch. Math.* 36 (1981) 305–312.

[H] E. Hamann, On the R-invariance of $R[X]$, *J. Algebra* 35 (1975) 1–16.

[M] H. Matsumura, *Commutative Algebra*, 2nd ed. Benjamin (1980).

[N] M. Nagata, *Local Rings*. Interscience (1962).

[O] N. Onoda, Subrings of finitely generated rings over a pseudogeometric ring, *Japan J. Math.* 10(1) (1984) 29–53.

[R-S] P. Russell and A. Sathaye, On finding and cancelling variables in $k[X, Y, Z]$, *J. Algebra* 57 (1979), 151–166.

[S] R. G. Swan, On seminormality, *J. Algebra* 67 (1980) 210–229.

Added in Proof. T. Asanuma has informed us that Question 4.5 has an affirmative answer in the following two cases: 1) A is a Krull domain; 2) A is noetherian and R is a local spot over a field.