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Inversion techniques and combinatorial identities – Jackson's q-analogue of the Dougall–Dixon Theorem and the dual formulae

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Abstract. As the common extension of theorems due to Gould and Hsu (1973) and Carlitz (1973), a general pair of reciprocal relations is established. By means of the embedding machinery, it is used to demonstrate that numerous q-hypergeometric identities are the dual relations of q-Saalschutz and Dougall-Dixon formulae. This fact serves as a natural reason for the existence of strange evaluations of basic hypergeometric series.

1. Introduction

As usual, denote by

$$\binom{n}{k} \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}$$

the binomial coefficient and the Gaussian q-binomial coefficient respectively. For the two complex sequences $\{a_k\}$ and $\{b_k\}$, define the ψ-polynomials by

$$\psi(x; n) = \prod_{i=0}^{n-1} (a_i + xb_i), \quad \psi(x; 0) = 1. \tag{1.1}$$

Then there exists a useful pair of inverse series relations

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \psi(k; n)\, g(k) \tag{1.2a}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a_k + kb_k}{\psi(n; k+1)} f(k) \tag{1.2b}$$

which was discovered by Gould and Hsu [18] in 1973. At the same time,

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Carlitz found its q-analogue which may be restated as follows:

$$f(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \psi(q^{-k}; n)\, g(k), \tag{1.3a}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{a_k + q^{-k} b_k}{\psi(q^{-n}; k+1)} f(k). \tag{1.3b}$$

Both (1.2) and (1.3) have been neglected, even though special versions were rediscovered by Andrews ([1], Bailey pair) and Gessel and Stanton ([15, 16], Lagrange inversions) and used to prove hypergeometric formulas. Their efficiency for confirming combinatorial identities, through the embedding machinery, was recognized in [7, 9].

As a natural extension of Carlitz' theorem, a more general pair of inverse series relations will be established in this paper. The development of inversion technique will be used to demonstrate numerous strange q-hypergeometric identities, by properly embedding the variants of the q-Saalschutz theorem and Jackson's q-analogue of Dougall-Dixon theorem in the members of this new pair of inversions. For convenience, these two q-series identities may be displayed as below:

$${}_3\Phi_2 \left[\begin{matrix} a,\ b,\ q^n \\ c,\ q^{1-n}ab/c \end{matrix}; q\right] = \left[\begin{matrix} c/a,\ c/b \\ c,\ c/ab \end{matrix}; q\right]_n \tag{1.4}$$

$$\begin{aligned} &{}_8\Phi_7 \left[\begin{matrix} a,\ qa^{1/2},\ -qa^{1/2},\ b,\ c,\ d,\ e,\ q^{-n} \\ a^{1/2},\ -a^{1/2},\ qa/b,\ qa/c,\ qa/d,\ qa/e,\ q^{1+n}a \end{matrix}; q\right] \\ &= \left[\begin{matrix} qa,\ qa/bc,\ qa/bd,\ qa/cd \\ qa/b,\ qa/c,\ qa/d,\ qa/bcd \end{matrix}; q\right]_n \end{aligned} \tag{1.5}$$

where the usual notation of q-shifted-factorial, Gaussian binomial coefficient, and basic hypergeometric series from the monographs by Bailey [3], Gasper and Rahman [14] and Slater [24] have been adopted and the fraction of q-factorials abbreviated as

$$\left[\begin{matrix} a,\ b,\ \cdots,\ c \\ u,\ v,\ \cdots,\ w \end{matrix}; q\right]_n = \frac{(a; q)_n (b; q)_n \cdots (c; q)_n}{(u; q)_n (v; q)_n \cdots (w; q)_n}.$$

The results exhibited in the present paper together with the previous one [7] demonstrate a remarkable fact that the most basic hypergeometric identities are the dual relations of only three q-formulas analogous to those named after Chu-Vandermonde-Gauss, Pfaff-Saalschutz and Dougall-Dixon. This serves as a natural reason for the existence of strange basic hypergeometric relations.

2. A new pair of reciprocal relations

As an extension of Carlitz theorem, this section will establish a general pair of reciprocal relations contained in the following

THEOREM. *With the ψ-polynomials defined by* (1.1), *we have the inverse series relations*:

$$f(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \psi(\lambda q^k; n)\psi(q^{-k}; n) \frac{1-\lambda q^{2k}}{(\lambda q^n; q)_{k+1}}\, g(k) \tag{2.1a}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{a_k+\lambda q^k b_k}{\psi(\lambda q^n;\, k+1)} \frac{a_k+q^{-k} b_k}{\psi(q^{-n};\, k+1)} (\lambda q^k; q)_n f(k) \tag{2.1b}$$

provided that the sequence-transforms involved are non-singular, i.e. $\psi(\lambda q^n; m+1)$, $\psi(q^{-n}; m+1)$ *and* $(\lambda q^n; q)_{m+1}$ *do not vanish for non-negative integers* $m \leqslant n$.

It is obvious that this pair of inversions will reduce to (1.3) when λ tends to infinity. Similar to the role of Carlitz inversions, (2.1) may be used, systematically, to deal with strange basic hypergeometric series through the so-called embedding technique (or creative telescoping) mentioned in the introduction. The exhibition demonstrated in the next three sections will show that this approach can not only certify the known q-hypergeometric formulas, but also create several new strange evaluations.

Proof of the theorem. Recalling the connection [23, p. 44–46] between inverse series relations and matrix inverse pairs, the reciprocity between (2.1a) and (2.1b) is equivalent to either of the following orthogonal relations:

$$\begin{aligned}(1-\lambda q^{2m}) \sum_{k=m}^{n} (-1)^{m+k} \begin{bmatrix} n-m \\ k-m \end{bmatrix} q^{\binom{n-k}{2}} & \frac{\psi(\lambda q^m;\, k)\psi(q^{-m};\, k)}{\psi(\lambda q^n;\, k+1)\psi(q^{-n};\, k+1)} \\ \times \frac{(\lambda q^n; q)_k}{(\lambda q^m; q)_{k+1}} (a_k+\lambda q^k b_k)(a_k+q^{-k} b_k) &= \delta_{m,n},\end{aligned} \tag{2.2a}$$

$$(a_n+\lambda q^n b_n)(a_n+q^{-n}b_n)\sum_{k=m}^{n}(-1)^{m+k}\begin{bmatrix}n-m\\k-m\end{bmatrix}q^{\binom{k-m}{2}}$$
$$\times\frac{\psi(\lambda q^k;\,n)\psi(q^{-k};\,n)}{\psi(\lambda q^k;\,m+1)\psi(q^{-k};\,m+1)}\,\frac{(\lambda q^k;\,q)_m}{(\lambda q^k;\,q)_{n+1}}(1-\lambda q^{2k})=\delta_{m,n}. \tag{2.2b}$$

The former simply follows from summand-splitting

$$(1-\lambda q^{m+n})\begin{bmatrix}n-m\\k-m\end{bmatrix}q^{\binom{n-k}{2}}\frac{\psi(\lambda q^m;\,k)\psi(q^{-m};\,k)}{\psi(\lambda q^n;\,k+1)\psi(q^{-n};\,k+1)}$$
$$\times\frac{(\lambda q^n;\,q)_k}{(\lambda q^m;\,q)_{k+1}}(a_k+\lambda q^k b_k)(a_k+q^{-k}b_k)$$
$$=\begin{bmatrix}n-m-1\\k-m\end{bmatrix}q^{\binom{n-k}{2}}\frac{\psi(\lambda q^m;\,k+1)\psi(q^{-m};\,k+1)}{\psi(\lambda q^n;\,k+1)\psi(q^{-n};\,k+1)}\,\frac{(\lambda q^n;\,q)_{k+1}}{(\lambda q^m;\,q)_{k+1}}$$
$$+\begin{bmatrix}n-m-1\\k-m-1\end{bmatrix}q^{\binom{n-k+1}{2}}\frac{\psi(\lambda q^m;\,k)\psi(q^{-m};\,k)}{\psi(\lambda q^n;\,k)\psi(q^{-n};\,k)}\,\frac{(\lambda q^n;\,q)_k}{(\lambda q^m;\,q)_k}$$

and diagonal-cancelling (but, the dual orthogonality (2.2b) is more difficult to confirm directly). This completes the proof of the theorem. □

The rotated version of the theorem may be stated as

COROLLARY. *Assume the conditions of the theorem. The system of equations*

$$F(n)=\sum_{k=n}^{N}(-1)^k\begin{bmatrix}k\\n\end{bmatrix}\psi(\lambda q^n;\,k)\psi(q^{-n};\,k)\,\frac{1-\lambda q^{2n}}{(\lambda q^k;\,q)_{n+1}}\,G(k),\quad (0\leqslant n\leqslant N) \tag{2.3a}$$

is equivalent to the system

$$G(n)=\sum_{k=n}^{N}(-1)^k\begin{bmatrix}k\\n\end{bmatrix}q^{\binom{k-n}{2}}\frac{a_n+\lambda q^n b_n}{\psi(\lambda q^k;\,n+1)}\,\frac{a_n+q^{-n}b_n}{\psi(q^{-k};\,n+1)}(\lambda q^n;\,q)_k F(k),$$
$$(0\leqslant n\leqslant N) \tag{2.3b}$$

where N is an arbitrary non-negative integer or infinity.

One pair of bi-basic inverse relations due to Bressoud [4] and Gasper [12] can also be specified from (2.1) and (2.3) by defining ψ-polynomials to be shifted-factorials.

PROPOSITION. *Let a and b be complex numbers such that* $1\text{-}ap^xq^y$, $1 - bp^xq^{-y}$ *and* $1 - q^xa/b$ *differ from zero for non-negative integers x and y. Then we have the inverse series relations*

$$F(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (aq^k;\, p)_n (bq^{-k};\, p)_n \frac{1 - q^{2k}a/b}{(q^na/b;\, q)_{k+1}} f(k) \tag{2.4a}$$

$$f(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1 - ap^kq^k}{(aq^n;\, p)_{k+1}} \frac{1 - bp^kq^{-k}}{(bq^{-n};\, p)_{k+1}} (q^ka/b;\, q)_n F(k) \tag{2.4b}$$

and their rotated forms

$$G(n) = \sum_{k=n}^{N} (-1)^k \begin{bmatrix} k \\ n \end{bmatrix} (aq^n;\, p)_k (bq^{-n};\, p)_k \frac{1 - q^{2n}a/b}{(q^ka/b;\, q)_{n+1}} g(k), \tag{2.5a}$$

$$g(n) = \sum_{k=n}^{N} (-1)^k \begin{bmatrix} k \\ n \end{bmatrix} q^{\binom{k-n}{2}} \frac{1 - ap^nq^n}{(aq^k;\, p)_{n+1}} \frac{1 - bp^nq^{-n}}{(bq^{-k};\, p)_{n+1}} (q^ka/b;\, q)_n G(k). \tag{2.5b}$$

Based on the theorem, the remaining part of the paper will display 18 examples which may be sketched as follows. If (2.1a) holds with the parameter λ and the sequences $\{a_i, b_i\}$ and $\{f(k), g(k)\}$ under a certain specification (which correspond to a known basic hypergeometric evaluation), then we have (2.1b) under the same specification (which can be rewritten as another hypergeometric evaluation). Or conversely, this statement is valid either by interchanging (2.1a) and (2.1b).

3. Embedding technique on balanced formulae

The transforms

$$(aq^k;\, q)_n = (aq^n;\, q)_k (a;\, q)_n / (a;\, q)_k \tag{3.0a}$$

$$(cq^{-k};\, q)_n = q^{-nk} (c;\, q)_n (q/c;\, q)_k / (q^{1-n}/c;\, q)_k \tag{3.0b}$$

will be used to demonstrate, by (2.4), the dual relations between the balanced summations (the Saalschutzian series) and the non-trivial q-hypergeometric evaluations. All the balanced formulae (cf. [8]) involved here may be regarded as the q-analogue of Bailey's identities [3, p. 30] and some of them have appeared in a slightly different version [21].

EXAMPLE 3.1. Saalschutz formula ⇔ the first non-trivial evaluation.

Rewrite the balanced formula

$$
{}_3\Phi_2\left[\begin{matrix} qu^2/v^2,\ uq^n,\ q^{-n} \\ q^{1-n}u/v,\ q^{1+n}u^2/v \end{matrix};q\right]=\left[\begin{matrix} qu^2/v,\ qu/v \\ v,\ v/u \end{matrix};q\right]_n\left[\begin{matrix} v,\ qv \\ qu^2/v,\ q^2u^2/v \end{matrix};q^2\right]_n \tag{3.1a}
$$

in the form of (2.4b)

$$
\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}q^{\binom{n-k}{2}}\frac{1-q^{2k}u^2/v}{(q^nu^2/v;\ q)_{k+1}}\,\frac{1-u/v}{(q^{-n}u/v;\ q)_{k+1}}(q^ku;\ q)_n\,\frac{1-u^2/v}{1-q^{2k}u^2/v}\\
&\quad\times[u,\ qu^2/v^2;\ q]_k=q^{\binom{n+1}{2}}\left[\begin{matrix} u,\ u^2/v,\ qu/v \\ v,\ qv/u \end{matrix};q\right]_n\left[\begin{matrix} v,\ qv \\ qu^2/v,\ q^2u^2/v \end{matrix};q^2\right]_n
\end{aligned} \tag{3.1b}
$$

whose dual relation

$$
\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}(q^ku^2/v;\ q)_n(q^{-k}u/v;\ q)_n\,\frac{1-q^{2k}u}{(q^nu;\ q)_{k+1}}\left[\begin{matrix} u,\ u^2/v,\ qu/v \\ v,\ qv/u \end{matrix};q\right]_k\\
&\quad\times\left[\begin{matrix} v,\ qv \\ qu^2/v,\ q^2u^2/v \end{matrix};q^2\right]q^{\binom{k+1}{2}}=\frac{1-u^2/v}{1-q^{2n}u^2/v}[u,\ qu^2/v^2;\ q]_n
\end{aligned} \tag{3.1c}
$$

is equivalent to a very-well poised evaluation

$$
\begin{aligned}
&{}_{10}\Phi_9\left[\begin{matrix} u,\ qu^{1/2},\ -qu^{1/2},\ qu/v,\ v^{1/2},\ -v^{1/2},\ q^{1/2}v^{1/2},\ -q^{1/2}v^{1/2},\ q^nu^2/v,\ q^{-n} \\ u^{1/2},\ -u^{1/2},\ v,\ qu/v^{1/2},\ -qu/v^{1/2},\ q^{1/2}u/v^{1/2},\ -q^{1/2}u/v^{1/2},\ q^{1-n}v/u,\ q^{1+n}u \end{matrix};q\right]\\
&\quad=\left[\begin{matrix} u/v^{1/2},\ -u/v^{1/2},\ qu,\ qu^2/v^2 \\ qu/v^{1/2},\ -qu/v^{1/2},\ u^2/v,\ u/v \end{matrix};q\right]_n.
\end{aligned} \tag{3.1d}
$$

EXAMPLE 3.2. Balanced formula ⇔ the second non-trivial evaluation.

Rewrite the balanced formula

$$
\begin{aligned}
&{}_4\Phi_3\left[\begin{matrix} u^2/v^2,\ -qu/v^{1/2},\ uq^n,\ q^{-n} \\ -u/v^{1/2},\ q^{1-n}u/v,\ q^{1+n}u^2/v \end{matrix};q\right]\\
&\quad=\left[\begin{matrix} v^{1/2},\ u/v^{1/2},\ qu^2/v,\ u/v \\ qv^{1/2},\ qu/v^{1/2},\ qv,\ v/u \end{matrix};q\right]_n\left[\begin{matrix} qv,\ q^2v \\ u^2/v,\ qu^2/v \end{matrix};q^2\right]_n
\end{aligned} \tag{3.2a}
$$

in the form of (2.4b)

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\frac{1-q^{2k}u^2/v}{(q^nu^2/v;\,q)_{k+1}}\frac{1-u/v}{(q^{-n}u/v;\,q)_{k+1}}$$
$$\times(q^ku;\,q)_n\frac{1-u/v^{1/2}}{1-q^ku/v^{1/2}}[u,\,u^2/v^2;\,q]_k$$
$$=q^{\binom{n+1}{2}}\begin{bmatrix}u,\ v^{1/2},\ u/v^{1/2},\ u^2/v,\ u/v\\ qv^{1/2},\ qu/v^{1/2},\ qv,\ qv/u\end{bmatrix};q\Bigg]_n\begin{bmatrix}qv,\ q^2v\\ u^2/v,\ qu^2/v\end{bmatrix};q^2\Bigg]_n \tag{3.2b}$$

whose dual relation

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}(q^ku^2/v;\,q)_n(q^{-k}u/v;\,q)_n\frac{1-q^{2k}u}{(q^nu;\,q)_{k+1}}$$
$$\times\left[\begin{matrix}u,\ v^{1/2},\ u/v^{1/2},\ u^2/v,\ u/v\\ qv^{1/2},\ qu/v^{1/2},\ qv,\ qv/u\end{matrix};q\right]_k\left[\begin{matrix}qv,\ q^2v\\ u^2/v,\ qu^2/v\end{matrix};q^2\right]_k q^{\binom{k+1}{2}}$$
$$=\frac{1-u/v^{1/2}}{1-q^nu/v^{1/2}}[u,\,u^2/v^2;\,q]_n \tag{3.2c}$$

is equivalent to a very-well poised evaluation

$${}_{10}\Phi_9\left[\begin{matrix}u,\ qu^{1/2},\ -qu^{1/2},\ u/v,\ v^{1/2},\ -qv^{1/2},\ q^{1/2}v^{1/2},\ -q^{1/2}v^{1/2},\ q^nu^2/v,\ q^{-n}\\ u^{1/2},\ -u^{1/2},\ qv,\ qu/v^{1/2},\ -u/v^{1/2},\ q^{1/2}u/v^{1/2},\ -q^{1/2}u/v^{1/2},\ q^{1-n}v/u,\ q^{1+n}u\end{matrix};q\right]$$
$$=\left[\begin{matrix}u/v^{1/2},\ qu,\ u^2/v^2\\ qu/v^{1/2},\ u^2/v,\ u/v\end{matrix};q\right]_n. \tag{3.2d}$$

EXAMPLE 3.3. Balanced formula ⇔ the third non-trivial evaluation.

Rewrite the balanced formula

$${}_4\Phi_3\left[\begin{matrix}u^2/v^2,\ -qu/v,\ uq^n,\ q^{-n}\\ -u/v,\ q^{1-n}u/v,\ q^{1+n}u^2/v\end{matrix};q\right]=q^n\left[\begin{matrix}qu^2/v,\ u/v\\ qv,\ v/u\end{matrix};q\right]_n\left[\begin{matrix}v,\ qv\\ qu^2/v,\ q^2u^2/v\end{matrix};q^2\right]_n \tag{3.3a}$$

in the form of (2.4b)

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\frac{1-q^{2k}u^2/v}{(q^nu^2/v;\,q)_{k+1}}\frac{1-u/v}{(q^{-n}u/v;\,q)_{k+1}}$$
$$\times(q^ku;\,q)_n\frac{1+q^ku/v}{1-q^{2k}u^2/v}[u,\,u^2/v^2;\,q]_k$$

$$= q^{n+\binom{n+1}{2}} \frac{1+u/v}{1-u^2/v} \left[\begin{matrix} u,\ u^2/v,\ u/v \\ qv,\ qv/u \end{matrix} ; q\right]_n \left[\begin{matrix} v,\ qv \\ qu^2/v,\ q^2u^2/v \end{matrix} ; q^2\right]_n \tag{3.3b}$$

whose dual relation

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (q^ku^2/v;\ q)_n (q^{-k}u/v;\ q)_n \frac{1-q^{2k}u}{(q^nu;\ q)_{k+1}} \left[\begin{matrix} u,\ u^2/v,\ u/v \\ qv,\ qv/u \end{matrix} ; q\right]_k$$

$$\times \left[\begin{matrix} v,\ qv \\ qu^2/v,\ q^2u^2/v \end{matrix} ; q^2\right]_k q^{k+\binom{k+1}{2}}$$

$$= \frac{1-u^2/v}{1+u/v} \frac{1+q^nu/v}{1-q^{2n}u^2/v} [u,\ u^2/v^2;\ q]_n \tag{3.3c}$$

is equivalent to a very-well poised evaluation

$${}_{10}\Phi_9 \left[\begin{matrix} u,\ qu^{1/2},\ -qu^{1/2},\ u/v,\ v^{1/2},\ -v^{1/2},\ q^{1/2}v^{1/2},\ -q^{1/2}v^{1/2},\ q^nu^2/v,\ q^{-n} \\ u^{1/2},\ -u^{1/2},\ qv,\ qu/v^{1/2},\ -qu/v^{1/2},\ q^{1/2}u/v^{1/2},\ -q^{1/2}u/v^{1/2},\ q^{1-n}v/u,\ q^{1+n}u \end{matrix} ; q^2\right]$$

$$= \left[\begin{matrix} u/v^{1/2},\ -u/v^{1/2},\ -qu/v,\ qu,\ u^2/v^2 \\ qu/v^{1/2},\ -qu/v^{1/2},\ -u/v,\ u^2/v,\ u/v \end{matrix} ; q\right]_n . \tag{3.3d}$$

EXAMPLE 3.4. Balanced formula ⇔ the fourth non-trivial evaluation.

Rewrite the balanced formulae

$${}_4\Phi_3 \left[\begin{matrix} u^2/v^2,\ qu/v^{1/2},\ uq^n,\ q^{-n} \\ u/v^{1/2},\ q^{1-n}u/v,\ q^{1+n}u^2/v \end{matrix} ; q\right]$$

$$= \left[\begin{matrix} qv^{1/2},\ qu/v^{1/2},\ qu^2/v,\ u/v \\ v^{1/2},\ u/v^{1/2},\ qv,\ v/u \end{matrix} ; q\right]_n \left[\begin{matrix} v,\ qv \\ qu^2/v,\ q^2u^2/v \end{matrix} ; q^2\right]_n \tag{3.4a}$$

in the form of (2.4b)

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-q^{2k}u^2/v}{(q^nu^2/v;\ q)_{k+1}} \frac{1-u/v}{(q^{-n}u/v;\ q)_{k+1}}$$

$$\times (q^ku;\ q)_n \frac{1-q^ku/v^{1/2}}{1-q^{2k}u^2/v} [u,\ u^2/v^2;\ q]_k$$

$$= q^{\binom{n-k}{2}}/(1+u/v^{1/2}) \left[\begin{matrix} u,\ qv^{1/2},\ qu/v^{1/2},\ u^2/v,\ u/v \\ v^{1/2},\ u/v^{1/2},\ qv,\ qv/u \end{matrix} ; q\right]_n \left[\begin{matrix} v,\ qv \\ qu^2/v, q^2u^2/v \end{matrix} ; q^2\right]_n$$

(3.4b)

whose dual relation

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}(q^ku^2/v;\,q)_n(q^{-k}u/v;\,q)_n\frac{1-q^{2k}u}{(q^nu;\,q)_{k+1}}$$
$$\times\begin{bmatrix}u,\ qv^{1/2},\ qu/v^{1/2},\ u^2/v,\ u/v\\ v^{1/2},\ u/v^{1/2},\ qv,\ qv/u\end{bmatrix}_k\begin{bmatrix}v,\ qv\\ qu^2/v,\ q^2u^2/v\end{bmatrix}_k q^{\binom{k+1}{2}}$$
$$=\frac{1-u^2/v}{1-u/v^{1/2}}\,\frac{1-q^nu/v^{1/2}}{1-q^{2n}u^2/v}\,[u,\ u^2/v^2;\ q]_n \tag{3.4c}$$

is equivalent to a very-well poised evaluation

$${}_{10}\Phi_9\begin{bmatrix}u,\ qu^{1/2},\ -qu^{1/2},\ u/v,\ q^{1/2}v^{1/2},\ -q^{1/2}v^{1/2},\ qv^{1/2},\ -v^{1/2},\ q^nu^2/v,\ q^{-n}\\ u^{1/2},\ -u^{1/2},\ qv,\ q^{1/2}u/v^{1/2},\ -q^{1/2}u/v^{1/2},\ u/v^{1/2},\ -qu/v^{1/2},\ q^{1-n}v/u,\ q^{1+n}u\end{bmatrix}$$
$$=\begin{bmatrix}qu,\ -u/v^{1/2},\ u^2/v^2\\ u^2/v,\ -qu/v^{1/2},\ u/v\end{bmatrix}_n. \tag{3.4d}$$

EXAMPLE 3.5. Balanced formula ⇔ the fifth non-trivial evaluation.

Rewrite the balanced formula

$${}_5\Phi_4\begin{bmatrix}q^{-1}u^2/v^2,\ qu/v^{1/2},\ -qu/v^{1/2},\ uq^n,\ q^{-n}\\ u/v^{1/2},\ -u/v^{1/2},\ q^{1-n}u/v,\ q^{1+n}u^2/v\end{bmatrix}$$
$$=\begin{bmatrix}qu^2/v,\ q^{-1}u/v\\ q^2v,\ v/u\end{bmatrix}_n\begin{bmatrix}qv,\ q^2v\\ u^2/v,\ qu^2/v\end{bmatrix}_n \tag{3.5a}$$

in the form of (2.4b)

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\frac{1-q^{2k}u^2/v}{(q^nu^2/v;\,q)_{k+1}}\,\frac{1-u/v}{(q^{-n}u/v;\,q)_{k+1}}(q^ku;\,q)_n[u,\ q^{-1}u^2/v^2;\ q]_k$$
$$=q^{\binom{n+1}{2}}\begin{bmatrix}u,\ u^2/v,\ q^{-1}u/v\\ q^2v,\ qv/u\end{bmatrix}_n\begin{bmatrix}qv,\ q^2v\\ u^2/v,\ qu^2/v\end{bmatrix}_n \tag{3.5b}$$

whose dual relation

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}(q^ku^2/v;\,q)_n(q^{-k}u/v;\,q)_n\frac{1-q^{2k}u}{(q^nu;\,q)_{k+1}}\begin{bmatrix}u,\ u^2/v,\ q^{-1}u/v\\ q^2v,\ qv/u\end{bmatrix}_k$$
$$\times\begin{bmatrix}qv,\ q^2v\\ u^2/v,\ qu^2/v\end{bmatrix}_k q^{\binom{k+1}{2}}=[u,\ q^{-1}u^2/v^2;\ q]_n \tag{3.5c}$$

is equivalent to a very-well poised evaluation

$$
{}_{10}\Phi_9\begin{bmatrix} u,\ qu^{1/2},\ -qu^{1/2},\ q^{-1}u/v,\ q^{1/2}v^{1/2},\ -q^{1/2}v^{1/2},\ qv^{1/2},\ -qv^{1/2},\ q^nu^2/v,\ q^{-n} \\ u^{1/2},\ -u^{1/2},\ q^2v,\ q^{1/2}u/v^{1/2},\ -q^{1/2}u/v^{1/2},\ u/v^{1/2},\ -u/v^{1/2},\ q^{1-n}v/u,\ q^{1+n}u \end{bmatrix};q\Bigg]
$$

$$
=\begin{bmatrix} qu,\ q^{-1}u^2/v^2 \\ u/v,\ u^2v \end{bmatrix};q\Bigg]_n. \tag{3.5d}
$$

In the similar way, one can enumerate the dual relations of the so called bi-basic formulas displayed in [21] (see also [14, §3.10]). The details are omitted here.

4. Strange evaluations associated with Jackson's q-Dougall-Dixon theorem

By means of (3.0) and transforms

$$(u;q)_{mk}=(u;q^m)_k(qu;q^m)_k\cdots(q^{m-1}u;q^m)_k \tag{4.0a}$$

$$(vq^n;q)_{mk}=(v;q)_{mk}(vq^{mk};q)_n/(v;q)_n \tag{4.0b}$$

$$(wq^{-n};q)_{mk}=q^{-mnk}(w;q)_{mk}(q/w;q)_n/(q^{1-mk}/w;q)_n \tag{4.0c}$$

Jackson's q-Dougall-Dixon theorem (1.5) will be embedded in (2.1a) and generate, through (2.1b), the dual formulas. From this process, several unexpected strange hypergeometric evaluations will be found.

Among the examples displayed in this section, some of their inverse relations (i.e. Examples 4.2, 4.3 and 4.6–4.9) are not balanced by the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$. Instead, it is replaced by non-symmetric forms $\begin{bmatrix} n \\ 2k \end{bmatrix}$, $\begin{bmatrix} n \\ 3k \end{bmatrix}$ and $\begin{bmatrix} n \\ 4k \end{bmatrix}$. This is due to the particular specification for sequence $g(\cdots)$. For example, the specification $g(2k+1)=0$ in Eq. (4.2b) comes from the special q-Dougall-Dixon formula (4.2a).

EXAMPLE 4.1. q-Dougall-Dixon theorem ⇔ Gessel and Stanton's evaluation.

The specialized q-Dougall-Dixon formula

$$
{}_8\Phi_7\begin{bmatrix} q^{1/2}ad, q^{5/4}(ad)^{1/2},\ -q^{5/4}(ad)^{1/2}, qd/b, q^{1/2}db, aq^{n/2}, aq^{(1+n)/2}, q^{-n} \\ q^{1/4}(ad)^{1/2},\ -q^{1/4}(ad)^{1/2}, q^{1/2}ab, qa/b, dq^{(3-n)/2}, dq^{(2-n/2}, q^{n+3/2}ad \end{bmatrix};q\Bigg]
$$

$$
=\begin{bmatrix} a/d,\ q^{3/2}ad \\ qa/b,\ q^{1/2}ab \end{bmatrix};q\Bigg]_n\begin{bmatrix} b,\ q^{1/2}/b \\ qd,\ q^{-1/2}/d \end{bmatrix};q^{1/2}\Bigg]_n \tag{4.1a}
$$

can be rewritten as

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}(q^ka;q^{1/2})_n(q^{-k-1/2}/d;q^{1/2})_n\frac{1-q^{2k+1/2}ad}{(q^{n+1/2}ad;q)_{k+1}}&\\ \times\begin{bmatrix}a,\ q^{1/2}a,\ q^{1/2}ad,\ q^{1/2}db,\ qd/b\\ qd,\ q^{3/2}d,\ q^{1/2}ab,\ qa/b\end{bmatrix};q\Bigg]_k q^{\binom{k+1}{2}}&\\ =\begin{bmatrix}a/d,\ q^{1/2}ad\\ qa/b,\ q^{1/2}ab\end{bmatrix};q\Bigg]_n\begin{bmatrix}a,\ b,\ q^{1/2}/b\\ qd\end{bmatrix};q^{1/2}\Bigg]_n&\end{aligned}\tag{4.1b}$$

whose dual version

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\frac{1-aq^{3k/2}}{(q^na;q^{1/2})_{k+1}}\frac{1-d^{-1}q^{-(1+k)/2}}{(q^{-n-1/2}/d;q^{1/2})_{k+1}}(q^{k+1/2}ad;q)_n&\\ \times\left[\begin{matrix}a/d,\ q^{1/2}ad\\ qa/b,\ q^{1/2}ab\end{matrix};q\right]_k\left[\begin{matrix}a,\ b,\ q^{1/2}/b\\ qd\end{matrix};q^{1/2}\right]_k&\\ =q^{\binom{n+1}{2}}\left[\begin{matrix}a,\ q^{1/2}a,\ q^{1/2}ad,\ q^{1/2}db,\ qd/b\\ qd,\ q^{3/2}d,\ q^{1/2}ab,\ qa/b\end{matrix};q\right]_n&\end{aligned}\tag{4.1c}$$

may be expressed as a strange terminating evaluation

$$\begin{aligned}\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},\ a/d,\ q^{n+1/2}ad\\ q,\ q^{1/2}ab,\ qa/b\end{matrix};q\right]_k\frac{1-aq^{3k/2}}{1-a}\left[\begin{matrix}a,\ b,\ q^{1/2}/b\\ q^{1/2}d,\ q^{-n}/d,\ q^{n+1/2}a\end{matrix};q^{1/2}\right]_k q^{k/2}&\\ =\left[\begin{matrix}q^{1/2}a\\ q^{1/2}d\end{matrix};q^{1/2}\right]_{2n}\left[\begin{matrix}q^{1/2}bd,\ qd/b\\ q^{1/2}ab,\ qa/b\end{matrix};q\right]_n&\end{aligned}\tag{4.1d}$$

which is due to Gessel and Stanton [16, Eq. (1.4)]. For its non-terminating version, see [12, Eq. (5.2)], [13, Eq. (5.1)] and [14, Eq. (3.8.12)].

EXAMPLE 4.2. q-Dougall-Dixon theorem ⇔ another Gessel and Stanton's evaluation.

The specialized q-Dougall-Dixon formula

$$\begin{aligned}{}_8\Phi_7\left[\begin{matrix}ad,\ q(ad)^{1/2},\ -q(ad)^{1/2},\ bd,\ q^{1/2}ad/b,\ aq^n,\ q^{-n/2},\ q^{-(n-1)/2}\\ (ad)^{1/2},\ -(ad)^{1/2},\ qa/b,\ q^{1/2}b,\ dq^{1-n},\ adq^{1+n/2},\ adq^{(1+n)/2}\end{matrix};q\right]&\\ =\left[\begin{matrix}b,\ q^{1/2}a/b\\ 1/d,\ q^{1/2}ad\end{matrix};q\right]_n\left[\begin{matrix}1/d,\ q^{1/2}ad\\ b,\ q^{1/2}a/b\end{matrix};q^{1/2}\right]_n&\end{aligned}\tag{4.2a}$$

can be rewritten as, after replacement $q \to q^2$

$$\sum_{k\geqslant 0}\begin{bmatrix} n\\ 2k\end{bmatrix}(q^{2k}a;\,q^2)_n(q^{-2k}/d;\,q^2)_n\frac{1-q^{4k}ad}{(q^nad;\,q)_{2k+1}}$$
$$\times\left[\begin{matrix} q,\ a,\ ad,\ bd,\ qad/b \\ qb,\ q^2a/b,\ q^2d\end{matrix};q^2\right]_k q^{\binom{2k+1}{2}}$$
$$=\left[\begin{matrix} 1/d,\ ad \\ b,\ qa/b\end{matrix};q\right]_n\left[\begin{matrix} a,\ b,\ qa/b \\ qad\end{matrix};q^2\right]_n \tag{4.2b}$$

whose dual version

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}q^{\binom{n-k}{2}}\frac{1-aq^{3k}}{(q^na;\,q^2)_{k+1}}\frac{1-d^{-1}q^{-k}}{(q^{-n}/d;\,q^2)_{k+1}}$$
$$\times(q^kad;\,q)_n\left[\begin{matrix} 1/d,\ ad \\ b,\ qa/b\end{matrix};q\right]_k\left[\begin{matrix} a,\ b,\ qa/b \\ qad\end{matrix};q^2\right]_k$$
$$=\begin{cases} 0, & (n-\text{odd})\\ q^{\binom{n+1}{2}}\left[\begin{matrix} q,\ a,\ ad,\ bd,\ qad/b \\ qb,\ q^2a/b,\ q^2d\end{matrix};q^2\right]_m, & (n=2m)\end{cases} \tag{4.2c}$$

may be expressed as another strange terminating evaluation

$$\sum_{k=0}^{n}\left[\begin{matrix} q^{-n},\ q/d,\ q^nad \\ q,\ b,\ qa/b\end{matrix};q\right]_k\frac{1-aq^{3k}}{1-a}\left[\begin{matrix} a,\ b,\ qa/b \\ qad,\ q^{2+n}a,\ q^{2-n}/d\end{matrix};q^2\right]_k q^k$$
$$=\begin{cases} 0, & (n-\text{odd})\\ \left[\begin{matrix} q,\ q^2a,\ bd,\ qad/b \\ d,\ qb,\ qad,\ q^2a/b\end{matrix};q^2\right]_m, & (n=2m)\end{cases} \tag{4.2d}$$

which is also due to Gessel and Stanton [16, Eq. (6.14)].

EXAMPLE 4.3. q-Dougall-Dixon theorem ⇔ the first new strange evaluation.

The specialized q-Dougall-Dixon formula

$${}_8\Phi_7\left[\begin{matrix} a,\ qa^{1/2},\ -qa^{1/2},\ a^2/b^2,\ bq^{n/2},\ bq^{(1+n)/2},\ q^{-n/2},\ q^{-(n-1)/2} \\ a^{1/2},\ -a^{1/2},\ qb^2/a,\ q^{1-n/2}a/b,\ q^{(1-n)/2}a/b,\ aq^{1+n/2},\ aq^{(1+n)/2}\end{matrix};q\right]$$
$$=(-b/a)^a\left[\begin{matrix} q^{1/2}b^2/a \\ q^{1/2}a\end{matrix};q\right]_n\left[\begin{matrix} q^{1/2}a,\ a/b \\ q^{1/2}b^2/a,\ b/a\end{matrix};q^{1/2}\right]_n \tag{4.3a}$$

can be rewritten as, after replacement $q \to q^2$

$$\sum_{k\geq 0}\begin{bmatrix} n\\ 2k\end{bmatrix}(q^{2k}b;\,q)_n(q^{-2k}b/a;\,q)_n\frac{1-q^{4k}a}{(q^n a;\,q)_{2k+1}}$$
$$\times\left[\begin{matrix} q,\ a,\ b,\ qba^2/b^2\\ qa/b,\ q^2a/b,\ q^2b^2/a\end{matrix};\,q^2\right]_k q^{\binom{2k+1}{2}}$$
$$=(-b/a)^n\left[\begin{matrix} qb^2/a\\ qa\end{matrix};\,q^2\right]_n\left[\begin{matrix} a,\ b,\ a/b\\ qb^2/a\end{matrix};\,q\right]_n \tag{4.3b}$$

whose dual version

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}}\frac{1-bq^{2k}}{(q^n b;\,q)_{k+1}}\,\frac{1-b/a}{(q^{-n}b/a;\,q)_{k+1}}$$
$$\times(q^k a;\,q)_n\left[\begin{matrix} qb^2/a\\ qa\end{matrix};\,q^2\right]_k\left[\begin{matrix} a,\ b,\ a/b\\ qb^2/a\end{matrix};\,q\right]_k (b/a)^k$$
$$=\begin{cases} 0, & (n-\text{odd})\\ q^{\binom{n+1}{2}}\left[\begin{matrix} q,\ a,\ b,\ bq,\ a^2/b^2\\ qa/b,\ q^2a/b,\ q^2b^2/a\end{matrix};\,q\right]_m, & (n=2m)\end{cases} \tag{4.3c}$$

may be expressed as the first new strange terminating evaluation in this section

$$\sum_{k=0}^{n}\left[\begin{matrix} q^{-n},\ aq^n,\ b,\ a/b\\ q,\ qb^2/a,\ q^{1+n}b,\ q^{1-n}b/a\end{matrix};\,q\right]_k\frac{1-bq^{2k}}{1-b}\left[\begin{matrix} qb^2/a\\ qa\end{matrix};\,q^2\right]_k(-qb/a)^k$$
$$=\begin{cases} 0, & (n-\text{odd})\\ \left[\begin{matrix} q,\ qb,\ q^2b,\ a^2/b^2\\ qa,\ a/b,\ q^2a/b,\ q^2b^2/a\end{matrix};\,q^2\right]_m, & (n=2m).\end{cases} \tag{4.3d}$$

EXAMPLE 4.4. q-Dougall-Dixon theorem ⇔ the second new strange evaluation.

The specialized q-Dougall-Dixon formula

$${}_8\Phi_7\left[\begin{matrix} a,\ qa^{1/2},\ -qa^{1/2},\ qa^2/b,\ b^{1/2}q^n,\ -b^{1/2}q^n,\ q^{-n},\ -q^{-n}\\ a^{1/2},\ -a^{1/2},\ b/a,\ q^{1-n}a/b^{1/2},\ -q^{1-n}a/b^{1/2},\ aq^{1+n},\ -aq^{1+n}\end{matrix};\,q\right]$$
$$=(b/qa^2)^n\left[\begin{matrix} q^2a^2,\ q^2a^2/b,\ -b/a,\ -qb/a\\ b/a^2,\ b^2/a^2,\ -qa,\ -q^2/a\end{matrix};\,q^2\right]_n \tag{4.4a}$$

can be rewritten as, after replacement $q^2 \to q$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-a^2q^{2k}}{(a^2q^n;\ q)_{k+1}} \frac{1-a^2/b}{(q^{-n}a^2/b;\ q)_{k+1}} (q^kb;\ q)_n$$

$$\times \left[\begin{matrix} -q^{1/2},\ a,\ q^{1/2}a^2/b \\ b/a \end{matrix} ;\ q^{1/2}\right]_n \frac{(b;\ q)_k}{1+aq^k} q^{-k/2}$$

$$= q^{\binom{n}{2}+n/2} \frac{(b/a^2)^n}{1+a} \left[\begin{matrix} a^2,\ b,\ qa^2/b,\ -b/a,\ q^{1/2}b/a \\ qb/a^2,\ b^2/a^2,\ -q^{1/2}a,\ -qa \end{matrix} ;\ q\right]_n \tag{4.4b}$$

whose dual version

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (q^ka^2;\ q)_n(q^{-k}a^2/b;\ q)_n \frac{1-q^{2k}b}{(q^nb;\ q)_{k+1}}$$

$$\times \left[\begin{matrix} a^2,\ b,\ qa^2/b,\ -b/a,\ q^{1/2}b/a \\ qb/a^2,\ b^2/a^2,\ -q^{1/2}a,\ -qa \end{matrix} ;\ q\right] \frac{(b/a^2)^k}{1+a} q^{\binom{k}{2}+k/2}$$

$$= q^{-n/2} \frac{(b;\ q)_n}{1+aq^n} \left[\begin{matrix} -q^{1/2},\ a,\ q^{1/2}a^2/b \\ b/a \end{matrix} ;\ q^{1/2}\right]_n$$

may be expressed as the second new strange terminating evaluation in this section

$${}_8\Phi_7 \left[\begin{matrix} b,\ qb^{1/2},\ -qb^{1/2},\ qa^2/b,\ -b/a,\ -q^{1/2}b/a,\ a^2q^n,\ q^{-n} \\ b^{1/2},\ -b^{1/2},\ b^2/a^2,\ -qa,\ -q^{1/2}a,\ q^{1-n}b/a^2,\ bq^{1+n} \end{matrix} ;\ q^{1/2}b/a^2\right]$$

$$= q^{-n/2} \left[\begin{matrix} -a,\ qb \\ -qa,\ a^2/b \end{matrix} ;\ q\right]_n \left[\begin{matrix} -q^{1/2},\ q^{1/2}a^2/b \\ -a,\ b/a \end{matrix} ;\ q^{1/2}\right]_n . \tag{4.4d}$$

EXAMPLE 4.5. q-Dougall-Dixon theorem ⇔ Gasper's strange evaluation.

The specialized q-Dougall-Dixon formula

$${}_8\Phi_7 \left[\begin{matrix} a^2b,\ qab^{1/2},\ -qab^{1/2},\ ab^2,\ aq^{n/3},\ aq^{(1+n)/3},\ aq^{(2+n)/3},\ q^{-n} \\ ab^{1/2},\ -ab^{1/2},\ qa/b,\ abq^{1-n/3},\ abq^{(2-n)/3},\ abq^{(1-n)/3},\ a^2bq^{1+n} \end{matrix} ;\ q\right]$$

$$= \left[\begin{matrix} qa^2b \\ qa/b \end{matrix} ;\ q\right]_n \left[\begin{matrix} b \\ (ab)^{-1} \end{matrix} ;\ q^{1/3}\right]_n \left[\begin{matrix} q^{1/3}/b \\ q^{1/3}ab \end{matrix} ;\ q^{1/3}\right]_{2n} \tag{4.5a}$$

can be rewritten as

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}(q^ka;q^{1/3})_n(q^{-k}a^{-1}b^{-1};q^{1/3})_n\frac{1-q^{2k}a^2b}{(q^na^2b;q)_{k+1}}\left[\begin{matrix}a^2b,\ ab^2\\ qa/b\end{matrix};q\right]_k$$
$$\times\left[\begin{matrix}a\\ q^{1/3}ab\end{matrix};q^{1/3}\right]_{3k}q^{\binom{k+1}{2}}$$
$$=[a,b;q^{1/3}]_n\left[\begin{matrix}a^2b\\ qa/b\end{matrix};q\right]_n\left[\begin{matrix}q^{1/3}/b\\ q^{1/3}ab\end{matrix};q^{1/3}\right]_{2n}\tag{4.5b}$$

whose dual version

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\frac{1-aq^{4k/3}}{(q^na;q^{1/3})_{k+1}}\,\frac{1-a^{-1}b^{-1}q^{-2k/3}}{(q^{-n}a^{-1}b^{-1};q^{1/3})_{k+1}}(q^ka^2b;q)_n$$
$$\times\left[\begin{matrix}a^2b\\ qa/b\end{matrix};q\right]_k\left[\begin{matrix}q^{1/3}/b\\ q^{1/3}ab\end{matrix};q^{1/3}\right]_{2k}[a,b;q^{1/3}]_k$$
$$=q^{\binom{n+1}{2}}\left[\begin{matrix}a^2b,\ a/b^2\\ qa/b\end{matrix};q\right]_n\left[\begin{matrix}a\\ q^{1/3}ab\end{matrix};q^{1/3}\right]_{3n}\tag{4.5c}$$

may be expressed as a strange terminating evaluation

$$\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},\ q^na^2b\\ q,\ qa/b\end{matrix};q\right]_k\frac{1-aq^{4k/3}}{1-a}\left[\begin{matrix}a,\ b\\ q^{n+1/3}a,\ q^{-n+1/3}a^{-1}b^{-1}\end{matrix};q^{1/3}\right]_k$$
$$\times\left[\begin{matrix}q^{1/3}b^{-1}\\ ab\end{matrix};q^{1/3}\right]_{2k}q^{k/3}=\left[\begin{matrix}ab^2\\ qa/b\end{matrix};q\right]_n\left[\begin{matrix}q^{1/3}a\\ ab\end{matrix};q^{1/3}\right]_{3n}\tag{4.5d}$$

which is due to Gasper [12, Eq. (5.22)]. See also [13, Eq. (1.2)].

EXAMPLE 4.6. q-Dougall-Dixon theorem ⇔ the third new strange evaluation.

The specialized q-Dougall-Dixon formula

$$ {}_8\Phi_7\left[\begin{matrix}a,\ qa^{1/2},\ -qa^{1/2},\ a/b,\ q^nab,\ q^{(1-n)/3},\ q^{(2-n)/3},\ q^{-n/3}\\ a^{1/2},\ -a^{1/2},\ qb,\ q^{1-n}/b,\ aq^{(2+n)/3},\ aq^{(1+n)/3},\ aq^{1+n/3}\end{matrix};q\right]$$
$$=\left[\begin{matrix}a\\ b\end{matrix};q\right]_n\left[\begin{matrix}aq^{1/3}\\ bq^{1/3}\end{matrix};q^{1/3}\right]_n\left[\begin{matrix}b\\ a\end{matrix};q^{1/3}\right]_{2n}\tag{4.6a}$$

can be rewritten as, after replacement $q \to q^3$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ 3k \end{bmatrix} (q^{3k}ab;\, q^3)_n (q^{-3k}b;\, q^3)_n \frac{1-q^{6k}a}{(q^n a;\, q)_{3k+1}}$$
$$\times \begin{bmatrix} a,\ ab,\ a/b \\ q^3,\ q^3b,\ q^3/b \end{bmatrix}_k (q;\, q)_{3k} q^{\binom{3k+1}{2}} = [a,\, ab;\, q^3]_n \begin{bmatrix} a \\ qb \end{bmatrix}; q \Big]_n \begin{bmatrix} b \\ a \end{bmatrix}; q \Big]_{2n} \tag{4.6b}$$

whose dual version

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-abq^{4k}}{(q^n ab;\, q^3)_{k+1}} \frac{1-bq^{2k}}{(q^{-n}b;\, q^3)_{k+1}}$$
$$\times (q^k a;\, q)_n \left[\begin{matrix} a \\ qb \end{matrix};\, q\right]_k \left[\begin{matrix} b \\ a \end{matrix};\, q\right]_{2k} [a,\, ab;\, q^3]_k$$
$$= \begin{cases} q^{\binom{n+1}{2}} \left[\begin{matrix} a,\ ab,\ a/b \\ q^3,\ q^3b,\ q^3/b \end{matrix};\, q^3\right]_m (q;\, q)_n, & (n = 3m) \\ 0, & \text{(otherwise)} \end{cases}$$

may be expressed as the third new strange terminating evaluation in this section

$$\sum_{k=0}^{n} \left[\begin{matrix} q^{-n},\ q^n a \\ q,\ qb \end{matrix};\, q\right]_k \frac{1-abq^{4k}}{1-ab} \left[\begin{matrix} a,\ ab \\ abq^{3+n},\ bq^{3-n} \end{matrix};\, q^3\right]_k \left[\begin{matrix} qb \\ a \end{matrix};\, q\right]_{2k} q^k$$
$$= \begin{cases} \left[\begin{matrix} q \\ a \end{matrix};\, q\right]_n \left[\begin{matrix} a,\ abq^3,\ a/b \\ q^3,\ q^3b,\ 1/b \end{matrix};\, q^3\right]_m, & (n = 3m) \\ 0, & \text{(otherwise).} \end{cases} \tag{4.6d}$$

EXAMPLE 4.7. q-Dougall-Dixon theorem $\Leftrightarrow$ the fourth new strange evaluation.

The specialized q-Dougall-Dixon formula

$${}_8\Phi_7\left[\begin{matrix} a,\ qa^{1/2},\ -qa^{1/2},\ a^2q^{n-1/2},\ q^{-n/4},\ q^{(1-n)/4},\ q^{(2-n)/4},\ q^{(3-n)/4} \\ a^{1/2},\ -a^{1/2},\ a^{-1}q^{-n+3/2},\ aq^{1+n/4},\ aq^{(3+n)/4},\ aq^{(2+n)/4},\ aq^{(1+n)/4} \end{matrix};\, q\right]$$
$$= \frac{(aq^{1/4};\, q^{1/4})_n (aq^{-1/2};\, q^{1/4})_{3n}}{(aq^{-1/4};\, q^{1/2})_n (aq^{-1/2};\, q)_n (a;\, q^{1/4})_{2n}} \tag{4.7a}$$

can be rewritten as, after replacement $q \to q^4$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ 4k \end{bmatrix} (q^{4k-2}a^2;\, q^4)_n (q^{-4k-2}a;\, q^4)_n \frac{1-q^{8k}a}{(q^n a;\, q)_{4k+1}} \times \begin{bmatrix} a,\ a^2q^{-2} \\ q^4,\ q^6a^{-1} \end{bmatrix};\, q^4 \Bigg]_k (q;\, q)_{4k} q^{\binom{4k+1}{2}}$$

$$= (-a/q;\, q^2)_n \frac{(aq^{-2};\, q)_{3n}}{(a;\, q)_{2n}} (a;\, q)_n \tag{4.7b}$$

whose dual version

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1-a^2q^{5k-2}}{(q^{n-2}a^2;\, q^4)_{k+1}} \frac{1-aq^{3k-2}}{(q^{-n-2}a;\, q^4)_{k+1}} (q^k a;\, q)_n (-a/q;\, q^2)_k \times \frac{(aq^{-2};\, q)_{3k}}{(a;\, q)_{2k}} (a;\, q)_k$$

$$= \begin{cases} q^{\binom{n+1}{2}} \left[\begin{matrix} a,\ a^2q^{-2} \\ q^4,\ q^6a^{-1} \end{matrix};\, q^4 \right]_m (q;\, q)_n, & (n = 4m) \\ 0, & (\text{otherwise}) \end{cases} \tag{4.7c}$$

may be expressed as the fourth new strange terminating evaluation in this section

$$\sum_{k=0}^{n} \frac{[q^{-n},\ q^n a;\, q]_k}{[a^2q^{2+n},\ q^{2-n}a;\, q^4]_k} \frac{1-a^2q^{5k-2}}{1-a^2q^{-2}} \frac{(a/q;\, q)_{3k}}{(q;\, q)_{2k}} \frac{(-a/q;\, q^2)_k}{(q;\, q)_k} q^k$$

$$= \begin{cases} \left[\begin{matrix} q \\ a \end{matrix};\, q \right]_n \left[\begin{matrix} a,\ q^2a^2 \\ q^4,\ q^2/a \end{matrix};\, q^4 \right]_m, & (n = 4m) \\ 0, & (\text{otherwise}). \end{cases} \tag{4.7d}$$

EXAMPLE 4.8. q-Dougall-Dixon theorem ⇔ the fifth new strange evaluation.

The specialized q-Dougall-Dixon formula

$${}_8\Phi_7 \left[\begin{matrix} q^{1/4}x^{3/2},\ q^{9/8}x^{3/4},\ -q^{9/8}x^{3/4},\ xq^{n/3},\ xq^{(1+n)/3},\ xq^{(2+n)/3},\ q^{(1-n)/2},\ q^{-n/2} \\ q^{1/8}x^{3/4},\ -q^{1/8}x^{3/4},\ q^{(15-4n)/12}x^{1/2},\ q^{(11-4n)/12}x^{1/2},\ q^{(7-4n)/12}x^{1/2},\ q^{(3+2n)/4}x^{3/2},\ q^{(5+2n)/4}x^{3/2} \end{matrix};\, q \right]$$

$$= (-1)^n x^{-n/2} q^{-(2n+n^2)/12} \left[\begin{matrix} q^{1/4}x^{1/2} \\ q^{1/12}x^{1/2},\ q^{5/12}x^{1/2} \end{matrix};\, q^{1/6} \right]_n \frac{(q^{3/4}x^{3/2};\, q^{1/2})_n (q^{1/12}x^{1/2};\, q^{1/3})_{2n}}{(q^{-1/4}x^{-1/2};\, q^{1/3})_n (q^{3/4}x^{3/2};\, q)_n} \tag{4.8a}$$

can be rewritten as, after replacement $q \to q^2$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ 2k \end{bmatrix} (q^{2k}x;\, q^{2/3})_n (x^{-1/2}q^{-2k-1/2};\, q^{2/3})_n \frac{1 - q^{2k+1/2}x^{3/2}}{(q^{n+1/2}x^{3/2};\, q)_{2k+1}}$$
$$\times \begin{bmatrix} x \\ q^{7/6}x^{1/2} \end{bmatrix}_{3k} ;\, q^{2/3} \Big]_{3k} [q,\, q^{1/2}x^{3/2};\, q^2]_k q^{\binom{2k+1}{2}}$$

$$= (-1)^n x^{-n/2} q^{-(2n+n^2)/6} \left[\begin{matrix} q^{1/2}x^{1/2} \\ q^{1/6}x^{1/2},\ q^{5/6}x^{1/2} \end{matrix} ;\, q^{1/3} \right]_n$$
$$\times \frac{(q^{1/2}x^{3/2};\, q)_n (x;\, q^{2/3})_n}{(q^{3/2}x^{3/2};\, q^2)_n} (q^{1/6}x^{1/2};\, q^{2/3})_{2n} \tag{4.8b}$$

whose dual version

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \frac{1 - xq^{5k/3}}{(xq^n;\, q^{2/3})_{k+1}} \frac{1 - x^{-1/2}q^{-(2k+3)/6}}{(x^{-1/2}q^{-n-1/2};\, q^{2/3})_{k+1}} (q^{k+1/2}x^{3/2};\, q)_n$$
$$\times \left[\begin{matrix} q^{1/2}x^{1/2} \\ q^{1/6}x^{1/2},\ q^{5/6}x^{1/2} \end{matrix} ;\, q^{1/3} \right]_k \frac{(q^{1/2}x^{3/2};\, q)_k (x;\, q^{2/3})_k}{(q^{3/2}x^{3/2};\, q^2)_k}$$
$$\times (q^{1/6}x^{1/2};\, q^{2/3})_{2k} x^{-k/2} q^{-(2k+k^2)/6}$$

$$= \begin{cases} 0, & (n - \text{odd}) \\ q^{\binom{n+1}{2}} \left[\begin{matrix} x \\ q^{7/6}x^{1/2} \end{matrix} ;\, q^{2/3} \right]_{3m} [q,\, q^{1/2}x^{3/2};\, q^2]_m, & (n = 2m) \end{cases} \tag{4.8c}$$

may be expressed as the fifth new strange terminating evaluation in this section

$$\sum_{k=0}^{n} \left[\begin{matrix} q^{-n} \\ q \end{matrix} ;\, q \right]_k \frac{1 - xq^{5k/3}}{1 - x} \frac{(q^{1/6}x^{1/2};\, q^{2/3})_{2k}}{[xq^{n+2/3},\, x^{-1/2}q^{-n+1/6};\, q^{2/3}]_k}$$
$$\times \frac{(x;\, q^{2/3})_k (q^{n+1/2}x^{3/2};\, q)_k}{(q^{1/6}x^{1/2};\, q^{1/3})_k (q^{3/2}x^{3/2};\, q^2)_k} (-1)^k x^{-k/2} q^{(2k-k^2)/6}$$

$$= \begin{cases} 0, & (n - \text{odd}) \\ \left[\begin{matrix} q^{2/3}x \\ q^{1/2}x^{1/2} \end{matrix} ;\, q^{2/3} \right]_{3m} \left[\begin{matrix} q \\ q^{3/2}x^{3/2} \end{matrix} ;\, q^2 \right]_m & (n = 2m). \end{cases} \tag{4.8d}$$

EXAMPLE 4.9. q-Dougall-Dixon theorem ⇔ the sixth new strange evaluation.

The specialized q-Dougall-Dixon formula

$$
{}_8\Phi_7\left[\begin{matrix} q^{1/4}y,\ q^{9/8}y^{1/2},\ -q^{9/8}y^{1/2},\ yq^{n/2},\ yq^{(1+n)/2},\ q^{-n/3},\ q^{(1-n)/3},\ q^{(2-n)/3} \\ q^{1/8}y^{1/2}, -q^{1/8}y^{1/2}, q^{(5-2n)/4}, q^{(3-2n)/4},\ yq^{(15+4n)/12}, yq^{(11+4n)/12}, yq^{(7+4n)/12} \end{matrix} ; q\right]
$$

$$
=(-1)^n q^{(2n+n^2)/12}\,\frac{1-q^{(1-2n)/4}}{1-q^{(3-2n)/12}}\,\frac{(q^{1/12};\ q^{1/6})_n}{(q^{1/4};\ q^{1/2})_n}\,\frac{(yq^{7/12};\ q^{1/3})_n(yq^{1/4};\ q)_n}{(yq^{1/4};\ q^{1/3})_{2n}} \tag{4.9a}
$$

can be rewritten as, after replacement $q \to q^3$

$$
\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ 3k \end{bmatrix}(yq^{3k};\ q^{3/2})_n(q^{-3k-3/4};\ q^{3/2})_n\,\frac{1-yq^{6k+3/4}}{(yq^{n+3/4};\ q)_{3k+1}}
$$

$$
\times\left[\begin{matrix} y \\ q^{9/4} \end{matrix} ;\ q^{3/2}\right]_{2k}[q,\ q^2,\ yq^{3/4};\ q^3]_k\, q^{\binom{3k+1}{2}}
$$

$$
=(-1)^n q^{-n+n^2/4}\,\frac{1-q^{3/4}}{1-q^{(3-2n)/4}}\,\frac{(q^{1/4};\ q^{1/2})_n(yq^{3/4};\ q)_n(y;\ q^{3/2})_n(yq^{3/4};\ q^3)_n}{(yq^{3/4};\ q)_{2n}} \tag{4.9b}
$$

whose dual version

$$
\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}\,\frac{1-yq^{5k/2}}{(yq^n;\ q^{3/2})_{k+1}}\,\frac{1-q^{(2k-3)/4}}{(q^{-n-3/4};\ q^{3/2})_{k+1}}\,(yq^{k+3/4};\ q)_n
$$

$$
\times\frac{1-q^{3/4}}{1-q^{(3-2k)/4}}\,\frac{(q^{1/4};\ q^{1/2})_k(yq^{3/4};\ q)_k(y;\ q^{3/2})_k(yq^{3/4};\ q^3)_k}{(yq^{3/4};\ q)_{2k}}\, q^{-k+k^2/4}
$$

$$
=\begin{cases} q^{\binom{n+1}{2}}\left[\begin{matrix} y \\ q^{9/4} \end{matrix} ;\ q^{3/2}\right]_{2m}[q,\ q^2,\ yq^{3/4};\ q^3]_m, & (n=3m) \\ 0, & (\text{otherwise}) \end{cases} \tag{4.9c}
$$

may be expressed as the last new strange terminating evaluation in this section

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}q^{-n}\\ q\end{bmatrix}_k;q\Big]_k\frac{1-yq^{5k/2}}{1-y}\frac{(q^{1/4};\ q^{1/2})_k(yq^{n+3/4};\ q)_k}{[yq^{n+3/2},\ q^{-n+3/4};\ q^{3/2}]_k}$$

$$\times\frac{(y;\ q^{3/2})_k(yq^{3/4};\ q^3)_k}{(yq^{3/4};\ q)_{2k}}q^{(2k+k^2)/4}$$

$$=\begin{cases}\begin{bmatrix}yq^{3/2}\\ q^{3/4}\end{bmatrix};q^{3/2}\Big]_{2m}\begin{bmatrix}q,\ q^2\\ yq^{7/4},\ yq^{11/4}\end{bmatrix};q^3\Big]_m, & (n=3m)\\ 0, & \text{(otherwise).}\end{cases}\tag{4.9d}$$

REMARK. Similarly, one can show that Jackson's q-Dougall-Dixon formula (1.5) with five free parameters, is self-reciprocal, i.e. the dual relation has exactly the same formation as the original one under the parameter replacement. In addition, for a very special case of q-Dougall-Dixon theorem

$${}_8\Phi_7\left[\begin{matrix}a^2q^{1/4},\ aq^{9/8}-aq^{9/8},\ aq^{n/4},\ aq^{(1+n)/4},\ aq^{(2+n)/4},\ aq^{(3+n)/4},\ q^{-n}\\ aq^{1/8},-aq^{1/8},aq^{(5-n)/4},aq^{(4-n)/4},aq^{(3-n)/4},\ aq^{(2-n)/4},\ a^2q^{n+5/4}\end{matrix};q\right]=\delta_{0,n}\tag{4.10}$$

its resulting dual relation is trivial.

5. Reversal embeddings on q-Dougall-Dixon theorem

In general, a terminating hypergeometric summation $\Sigma_{k=0}^{n}T(n,k)=S(n)$, cannot be telescoped into one of the relations stated in (2.1). But occasionally, its reversal $\Sigma_{k=0}^{n}T(n,n-k)=S(n)$ may be restated as one member of (2.1). In this case, the dual relation will create some "mysterious-looking" formulas. By means of (3.0), (4.0) and transforms

$$(xq^{n-mn};\ q)_{mn-mk}=(-x)^{mn+mk-n}q^{\binom{1+mk}{2}-\binom{1+mn-n}{2}}$$
$$\times(xq^{-mk};\ q)_n(q/x;\ q)_{mn-n}/(q/x;\ q)_{mk}\tag{5.0a}$$

$$(zq^{-mn-n};\ q)_{mn-mk}=(-z)^{mn-mk}q^{\binom{1+mk+n}{2}-\binom{1+mn+n}{2}}$$
$$\times(q^{1+mk}/z;\ q)_n^{-1}(q/z;\ q)_{mn+n}/(q/z;\ q)_{mk}\tag{5.0b}$$

some remarkable examples are demonstrated as follows.

EXAMPLE 5.1. q-Dougall-Dixon theorem ⇔ one very strange new evaluation.

The reversal of q-Dougall-Dixon formula

$$
{}_8\Phi_7\left[\begin{matrix} A^{-1}q^{-2n},\ A^{-1/2}q^{1-n},\ -A^{-1/2}q^{1-n},\ q^{-(1+n)/2},\ q^{-n/2},\ q^{-n}u/A,\ q^{-n}v/A,\ q^{-n+3/2}u^{-1}v^{-1} \\ A^{-1/2}q^{-n},\ -A^{-1/2}q^{-n},\ A^{-1}q^{(3-3n)/2},\ A^{-1}q^{(2-3n)/2},\ q^{1-n}/u,\ q^{1-n}/v,\ q^{-n-1/2}uv/A \end{matrix};q\right]
$$

$$
= q^{\binom{n+1}{2}/2}\frac{(A;\,q)_{2n}(q^{-1/2}A;\,q)_n}{(q^{-1/2}A;\,q^{1/2})_{3n}}\frac{[q^{-1/2}u,\ q^{-1/2}v,\ qA/uv;\ q^{1/2}]_n}{[u,\ v,\ q^{3/2}A/uv;\ q]_n} \tag{5.1a}
$$

can be expressed as, after adding some extra-zero terms

$$
\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}(q^{k-1/2}A;\,q^{1/2})_n(q^{-k-1/2};\,q^{1/2})_n\frac{1-Aq^{2k}}{(Aq^n;\,q)_{k+1}}
$$

$$
\times\left[\begin{matrix} u,\ v,\ A,\ Aq^{-1/2},\ q^{3/2}A/uv \\ q^{3/2},\ qA/u,\ qA/v,\ q^{-1/2}uv \end{matrix};q\right]_k q^{\binom{k+1}{2}}
$$

$$
= \left[\begin{matrix} q,\ A,\ q^{-1/2}A \\ qA/u,\ qA/v,\ q^{-1/2}uv \end{matrix};q\right]_n\left[\begin{matrix} q^{-1/2}u,\ q^{-1/2}v,\ qA/uv \\ q \end{matrix};q^{1/2}\right]_n \tag{5.1b}
$$

whose dual relation

$$
\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}q^{\binom{n-k}{2}}\frac{1-Aq^{(3k-1)/2}}{(Aq^{n-1/2};\,q^{1/2})_{k+1}}\frac{1-q^{-(k+1)/2}}{(q^{-n-1/2};\,q^{1/2})_{k+1}}(Aq^k;\,q)_n
$$

$$
\times\left[\begin{matrix} q,\ A,\ q^{-1/2}A \\ qA/u,\ qA/v,\ q^{-1/2}uv \end{matrix};q\right]_k\left[\begin{matrix} q^{-1/2}u,\ q^{-1/2}v,\ qA/uv \\ q \end{matrix};q^{1/2}\right]_k
$$

$$
= q^{\binom{n+1}{2}}\left[\begin{matrix} u,\ v,\ A,\ Aq^{-1/2},\ q^{3/2}A/uv \\ q^{3/2},\ qA/u,\ qA/v,\ q^{-1/2}uv \end{matrix};q\right]_n
$$

can be reformulated in q-series

$$
\sum_{k=0}^{n}\left[\begin{matrix} q^{-n},\ q^nA,\ q^{-1/2}A \\ qA/u,\ qA/v,\ q^{-1/2}uv \end{matrix};q\right]_k\frac{1-Aq^{(3k-1)/2}}{1-Aq^{-1/2}}
$$

$$
\times\left[\begin{matrix} q^{-1/2}u,\ q^{-1/2}v,\ qA/uv \\ q^{1/2},\ Aq^n,\ q^{-n} \end{matrix};q^{1/2}\right]_k q^{k/2}
$$

$$
= \left[\begin{matrix} u,\ v,\ Aq^{1/2},\ q^{3/2}A/uv \\ q^{1/2},\ qA/u,\ qA/v,\ q^{-1/2}uv \end{matrix};q\right]_n \tag{5.1d}
$$

which seems to be new.

EXAMPLE 5.2. q-Dougall-Dixon theorem ⇔ Gasper and Rahman's two very strange evaluations.

For $\lambda = 1$ and 2, the reversal of q-Dougall-Dixon formula

$$
{}_8\Phi_7\left[\begin{matrix} a^{-2/3}q^{-2n-\lambda/3}, a^{-1/3}q^{1-n-\lambda/6}, -a^{-1/3}q^{1-n-\lambda/6}, q^{-(\lambda+2n)/3}, q^{-(\lambda+2n-1)/3}, q^{-(\lambda+2n-2)/3}, \\ a^{-2/3}b^{1/3}q^{-n}, a^{-2/3}b^{-1/3}q^{-n+\lambda/3} \\ a^{-1/3}q^{-n-\lambda/6}, -a^{-1/3}q^{-n-\lambda/6}, a^{-2/3}q^{1-4n/3}, a^{-2/3}q^{(2-4n)/3}, a^{-2/3}q^{(1-4n)/3}, \\ b^{-1/3}q^{1-n-\lambda/3}, b^{1/3}q^{1-n-2\lambda/3} \end{matrix} ; q\right]
$$

$$
= q^{(\lambda n-n+n^2)/3}\frac{(a^{2/3}q^{(1-\lambda)/3};\ q^{1/3})_{2n}(a^{2/3}q^{\lambda/3};\ q)_{2n}}{(a^{2/3};\ q^{1/3})_{4n}}\frac{[b^{1/3},\ b^{-1/3}q^{\lambda/3};\ q^{1/3}]_n}{[b^{1/3}q^{\lambda/3},\ b^{-1/3}q^{2\lambda/3};\ q]_n} \tag{5.2a}
$$

can be expressed as, after adding some extra-zero terms

$$
\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}(a^{2/3}q^k;\ q^{1/3})_n(q^{-k-\lambda/3};\ q^{1/3})_n\frac{1-a^{2/3}q^{2k+\lambda/3}}{(a^{2/3}q^{n+\lambda/3};\ q)_{k+1}}
$$

$$
\times\left[\begin{matrix} a^{2/3} \\ q^{(1+\lambda)/3} \end{matrix} ; q^{1/3}\right]_{3k}\left[\begin{matrix} q,\ b^{1/3}q^{\lambda/3},\ b^{-1/3}q^{2\lambda/3} \\ qa^{2/3}b^{-1/3},\ a^{2/3}b^{1/3}q^{1-\lambda/3} \end{matrix} ; q\right]_k q^{\binom{k+1}{2}}
$$

$$
=\left[\begin{matrix} a^{2/3}q^{(1-\lambda)/3} \\ q^{(1+\lambda)/3} \end{matrix} ; q^{1/3}\right]_{2n}\left[\begin{matrix} q,\ a^{2/3}q^{\lambda/3} \\ qa^{2/3}b^{-1/3},\ a^{2/3}b^{1/3}q^{1-\lambda/3} \end{matrix} ; q\right]_n
$$

$$
\times[b^{1/3}, b^{-1/3}q^{\lambda/3};\ q^{1/3}]_n \tag{5.2b}
$$

whose dual relation

$$
\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}q^{\binom{n-k}{2}}\frac{1-a^{2/3}q^{4k/3}}{(a^{2/3}q^n;\ q^{1/3})_{k+1}}\frac{1-q^{-(2k+\lambda)/3}}{(q^{-n-\lambda/3};\ q^{1/3})_{k+1}}(a^{2/3}q^{k+\lambda/3};\ q)_n
$$

$$
\times\left[\begin{matrix} a^{2/3}q^{(1-\lambda)/3} \\ q^{(1+\lambda)/3} \end{matrix} ; q^{1/3}\right]_{2k}\left[\begin{matrix} q,\ a^{2/3}q^{\lambda/3} \\ qa^{2/3}b^{-1/3},\ a^{2/3}b^{1/3}q^{1-\lambda/3} \end{matrix} ; q\right]_k
$$

$$
\times[b^{1/3}, b^{-1/3}q^{\lambda/3};\ q^{1/3}]_k
$$

$$
= q^{\binom{n+1}{2}}\left[\begin{matrix} a^{2/3} \\ q^{(1+\lambda)/3} \end{matrix} ; q^{1/3}\right]_{3n}\left[\begin{matrix} q,\ b^{1/3}q^{\lambda/3},\ b^{-1/3}q^{2\lambda/3} \\ qa^{2/3}b^{-1/3},\ a^{2/3}b^{1/3}q^{1-\lambda/3} \end{matrix} ; q\right]_n \tag{5.2c}
$$

can be reformulated in q-series

$$\begin{aligned}&\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},\ a^{2/3}q^{n+\lambda/3}\\ qa^{2/3}b^{-1/3},\ q^{1-\lambda/3}a^{2/3}b^{1/3}\end{matrix};q\right]_k\frac{1-a^{2/3}q^{4k/3}}{1-a^{2/3}}\\ &\times\left[\begin{matrix}a^{2/3}q^{(1-\lambda)/3}\\ q^{\lambda/3}\end{matrix};q^{1/3}\right]_{2k}\left[\begin{matrix}b^{1/3},\ q^{\lambda/3}b^{-1/3}\\ a^{2/3}q^{n+1/3},\ q^{-n+(1-\lambda)/3}\end{matrix};q\right]_k q^{k/3}\\ &=\left[\begin{matrix}a^{2/3}q^{1/3}\\ q^{\lambda/3}\end{matrix};q^{1/3}\right]_{3n}\left[\begin{matrix}q,\ b^{1/3}q^{\lambda/3},\ b^{-1/3}q^{2\lambda/3}\\ a^{2/3}q^{\lambda/3},\ qa^{2/3}b^{-1/3},\ a^{2/3}b^{1/3}q^{1-\lambda/3}\end{matrix};q\right]_n\end{aligned}\tag{5.2d}$$

which yields terminating Gasper and Rahman's formulas [13, Eqs. (1.8, 4.5)], respectively, for $\lambda = 1$ and 2. See also [14, Eq. (3.8.18), Ex. 3.32].

EXAMPLE 5.3. q-Dougall-Dixon theorem ⇔ three very strange new evaluations.

For $\lambda = 1$, 2 and 3, the reversal of q-Dougall-Dixon formula

$$\begin{aligned}&{}_8\Phi_7\left[\begin{matrix}xq^{-2n},x^{1/2}q^{1-n},-x^{1/2}q^{1-n},q^{-(\lambda+3n)/4},q^{-(\lambda+3n-1)/4},q^{-(\lambda+3n-2)/4},q^{-(\lambda+3n-3)/4},x^2q^{\lambda-n-1/2}\\ x^{1/2}q^{-n},\ -x^{1/2}q^{-n},\ xq^{(4+\lambda-5n)/4},\ xq^{(3+\lambda-5n)/4},\ xq^{(2+\lambda-5n)/4},\ xq^{(1+\lambda-5n)/4},\ x^{-1}q^{-\lambda-n+3/2}\end{matrix};q\right]\\ &=q^{(2\lambda n-3n+3n^2)/8}\frac{(xq^{(3\lambda-2)/4};\ q^{1/4})_n(x^{-1}q^{(3-2\lambda)/4};\ q^{1/2})_n}{(xq^{\lambda-1/2};\ q)_n}\frac{(x^{-1};\ q)_{2n}(x^{-1}q^{(1-2\lambda)/4};\ q^{1/4})_{2n}}{(x^{-1}q^{-\lambda/4};\ q^{1/4})_{5n}}\end{aligned}\tag{5.3a}$$

can be expressed as, after adding some extra-zero terms

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\left[\begin{matrix}n\\ k\end{matrix}\right](x^{-1}q^{k-\lambda/4};\ q^{1/4})_n(q^{-k-\lambda/4};\ q^{1/4})_n\frac{1-x^{-1}q^{2k}}{(x^{-1}q^n;\ q)_{k+1}}\\ &\times\left[\begin{matrix}x^{-1}q^{-\lambda/4}\\ q^{(1+\lambda)/4}\end{matrix};q^{1/3}\right]_{4k}\left[\begin{matrix}q,\ xq^{\lambda-1/2}\\ x^{-2}q^{-\lambda+3/2}\end{matrix};q\right]_k q^{\binom{k+1}{2}}\\ &=\left[\begin{matrix}q,\ x^{-1}\\ x^{-2}q^{-\lambda+3/2}\end{matrix};q\right]_n\frac{(x^{-1}q^{(1-2\lambda)/4};\ q^{1/4})_{2n}}{(q^{(1+\lambda)/4};\ q^{1/4})_{3n}}\\ &\times(xq^{(3\lambda-2)/4};\ q^{1/4})_n(x^{-1}q^{(3-2\lambda)/4};\ q^{1/2})_n\end{aligned}\tag{5.3b}$$

whose dual relation

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\frac{1-x^{-1}q^{(5k-\lambda)/4}}{(x^{-1}q^{n-\lambda/4};\,q^{1/4})_{k+1}}\,\frac{1-q^{-(3k+\lambda)/4}}{(q^{-n-\lambda/4};\,q^{1/4})_{k+1}}\,(x^{-1}q^k;\,q)_n\\
&\quad\times\left[\begin{matrix}q,\ x^{-1}\\ x^{-2}q^{-\lambda+3/2}\end{matrix};q\right]_k\frac{(x^{-1}q^{(1-2\lambda)/4};\,q^{1/4})_{2k}}{(q^{(1+\lambda)/4};\,q^{1/4})_{3k}}\\
&\quad\times(xq^{(3\lambda-2)/4};\,q^{1/4})_k(x^{-1}q^{(3-2\lambda)/4};\,q^{1/2})_k\\
&=q^{\binom{n+1}{2}}\left[\begin{matrix}x^{-1}q^{-\lambda/4}\\ q^{(1+\lambda)/4}\end{matrix};q^{1/4}\right]_{4n}\left[\begin{matrix}q,\ xq^{\lambda-1/2}\\ x^{-2}q^{-\lambda+3/2}\end{matrix};q\right]_n
\end{aligned}\tag{5.3c}$$

can be reformulated in q-series

$$\begin{aligned}
&\sum_{k=0}^{n}\left[\begin{matrix}q^{-n},\ x^{-1}q^{n}\\ x^{-2}q^{-\lambda+3/2}\end{matrix};q\right]_k\left[\begin{matrix}xq^{(3\lambda-2)/4}\\ x^{-1}q^{n+(1-\lambda)/4},\ q^{-n+(1-\lambda)/4}\end{matrix};q^{1/4}\right]_k\\
&\quad\times\frac{1-x^{-1}q^{(5k-\lambda)/4}}{1-x^{-1}q^{-\lambda/4}}\,\frac{(x^{-1}q^{(1-2\lambda)/4};\,q^{1/4})_{2k}}{(q^{\lambda/4};\,q^{1/4})_{3k}}\,(q^{(3-2\lambda)/4};\,q^{1/2})_k\,q^{k/4}\\
&=\left[\begin{matrix}x^{-1}q^{(1-\lambda)/4}\\ q^{\lambda/4}\end{matrix};q^{1/4}\right]_{4n}\left[\begin{matrix}q,\ xq^{\lambda-1/2}\\ x^{-1},\ x^{-2}q^{-\lambda+3/2}\end{matrix};q\right]_n
\end{aligned}\tag{5.3d}$$

which is the unified version of three new terminating strange evaluations corresponding to $\lambda = 1$, 2 and 3.

EXAMPLE 5.4. q-Dougall-Dixon theorem $\Leftrightarrow$ two very strange new evaluations.

For $\lambda = 1$ and 2, the reversal of q-Dougall-Dixon formula

$$\begin{aligned}
&{}_8\Phi_7\left[\begin{matrix}q^{-\lambda-2n+1/2},q^{-n+(5-2\lambda)/4},-q^{-n+(5-2\lambda)/4},q^{-(2\lambda+4n)/5},q^{-(2\lambda+4n-1)/5},q^{-(2\lambda+4n-2)/5},\\ q^{-(2\lambda+4n-3)/5},q^{-(2\lambda+4n-4)/5}\\ q^{-n+(1-2\lambda)/4},-q^{-n+(1-2\lambda)/4},q^{(15-6\lambda-12n)/10},q^{-(13-6\lambda-12n)/10},q^{-(11-6\lambda-12n)/10},\\ q^{-(9-6\lambda-12n)/10},q^{-(7-6\lambda-12n)/10}\end{matrix};q\right]\\
&=q^{2(\lambda n-n+n^2)/5}\frac{(q^{\lambda-1/2};\,q)_{2n}}{(q^{(6\lambda-5)/10};\,q^{1/5})_{6n}}\,[q^{(2\lambda+1)/10},\,q^{(2\lambda-3)/10};\,q^{1/5}]_{2n}
\end{aligned}\tag{5.4a}$$

can be expressed as, after adding some extra-zero terms

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}(q^{k+(6\lambda-5)/10};\,q^{1/5})_n(q^{-k-2\lambda/5};\,q^{1/5})_n\frac{1-q^{2k+\lambda-1/2}}{(q^{n+\lambda-1/2};\,q)_{k+1}}\\
&\quad\times\left[\begin{matrix}q^{(6\lambda-5)/10}\\ q^{(1+2\lambda)/5}\end{matrix};q^{1/5}\right]_{5k}(q;\,q)_k\,q^{\binom{k+1}{2}}\\
&=\frac{[q^{(2\lambda+1)/10},\,q^{(2\lambda-3)/10};\,q^{1/5}]_{2n}}{(q^{(1+2\lambda)/5};\,q^{1/5})_{4n}}\,[q,\,q^{\lambda-1/2};\,q]_n
\end{aligned}\tag{5.4b}$$

whose dual relation

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}&\frac{1-q^{(12k+6\lambda-5)/10}}{(q^{n+(6\lambda-5)/10};\ q^{1/5})_{k+1}}\,\frac{1-q^{-(4k+2\lambda)/5}}{(q^{-n-2\lambda/5};\ q^{1/5})_{k+1}}\,(q^{k+\lambda-1/2};q)_n\\ &\times\frac{[q^{(2\lambda+1)/10},\ q^{(2\lambda-3)/10};\ q^{1/5}]_{2k}}{(q^{(1+2\lambda)/5};\ q^{1/5})_{4k}}\,[q,q^{\lambda-1/2};q]_k\\ =q^{\binom{n+1}{2}}&\left[\begin{matrix}q^{(6\lambda-5)/10}\\ q^{(2\lambda+1)/5}\end{matrix};q^{1/5}\right]_{5n}(q;q)_n\end{aligned}\tag{5.4c}$$

can be reformulated in q-series

$$\begin{aligned}\sum_{k=0}^{n}&\left[\begin{matrix}q^{-n},\ q^{n+\lambda-1/2}\\ q^{n+(6\lambda-3)/10},\ q^{-n+(1-2\lambda)/5}\end{matrix};q^{1/5}\right]_k\frac{1-q^{(12k+6\lambda-5)/10}}{1-q^{(6\lambda-5)/10}}\\ &\times\frac{[q^{(2\lambda+1)/10},\ q^{(2\lambda-3)/10};\ q^{1/5}]_{2k}}{(q^{2\lambda/5};\ q^{1/5})_{4k}}\,q^{k/5}\\ =&\left[\begin{matrix}q^{(6\lambda-3)/10}\\ q^{2\lambda/5}\end{matrix};q^{1/5}\right]_{5n}\left[\begin{matrix}q\\ q^{\lambda-1/2}\end{matrix};q\right]_n\end{aligned}\tag{5.4d}$$

which is the unified version of two new terminating strange evaluations corresponding to $\lambda = 1$ and 2.

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