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Uniformization of rigid subanalytic sets

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Introduction

In a first paper [Sch 1] we introduced the notion of a (rigid) strongly subanalytic set as follows. Let K be an algebraically closed field endowed with a complete non trivial ultrametric norm $|\cdot|$, let R be its valuation ring and let A be a reduced affinoid algebra over K . We will always tacitly assume that all affinoid algebras are reduced, so that A is a subring of the ring of all K -valued functions on $\mathrm{Sp} A$ (the associated affinoid space). A subset of $\mathrm{Sp} A$ is called (globally) semianalytic if it can be described by disjunctions and conjunctions of inequalities of the form $|f(x)| \leq |g(x)|$ or $|f(x)| < |g(x)|$, where $f, g \in A$. A subset of $\mathrm{Sp} A \times R^m$ is called (globally) strongly semianalytic in the R^m -direction if the describing functions are overconvergent in the R^m variables, that is, still are convergent on $\mathrm{Sp} A \times D$, where $D \subset \mathbb{A}^m$ is a disk, depending on the functions, of radius strictly bigger than one. A projection of such a set along the R^m -variables is then called strongly subanalytic.

If we allow the describing functions in the definition of a semianalytic set to be strongly \mathbf{D} -functions, then we get strongly \mathbf{D} -semianalytic sets. By a strongly \mathbf{D} -function we mean a function which can be constructed by successively composing with affinoid functions (that is, elements of A) and with the \mathbf{D} -operator (from R^2 to R)

$$\mathbf{D}(a, b) = \begin{cases} a/b & \text{if } |a| \leq |b| \text{ and } b \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

but by allowing only substitution of this \mathbf{D} -operator in overconvergent variables. The number of times \mathbf{D} is used, is called the complexity of the \mathbf{D} -function. In [Sch 1] we then prove that strongly subanalytic and strongly \mathbf{D} -semianalytic are the same. For more detailed definitions we refer to [Sch 1].

In this paper we want to make plausible that the restriction on the overconvergence is a kind of relative compactness requirement, which also occurs in the real case (see for instance [BM] or [Hi 1]). Indeed, we will prove that the strongly subanalytic sets are exactly those sets which are the image under a proper map of a semianalytic set, see (2.5). Moreover, strongly subanalyticity is preserved under proper maps, see (2.6). For technical reasons we introduce the notion of a globally proper map as follows. Let $X \xrightarrow{\varphi} Y$ be a separated map of rigid analytic varieties, then we call φ *globally proper* if, for each admissible affinoid covering $\mathcal{Y} = \{Y_i\}_i$ of Y , and, for each i , there exist two finite admissible affinoid coverings, $\bar{\mathcal{X}}_i = \{\bar{X}_{ij}\}_j$ and $\mathcal{X}_i = \{X_{ij}\}_j$, of $\varphi^{-1}(Y_i)$, such that, for each i and j , we have that

$$\bar{X}_{ij} \Subset_{\mathcal{Y}} X_{ij}.$$

(For the notion of a relatively compact subset, we refer to [BGR, 9.6.2]). Note the difference with the definition of proper maps, where only the existence of one such an \mathcal{Y} is required. Nevertheless it should be clear that given a proper map $X \xrightarrow{\varphi} Y$, we always can find an admissible affinoid covering $\mathcal{Y} = \{Y_i\}_i$ of Y , such that the restricted maps $\varphi^{-1}(Y_i) \xrightarrow{\varphi} Y_i$ are globally proper. In [Sch 2] we prove that a map from a proper rigid analytic variety is always globally proper and that the composition $\beta\alpha$ of a proper (respectively, globally proper) map β and a globally proper map α is again proper (respectively, globally proper). This last statement could also be derived from the fact that the composition of proper maps is again proper, but the proof of this fact is far from trivial, whereas ours is straightforward. Moreover, finite maps and blowing-up maps are globally proper.

Using the results of our first paper together with section two, we obtain in the next section a theorem on the existence of bounds of proper maps. More precisely, let $N \xrightarrow{\varphi} M$ be a proper map with M quasi-compact and $S \subset N$ a strongly subanalytic set. Then there exists a natural number A such that, for all $x \in M$, the fibers $S_x = S \cap \varphi^{-1}(x)$ either are infinite or contain at most A elements, see (3.5). Similar (and more general) results can be found in [Lip 2] and [Lip 3], but the proofs are substantially more complicated, which is the reason why we publish ours here. Also Bartenwerfer has stronger results in the case of an analytic set, see [Bar].

In the last section we obtain a Uniformization Theorem of strongly subanalytic sets, by using a rigid analytic version of Hironaka’s Embedded Resolution of Singularities. Since however it is not known yet whether Hironaka’s theorem also holds in positive characteristic, we have to restrict ourselves to the case that K has characteristic zero. We introduce the notion of the *global voûte* \mathcal{E}_M over M , where M is an affinoid manifold, as

follows. By a *local blowing-up map*, we mean the composition of a blowing-up map and an open immersion. Now, an element $e \in \mathcal{E}_M$ is a family of compositions of finitely many local blowing-up maps with target space M with respect to regular centers of codimension two or greater, such that they ‘cover up’ the whole of M . For a precise definition, see (4.3). This covering up property should justify the name *global voûte*. This being defined, we can state the Uniformization Theorem.

UNIFORMIZATION. *Let M be an affinoid manifold and $S \subset M$ strongly subanalytic in M . Then there exists an $e \in \mathcal{E}_M$, such that, for each $h \in e$, with $\mathbf{M} \xrightarrow{h} M$, we have that $h^{-1}(S)$ is semianalytic in \mathbf{M} .*

See (4.4).

I want to thank Tom Denef for the valuable ideas and advice he has given me. He suggested to me the use of proper maps in the description of strongly subanalytic sets.

1. Semianalytic sets and closed immersions

1.1. **LEMMA.** *Let $X \xrightarrow{\theta} Y$ be a closed immersion of affinoid varieties. If $S \subset X$ is globally semianalytic in X , then $\theta(S)$ is globally semianalytic in Y .*

Proof. Let $X = \text{Sp } A$ and $Y = \text{Sp } B$, and let $\theta^*: B \rightarrow A$ be the corresponding (surjective) morphism. As S is a finite union of basic subsets of X , we may assume, without loss of generality, that S is already a basic subset in X . Let $g_i, h_i \in A$ and $\diamond_i \in \{\leq, <\}$, such that

$$S = \{x \in X \mid \forall i: |g_i(x)| \diamond_i |h_i(x)|\}.$$

Since θ^* is surjective, we can find elements $G_i, H_i \in B$ such that $g_i = \theta^*(G_i)$ and $h_i = \theta^*(H_i)$. Let T be the basic subset defined by these functions,

$$T \stackrel{\text{def}}{=} \{y \in Y \mid \forall i: |G_i(y)| \diamond_i |H_i(y)|\}.$$

Since for an arbitrary $p \in B$ and $x \in X$, we have that

$$(\theta^*(p))(x) = p(\theta(x)),$$

one directly verifies that $\theta(S) = T \cap \theta(X)$. But since θ is a closed immersion, we have that $\theta(X)$ is an analytic, hence semianalytic subset of Y . □

1.2. **LEMMA.** *Let $X \xrightarrow{\theta} Y$ be a closed immersion of affinoid varieties and \mathcal{X} a finite admissible affinoid covering of X . Then there exists a finite admissible affinoid covering \mathcal{Y} of Y , such that $\theta^{-1}(\mathcal{Y})$ refines \mathcal{X} .*

REMARK. If $X \xrightarrow{\theta} Y$ is a map of rigid analytic varieties and $\mathcal{Y} = \{Y_i\}_i$ an

admissible affinoid covering of Y , then we mean by the *inverse image covering* of \mathcal{Y} , the admissible affinoid covering $\theta^{-1}(\mathcal{Y}) = \{\theta^{-1}(Y_i)\}_i$.

Proof. Let $X = \text{Sp } A$ and $Y = \text{Sp } B$. Let $\theta^*: B \rightarrow A$ be the corresponding surjective morphism and $\mathfrak{a} = \ker \theta^*$ its kernel. By [BGR, 8.2.2. Lemma 2], we know that there exist $f_i \in A$, such that $\mathcal{C}(\mathfrak{f}) \stackrel{\text{def}}{=} \{U_i(\mathfrak{f})\}_i$ refines \mathcal{X} , where $\mathfrak{f} = \{f_1, \dots, f_s\}$, with $(f_1, \dots, f_s)A = (1)$ and

$$U_i(\mathfrak{f}) \stackrel{\text{def}}{=} \{x \in \text{Sp } A \mid \forall j: |f_j(x)| \leq |f_i(x)|\}.$$

Let $F_i \in B$ be a lifting of f_i , i.e. $\theta^*(F_i) = f_i$. Then there exists an $F_{s+1} \in \mathfrak{a}$, such that

$$(F_1, \dots, F_{s+1})B = (1). \tag{1}$$

Put $\mathfrak{F} = \{F_1, \dots, F_{s+1}\}$, then $\mathcal{C}(\mathfrak{F})$ is the claimed admissible affinoid covering of Y . Indeed, for $i = 1, \dots, s$, we clearly have $U_i(\mathfrak{f}) = \theta^{-1}(U_i(\mathfrak{F}))$. But the image of F_{s+1} in A is zero so that using (1), we must have that $\theta^{-1}(U_{s+1}(\mathfrak{F})) = \emptyset$. \square

1.3. PROPOSITION. *Let $X \xrightarrow{\theta} Y$ be a closed immersion of rigid analytic varieties. If $S \subset X$ is semianalytic in X , then $\theta(S)$ is semianalytic in Y .*

Proof. We easily reduce to the affinoid case. Since S is semianalytic in X , there exists a finite admissible affinoid covering $\mathcal{X} = \{X_i\}_i$ of X , such that, for each i , $S \cap X_i$ is globally semianalytic in X_i . By the previous lemma we can find an admissible affinoid covering $\mathcal{Y} = \{Y_j\}_j$ of Y , such that $\theta^{-1}(\mathcal{Y})$ refines \mathcal{X} . Therefore, for each j , we have that $S \cap \theta^{-1}(Y_j)$ is globally semianalytic in $\theta^{-1}(Y_j)$. Now, since θ is a closed immersion, also $\theta^{-1}(Y_j) \rightarrow Y_j$ is a closed immersion by base-change. Therefore, by (1.1), $\theta(S) \cap Y_j$ is globally semianalytic in Y_j , for each j . \square

REMARK. Note that we did not use quantifier elimination for this result.

2. Strongly subanalytic sets and proper maps

2.1. PROPOSITION. *Let $S \subset \mathbb{R}^m$ be strongly semianalytic in all variables. Then S is semianalytic in \mathbb{P}^m .*

Proof. Let $f_1, \dots, f_s \in K\langle\langle X \rangle\rangle$, generating the unit ideal, such that, for each i , we have that $S \cap U_i(\mathfrak{f})$ is globally strongly semianalytic (in all variables), where $X = (X_1, \dots, X_m)$ and $\mathfrak{f} = \{f_1, \dots, f_s\}$. Choose now $0 \neq \pi \in \wp$, such that all the functions which occur in the description of one of the $S \cap U_i(\mathfrak{f})$ and all the f_i still converge on the set

$$U_0 = \{x = (x_1, \dots, x_m) \in \mathbb{A}^m \mid \forall i: |x_i| \leq 1/|\pi|\}.$$

If we therefore set $F_i = f_i(\pi^{-1}X) \in K\langle X \rangle$ and $\mathfrak{F} = \{F_1, \dots, F_s\}$, then we have that $\mathcal{C}(\mathfrak{F})$ is a rational covering of U_0 , such that, for all i , $S \cap U_i(\mathfrak{F})$ is globally semianalytic in $U_i(\mathfrak{F})$, hence $S \cap U_0$ is semianalytic in U_0 . Let

$$U_i = \{\xi = (\xi_0, \dots, \xi_m) \in \mathbb{P}^m \mid \forall j: |\xi_i| \geq |\xi_j|/|\pi|\},$$

for $i = 1, \dots, m$. Then each U_i is affinoid and

$$U_i \cap \mathbb{A}^m \subset \{x = (x_1, \dots, x_m) \in \mathbb{A}^m \mid |x_i| \geq 1/|\pi|\},$$

and so $U_i \cap S = \emptyset$, since $1/|\pi| > 1$ and $S \subset R^m$.

Hence $\{U_0, \dots, U_m\}$ is a finite admissible affinoid covering of \mathbb{P}^m and each $S \cap U_i$ is semianalytic. □

REMARK. This need not to be the case, if S is only semianalytic, since S is described then by functions which only converge on R^m . But then, for an arbitrary admissible affinoid covering $\mathcal{X} = \{X_i\}_i$ of \mathbb{P}^m , there will be an i such that $S \cap X_i \neq \emptyset$ and $X_i \not\subset R^m$, so that functions to describe $S \cap X_i$ should converge also outside R^m .

2.2. COROLLARY. *Let M be an affinoid variety and $S \subset M \times R^m$ be strongly semianalytic in the R^m -direction. Then S is semianalytic in $M \times \mathbb{P}^m$.*

Proof. The proof is the same as above, just replace K by the affinoid algebra A of M . □

2.3. PROPOSITION. *Let M be an affinoid variety and $S \subset M$ be strongly subanalytic. Then there exists a quasi-compact rigid analytic variety N , a globally proper map $N \xrightarrow{\varphi} M$ and a semianalytic subset T of N , such that $\varphi(T) = S$.*

Proof. Let $T \subset M \times R^m$ be strongly semianalytic in the R^m -direction, such that

$$\pi(T) = S,$$

where $M \times R^m \xrightarrow{\pi} M$ is the projection on the first factor. If we set $N = M \times \mathbb{P}^m$, then N is quasicompact. Let

$$N \xrightarrow{\varphi} M$$

be the extended projection map, then φ is globally proper since $\varphi|_M$ is an isomorphism and \mathbb{P}^m is proper. But by (2.2) T is semianalytic in N and clearly $\varphi(T) = \pi(T) = S$. □

REMARK. By the Quantifier Elimination in [Sch 1, Theorem (5.2)], we know that S is globally strongly \mathbf{D} -semianalytic, so that we can find a

globally strongly semianalytic subset T in the R^n -direction of $M \times R^n$, such that the projection π induces a bijection between T and S and hence we can take $\varphi|_T$ to be bijective in the proposition.

2.4. LEMMA. *Let N and M be affinoid varieties and $N \xrightarrow{\varphi} M$ a map of affinoid varieties. Let $U \in_M N$ be relative compact and let $W \subset N$ be semianalytic in N . Then $\varphi(U \cap W)$ is strongly subanalytic in M .*

Proof. Let $N = \text{Sp } A$ and $M = \text{Sp } B$, and let $\varphi^*: B \rightarrow A$ be the morphism corresponding to φ . Since $U \in_M N$, there exists by definition an affinoid generating system $\{\mu_1, \dots, \mu_n\}$ of A over B , such that

$$U \subset \{z \in N \mid \forall i: |\mu_i(z)| < 1\}.$$

Due to the Maximum Modulus Principle, we can even find $0 \neq c \in \wp$, such that

$$U \subset \{z \in N \mid \forall i: |\mu_i(z)| \leq |c|\}. \tag{1}$$

Let $\theta: N \hookrightarrow M \times R^n$ be the closed immersion induced by the affinoid generating system, given by $x \mapsto (\varphi(x), \mu_1(x), \dots, \mu_n(x))$, so that the following diagram is commutative

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & M \\ \theta \downarrow & & \parallel \\ M \times R^n & \xrightarrow{\pi} & M. \end{array} \tag{2}$$

Since U is an affinoid subdomain of N , it is semianalytic and hence also $U \cap W$ is semianalytic in N . Since θ is a closed immersion, we know by (1.3) that $\theta(U \cap W)$ is semianalytic in $M \times R^n$. Hence there exists a rational covering $\mathcal{C}(\mathfrak{f})$ of $M \times R^n$, such that, for each i ,

$$\theta(U \cap W) \cap U_i(\mathfrak{f})$$

is globally semianalytic in $U_i(\mathfrak{f})$, where $\mathfrak{f} = \{q_1, \dots, q_s\}$ with $q_i \in B\langle Y \rangle$ generating the unit ideal in $B\langle Y \rangle$. So, for each i , there exists a finite number of basic subsets $B_{ij} \subset U_i(\mathfrak{f})$, such that

$$\theta(U \cap W) \cap U_i(\mathfrak{f}) = \bigcup_{j=1}^t B_{ij}. \tag{3}$$

Hence, for each i and j , there exist finitely many $g_{ijk}, h_{ijk} \in B\langle Y, q/q_i \rangle$, where $Y = (Y_1, \dots, Y_n)$, and symbols $\diamond_k \in \{\leq, <\}$, such that

$$B_{ij} = \{(x, y) \in M \times R^n \mid \forall k: |g_{ijk}(x, y)| \triangleleft_k |h_{ijk}(x, y)|\}.$$

Let $Q_i = q_i(cY) \in B\langle\langle Y \rangle\rangle$ and $\mathfrak{F} = \{Q_1, \dots, Q_s\}$, then $\mathcal{C}(\mathfrak{F})$ is a strongly rational covering of $M \times R^n$ in the R^n -direction.

Define

$$G_{ijk} = g_{ijk}(cY) \text{ and } H_{ijk} = h_{ijk}(cY),$$

so that we have that $G_{ijk}, H_{ijk} \in B\langle Q/Q_i \rangle\langle\langle Y \rangle\rangle$ and define S_{ij} to be the corresponding basic subset, that is to say,

$$S_{ij} \stackrel{\text{def}}{=} \{(x, y) \in M \times R^n \mid \forall k: |G_{ijk}(x, y)| \triangleleft_k |H_{ijk}(x, y)|\}.$$

So, S_{ij} is strongly basic subset in the R^n -direction and hence, if we call

$$S_i \stackrel{\text{def}}{=} \bigcup_j S_{ij},$$

then S_i is globally strongly semianalytic in the R^n -direction. So put

$$S = \bigcup_i S_i \subset M \times R^n,$$

then we claim that

$$S \cap U_i(\mathfrak{F}) = S_i. \tag{4}$$

Indeed, first of all note that, by construction, we have that $(x, y) \in U_i(\mathfrak{F})$, if and only if, $(x, cy) \in U_i(\mathfrak{f})$ and that therefore also $(x, y) \in S_{ij}$, if and only if, $(x, cy) \in B_{ij}$. Take now $(x, y) \in S \cap U_i(\mathfrak{F})$, so $(x, cy) \in U_i(\mathfrak{f})$. Moreover, there exists an i_0 , such that $(x, y) \in S_{i_0}$, and therefore there exists a j_0 , such that $(x, y) \in S_{i_0 j_0}$, hence $(x, cy) \in B_{i_0 j_0} \subset \theta(U \cap W)$ by (3). And thus,

$$(x, cy) \in \theta(U \cap W) \cap U_i(\mathfrak{f}) = \bigcup_j B_{ij},$$

hence there exists a j , such that $(x, cy) \in B_{ij}$ and therefore $(x, y) \in S_{ij} \subset S_i$, which proves the claim (4).

So, we have proved that S is strongly semianalytic in the R^n -direction and therefore $\pi(S)$ is strongly subanalytic. We now claim that

$$\pi(S) = \varphi(U \cap W).$$

The inclusion \subset is easy, so we only check \supset . Let $x \in \varphi(U \cap W)$, hence there exists a $z \in U \cap W$ with $\varphi(z) = x$. Let $y_i = \mu_i(z)$ and if we put $y = (y_1, \dots, y_n)$, then $\theta(z) = (x, y) \in \theta(U \cap W)$. Moreover $|y_i| \leq |c|$ by (1), hence there exist $y'_i \in R$ such that $y_i = cy'_i$. One therefore verifies immediately that $(x, y') \in S$ and therefore $x \in \pi(S)$. \square

REMARK. Note that we actually proved the following sharper result. If we define the following map

$$U \cap W \xrightarrow{\tau} S: z \mapsto (x, y/c),$$

where $\theta(z) = (x, y)$, then τ is a bijection between this two sets and moreover, by the commutative diagram (2), we get a commutative diagram

$$\begin{array}{ccc} U \cap W & \xrightarrow{\varphi} & M \\ \tau \downarrow & & \parallel \\ S & \xrightarrow{\pi} & M. \end{array} \tag{5}$$

2.5. THEOREM. *Let M be an affinoid variety and $S \subset M$. Then the following are equivalent*

- (1) S is strongly subanalytic in M ,
- (2) there exist a rigid analytic variety N , a proper map $N \xrightarrow{\rho} M$ of rigid analytic varieties and a semianalytic subset T of N , such that $S = \rho(T)$.

Proof. (1) \Rightarrow (2). This is (2.3).

(2) \Rightarrow (1). Suppose first that ρ is globally proper. Since the singleton $\{M\}$ is an admissible affinoid covering of M , we know that there exist finite admissible affinoid coverings $\bar{\mathcal{X}} = \{\bar{X}_i\}_i$ and $\mathcal{X} = \{X_i\}_i$ of N , such that, for each i ,

$$\bar{X}_i \Subset_M X_i.$$

Then $T \cap X_i$ is semianalytic in X_i , hence by (2.4), $\rho(T \cap \bar{X}_i)$ is strongly subanalytic in M and since $\bar{\mathcal{X}}$ is a covering of N , we have that

$$\rho(T) = \bigcup_i \rho(T \cap \bar{X}_i),$$

and hence is strongly subanalytic in M .

Suppose now that ρ is just proper, then there exists an admissible affinoid covering $\mathcal{Y} = \{Y_i\}_i$ of M , such that the restrictions $\rho^{-1}(Y_i) \xrightarrow{\rho} Y_i$ are globally proper. So, by what we already proved, we have that each $\rho(T) \cap Y_i$ is strongly subanalytic in Y_i , hence $\rho(T)$ is locally strongly subanalytic in M and hence by [Sch 1, Proposition (4.2)] strongly subanalytic in M . \square

2.6. THEOREM. Let $N \xrightarrow{\varphi} M$ be a proper map of rigid analytic varieties and $S \subset N$ strongly subanalytic in N . Then $\varphi(S)$ is strongly subanalytic in M .

Proof. One easily reduces to the case that M is affinoid. Arguing in the same way as in the proof of (2.5), we also may assume that φ is globally proper. So there exist finite admissible affinoid coverings $\bar{\mathcal{X}} = \{\bar{X}_i\}_i$ and $\mathcal{X} = \{X_i\}_i$ of N , such that, for each i ,

$$\bar{X}_i \Subset_M X_i.$$

Moreover, each $S \cap X_i$ is strongly subanalytic in X_i , hence by (2.3) there exist a rigid analytic variety M_i , a globally proper map

$$M_i \xrightarrow{\rho_i} X_i$$

and a semianalytic subset $T_i \subset M_i$, such that

$$\rho_i(T_i) = S \cap X_i. \tag{1}$$

Since ρ_i is globally proper, there exist two finite admissible affinoid coverings $\bar{\mathcal{Y}}_i = \{\bar{Y}_{ij}\}_j$ and $\mathcal{Y}_i = \{Y_{ij}\}_j$ of M_i , such that, for each j ,

$$\bar{Y}_{ij} \Subset_{X_i} Y_{ij}.$$

One easily shows that, for each i and j ,

$$\bar{Y}_{ij} \cap \rho_i^{-1}(\bar{X}_i) \Subset_M Y_{ij}.$$

Moreover, $T_i \cap Y_{ij}$ is semianalytic in Y_{ij} , so that if we set, for each i and j ,

$$S_{ij} \stackrel{\text{def}}{=} \varphi \rho_i(\bar{Y}_{ij} \cap \rho_i^{-1}(\bar{X}_i) \cap T_i)$$

then from (2.4), we know that S_{ij} is strongly subanalytic in M (note also that $\varphi \rho_i$ is globally proper). Since $\bar{\mathcal{Y}}_i$ is a covering of M_i , we have that

$$\begin{aligned} \bigcup_{i,j} \rho_i(\bar{Y}_{ij} \cap \rho_i^{-1}(\bar{X}_i) \cap T_i) &= \bigcup_i \rho_i(\rho_i^{-1}(\bar{X}_i) \cap T_i) \\ &= \bigcup_i (\bar{X}_i \cap \rho_i(T_i)) \\ &= S, \end{aligned}$$

by (1) and since $\bar{\mathcal{X}}$ is a covering of N . Therefore,

$$\varphi(S) = \bigcup_{ij} S_{ij}$$

is strongly subanalytic in M . □

3. Existence of bounds

3.1. DEFINITION. Let $N \xrightarrow{\varphi} M$ be a map of rigid analytic varieties and $S \subset N$ an arbitrary subset. Then, for $x \in M$, we call the *fiber of S in x (with respect to φ)*, the set

$$S_x \stackrel{\text{def}}{=} \{y \in S \mid \varphi(y) = x\},$$

in other words $S_x = S \cap \varphi^{-1}(x)$.

We will say that S has *bounded fibers (with respect to φ)*, if there exists a bound $A \in \mathbb{N}$, such that, for all $x \in M$ with finite fiber S_x , we have that $\text{card}(S_x) \leq A$. Another way to express this is by writing that, for all $x \in M$, we have that

$$\text{card}(S_x) \in \{1, \dots, A\} \cup \{\infty\}.$$

3.2. LEMMA. Let $g, h \in R[T]$ be polynomials in one variable T over R and $\diamond \in \{\leq, <\}$ a symbol. Let

$$W \stackrel{\text{def}}{=} \{t \in R \mid |g(t)| \diamond |h(t)|\}.$$

If W is finite, we have the following estimates

- (1) if g is non-zero, then $\text{card}(W) \leq \text{deg}(g)$,
- (2) if $g = 0$ and h is non-zero, then $\text{card}(W) \leq \text{deg}(h)$.

Proof. First of all note that for an arbitrary $f \in A\langle X \rangle$, where A is an affinoid algebra and $X = (X_1, \dots, X_n)$, and, for all $x, y \in \text{Sp } A$, we have that

$$|f(x) - f(y)| \leq |f| \cdot |x - y|,$$

see [BGR, 7.2.1. Proposition 1]. This implies that for a given $x \in \text{Sp } A$ with $f(x) \neq 0$ and for a δ with $0 < \delta < |f(x)|/|f|$, we have, for all $y \in \text{Sp } A$ with $|x - y| \leq \delta$, that

$$|f(x)| = |f(y)|. \tag{3}$$

Now, we may of course assume that W is non-empty. First of all, suppose that g is non-zero. There are two cases we will consider. Either for all $t \in W$, we

have that $g(t) = 0$ and hence $\text{card}(W) \leq \text{deg}(g)$. Or otherwise, let $t_0 \in W$ with $g(t_0) \neq 0$, hence also $h(t_0) \neq 0$ and we can find

$$0 < \delta < \min\{|g(t_0)|/|g|, |h(t_0)|/|h|\}.$$

Therefore, by (3), we have, for all $t \in R$ with $|t - t_0| \leq \delta$, that $|g(t)| = |g(t_0)|$ and $|h(t)| = |h(t_0)|$, which implies that $t \in W$, hence W is infinite.

Suppose now that $g = 0$, then either, for all $t \in W$, we have that $h(t) = 0$, hence $\text{card}(W) \leq \text{deg}(h)$ or there exists a $t_0 \in W$, with $h(t_0) \neq 0$. As before, we take

$$0 < \delta < |h(t_0)|/|h|$$

and then, for all $t \in R$ with $|t - t_0| \leq \delta$, we have that $|h(t)| = |h(t_0)|$, hence $t \in W$ and therefore W is infinite. □

3.3. THEOREM. *Let M be a quasi-compact rigid analytic variety and let $S \subset M \times R^m$. If S is strongly \mathbf{D} -semianalytic in the R^m -direction, then S has bounded fibers with respect to π , where $M \times R^m \xrightarrow{\pi} M$ is the projection map.*

Proof. Note that if $B_1, \dots, B_s \subset M \times R^m$ have bounded fibers, then also their union, because we just have to take as a bound the sum of all the bounds of the B_i . Hence, since M is quasi-compact and therefore has a finite admissible affinoid covering, we already may assume that M is affinoid. We now do induction on m , the case $m = 0$ being trivial. By what we have remarked above, we even may assume that S is a strongly \mathbf{D} -basic subset in the R^m -direction. From (4) of the proof of the basic lemma [Sch 1, (5.1)], we can find a strongly \mathbf{D} -basic subset $C \subset M \times R^m$ in the R^m -direction, which is algebraic in the Y_m -direction (that is, the describing functions are polynomial in Y_m), and a (Weierstrass) automorphism τ of the Y -variables, such that, for $(x, y) \in M \times R^m$,

$$(x, y) \in S \Leftrightarrow (x, \tau(y)) \in C.$$

Hence, for all $x \in M$, we have that $\text{card}(S_x) = \text{card}(C_x)$, so that we may replace S by C . From loc. cit. we know that $\theta(C)$ is globally strongly \mathbf{D} -semianalytic in the R^{m-1} -direction, where $M \times R^m \xrightarrow{\theta} M \times R^{m-1}$ is the projection map in the Y_m -direction. By induction therefore, $\theta(C)$ has bounded fibers with respect to $\tilde{\pi}$, where

$$M \times R^{m-1} \xrightarrow{\tilde{\pi}} M$$

is the projection map. So, there exists an $A_1 \in \mathbb{N}$, such that, for all $x \in M$,

$$\text{card}(\theta(C)_x) \in \{1, \dots, A_1\} \cup \{\infty\}. \tag{1}$$

For $(x, y') \in M \times R^{m-1}$, we consider the fiber $C_{(x, y')}$ of C with respect to θ . I claim that it is enough to prove now that C has bounded fibers with respect to θ , in other words, that there exists an $A_2 \in \mathbb{N}$, such that, for an arbitrary point $(x, y') \in M \times R^{m-1}$, we have that

$$\text{card}(C_{(x, y')}) \in \{1, \dots, A_2\} \cup \{\infty\}.$$

Indeed, then $A = A_1 A_2$ is a bound for C with respect to π .

Again, by our remark above, we may suppose that C is a strongly \mathbf{D} -basic subset in the R^m -direction, which is algebraic in the Y_m -direction, and then the existence of the bound A_2 follows immediately from the previous Lemma (3.2), just take $A_2 = \max\{\deg_{Y_m}(g)\}$, where the maximum runs over all non-zero describing functions g of C , which, by construction, are polynomials in Y_m . \square

3.4. COROLLARY. *Let M be a quasi-compact rigid analytic variety and let $S \subset M \times R^m$. If S is strongly \mathbf{D} -semianalytic in the R^m -direction in $M \times R^m$, then the set*

$$W \stackrel{\text{def}}{=} \{x \in M \mid S_x \text{ is finite}\}$$

is strongly subanalytic in M .

Proof. Easy. \square

3.5. THEOREM. *Let $N \xrightarrow{\varphi} M$ be a proper map of rigid analytic varieties and suppose that M is quasi-compact. If $S \subset N$ is strongly subanalytic in N , then S has bounded fibers with respect to φ .*

Proof. Since M is quasi-compact, we can find a finite admissible affinoid covering of M and so, by using [BGR, 9.6.2. Proposition 3], we can reduce to the case that M is affinoid. Then by using the definition of a proper map, one readily reduces to the following case.

SPECIAL CASE. *Let $N \xrightarrow{\varphi} M$ be a map of affinoid varieties and $U \Subset_M N$ a relative compact in N over M . If $W \subset U$ is strongly subanalytic in N , then W has bounded fibers with respect to φ .*

Let us now prove this statement. Making use in the right way of the remark after (2.3), we can reduce to the case that W is semianalytic in N .

In this case, we obtain from the remark after (2.4), a strongly semianalytic subset $S \subset M \times R^n$ in the R^n -direction and a bijection $W \xrightarrow{\tau} S$, such that the following diagram is commutative

$$\begin{array}{ccc} W & \xrightarrow{\varphi} & M \\ \tau \downarrow & & \parallel \\ S & \xrightarrow{\pi} & M, \end{array} \tag{1}$$

where $M \times R^n \xrightarrow{\pi} M$ is the projection map.

Now, having established this, we apply the foregoing theorem (3.3) to the set S , proving that S has bounded fibers with respect to the projection π . Using that τ is a bijection and the commutativity of diagram (1), it is an exercise to get that τ induces a bijection, for each $x \in M$, between the fibers S_x and W_x , proving our claim. \square

4. Uniformization of strongly subanalytic sets

4.0. In this section we require that $\text{char } K = 0$, since we do not have an embedded resolution of singularities in positive characteristic yet. The only place where we need this restriction is therefore in the following theorem.

4.1. THEOREM (Embedded Resolution in Rigid Analytic Geometry, local version). *Let $M = \text{Sp } A$ be an affinoid manifold and $f \in A$, with $f \neq 0$. Then there exist a finite admissible affinoid covering $\mathcal{X} = \{X_i\}_i$ of M , rigid analytic manifolds M_i and maps $M_i \xrightarrow{h_i} X_i$ of rigid analytic varieties, such that*

- (i) *each h_i is a composition of finitely many blowing-up maps with respect to closed analytic subsets (which are rigid analytic manifolds) of codimension ≥ 2 ,*
- (ii) *$h_i^{-1}(V(f) \cap X_i)$ has normal crossing in M_i ,*

where $V(f)$ denotes the closed subset of M defined by $f = 0$.

REMARK. We call a rigid analytic variety M a *rigid analytic manifold* if it is quasi-compact, connected and the local ring $\mathcal{O}_{M,x}$ at each point $x \in M$ is regular.

Proof. See [Sch 2, Chapter III, Theorem (4.1.4)]. The idea of the proof is as follows.

Let x be a point of M and \mathfrak{m} the corresponding maximal ideal. Apply Hironaka’s embedded resolution (see [Hi 1, p. 146, Corollary 3]) to $A_{\mathfrak{m}}$. After taking an ‘analytification’ of this scheme-theoretic resolution, we can take a small (affinoid) neighbourhood U of x , such that the theorem is true over U . We even can do this around each point, in such a way that the affinoid neighbourhoods constitute an admissible affinoid covering of M . Hence taking a finite subcovering finishes the proof. See also [Sch 4]. \square

4.2. LEMMA. *Let $M = \text{Sp } A$ be an affinoid manifold and $f, g \in A$. Then there exist a finite admissible affinoid covering $\mathcal{X} = \{X_i\}_i$ of M , rigid analytic manifolds M_i and maps*

$$M_i \xrightarrow{h_i} X_i,$$

which are compositions of finitely many blowing-up maps with respect to regular

centers of codimension greater than or equal to 2, and, for each j , a finite admissible affinoid covering $\mathcal{Y}_j = \{Y_{ij}\}_i$ of \mathbf{M}_i , such that we have, for each i and j , that either $\theta_{ij}(f)$ divides $\theta_{ij}(g)$ or $\theta_{ij}(g)$ divides $\theta_{ij}(f)$ in B_{ij} , where $\theta_{ij}: A_i \rightarrow B_{ij}$ is the morphism corresponding to $h_i|_{Y_j}$ and A_i (respectively, B_{ij}) are the affinoid algebras of X_i (respectively, Y_{ij}).

Proof. We may of course assume that f and g are non-zero and non-equal. Now apply (4.1) to $F = fg(f - g)$. Hence, we get an admissible affinoid covering $\mathcal{X} = \{X_i\}_i$ of M and maps

$$\mathbf{M}_i \xrightarrow{h_i} X_i$$

of the required type, such that $h_i^{-1}(V(F))$ has normal crossings. Let A_i denote the affinoid algebra of X_i . Fix i , let $U = \text{Sp } B \subset \mathbf{M}_i$ be an admissible affinoid of \mathbf{M}_i and take a point $x \in U$. Let \mathfrak{m}_x denote the maximal ideal of B corresponding to the point x . Since $h_i^{-1}(V(F))$ has normal crossings at x , we can find a regular system of parameters $\{\xi_1, \dots, \xi_n\}$ of $B_{\mathfrak{m}_x}$, where $n = \dim(M) = \dim(\mathbf{M}_i)$, units $u, v, w \in B$ and multi-indices $\alpha, \beta, \gamma \in \mathbb{N}^n$, such that in the local ring $B_{\mathfrak{m}_x}$ we have

$$\begin{aligned} \theta(f) &= u\xi^\alpha, \\ \theta(g) &= v\xi^\beta, \\ \theta(f - g) &= w\xi^\gamma, \end{aligned} \tag{1}$$

where $A_i \xrightarrow{\theta} B$ is the morphism corresponding to $h_i|_U$. When we give \mathbb{N}^n the following partial ordering given by

$$(\alpha_1, \dots, \alpha_n) \leq (\beta_1, \dots, \beta_n),$$

if and only if, for all i , we have that $\alpha_i \leq \beta_i$, then it is an exercise that, with the notations of above, either $\alpha \leq \beta$ or $\beta \leq \alpha$, since $B_{\mathfrak{m}_x}$ is a unique factorisation domain, see for instance [BM, Lemma 4.7]. Hence, we can find around each point x of U a Zariski open subset in which (1) holds with either $\alpha \leq \beta$ or $\beta \leq \alpha$. The thus obtained covering of U by Zariski opens is automatically admissible, see [BGR, 9.1.4. Corollary 7]. Covering each of these Zariski opens by an admissible affinoid covering and taking a finite refinement out of the union of all the admissible affinoids involved, we conclude that we can find a finite admissible affinoid covering of U , such that on each admissible open we have the relations (1), with $\alpha \leq \beta$ or vice versa. From this the lemma follows immediately. \square

4.3. DEFINITION. Let M be a rigid analytic variety, $U \subset M$ an admissible affinoid open of M and $Z \subset U$ a closed analytic subset in U . Let

$$U \xrightarrow{h} U$$

denote the blowing-up of U with center Z and π the composition of h and the open immersion $U \hookrightarrow M$, then we call the triple (π, Z, U) , or π , for short, a *local blowing-up* of M . In honour of Hironaka’s terminology from [Hi 2], we will introduce the notion of a *global voûte* over M , where M is a quasi-compact rigid analytic variety and we will denote this gadget by \mathcal{E}_M . The definition is given by recursion as follows.

An element $e \in \mathcal{E}_M$ consists of a certain finite family of maps

$$M_i \xrightarrow{h_i} M,$$

each of which is a composition of finitely many local blowing-up maps. At the first stage, let $\mathcal{X} = \{X_i\}_{i=1,\dots,r}$ be a finite admissible affinoid covering of M and $M_i \xrightarrow{h_i} X_i$ a blowing-up map, for each i . If π_i denotes the composition

$$M_i \xrightarrow{h_i} X_i \hookrightarrow M,$$

then the collection $e = \{\pi_1, \dots, \pi_r\}$ is an element of \mathcal{E}_M . Suppose now that we have already defined $e \in \mathcal{E}_M$, with $e = \{\pi_1, \dots, \pi_r\}$ and each π_j a composition of finitely many local blowing-up maps,

$$M_j \xrightarrow{\pi_j} M.$$

Let, for each j , $\mathcal{Y}_j = \{Y_{ij}\}_{i=1,\dots,s}$ be a finite admissible affinoid covering of M_j (it should be noticed that the blowing-up of a quasi-compact rigid analytic variety is again quasi-compact). Let, for each i and j ,

$$N_{ij} \xrightarrow{h_{ij}} Y_{ij}$$

be a blowing-up map and let π_{ij} denote the composition map

$$N_{ij} \xrightarrow{h_{ij}} Y_{ij} \hookrightarrow M_j.$$

Then the collection

$$\tilde{e} \stackrel{\text{def}}{=} \{\pi_j \circ \pi_{ij} \mid i = 1, \dots, s \text{ and } j = 1, \dots, r\}$$

belongs to \mathcal{E}_M .

Note that in particular we have, for each $e = \{\pi_1, \dots, \pi_r\}$, that

$$\bigcup_{j=1}^r \pi_j(M_j) = M,$$

where $M_j \xrightarrow{\pi_j} M$. This justifies our use of the term ‘global’.

We will mainly be interested in blowing-up maps with respect to a closed center of codimension ≥ 2 , since blowing-up along a closed subset of codimension 1 is an isomorphism. Moreover all the centers involved will be regular (that is, rigid analytic manifolds). We therefore agree to take these restrictions within our definition of global voûte.

4.4. THEOREM (Uniformization). *Let M be an affinoid manifold and $S \subset M$ strongly subanalytic in M . Then there exists an $e \in \mathcal{E}_M$, such that, for each $(M \xrightarrow{h} M) \in e$, we have that $h^{-1}(S)$ is globally semianalytic in M .*

Proof. By the Quantifier Elimination in [Sch 1, Theorem (5.2)], we know that S is globally strongly \mathbf{D} -semianalytic. Let $\kappa(S)$ denote the complexity of S , that is, the sum of the complexities of all describing \mathbf{D} -functions of S , as defined in the introduction. We will do induction on $\kappa(S)$. If $\kappa(S) = 0$, then S is globally semianalytic and the theorem is clear, so suppose $\kappa = \kappa(S) > 0$ and the theorem been proved for all globally strongly \mathbf{D} -semianalytic subsets T of an affinoid manifold with $\kappa(T) < \kappa$.

We prove the following assertion.

ASSERTION. *Let $S_i \subset N$ be globally strongly \mathbf{D} -semianalytic subsets of an affinoid manifold N , such that, for each i , $\kappa(S_i) < \kappa$. Then the theorem also holds for $\bigcup_{i=1}^r S_i$.*

Proof of the assertion. We prove this by induction on the number r of globally strongly \mathbf{D} -semianalytic subsets. If $r = 1$, this is just our induction hypothesis of the main proof. So assume $r > 1$, then we can apply the induction hypothesis of the main proof on S_r to get $e \in \mathcal{E}_M$, with $e = \{\pi_1, \dots, \pi_s\}$, where $M_i \xrightarrow{\pi_i} M$ are such that $\pi_i^{-1}(S_r)$ is globally semianalytic in M_i . Let

$$S' \stackrel{\text{def}}{=} \bigcup_{i=1}^{r-1} S_i,$$

hence $S'_i \stackrel{\text{def}}{=} \pi_i^{-1}(S')$ is globally strongly \mathbf{D} -semianalytic in M_i , for each i , and we can apply our induction hypothesis on r to this set. So, there exists an $e_i \in \mathcal{E}_{M_i}$, such that, for each $h \in e_i$, we have that $h^{-1}(S'_i)$ is globally semianalytic (in the ambient space). However, since the inverse image of a globally semianalytic subset remains globally semianalytic, also $h^{-1}(\pi_i^{-1}(S_r))$ is globally semianalytic. It should now be clear how to construct from the given e and e_i an element $e' \in \mathcal{E}_M$ with the required properties. \square

Now, continuing the proof of the theorem, we know that by the construction of strongly \mathbf{D} -functions, it is possible to find $f, g \in A$, $\pi \in \wp$ and a globally strongly \mathbf{D} -semianalytic subset $\tilde{S} \subset M \times R$, with $\kappa(\tilde{S}) = \kappa - 1$, such that, for $x \in M$, we have that

$$x \in S \Leftrightarrow (x, \pi \mathbf{D}(f(x), g(x))) \in \tilde{S}. \tag{1}$$

By (4.2), we can find a finite admissible affinoid covering $\mathcal{X} = \{X_i\}_i$ of M , rigid analytic manifolds \mathbf{M}_i and maps

$$\mathbf{M}_i \xrightarrow{h_i} X_i,$$

which are compositions of finitely many blowing-up maps with respect to centers of codimension greater than or equal to 2, and admissible affinoid coverings $\mathcal{Y}_i = \{Y_{ij}\}_j$ of \mathbf{M}_i , such that we have, for each i and j , that either $\theta_{ij}(f)$ divides $\theta_{ij}(g)$ or conversely, $\theta_{ij}(g)$ divides $\theta_{ij}(f)$, where

$$A_i \xrightarrow{\theta_{ij}} B_{ij}$$

are the morphisms corresponding to $h_i|_{Y_{ij}}$ and A_i (respectively, B_{ij}) are the affinoid algebras corresponding to X_i (respectively, Y_{ij}).

We split up in two cases. First assume that $\theta_{ij}(f)$ divides $\theta_{ij}(g)$. So, let $v_{ij} \in B_{ij}$ be such that

$$\theta_{ij}(g) = v_{ij} \theta_{ij}(f).$$

Hence, for $y \in Y_{ij}$ and $x = h_i(y)$, we have that

$$\mathbf{D}(f(x), g(x)) = \begin{cases} \frac{1}{v_{ij}(y)} & \text{if } |v_{ij}(y)| = 1 \text{ and } g(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Let

$$Y_{ij}^{(0)} \stackrel{\text{def}}{=} \{y \in Y_{ij} \mid |v_{ij}(y)| \geq \pi\} \quad \text{and}$$

$$Y_{ij}^{(1)} \stackrel{\text{def}}{=} \{y \in Y_{ij} \mid |v_{ij}(y)| \leq \pi\},$$

so that $\{Y_{ij}^{(0)}, Y_{ij}^{(1)}\}$ is an admissible affinoid covering of Y_{ij} . Define

$$\begin{aligned} S_{ij}^{(0)} \stackrel{\text{def}}{=} & \{y \in Y_{ij}^{(0)} \mid |v_{ij}(y)| \geq 1, \theta_{ij}(g)(y) \neq 0, (h_i(y), \pi/v_{ij}(y)) \in \tilde{S}\} \\ & \cup \{y \in Y_{ij}^{(0)} \mid |v_{ij}(y)| < 1, (h_i(y), 0) \in \tilde{S}\} \\ & \cup \{y \in Y_{ij}^{(0)} \mid \theta_{ij}(g)(y) = 0, (h_i(y), 0) \in \tilde{S}\}. \end{aligned}$$

Then $S_{ij}^{(0)}$ is globally strongly \mathbf{D} -semianalytic in $Y_{ij}^{(0)}$. Define also

$$S_{ij}^{(1)} \stackrel{\text{def}}{=} \{y \in Y_{ij}^{(1)} \mid (h_i(y), 0) \in \tilde{S}\},$$

so that $S_{ij}^{(1)}$ is globally strongly **D**-semianalytic in Y_{ij} .

For the second case, assume that $\theta_{ij}(g)$ divides $\theta_{ij}(f)$. Hence there exists $v_{ij} \in B_{ij}$, such that

$$\theta_{ij}(f) = v_{ij}\theta_{ij}(g).$$

Now, for $y \in Y_{ij}$ and $x = h_i(y)$, we have that

$$D(f(x), g(x)) = \begin{cases} v_{ij}(y) & \text{if } f(x) \neq 0, \\ 0 & \text{if } f(x) = 0. \end{cases} \tag{3}$$

We define in the same manner as above

$$S_{ij} \stackrel{\text{def}}{=} \{y \in Y_{ij} \mid \theta_{ij}(f)(y) \neq 0, (h_i(y), \pi v_{ij}(y)) \in \tilde{S}\} \\ \cup \{y \in Y_{ij} \mid \theta_{ij}(f)(y) = 0, (h_i(y), 0) \in \tilde{S}\},$$

so that S_{ij} is globally strongly **D**-semianalytic in Y_{ij} .

Hence, after replacing in the first case Y_{ij} by the pair $\{Y_{ij}^{(0)}, Y_{ij}^{(1)}\}$, we still end up with a finite admissible affinoid covering of M_i and after re-indexing, we find globally strongly **D**-semianalytic subsets $S_{ij} \subset Y_{ij}$, which are finite unions of globally strongly **D**-semianalytic subsets of complexity less than κ , with the property that

$$h_i^{-1}(S \cap X_i) \cap Y_{ij} = S_{ij}.$$

Indeed, this follows immediately from (1), (2) and (3). We can now finish the proof by applying the assertion to each S_{ij} and constructing from all the data an element of the global voûte with the required properties. □

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