JOACHIM HILGERT
KARL-HERMANN NEEB
WERNER PLANK

Symplectic convexity theorems and coadjoint orbits

Compositio Mathematica, tome 94, nº 2 (1994), p. 129-180
<http://www.numdam.org/item?id=CM_1994__94_2_129_0>

Compositio Mathematica **94:** 129–180, 1994.

Symplectic Convexity Theorems and Coadjoint Orbits

JOACHIM HILGERT*, **
Department of Mathematics,
University of Wisconsin,
Madison, WI 53706, U.S.A.

and

KARL-HERMANN NEEB and WERNER PLANK
Fachbereich Mathematik,
Arbeitsgruppe 5,
Technische Hochschule Darmstadt,
Schlossgartenstr. 7,
D-64289 Darmstadt, FRG

Received 3 June 1993; accepted in final form 3 September 1993

§ 0. Introduction

In 1923 Schur proved that the diagonal entries $a = (a_1, \ldots, a_n)$ of a Hermitean $n \times n$-matrix with eigenvalues $r = (r_1, \ldots, r_n)$ are contained in the convex hull of $\mathcal{S}_n.r$, where $\mathcal{S}_n$ is the symmetric group acting on $\mathbb{C}^n$ by permutation of coordinates. 31 years later Horn proved that each point of the convex hull can be obtained this way. In 1973 Kostant published a seminal paper in which he interpreted the Schur-Horn result as a property of adjoint orbits of the unitary group and generalized it to arbitrary compact Lie groups. More precisely, he proved that for an element X in a maximal abelian subspace $\mathfrak{t}$ in the Lie algebra $\mathfrak{k}$ of a compact Lie group K one has

$$\mathrm{pr}_{\mathfrak{t}}(\mathrm{Ad}\, K.X) = \mathrm{conv}\, \mathcal{W}.X,$$

where $\mathrm{pr}_{\mathfrak{t}}: \mathfrak{k} \to \mathfrak{t}$ is the orthogonal projection (w.r.t. the Killing form) and $\mathcal{W}$ is the Weyl group associated to the pair $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. In turn Atiyah and, independently, Guillemin and Sternberg, in 1982 gave an interpretation of Kostant's theorem as a special case of a theorem on the image of the momentum map of a Hamiltonian torus action. In that context one has a symplectic manifold (M, ω) and a smooth action $\sigma: G \times M \to M$ of a Lie group G on M which preserves the form ω. The

* Supported by a DFG Heisenberg-grant.
** *Present address*: Mathematisches Institut, TU Clausthal, 38678 Clausthal, FRG

space $C^\infty(M)$ carries a Lie algebra structure given by the Poisson bracket

$$\{f, h\} = \omega(\mathcal{X}_f, \mathcal{X}_h),$$

where the vector fields $\mathcal{X}_f$ and $\mathcal{X}_h$ correspond to df and dh under the isomorphism $TM \cong T^*M$ coming from ω. Moreover σ induces a natural homomorphism $\dot{\sigma}$ from the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathcal{V}(M)$ of vector fields on M. The action σ is called Hamiltonian if there exists a Lie algebra homomorphism $\lambda\colon \mathfrak{g} \to C^\infty(M)$ such that

$$\dot{\sigma}(X) = \mathcal{X}_{\lambda(X)} \qquad \forall X \in \mathfrak{t}.$$

The $\lambda(X)$ are called Hamiltonian functions. Given λ one defines the moment map $\Phi\colon M \to \mathfrak{g}^*$ via

$$\langle \Phi(m), X \rangle = \lambda(X)(m) \qquad \forall X \in \mathfrak{g}.$$

The Atiyah-Guillemin-Sternberg (AGS) theorem then reads: If M is compact and T is a torus, then

$$\Phi(M) = \operatorname{conv} \Phi(\operatorname{Fix}(M)),$$

where $\operatorname{Fix}(M)$ is the set of T-fixed points in M.

In Kostant's situation the symplectic manifold is the adjoint orbit which can be identified with a coadjoint orbit carrying a natural symplectic form. The group is the maximal torus T in K with Lie algebra $\mathfrak{t}$ and the action is the coadjoint action. Then it is not hard to see that the corresponding moment map is the natural map $\operatorname{pr}_{\mathfrak{t}^*}\colon \mathfrak{k}^* \to \mathfrak{t}^*$ restricted to the coadjoint orbit. Some standard Lie theoretic arguments show that for $M = \operatorname{Ad}^*(K).\alpha$ the set $\operatorname{Fix}(M)$ coincides with the Weyl group orbit of α.

Since complex flag manifolds can be viewed as certain compact coadjoint orbits, the AGS-theorem proves convexity properties of complex flag manifolds. On the other hand Kostant had proved analogous results for real flag manifolds. In order to give a symplectic interpretation for those, Duistermaat in 1983 proved a convexity theorem for fixed point sets of involutions τ on symplectic manifolds which satisfy $\tau^*\omega = -\omega$.

All the symplectic convexity theorems mentioned so far were proved applying some Morse theory to a generic component function of the moment map. At that point it was essential to assume that the symplectic manifold was compact. Nevertheless, using Kostant's theorem Paneitz in 1984 and Olafsson in 1990 were able to prove convexity theorems which can be interpreted as symplectic convexity theorems for certain non-compact coadjoint orbits (with involution). It turns out that it actually is enough to assume that the moment map Φ is proper, i.e., the inverse images of compact sets are compact. To prove that, one first proves a local convexity theorem using a suitable normal form for Hamiltonian torus actions.

Here by a local convexity theorem for a map $\Psi: X \to V$ from some space X to a vector space V we mean the existence of a collection of closed convex cones C_x in V with vertex $\Psi(x)$ and open neighborhoods U_x of x in X such that

(O) $\Psi: U_x \to C_x$ is an open map,

and

(LC) $\Psi^{-1}(\Psi(u)) \cap U_x$ is connected for all $u \in U_x$.

The next step is to establish a very general principle (we call it the Lokal-global-Prinzip) which shows that the local convexity theorem together with the properness always gives rise to a global convexity theorem. The basic idea for the proof of the Lokal-global-Prinzip (which we borrow from the paper [CDM88] by Condevaux, Dazord and Molino) is to factor the connected components of the fibers of Ψ to obtain a quotient space $\widetilde{X}$ which is Hausdorff and locally homeomorphic to closed convex cones thanks to (O) and (LC). Given a Euclidean metric on V one can use the local homeomorphisms to define a metric on $\widetilde{X}$ via the length of curves. Then the properness of Ψ guarantees the existence of shortest curves connecting two points in a very similar way one proves the Hopf-Rinow theorem in Riemannian geometry. When projected to V via Ψ, these curves give straight lines from which one easily deduces the convexity of $\Psi(X)$.

This line of argument gives strengthened versions of the AGS-theorem as well as the Duistermaat theorem. Moreover it has the advantage that in order to derive Duistermaats theorem one no longer needs to essentially redo the proof of the AGS theorem but only to establish the right local convexity theorem and then apply the Lokal-global-Prinzip. When applied to coadjoint orbits the strengthened convexity theorems, just as in Kostant's case, yield an "abstract" convexity statement. In order to give a "concrete" description of that convex image one again has to use Lie theoretic arguments. The situation becomes more complicated for non-compact coadjoint orbits because it is no longer clear that the convex image is spanned by extreme points. In fact, it turns out that the image of the moment map which is closed, convex and locally polyhedral is always a sum of the convex hull of its extreme points and a convex cone which may be interpreted as the cone of limit directions of the set. In terms of Lie theory the extreme points come up as a Weyl group orbit whereas the limit cone is given by certain roots. As special cases one finds Paneitz's and Olafsson's theorems.

So far we only considered torus actions. Examples show that one may have non-convex images of the moment map for Hamiltonian actions of non-abelian groups. On the other hand Kirwan showed in 1984 that for a compact Lie group K with maximal torus T the intersection of $\Phi(M) \subseteq \mathfrak{k}^*$ with a Weyl chamber $\mathfrak{t}^*_+ \subseteq \mathfrak{t}^* \subseteq \mathfrak{k}^*$ (via the Killing form) is always convex. We show here that $\Phi(M)$ is convex if and only if $\Phi(M) \cap \mathfrak{t}^*$ is convex, a fact that has been proven in the special case of projective unitary representations by Arnal and Ludwig in [AL92]. We also give a proof of a strengthened version of Kirwan's theorem for non-compact manifolds following ideas of [CDM88]. Here the crucial point is to construct a T-invariant symplectic submanifold M_F of M whose associated moment map $\Phi_T: M_F \to \mathfrak{t}^*$

has an image which is dense in $\Phi(M) \cap \mathfrak{t}_+^*$. Unfortunately (and contrary to an assertion in [CDM88]) the map Φ_T in general is *not* proper on M_F. But writing M_F as a union of sets on which it is and using the full generality of the Lokal-global-Prinzip one can still prove the convexity of $\Phi_T(M_F)$ and hence the strengthened Kirwan theorem.

Kostant's paper [Ko73] also contains convexity theorems for Iwasawa projections which recently have been given a symplectic interpretation in the context of Poisson Lie groups by Lu and Ratiu ([LR91]). Similarly, convexity theorems for non-linear projections appear in [Ne92] and [vdB86]. Whereas a symplectic interpretation for the results from [Ne92] is available and will be discussed in [HiNe93b], the non-linear convexity theorem due to van den Ban which also generalizes Konstant's non-linear convexity theorem still lacks such an interpretation.

The authors thank the referee for many helpful comments and for pointing out an error in a previous version of the manuscript.

The paper is organized as follows:

§ 1. Closed convex sets

§ 2. Local convexity theorems

§ 3. The "Lokal-global-Prinzip" for convexity theorems

§ 4. Hamiltonian torus actions

§ 5. Applications to coadjoint orbits

§ 6. Hamiltonian actions of compact Lie groups

In the first section we collect some results on closed convex sets in finite dimensional real vector spaces. These results are mostly of an elementary nature but nevertheless crucial for the rest of the paper. The main difficulty comes from the fact that we also have to consider non-compact convex sets which are not as simple as compact ones.

In the second section we describe a local normal form for a Hamiltonian torus action on a symplectic manifold and its moment mapping. This normal form then leads directly to a local convexity theorem. To pave the way to Duistermaat's result, we also obtain a normal form which takes into account the presence of an antisymplectic involution which anticommutes with the torus action. In this case we also get a corresponding local convexity theorem. This section consists of general symplectic geometry.

The third section is of a topological nature. It only depends on some lemmas in Section 1. Here it is proved how one obtains a global convexity theorem from a local convexity theorem under the assumption that the moment map is proper. This result is at the heart of the paper.

In Section 4 we formulate the explicit convexity theorems for Hamiltonian torus actions and some generalizations which we obtain by combining the "Lokal-global-Prinzip" with the local theorem from Section 2. We also formulate a non-compact version of Duistermaat's Convexity Theorem which we also obtain by the "Lokal-global-Prinzip".

Section 5 contains the applications of these results in the simplest case, namely for coadjoint orbits in the dual of a Lie algebra $\mathfrak{g}$. We call a coadjoint orbit *admissible* if it is closed and its convex hull contains no lines. It turns out that this is a class of coadjoint orbits where our results apply most naturally to the Hamiltonian action of the adjoint group associated to a compactly embedded Cartan algebra $\mathfrak{t}$. We obtain a complete description of the set of admissible coadjoint orbits for Lie algebras which contain a compactly embedded Cartan algebra and the Convexity Theorem for coadjoint orbits which applies to those includes the corresponding well known result for compact Lie algebras and the Convexity Theorem of Paneitz for Hermitean simple Lie algebras (even for Hermitean simple Lie algebras we obtain a stronger result).

Having the non-compact version of Duistermaat's Theorem at hand we also obtain convexity theorems for coadjoint orbits associated to symmetric spaces. Here we prove a result for general Lie algebras containing a compactly embedded Cartan algebra and show how it specializes to Kostant's Linear Convexity Theorem for the Cartan decomposition of a semisimple Lie algebra and Olafsson's Convexity Theorem for irreducible regular symmetric spaces.

The last section contains the aforementioned strengthened version of Kirwan's Convexity Theorem for Hamiltonian actions of compact groups. These last two sections rest on the results of Sections 1 and 4 but they are independent of each other.

§ 1. Closed convex sets

Let V be a finite dimensional real vector space and $C \subseteq V$ a closed convex set. For $x \in C$ we define the *subtangent wedge* $L_x(C) := \overline{\mathbf{R}^+(C - x)}$ (cf. [HHL89, Ch. I]). Note that this set deserves to be called a *wedge* since it is a closed convex cone in V. It follows immediately from the definition that

$$C \subseteq x + L_x(C) \qquad \forall x \in C.$$

We say that a closed convex set C is *locally polyhedral* if for every $x \in C$ there exists a neighborhood U such that $U \cap C = U \cap (x + L_x(C))$ and $L_x(C)$ is polyhedral. We remark that the condition that $L_x(C)$ is polyhedral is superfluous (cf. [Ne93f]). It follows from the condition that C is locally polyhedral. The edge $L_x(C) \cap -L_x(C)$ will be denoted by $H_x(C)$. For a closed convex set C we write $C^\star := \{\omega \in V^*\colon \omega(C) \subseteq \mathbf{R}^+\}$ for the dual cone.

PROPOSITION 1.1.

(i) $\lim(C) := \{v \in V\colon C + v \subseteq C\}$ *is a closed convex cone in* V.

(ii) $H(C) := \{v \in V\colon C + v = C\}$ *is a vector subspace of* V.

(iii) Let $x \in C$ *and suppose that* D *is a convex cone in* V *with* $x + D \subseteq C$. *Then* $D \subseteq \lim(C)$.

Proof. (i) It is clear that $\lim(C)$ is a closed convex set which is a subsemigroup of V. This means that $\lim(C)$ is a closed convex cone.
(ii) is an immediate consequence of (i).
(iii) Let $y \in C$ and $d \in D$. Then

$$t\left(x + \frac{1}{t}d\right) + (1-t)y \in C$$

for all $t > 0$. As t goes to 0, we find that $y + d \in C$, hence that $C + D \subseteq C$, i.e., $D \subseteq \lim(C)$. □

DEFINITION 1.2. Let C be a closed convex set. A subset $F \subseteq C$ is said to be *extremal* if $tx + (1-t)y \in F$, $t \in]0,1[$ and $x, y \in C$ implies that $x, y \in F$.

A *face* is a convex extremal set, an *extreme point* is a point $e \in C$ such that $\{e\}$ is a face, and an *extremal ray* is a face which is a half-line. We note that, if C is locally polyhedral, a point $x \in C$ is an *extreme point* if and only if the wedge $L_x(C)$ is pointed. We write $\mathrm{Ext}(C)$ for the set of extreme points of C and $\mathrm{Rext}(C)$ for the set of extremal rays. Note that $\mathrm{Ext}(F) = \mathrm{Ext}(C) \cap F$ holds for every face F of C.

An *exposed face* F is a set $F = \{c \in C : \omega(c) = \min \omega(C)\}$, where $\omega \in V^*$ is a linear functional which attains its minimum on C. It is clear that every exposed face is a face and that every face of a face is a face. We also note that $H(C) = H(F)$ and $\lim(F) \subseteq \lim(C)$ for every face F of C.

We write $\mathrm{algint}\, C$ for the interior of C relative to the affine subspace generated by C and call this set the *algebraic interior of* C. □

LEMMA 1.3. *Let C be a closed convex set. Then the following assertions hold:*

(i) If a face F intersects the algebraic interior, then $F = C$.

(ii) Every face is closed.

(iii) Every face of minimal dimension is an affine subspace $f + H(C)$ for every $f \in F$.

Proof. (i) Let $f \in F \cap \mathrm{algint}(C)$ and $c \in C$. Then there exists $t > 1$ such that $c' := c + t(f - c) \in C$. Then $f = \frac{1}{t}c' + \frac{t-1}{t}c$ yields that $c \in F$.
(ii) Let E denote the affine subspace generated by F. Then $C \cap E$ is a closed convex set containing F and F is a face of $C \cap E$. Since F generates E, it intersects the algebraic interior of $E \cap C$ which is a dense subset. Thus $F = E \cap C$ by (i) and therefore F is closed.
(iii) Let F be a face of minimal dimension in C. Then F is a closed convex set without any faces of lower dimension. It follows in particular that F has no exposed faces. In view of the Separation Theorem of Hahn-Banach, F is open in the affine subspace it generates (otherwise we find non-trivial support hyperplanes), and therefore F is an affine space.

On the other hand $f + H(C) \subseteq F$ holds for every $f \in F$ because $f = \frac{1}{2}(f+v)+\frac{1}{2}(f-v)$ for $v \in H(C)$. Hence $F = f+H(C)$ follows from $v \in H(C)$ for every $v \in V$ with $f + \mathbf{R}v \subseteq F$. □

The following lemma is also contained in [Le80, p.41]. It is an immediate consequence of Lemma 1.3.

LEMMA 1.4. *The following statements are equivalent.*
(1) C contains an affine subspace of positive dimension.
(2) $H(C) \neq \{0\}$.
(3) The set of extremal points $\mathrm{Ext}(C)$ *of C is empty.* □

We define the set $\widetilde{\mathrm{Ext}}(C)$ as the union of all faces of minimal dimension (cf. Lemma 1.3(iii)). This is the set of the "most extremal points" of C. Note that $\widetilde{\mathrm{Ext}}(C) = \mathrm{Ext}(C)$ if and only if $H(C) = \{0\}$ (cf. Lemma 1.4).

PROPOSITION 1.5. *Let C be a closed convex set. Then*

$$C = \mathrm{conv}\,\widetilde{\mathrm{Ext}}(C) + \lim(C).$$

Proof. The inclusion "$\supseteq$" follows immediately from Proposition 1.1.

For the converse inclusion we first note that, since both sides are invariant under translations with elements in the subspace $H(C)$, we may factor this subspace and therefore assume that $H(C) = \{0\}$. Then, according to Klee's theorem ([Le80, Satz 4.3])

$$C = \mathrm{conv}\,\big(\mathrm{Ext}(C) \cup \mathrm{Rext}(C)\big).$$

To see that this implies the assertion, note that for every ray $F = x + \mathbf{R}^+ v \in \mathrm{Rext}(C)$ we have that $x \in \mathrm{Ext}(F) \subseteq \mathrm{Ext}(C)$ and $v \in \lim(F) \subseteq \lim(C)$. □

LEMMA 1.6. *Let C be a closed convex subset of V. Then*

$$C = \bigcap_{x\in C} (x + L_x(C)) \quad \textit{and} \quad \lim(C) = \bigcap_{x\in C} L_x(C).$$

Proof. For the first equality the inclusion "$\subseteq$" is clear. Suppose that $y \notin C$. We identify V with $\mathbf{R}^n$ which we endow with the Euclidean scalar product. Let $x \in C$ be a best approximation for y in C and $\omega(z) := \langle x-y, z\rangle$. Then $\omega(z-x) \geq 0$ for $z \in C$. Hence $\omega \in L_x(C)^\star$ and therefore $y \notin x + L_x(C)$.

For the second equality the inclusion "$\subseteq$" is also clear since $C \subseteq x + L_x(C)$ yields $\lim(C) \subseteq \lim\,(x + L_x(C)) = L_x(C)$ for all $x \in C$. Now let $v \in \bigcap_{x\in C} L_x(C)$. Then, in view of the first equality,

$$C + v = \bigcap_{x\in C} (x + L_x(C) + v) \subseteq \bigcap_{x\in C} (x + L_x(C)) = C.$$

Hence $v \in \lim(C)$. □

LEMMA 1.7. *Let $E \subseteq C$ and $x \in \operatorname{conv} E$. Then* $\bigcap_{e \in E} (e + L_e(C)) \subseteq x + L_x(C)$.

Proof. First we assume that $E = \{x_0, x_1\}$. Let $y \in C$ and $v \in L_y(C)$. We note that $t(C - y) \subseteq C - y$ for $t \leq 1$ since $tC + (1-t)y \subseteq C$. Hence $L_y(C) = \overline{\bigcup_{n \in \mathbf{N}} n(C - y)}$. We identify V with a real Hilbert space. Then we find in each of the closed convex sets $n(C - y)$ a best approximation $v_n = n(c_n - y)$ for v. Since the family $n(C - y)$ is increasing and v is contained in the closure of its union, it follows that $v_n \to v$.

Now let $v \in (x_0 + L_{x_0}(C)) \cap (x_1 + L_{x_1}(C))$. Then there exist sequences $c_n^0, c_n^1 \in C$ such that

$$v - x_0 = \lim_{n \to \infty} n(c_n^0 - x_0), \quad v - x_1 = \lim_{n \to \infty} n(c_n^1 - x_1).$$

Suppose that $x = tx_0 + (1-t)x_1$ with $t \in [0, 1]$. Then

$$\begin{aligned} v - x &= t(v - x_0) + (1-t)(v - x_1) = \lim_{n \to \infty} tn(c_n^0 - x_0) + (1-t)n(c_n^1 - x_1 \\ &= \lim_{n \to \infty} n(tc_n^0 + (1-t)c_n^1 - tx_0 - (1-t)x_1) \\ &= \lim_{n \to \infty} n(tc_n^0 + (1-t)c_n^1 - x) \in L_x(C). \end{aligned}$$

Thus $(x_0 + L_{x_0}(C)) \cap (x_1 + L_{x_1}(C)) \subseteq x + L_x(C)$.

For a general subset $E \subseteq C$, it clearly suffices to assume that E is finite since x is already contained in the convex hull of a finite subset. Now we can use the two-element case to argue by induction. □

PROPOSITION 1.8. *Let C be a closed convex set. Then*

$$C = \operatorname{conv} \widetilde{\operatorname{Ext}}(C) + \lim C = \bigcap_{x \in \widetilde{\operatorname{Ext}}(C)} (x + L_x(C))$$

and $\lim(C) = \bigcap_{x \in \widetilde{\operatorname{Ext}}(C)} L_x(C)$.

Proof. The first equality is Proposition 1.5. To prove the second equality, we write C' for the right hand side and note that $C \subseteq C'$ holds trivially because of Lemma 1.6. Now C' is a closed convex set and

$$\lim(C) \subseteq \bigcap_{x \in \widetilde{\operatorname{Ext}}(C)} L_x(C) \subseteq \lim(C').$$

So we only have to show that $C = C'$. If $y = x + v$ with $x \in C$ and $v \in \lim(C)$, then $\mathbf{R}v \subseteq L_y(C)$ and therefore

$$y + L_y(C) = x + v + L_{x+v}(C) = x + L_{x+v}(C) \supseteq x + L_{x+v}(C + v) = x + L_x(C).$$

Hence the first equality and Lemma 1.6 tell us that

$$C = \bigcap_{x \in \operatorname{conv} \widetilde{\operatorname{Ext}}(C)} (x + L_x(C)),$$

so that the assertion follows from Lemma 1.7. □

Proposition 1.8 says that in order to calculate C it is enough to know $\lim(C)$ and $\widetilde{\operatorname{Ext}}(C)$. Moreover the second part tells us that in order to calculate $\lim(C)$, we only have to know the subtangent cones $L_x(C)$ in the most extremal points. It follows in particular that the locally polyhedral set that is associated to any local convexity data via the "Lokal-global-Prinzip" (cf. § 3), we only have to know the local convexity data for the most extremal points. This will be useful in the applications to coadjoint orbits in § 5.

In the remainder of this section we will deduce some results on convex sets which will be useful in § 5 to describe the set of those coadjoint orbits to which the Convexity Theorem from § 4 applies.

Let C be a closed convex set in V. We define

$$B(C) := \{\omega \in V^*: \inf \omega(C) > -\infty\}.$$

LEMMA 1.9. *The set $B(C)$ is a convex cone which satisfies*

$$\operatorname{algint} \lim(C)^\star \subseteq B(C) \subseteq \lim(C)^\star. \tag{1.1}$$

Proof. That $B(C)$ is convex and invariant under multiplication with non-negative scalars is trivial. Moreover since $C + \lim(C) = C$, it is clear that every element in $B(C)$ must be non-negative on $\lim(C)$, i.e., $B(C) \subseteq \lim(C)^\star$.

In view of Lemma 1.6, we have that $\lim(C)^\star = \overline{\sum_{x \in C} L_x(C)^\star}$. Hence it suffices to show that $L_x(C)^\star \subseteq B(C)$ holds for all $x \in C$ to see that $B(C)$ contains $\operatorname{algint}\big(\lim(C)\big)^\star$. Here we use that a dense convex subset of a convex set contains the algebraic interior.

If $x \in C$, then $C \subseteq x + L_x(C)$, so that $\omega \in L_x(C)^\star$ implies that $\omega(C) \subseteq [\omega(x), \infty[$. Now $L_x(C)^\star \subseteq B(C)$ and the lemma is proved. □

REMARK 1.10. To see that the cone $B(C)$ need not be closed, let $C := \{(x, y) \in \mathbb{R}^2 : y \geq e^x\}$. Then $B(C) = \{(0,0)\} \cup \{(\lambda, \mu) : \mu > 0, \lambda \leq 0\}$. □

PROPOSITION 1.11. *Let C be a closed convex set. Then the following are equivalent:*

(1) $\lim(C)$ *is pointed.*

(2) C contains no lines.

(3) $\operatorname{int} B(C) \neq \emptyset$.

(4) $C \times \{1\} \subseteq V \times \mathbb{R}$ lies in a pointed closed convex cone.

Proof. (1) $\Leftrightarrow$ (2): This is part of Lemma 1.4.
(1) $\Leftrightarrow$ (3): This follows immediately from (1.1) and the fact that $\lim(C)$ is pointed if and only if its dual cone is generating.
(3) $\Rightarrow$ (4): To see that (4) holds, we have to show that the set

$$D := \{(\omega, t) \in (V \times \mathbf{R})^*: (\omega, t)(C \times \{1\}) = \omega(C) + t \subseteq \mathbf{R}^+\}$$

has interior points. The set D is a closed convex cone in $(V \times \mathbf{R})^*$. Let $\pi: (V \times \mathbf{R})^* \to V^*$ denote the restriction mapping. Then it is clear that $\pi(D) = B(C)$. Let $d \in \operatorname{algint}(D)$ and K be a closed convex neighborhood of d in the affine subspace generated by D. Then $\pi(d)$ is contained in the algebraic interior of $B(C)$ and $\pi(K)$ is a neighborhood of $\pi(d)$ in $B(C)$, hence also in V^* since $B(C)$ is generating by assumption (3).

Suppose that D is not generating. Then $\dim D = \dim V$ and D generates a hyperplane in $(V \times \mathbf{R})^\star$ which is mapped isomorphically onto $V^\star$. This contradicts the obvious fact that $\{0\} \times \mathbf{R}^+ \subseteq D$. Hence D is generating and (4) follows.
(4) $\Rightarrow$ (2): If $C \times \{1\}$ lies in a pointed closed convex cone, then it contains no lines. □

In the following we call a mapping $\varphi: X \to Y$ between Hausdorff spaces *proper* if it is closed and the fibers $\varphi^{-1}(y)$ are compact (cf. [Bou71, Ch. 1, § 10, Th. 1]).

PROPOSITION 1.12. *Let $\omega \in B(C)$. Then $H(C) \subseteq \ker\omega$ so that ω factors to a linear functional $\overline{\omega}$ on $V/H(C)$. The following are equivalent:*
(1) $\omega \in \operatorname{algint} B(C)$.
(2) $\omega \in \operatorname{algint}\lim(C)^\star$.
(3) $\overline{\omega}$ is proper on $C/H(C)$ and $\omega \in B(C)$.

Proof. Since ω is bounded on the vector space $H(C)$, it vanishes on $H(C)$, i.e., $H(C) \subseteq \ker\omega$.
(1) $\Leftrightarrow$ (2): This is a consequence of Lemma 1.9.
(1) $\Rightarrow$ (3): First we set $C' := C/H(C) \subseteq V' := V/H(C)$. Let $\pi: V'^* \times \mathbf{R} \to V'^*$ denote the restriction mapping and

$$D := \{(\overline{\omega}, t) \in (V' \times \mathbf{R})^*: (\overline{\omega}, t)(C' \times \{1\}) = \overline{\omega}(C') + t \subseteq \mathbf{R}^+\}$$

as in the proof of Proposition 1.11. Then $\pi(D) = B(C')$ and therefore $\pi(\operatorname{int} D) = \pi(\operatorname{int} B(C'))$ since π is an open mapping and $\pi(\operatorname{int} D)$ is a dense convex subset of $B(C')$. Hence there exists $t \in \mathbf{R}$ with $(\overline{\omega}, t) \in \operatorname{int} D$. Then $(\overline{\omega}, t) \in \operatorname{int} \widetilde{C}'$ with $\widetilde{C}' := \operatorname{cone}(C' \times \{1\})^\star$. Therefore this functional is proper on the pointed cone $\widetilde{C}'$, hence proper on the closed subset $C' \times \{1\}$. It follows that $\overline{\omega}$ is proper on C'.
(3) $\Rightarrow$ (2): If $\overline{\omega}$ is proper on C and $x \in C'$, then it is also proper on the closed subset $x + \lim(C')$. This means that $\overline{\omega}$ is a proper function on $\lim(C')$, hence $\overline{\omega} \in \operatorname{int}\lim(C')^\star$. This property translates into $\omega \in \operatorname{algint}\lim(C)^\star$. □

COROLLARY 1.13. *For a functional $\omega \in V^*$ the following are equivalent:*

(1) $\omega \in \operatorname{int} B(C)$.

(2) $\omega \in \operatorname{int} \lim(C)^\star$.

(3) ω *is proper on* C *and in* $B(C)$.

Proof. In view of Proposition 1.12, we only have to note that (1)-(3) are false if $B(C)$ has no interior points. On the other hand if $B(C)$ has non-empty interior, this is simply Proposition 1.12. □

The following simple lemma will be useful to check properness of mappings throughout the whole paper. Its proof is an easy exercise in topology.

LEMMA 1.14. *Let* X, Y, *and* Z *be locally compact spaces and* $p: X \to Y$, $q: Y \to Z$ *continuous mappings such that* $q \circ p$ *is proper. Then* p *is proper, the restriction of* q *to* $p(X)$ *is proper, and* $p(X)$ *is closed.* □

LEMMA 1.15. *Let* $C \subseteq V$ *be a closed convex set and* $\varphi: V \to V'$ *a linear mapping. Then the following are equivalent:*

(1) $\varphi|_C: C \to \varphi(C)$ *is a proper mapping.*

(2) $\ker \varphi \cap H(C) = \{0\}$ *and there exists* $\omega \in \operatorname{algint} B(C)$ *such that* $\ker \varphi \subseteq \ker \omega$.

Proof. (1) $\Rightarrow$ (2): If the restriction of φ to C is proper, then $\varphi(C)$ is closed. Take $\omega' \in \operatorname{algint} B(\varphi(C))$ and set $\omega := \omega' \circ \varphi$. Then $\ker \varphi \subseteq \ker \omega$.

We claim that $\omega \in \operatorname{algint} B(C)$. That $\omega \in B(C)$ is clear. We write C as $C = H(C) \times C_1$, where $V = H(C) \times V_1$ is a corresponding direct decomposition. Since φ is proper on $H(C)$, we have that $\ker \varphi \cap H(C) = \{0\}$, and therefore there exists a direct decomposition $V' = \varphi(H(C)) \times V_1'$ with $\varphi(C_1) \subseteq V_1'$. Since C_1 is closed in C, the restriction of φ to C_1 is proper, hence $\varphi(C_1)$ is closed and, moreover, the linear functional ω' is proper on C_1 by Corollary 1.13. We conclude that $\omega|_{C_1}$ is proper, i.e., that $\omega \in \operatorname{algint} B(C)$ (Proposition 1.12).
(2) $\Rightarrow$ (1): Again we write C as $C = H(C) \times C_1$. If $\omega \in \operatorname{algint} B(C)$, then ω is proper on C_1 (Proposition 1.12) and since ω factors to $\omega' \in V^*$ with $\omega = \omega' \circ \varphi$, we conclude with Lemma 1.14 that φ is proper on C_1.

In view of the fact that $\varphi(C) = \varphi(H(C)) \times \varphi(C_1)$, where φ is injective on $H(C)$, the mapping φ is equivalent to $\varphi|_{H(C)} \times \varphi|_{C_1}$. Both mappings are proper, so the same holds for φ. □

PROPOSITION 1.16. *Let* C *be a closed convex set in* V *and* $\varphi: V \to V'$ *linear such that* φ *is proper on* C. *Then the following assertions hold:*

(i) If C *is locally polyhedral, then* $\varphi(C)$ *is locally polyhedral.*

(ii) $\operatorname{Ext}(\varphi(C)) = \varphi(\operatorname{Ext}(C))$.

Proof. (i) First we note that $\varphi(C)$ is closed because φ is proper on C. Let $x' \in C' := \varphi(C)$ and $x \in C$ with $x' = \varphi(x)$. Further let U' be a compact polyhedron in V' which contains x' in its interior. Then the set $U := \varphi^{-1}(U') \cap C$ is compact and convex. Moreover, the set $\varphi^{-1}(U')$ is locally polyhedral so that the same follows for U because C is locally polyhedral. Thus U is a polyhedron. Now

$U' = \varphi(U)$ is the image of a polyhedron, hence a polyhedron. This shows that C' is locally polyhedral.
(ii) Let $x' \in \text{Ext}\,(\varphi(C))$. Then the compactness of $\varphi^{-1}(x') \cap C$ shows that this set contains an extreme point x. Then $\varphi(x) = x'$ and since $\varphi^{-1}(x') \cap C$ is a face of C, the point x is even extremal in C. □

The following lemma will be needed for the Local Convexity Theorem in § 2.

LEMMA 1.17. *Let $\alpha: V \to V'$ a linear mapping of finite dimensional vector spaces and $C \subseteq V$ a polyhedral cone. Then $\alpha: C \to \alpha(C)$ is an open mapping.*

Proof. We prove this lemma by induction over $\dim C$. It is clear if $\dim C = 0$. If C is not pointed and $H(C) := C \cap -C$, then $V = H(C) \times V_1$. We set $V'' := V'/\alpha(H(C))$. Then α induces a mapping $\alpha': V_1 \to V''$ which maps $C \cap V_1$ onto $(\alpha(C) + V'')/V''$. If this mapping is open, then α is also open because $V' \cong V'' \oplus \alpha(H(C))$ and the restriction of α to $H(C)$ is trivially an open mapping onto its image. Hence we may w.l.o.g. assume that C is pointed.

Let $C = \sum_{e \in E} \mathbf{R}^+ e$, where E is a finite set. If $U \subseteq C$ is a neighborhood of 0, then there exists for each $e \in E$ a $\lambda_e > 0$ such that $\lambda_e e \in U$. Then $\alpha(U)$ contains $\sum_{e \in E} [0, \lambda_e] \alpha(e)$ which is a neighborhood of 0 in $\alpha(C)$.

To see that α is also open in other points of C, let $x \in C \setminus \{0\}$. Then there exists a neighborhood U of x with $U \cap C \subseteq x + L_x(C)$. Since the cone $L_x(C)$ is polyhedral with non-trivial edge, the mapping

$$\alpha: L_x(C) = C - \mathbf{R}^+ x \to L_{\alpha(x)}(\alpha(C)) = \alpha(C) - \mathbf{R}^+ \alpha(x)$$

is open. Hence α is open in x because $\alpha(C)$ is also a polyhedral cone. This proves the assertion in the case where C is pointed, hence the proof is complete. □

We note that it is false in general that a linear map which maps a closed convex cone W_1 onto another closed convex cone W_2 is an open mapping.

EXAMPLE 1.18. The following counterexample is due to J. D. Lawson. We set

$$C := \{(x, y, z) \in \mathbf{R}^3 : z \geq \sqrt{x^2 + y^2}\}.$$

Let $\gamma: [0,1] \to \mathbf{R}^6, \quad t \mapsto (\cos 2\pi t, t\cos 2\pi t, \sin 2\pi t, t \sin 2\pi t, 1, t)$. Then $\gamma([0,1])$ is a compact set whose convex hull K does not contain the origin. Let

$$\pi: \mathbf{R}^6 \to \mathbf{R}^3, \quad (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1 - x_2, x_3 - x_4, x_5 - x_6).$$

Then π maps the convex cone $\widetilde{C} := \mathbf{R}^+ K$ onto $\mathbf{R}^+ \pi(\widetilde{K}) = C$. But $\pi([0,1]\widetilde{K})$ is not a neighborhood of 0 in C and $[0,1]\widetilde{K}$ is a neighborhood of 0 in $\widetilde{C}$. □

§ 2. Local convexity theorems

LEMMA 2.1. *Let (M, ω) be a symplectic manifold, $\sigma : T \times M \to M$ a Hamiltonian action of a torus T on M which is given by the Lie algebra homomorphism*

$\lambda\colon \mathfrak{t} \to C^\infty(M)$, and $m_0 \in M$. Then there exists a T-invariant neighborhood $\mathcal{U} \subset M$ of $T.m_0$, a subtorus T_1 of T and a symplectic vector space V with the following properties:

(i) $T = T_0 \times T_1$, where $T_0 := (T_{m_0})_0$ is the connected component of the stabilizer T_{m_0}.

(ii) There exists a symplectic covering of a T-invariant open subset $\mathcal{U}'$ of $T_1 \times \mathfrak{t}_1^* \times V \cong T^*(T_1) \times V$ onto $\mathcal{U}$ under which σ gets transformed into the action

$$(T_0 \times T_1) \times ((T_1 \times \mathfrak{t}_1^*) \times V) \to ((T_1 \times \mathfrak{t}_1^*) \times V)$$
$$((t_0, t_1), (t_1', \beta, v)) \mapsto (t_1 t_1', \beta, \pi(t_0)v),$$

where

$\pi\colon T_0 \to \mathrm{Sp}(V)$ is a linear symplectic representation.

(iii) If, in addition, τ is an antisymplectic involution on M, i.e., $\tau^*\omega = -\omega$, $\tau(m_0) = m_0$, and all Hamiltonian functions $\lambda(X)$, $X \in \mathfrak{t}$ are τ-invariant, then the covering in (ii) *can be chosen τ-equivariant, where τ acts on $T^*(T_1) \times V$ by*

$$\tau.(t_1', \beta, v) = (t_1'^{-1}, \beta, \tau_V.v),$$

and τ_V is an antisymplectic involution on V.

Proof. (i) Let $m_0 \in M$ and $T_0 := (T_{m_0})_0$ denote the connected component of the stabilizer T_{m_0}. The group T_0 is a subtorus of T, hence there exists another subtorus T_1 such that $T \cong T_0 \times T_1$. Note that $T_{m_0} = T_0\Gamma$, where $\Gamma := (T_1 \cap T_{m_0})$ is a finite group, and the orbit $T.m_0$ is isomorphic to T_1/Γ.

(ii), (iii) To treat (ii) and (iii) simultaneously, let $K := T$ if we are in the situation, where there is no τ. Otherwise let τ act on T by $\tau.t := t^{-1}$ and form the semidirect product $K := T \rtimes \{\mathbf{1}, \tau\}$. To see that this leads to an action of K on M, let $X \in \mathfrak{t}$. Then $\lambda(X) \circ \tau = \tau^*\lambda(X) = \lambda(X)$ by assumption. Hence $\tau^* d\lambda(X) = d\lambda(X)$. If $\dot\sigma(X)$ denotes the corresponding Hamiltonian vector field with $i_{\dot\sigma(X)}\omega = -d\lambda(X)$, then $\tau^*\omega = -\omega$ yields that $\tau_*\dot\sigma(X) = -\dot\sigma(X)$. We conclude that $\tau(t.m) = t^{-1}.\tau(m)$ for $t \in T$. Hence we obtain an action of K on M such that $k^*\omega \in \{\pm\omega\}$ for all $k \in K$.

To obtain the normal form for our Hamiltonian torus action, we will use Theorem 41.2 in [GS84]. We set $V := T_{m_0}(T.m_0)^\perp / T_{m_0}(T.m_0)$. This is a symplectic vector space which carries a linear action π of the group K_{m_0}, where the involution $\tau_V := \pi(\tau)$ is antisymplectic.

Let $\widetilde{M}_1 := T^*(T_1) \times V \cong T_1 \times \mathfrak{t}_1 \times V$ with the action of T given by

$$((t_0, t_1), (t_1', \beta, v)) \mapsto (t_1 t_1', \beta, \pi(t_0)v)$$

and endowed with the natural symplectic product structure $\widetilde{\omega}_1$. The finite group Γ acts on $\widetilde{M}_1$ by

$$\gamma.(t_1', \beta, v) := (t_1'\gamma, \beta, \pi(\gamma).v),$$

the orbit mapping $\rho: \widetilde{M_1} \to M_1 := M_1/\Gamma$ is a finite covering, and there exists a symplectic structure ω_1 on M_1 such that $\rho^*\omega_1 = \widetilde{\omega}_1$. Moreover, since K permutes the Γ-orbits in $\widetilde{M_1}$, the action of K on $\widetilde{M_1}$ induces an action of K on M_1 given by $k.\rho(m) := \rho(k.m)$ for $m \in \widetilde{M_1}$. This action is Hamiltonian for T and $\tau^*.\omega_1 = -\omega_1$.

The stabilizer in T of the point $[(1,0,0)]$ in M_1 coincides with T_{m_0}, and the linear action of T_{m_0} on the tangent space $T_{m_0}(M_1)$ is equivalent to the action on $T_{m_0}(M)$. Hence Theorem 41.2 in [GS84] provides a symplectic isomorphism of a T-invariant neighborhood of $\rho(T_1 \times \{(0,0)\})$ with a T-invariant neighborhood of $T.m_0$ in M which is T-equivariant. This proves (ii).

To see that (iii) can also be obtained along the same way, we first note that the proof of the equivariant Darboux-Weinstein Theorem ([GS84, § 22, p.155]) carries over almost word by word to an action of a compact group K acting on a symplectic manifold (M, ω) such that there exists a continuous homomorphism $\varepsilon: K \to \{1, -1\}$ with $k^*\omega = \varepsilon(g)$ for all $k \in K$.

Thus we also obtain the appropriate generalization of the Isotropic Embedding Theorem ([GS84, Th. 39.1]) to this more general setting. Finally the argumentation in [GS84, pp.324–326] shows that Theorem 41.2 in [GS84] remains true for our group $K = T \rtimes \{\mathbf{1}, \tau\}$. Therefore the local isomorphism from above can also be chosen τ-equivariant. This proves (iii). $\square$

Normal forms as in Lemma 2.1 are a standard tool in symplectic geometry. The second part of Lemma 2.1 concerning the normal form with involution is for the case $T = \{\mathbf{1}\}$ due to Meyer ([Me81]) and in the case where m_0 is a fixed point of T due to Duistermaat ([Dui83]). It follows immediately from the normal form that the manifold Q of τ-fixed points in M is a submanifold which is also due to Meyer ([Me81]).

The application of the equivariant version of the Darboux-Weinstein lemma for compact groups which do necessarily preserve the symplectic structure is in fact a combination of the cited result in [GS84] and of Lemma 2.3 in [Dui83].

LEMMA 2.2. *Let σ be the Hamiltonian action as in the lemma above and $\pi: T_0 \to \mathrm{Sp}(V)$ the corresponding symplectic action of the torus T_0 on the symplectic vector space (V, ω_V). Then there exists a complex structure I on V such that $\langle v, w\rangle := \omega_V(Iv, w)$ defines a positive definite scalar product on V. Then $V = \bigoplus_{\alpha \in \mathcal{P}_V} V_\alpha$, where $V_\alpha := \{v \in V: (\forall Y \in \mathfrak{t}_0) Y.v = \alpha(Y) Iv\}$ and $\mathcal{P}_V := \{\alpha \in \mathfrak{t}_0^*: V_\alpha \neq \{0\}\}$. The moment map for σ is given by*

$$\begin{aligned} \Phi : T^*(T_1) \times V &\to \mathfrak{t}_1^* \times \mathfrak{t}_0^* \cong \mathfrak{t}^* \\ ((t_1, \beta), v) &\mapsto \Phi(1,0,0) + \left(\beta, \frac{1}{2} \sum_{\alpha \in \mathcal{P}_V} \|v_\alpha\|^2 \alpha\right). \end{aligned}$$

If, in addition, τ_V is an antisymplectic involution on V which leaves all the Hamiltonian functions $\lambda(X)(v) := \frac{1}{2}\omega(X.v, v)$, $X \in \mathfrak{t}_0$ invariant, then τ_V is antilinear, i.e., it anticommutes with I, and it leaves all subspaces V_α invariant.

Proof. We decompose the action into two parts

$$\sigma_1 : T_1 \times T^*(T_1) \to T^*(T_1), \quad (t_1, (t_1', \beta)) \mapsto (t_1 t_1', \beta)$$

and

$$\sigma_2 : T_0 \times V \to V, \quad (t_0, v) \mapsto \pi(t_0)v.$$

It follows from [GS84, Th. 41.2] that the moment map for the action of T is, up to a constant, the sum of the moment maps corresponding to the Hamiltonian actions σ_1 and σ_2. The first one is the action by left multiplication on the cotangent bundle, hence the moment map is simply given by the projection $T^*(T_1) \to \mathfrak{t}_1^* \subseteq \mathfrak{t}^*$, where we used the splitting $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ to identify $\mathfrak{t}_1^*$ with a subspace of $\mathfrak{t}^*$.

In the second case the corresponding moment map is given by

$$\Phi_V(v)(Y) = \frac{1}{2}\omega_V(Y.v, v),$$

where ω_V is the symplectic form on V. We can choose a complex structure $I: V \to V$ such that the form $(v, w) \mapsto \omega_V(Iv, w)$ defines a positive definite scalar product on V which is invariant under T_0 and τ_V. This follows by a slight generalization concerning the presence of τ_V of the argument given in ([Ne93a, Lemma II.31]). Since $\tau_V^*\omega_V = -\omega_V$, the mapping τ_V anticommutes with the complex structure I.

There is a finite set $\mathcal{P}_V$ of linear functionals $\alpha \in \mathfrak{t}_0^*$ and a decomposition $V = \bigoplus_\alpha V_\alpha$ which is orthogonal with respect to $\omega_V(I\cdot, \cdot)$ of V, where

$$V_\alpha = \{v \in V : (\forall Y \in \mathfrak{t}_0)\, Y.v = \alpha(Y)Iv\}.$$

Since τ_V anticommutes with I and the action of $\mathfrak{t}_0$, it leaves the subspaces V_α invariant. Then

$$\frac{1}{2}\omega_V(Y.v_\alpha, v_\alpha) = \frac{1}{2}\alpha(Y)\omega_V(Iv_\alpha, v_\alpha) = \frac{1}{2}\alpha(Y)\|v_\alpha\|^2$$

for $v_\alpha \in V_\alpha$ gives us the corresponding moment map for the second action as $v \mapsto \frac{1}{2}\sum_{\alpha\in\mathcal{P}_V} \|v_\alpha\|^2\alpha$. Composing these two moment maps we get the lemma. □

THEOREM 2.3. (The local convexity theorem for Hamiltonian torus actions) *Let $\sigma: T \times M \to M$ be a Hamiltonian action of a torus T on a symplectic manifold M and $m_0 \in M$. Then there exist an arbitrarily small open neighborhood $\mathcal{U}$ of m_0 and a polyhedral cone $C_{m_0} \subset \mathfrak{t}^*$ with vertex $\Phi(m_0)$ such that the following is true:*

(i) $\Phi(\mathcal{U})$ is an open neighborhood of $\Phi(m_0)$ in C_{m_0}.

(ii) $\Phi : \mathcal{U} \to C_{m_0}$ *is an open map*

(iii) $\Phi^{-1}(\Phi(u)) \cap \mathcal{U}$ *is connected for all* $u \in \mathcal{U}$.

(iv) If $\mathfrak{t}_0$ *is the Lie algebra of the stabilizer* T_0 *of* m_0, *then* $C_{m_0} = \Phi(m_0) + \mathfrak{t}_0^\perp + \operatorname{cone} \mathcal{P}_V$, *where* $\mathcal{P}_V$ *is the set of all* $\mathfrak{t}$*-weights on the vector space* $V := T_{m_0}(M)^\perp / T_{m_0}(M)$ *as described in* Lemma 2.2.

If, in addition, τ *is an antisymplectic involution on* M *which leaves all Hamiltonian functions associated to the action of* T *invariant, then the assertions* (i) *and* (ii) *of the proposition remain true for the manifold* $Q := \{m \in M : \tau(m) = m\}$ *of fixed points of* τ *and the same cones* C_m, $m \in Q$.

Proof. We take the local coordinate representation of the moment map in a neighborhood $\mathcal{U} = T_1 \times B_{\mathfrak{t}_1^*} \times B_V$, where $B_{\mathfrak{t}_1^*}$ and B_V are small neighborhoods of 0 in $\mathfrak{t}_1^*$ and V respectively. We define the polyhedral cone $C_{m_0} := \Phi(m_0) + \mathfrak{t}_1^* + \operatorname{cone} \mathcal{P}_V$, where $\operatorname{cone} \mathcal{P}_V$ denotes the cone generated by the finite set $\mathcal{P}_V = \{\alpha_1, \ldots, \alpha_n\}$. Note that this definition is compatible with (iv). We decompose Φ in two maps $\varphi_1 : (t_1, \beta, v) \mapsto (\beta, (\|v_\alpha\|^2))$ and

$$\varphi_2 : (\beta, a_1, \ldots, a_n) \mapsto \left(\beta, \frac{1}{2} \sum a_j \alpha_j\right)$$

such that $\Phi = \varphi_2 \circ \varphi_1 + \Phi(m_0)$. The map $\varphi_1 : \mathcal{U} \to B_{\mathfrak{t}_1^*} \times (\mathbf{R}^+)^n$ satisfies the assertions, so that we only have to check this for the linear map

$$\widetilde{\varphi}_2 : (\mathbf{R}^+)^n \to \operatorname{cone} \mathcal{P}_V, (a_1, \ldots, a_n) \mapsto \frac{1}{2} \sum a_j \alpha_j.$$

This is a linear mapping between polyhedral cones, and therefore the openness of $\widetilde{\varphi}_2$ follows from Lemma 1.17. That inverse images of points are connected follows from the fact that they are intersections of affine subspaces with the cone $(\mathbf{R}^+)^n$, hence convex.

Now let us assume that we have, in addition, an antisymplectic involution with the above stated properties. We have already noted in Lemma 2.2 that the subspaces V_α are invariant under τ and that τ anticommutes with the complex structure. This means that τ defines a complex conjugation on each space V_α with respect to the subspace $V_\alpha^+ := \{v \in V_\alpha : \tau(v) = v\}$. We choose the neighborhood B_V in V such that it is a sum of circular τ-invariant neighborhoods B_{V_α} of 0 in V_α. Then it follows from $\tau.(t_1, \beta, v) = (t_1^{-1}, \beta, \tau_V.v)$ that $\Phi(\mathcal{U}) = \Phi(\mathcal{U} \cap Q)$ holds, hence that (i) is satisfied for $\mathcal{U}_Q := \mathcal{U} \cap Q$. It also follows from the special choice of $\mathcal{U}$ that $\varphi_1 |_{\mathcal{U}_Q}$ is open, hence that (i) and (ii) remain true for the restriction $\Phi |_{\mathcal{U}_Q}$. □

In general (iii) is false for the submanifold Q on M. This comes from the fact that the vector space V_α^+ is a real form of the complex vector space V_α. Hence the spheres in V_α^+ are disconnected if V_α is a one-dimensional complex vector space.

§ 3. The "Lokal-global-Prinzip" for convexity theorems

In this section we prove a general theorem that allows to pass from a local convexity theorem to a global one if the convex sets involved are of polyhedral type. The material of this section is to a large extent a reworking of [CDM88].

DEFINITION 3.1. Let X be a connected Hausdorff topological space and V a finite dimensional vector space. A continuous map $\Psi: X \to V$ is called *locally fiber connected* if for each $x \in X$ there exist arbitrarily small neighborhoods U of x such that

(LC) $\Psi^{-1}(\Psi(u)) \cap U$ is connected for all $u \in U$.

If $\Psi: X \to V$ is locally fiber connected we define an equivalence relation $\sim$ on X by saying $x \sim y$ if $\Psi(x) = \Psi(y)$ and x and y belong to the same connected component of $\Psi^{-1}(\Psi(x))$. The topological quotient space $\widetilde{X} := X/\sim$ is called the Ψ-quotient of X. The quotient map will be denoted by $\pi: X \to \widetilde{X}$ and the map induced on $\widetilde{X}$ by Ψ by $\widetilde{\Psi}: \widetilde{X} \to V$. For $x \in X$ we write $E_x := \pi^{-1}(x) = \{y \in X: y \sim x\}$ for the equivalence class of x. □

PROPOSITION 3.2. *Let $\Psi: X \to V$ be a locally fiber connected map. Suppose that Ψ is a proper mapping, i.e., Ψ is closed and the sets $\Psi^{-1}(v)$, $v \in V$ are compact. Then the following assertions hold:*

(i) The fibers $\Psi^{-1}(v)$ are locally connected.

(ii) $\widetilde{\Psi}^{-1}(v)$ is finite for any $v \in V$.

(iii) For every $x \in X$ there exists an open neighborhood U_x satisfying (LC) *such that in addition U_x is relatively compact and $\overline{U}_x$ intersects only one component of $\Psi^{-1}(\Psi(x))$.*

Proof. (i) This follows from (LC).

(ii) Let $v \in V$. Since the fiber $\Psi^{-1}(v)$ is compact by assumption and locally connected, it has at most finitely many connected components. This means that $\widetilde{\Psi}^{-1}(v)$ is finite.

(iii) Let K be a compact neighborhood of $\Psi(x)$ in V. Then $\Psi^{-1}(K)$ is a compact set since Ψ is proper ([tD91, p.373]). According to (ii), the fiber $\Psi^{-1}(\Psi(x))$ consists of finitely many compact sets. Hence the fact that we find U_x arbitrarily small satisfying (LC) implies (iii).

DEFINITION 3.3. Let $\Psi: X \to V$ be a locally fiber connected map. A map $x \mapsto C_x$ which associates to each point in $x \in X$ a closed convex cone C_x with vertex $\Psi(x)$ in V is called *local convexity data* if for each $x \in X$ there exists an arbitrarily small open neighborhood U_x of x such that

(O) $\Psi: U_x \to C_x$ is an open map.

(LC) $\Psi^{-1}(\Psi(u)) \cap U_x$ is connected for all $u \in U_x$. □

REMARK 3.4. (i) A convex cone is uniquely determined by its intersection with a neighborhood of its vertex. Therefore the C_x are uniquely determined once the U_x are fixed. In fact, more is true. If $U_x' \subseteq U_x$ is an open neighborhood of x, then $\Psi(U_x')$ still generates the same cone because of (O). Thus we actually have a map $x \to C_x$ which does not depend on the *choice* of the neighborhoods U_x.
(ii) The concept of local convexity data is *not* the most general one might face in this context, since it does for instance not model a spherical ball locally. □

LEMMA 3.5. *Let $\Psi: X \to V$ be a locally fiber connected map with local convexity data $(C_x)_{x \in X}$. Suppose that Ψ is proper. Then the following assertions hold:*

(i) C_x only depends on the equivalence class E_x of x, so we have a well defined mapping $\pi(x) \mapsto C_{\pi(x)}$.
(ii) For every $x \in X$ there exists an open relatively compact neighborhood $\mathcal{U}_{\pi(x)}$ of E_x such that $\mathcal{U}_{\pi(x)}$ satisfies (LC).
(iii) $\widetilde{X}$ is a Hausdorff space.
(iv) If $\mathcal{U}_{\pi(x)}$ is chosen as in (ii), *then $\widetilde{\mathcal{U}}_{\pi(x)} = \pi\left(\mathcal{U}_{\pi(x)}\right)$ is a neighborhood of $\pi(x)$ in $\widetilde{X}$ such that $\widetilde{\Psi}: \widetilde{\mathcal{U}}_{\pi(x)} \to C_{\pi(x)}$ is a homeomorphism onto its open image.*
(v) Suppose that $\widetilde{\Psi}^{-1}(v) = \{\pi(x_1), \ldots, \pi(x_m)\}$. Then there exist pairwise disjoint neighborhoods $\widetilde{\mathcal{U}}_l$ of $\widetilde{x}_l := \pi(x_l)$ and convex open neighborhoods B_l of $\widetilde{\Psi}(\widetilde{x}_l)$ such that $B_l \cap C_{\widetilde{x}_l} = \Psi(\widetilde{\mathcal{U}}_l)$.

Proof. Let $(U_x)_{x \in X}$ be a family of open sets satisfying (O) and (LC) which in addition are locally compact and $\overline{U}_x$ intersects only one component of $\Psi^{-1}(\Psi(x))$ (Proposition 3.2(iii)).
(i) Fix a $v \in \Psi(X)$. Then, for $x \in \Psi^{-1}(v)$ and $y \in U_x \cap \Psi^{-1}(v)$ we claim that $C_y = C_x$. That $C_y \subseteq C_x$ follows from (ii) since a neighborhood of y in C_y is contained in C_x. On the other hand the fact that $\Psi \mid_{U_x}$ is an open mapping yields that there exist arbitrary small neighborhoods of y which are mapped onto neighborhoods of x in C_x. Hence $C_x \subseteq C_y$.

Therefore the map $x \to C_x$ is locally constant on $\Psi^{-1}(v)$ and hence constant on the connected components of $\Psi^{-1}(v)$.
(ii) The U_x form an open covering of the compact set $\Psi^{-1}(\Psi(x))$. Thus we can find a finite subcovering. The compactness of E_x shows that E_x is covered by finitely many of the U_y, $y \in E_x$ say $U_1, \ldots, U_k$, where $U_j := U_{y_j}$. The set

$$\bigcap_{E_x \cap U_i \cap U_j \neq \emptyset} \Psi(U_i \cap U_j)$$

is an open neighborhood of v in $C_{\pi(x)}$ since $\pi(U_i \cap U_j)$ is an open neighborhood of $\pi(x)$ if $E_x \cap U_i \cap U_j \neq \emptyset$ and $\widetilde{\Psi}: \pi(U_j) \to C_{\pi(x)}$ is an open mapping. Similarly we may now choose a convex open neighborhood $B_{\pi(x)}$ of $\pi(x) \in V$ such that

$$\Omega_{\pi(x)} := B_{\pi(x)} \cap C_{\pi(x)} \subseteq \left(\bigcap_{E_x \cap U_i \cap U_j \neq \emptyset} \Psi(U_i \cap U_j) \right) \cap \left(\bigcap_{j=1}^{k} \Psi(U_j) \right).$$

We use $\Omega_{\pi(x)}$ to reduce the U_j in size via $U_j' := U_j \cap \Psi^{-1}(\Omega_{\pi(x)})$ and define $\mathcal{U}_{\pi(x)} := \bigcup_{j=1}^{k} U_j'$. Note that, according to the relative compactness of the sets U_x, this set is relatively compact.

To prove (ii), we have to show that for every $u \in \mathcal{U}_{\pi(x)}$ the set $\Psi^{-1}(\Psi(u)) \cap \mathcal{U}_{\pi(x)}$ is connected.

First we note that for $u \in U_j'$ we have

$$\Psi^{-1}(\Psi(u)) \cap U_j' = \Psi^{-1}(\Psi(u)) \cap U_j.$$

It follows in particular that these sets are connected because the U_j satisfy hypothesis (LC).

To prove the assertion, we also note that for each $u \in \mathcal{U}_{\pi(x)}$ and each pair i, j with $E_x \cap U_i \cap U_j \neq \emptyset$ we find a $u_{ij} \in U_i \cap U_j$ with $\Psi(u) = \Psi(u_{ij})$. This shows that $u_{ij} \in U_i' \cap U_j'$ and hence

$$\left(U_i \cap \Psi^{-1}(\Psi(u))\right) \cap \left(U_j \cap \Psi^{-1}(\Psi(u))\right) \neq \emptyset.$$

The connectedness of E_x shows that for any pair i, j we may find a sequence $i = i_1, \ldots, i_m = j$ such that $E_x \cap U_{i_r} \cap U_{i_{r+1}} \neq \emptyset$ for $r \in \{1, \ldots, m-1\}$ and hence, by the argument just made, also

$$\left(U_{i_r} \cap \Psi^{-1}(\Psi(u))\right) \cap \left(U_{i_{r+1}} \cap \Psi^{-1}(\Psi(u))\right) \neq \emptyset \quad \forall r \in \{1, \ldots, m-1\}.$$

Now (ii) follows since

$$\mathcal{U}_{\pi(x)} \cap \Psi^{-1}(\Psi(u)) = \bigcup_{j=1}^{k} \left(U_j' \cap \Psi^{-1}(\Psi(u))\right) = \bigcup_{j=1}^{k} \left(U_j \cap \Psi^{-1}(\Psi(u))\right)$$

and each set $U_j \cap \Psi^{-1}(\Psi(u))$ is connected.

(iii) In view of [tD91, p.376], we have to show that $\sim$ is a closed equivalence relation on X.

Let $(x_i, y_i)_{i \in I}$ be a net with $x_i \sim y_i$ converging to (x, y) in $X \times X$. We have to show that $x \sim y$. We choose $\mathcal{U}_{\pi(x)}$ as in (ii). Then the construction of the U_j shows that $\mathcal{U}_{\pi(x)}$ is relatively compact and $\overline{\mathcal{U}}_{\pi(x)}$ intersects $\Psi^{-1}(\Psi(x))$ in E_x (cf. Proposition 3.2(iii)).

We may w.l.o.g. assume that $x_i \in \mathcal{U}_{\pi(x)}$ holds for all $i \in I$. If there exists an $i_0 \in I$ such that $y_i \in \mathcal{U}_{\pi(x)}$ whenever $i \geq i_0$, then $y = \lim y_i \in \overline{\mathcal{U}_{\pi(x)}}$ and $\Psi(y) = \Psi(x)$, so that

$$y \in \Psi^{-1}(\Psi(x)) \cap \overline{\mathcal{U}_{\pi(x)}} = E_x$$

which yields that $y \sim x$.

Otherwise there exists a subnet $(x_{i_j}, y_{i_j})_{j\in J}$ with $y_{i_j} \notin \overline{\mathcal{U}_{\pi(x)}}$. Then we may w.l.o.g. assume that $y_i \notin \overline{\mathcal{U}_{\pi(x)}}$ for all $i \in I$. Since E_{x_i} is connected and it contains x_i and y_i, we find $z_i \in E_{x_i} \cap \partial\mathcal{U}_{\pi(x)}$. Passing to a subnet, we may assume that z_i converges to an element $z \in \partial\mathcal{U}_{\pi(x)}$. Then it follows as above that $z \in E_x \cap \partial\mathcal{U}_{\pi(x)}$ contradicting the fact that E_x lies in the open set $\mathcal{U}_{\pi(x)}$.

(iv) We note first that the map is continuous. But its restrictions to any of the $\pi(U_j')$ are also open maps according to hypothesis (O), whence the map is also open. Thus in order to prove the claim, it only remains to show that the map is injective. But we know that $\mathcal{U}_{\pi(x)} \cap \Psi^{-1}(\Psi(u))$ is connected for all $u \in \mathcal{U}_{\pi(x)}$. So if $\Psi(u_1) = \Psi(u_2)$ for $u_1, u_2 \in \mathcal{U}_{\pi(x)}$, then we see that $u_1 \sim u_2$, whence $\pi(u_1) = \pi(u_2)$.

(v) We repeat the construction from the proof of (ii) for all the $x_1, \ldots, x_m$ and set $\widetilde{\mathcal{U}}_l := \pi(\mathcal{U}_{\pi(x_l)})$. Choosing the neighborhoods of the E_{x_l} at the beginning of the construction small enough we may, according to (iii), assume that the $\widetilde{\mathcal{U}}_l$ are pairwise disjoint. Now the assertion follows from the convexity of the set $\Omega_{\pi(x)}$. □

DEFINITION 3.6. Let $\Psi: X \to V$ be a locally fiber connected map. We call a continuous map $\gamma: [0,1] \to \widetilde{X}$ a *regular curve* connecting $\widetilde{x}_0$ and $\widetilde{x}_1$ if $\gamma(i) = \widetilde{x}_i$ for $i = 0, 1$ and $\widetilde{\Psi} \circ \gamma$ is piecewise differentiable. □

REMARK 3.7. Let $\Psi: X \to V$ and $(C_x)_{x\in X}$ be as in Lemma 3.5. Since $\widetilde{X}$ is connected and $\widetilde{\Psi}$ locally is a homeomorphism onto a connected open subset of a convex cone, any two points in $\widetilde{X}$ can be connected by a regular curve. We define $d(\widetilde{x}, \widetilde{y})$ to be the infimum of the lengths $l(\widetilde{\Psi} \circ \gamma)$ of all the curves $\widetilde{\Psi} \circ \gamma$ with γ a regular curve connecting $\widetilde{x}$ and $\widetilde{y}$. Here the length of a curve $[0,1] \to V$ is calculated with respect to an arbitrary but fixed Euclidean metric d_V on V. Obviously d is symmetric and satisfies the triangle inequality. Moreover it is clear that $d_V(\widetilde{\Psi}(\widetilde{x}), \widetilde{\Psi}(\widetilde{y})) \leq d(\widetilde{x}, \widetilde{y})$. □

PROPOSITION 3.8. *Let $\Psi: X \to V$ be a locally fiber connected map with local convexity data $(C_x)_{x\in X}$ and d_V a Euclidean metric on V. Suppose that $\Psi^{-1}(\Psi(x))$ is compact for all $x \in X$. Then*

(i) *$d: \widetilde{X} \times \widetilde{X} \to \mathbf{R}$ is a metric.*

(ii) *The metric d defines the topology of $\widetilde{X}$.*

Proof. (i) In view of Remark 3.7, it only remains to show that $d(\widetilde{x}, \widetilde{y}) = 0$ implies $\widetilde{x} = \widetilde{y}$. Thus we assume that $d(\widetilde{x}, \widetilde{y}) = 0$. Then $v := \widetilde{\Psi}(\widetilde{x}) = \widetilde{\Psi}(\widetilde{y})$ and since $\widetilde{X}$ is Hausdorff we find finitely many disjoint open sets $\widetilde{\mathcal{U}}_l$ in $\widetilde{X}$ covering the finite set $\widetilde{\Psi}^{-1}(v)$ in such a way that each $\widetilde{\mathcal{U}}_l$ contains only one element $\widetilde{x}_l$ of $\widetilde{\Psi}^{-1}(v)$. Moreover, using Lemma 3.5(iii), we may assume that each $\widetilde{\Psi}(\widetilde{\mathcal{U}}_l)$ contains the intersection of $C_{\widetilde{x}_l}$ with an ε-ball around $\widetilde{x}_l$. Thus any regular curve γ in $\widetilde{X}$ which starts at $\widetilde{x} = \widetilde{x}_{l_o}$ and leaves $\widetilde{\mathcal{U}}_{l_o}$ satisfies $l(\widetilde{\Psi} \circ \gamma) \geqslant \varepsilon$. In other words, any

regular curve γ connecting $\widetilde{x}$ and $\widetilde{y}$ which satisfies $l(\widetilde{\Psi} \circ \gamma) < \varepsilon$ has an image completely contained in $\widetilde{\mathcal{U}}_{l_o}$. This shows that $\widetilde{x} = \widetilde{y}$.
(ii) Fix an $x \in X$ and recall from Lemma 3.5 the homeomorphism

$$\widetilde{\Psi}: \widetilde{\mathcal{U}}_{\pi(x)} \to \Omega_{\pi(x)} \subseteq C_{\pi(x)} \tag{$*$}$$

Note that the convex set $\Omega_{\pi(x)}$ is a metric space with respect to the metric d_V. We show that $(*)$ is an isometry with respect to the metrics d and d_V. If now $\widetilde{y}_1, \widetilde{y}_2 \in \widetilde{\mathcal{U}}_{\pi(x)}$ and L is the straight line connecting $\widetilde{\Psi}(\widetilde{y}_1)$ and $\widetilde{\Psi}(\widetilde{y}_2)$, then the inverse image of this line in $\widetilde{\mathcal{U}}_{\pi(x)}$ is a regular curve γ with

$$l(\widetilde{\Psi} \circ \gamma) = d_V(\widetilde{\Psi}(\widetilde{y}_1), \widetilde{\Psi}(\widetilde{y}_2)).$$

Since we have already remarked that $\widetilde{\Psi}$ is a contraction with respect to d and d_V, this proves our claim about $(*)$. But since d_V defines the topology of $\Omega_{\pi(x)}$ this proves also (ii). □

LEMMA 3.9. *Let $\Psi: X \to V$ be a locally fiber connected map with local convexity data $(C_x)_{x\in X}$ and d_V a Euclidean metric on V. Suppose that Ψ is proper. Then for any $\widetilde{x} \in \widetilde{X}$ and any $r \in \mathbf{R}^+$ the ball $B_r(\widetilde{x}) := \{\widetilde{y} \in \widetilde{X}: d(\widetilde{x}, \widetilde{y}) \leq r\}$ is compact.*

Proof. It follows from Proposition 3.8 that $B_r(\widetilde{x})$ is closed. Its image under $\widetilde{\Psi}$ is contained in the closed ball $B_r\,(\Psi(x))$ of radius r about $\Psi(x)$. Since Ψ is proper and $\widetilde{X}$ is Hausdorff, $\widetilde{\Psi}$ is proper, and therefore $B_r(\widetilde{x})$ is compact. □

THEOREM 3.10. (The "Lokal-global-Prinzip" for convexity theorems) *Let $\Psi: X \to V$ be a locally fiber connected map with local convexity data $(C_x)_{x\in X}$. Suppose that Ψ is proper. Then $\Psi(X)$ is a closed locally polyhedral convex subset of V, the fibers $\Psi^{-1}(v)$ are all connected, $\Psi: X \to \Psi(X)$ is an open mapping, and $C_x = \Psi(x) + L_{\Psi(x)}(\Psi(X))$ holds for all $x \in X$.*

Proof. Fix $\widetilde{x}_0, \widetilde{x}_1 \in \widetilde{X}$ and set $c := d(\widetilde{x}_0, \widetilde{x}_1)$. Then for any $n \in \mathbb{N}$ there exists a regular curve γ_n connecting $\widetilde{x}_0$ and $\widetilde{x}_1$ with $l(\widetilde{\Psi} \circ \gamma_n) \leq c + \frac{1}{n}$. Let $\widetilde{x}_{1/2}^{(n)}$ be the midpoints of the curves γ_n. This means that the pieces of γ_n from $\widetilde{x}_0$ to $\widetilde{x}_{1/2}^{(n)}$ and from $\widetilde{x}_{1/2}^{(n)}$ to $\widetilde{x}_1$ have equal length when projected via $\widetilde{\Psi}$. These midpoints obviously are contained in the ball $B(2c, \widetilde{x}_0)$ which is compact and hence they have an accumulation point $\widetilde{x}_{1/2}$. This point satisfies

$$d(\widetilde{x}_0, \widetilde{x}_{1/2}) = d(\widetilde{x}_{1/2}, \widetilde{x}_1) = \frac{c}{2}.$$

We repeat this process for the pairs of points $(\widetilde{x}_0, \widetilde{x}_{1/2}), (\widetilde{x}_{1/2}, \widetilde{x}_1)$ to obtain points $\widetilde{x}_{1/4}, \widetilde{x}_{3/4}$ satisfying

$$d(\widetilde{x}_0, \widetilde{x}_{1/4}) = d(\widetilde{x}_{1/4}, \widetilde{x}_{1/2}) = d(\widetilde{x}_{1/2}, \widetilde{x}_{3/4}) = d(\widetilde{x}_{3/4}, \widetilde{x}_1) = \frac{c}{4}.$$

Inductively we find points $\widetilde{x}_{n/2^m}$ for $0 \leq n \leq 2^m$ such that

$$d(\widetilde{x}_{n/2^m}, \widetilde{x}_{n'/2^{m'}}) = c\left|\frac{n}{2^m} - \frac{n'}{2^{m'}}\right|.$$

Thus we can extend the map $n/2^m \mapsto \widetilde{x}_{n/2^m}$ to a continuous map $\gamma\colon [0,1] \to \widetilde{X}$ with

$$d(\gamma(t), \gamma(t')) = c|t - t'|.$$

This means that, if d_V denotes the metric on V, locally we have

$$d_V(\widetilde{\Psi} \circ \gamma(t), \widetilde{\Psi} \circ \gamma(t')) = c|t - t'|$$

which can only happen if $\widetilde{\Psi} \circ \gamma$ is a straight line. Thus $\widetilde{\Psi} \circ \gamma$ actually is the straight line connecting $\widetilde{\Psi}(\widetilde{x}_0)$ and $\widetilde{\Psi}(\widetilde{x}_1)$ which proves the convexity of $\Psi(X) = \widetilde{\Psi}(\widetilde{X})$.

Since the image of a proper map is closed by definition, it only remains to show that the $\Psi^{-1}(v)$ are all connected. To show this, it is enough to show that $\widetilde{\Psi}$ is injective since the equivalence classes of $\sim$ are connected. So we assume that $\widetilde{\Psi}(\widetilde{x}_0) = \widetilde{\Psi}(\widetilde{x}_1)$. We construct a regular curve γ connecting $\widetilde{x}_0$ and $\widetilde{x}_1$ as in the previous paragraph. Then $0 = d_V(\widetilde{\Psi} \circ \gamma(0), \widetilde{\Psi} \circ \gamma(1)) = c = d(\widetilde{x}_0, \widetilde{x}_1)$ so that $\widetilde{x}_0 = \widetilde{x}_1$.

In view of what we have already shown, $\widetilde{\Psi}$ is a homeomorphism $\widetilde{X} \to \Psi(X)$ because it is continuous, injective, and closed. According to Lemma 3.5(iv), the projection $\pi\colon X \to \widetilde{X}$ is an open mapping, hence the same is true for $\Psi = \widetilde{\Psi} \circ \pi$.

To prove the last assertion, we first note that $C_x \subseteq \Psi(x) + L_{\Psi(x)}(\Psi(X))$ follows from the fact that $\Psi(X)$ contains a neighborhood of $\Psi(x)$ in C_x. Since, on the other hand, this neighborhood is also a neighborhood of x in $\Psi(X)$, equality follows. □

COROLLARY 3.11. *Let V be a finite dimensional real vector space and $X \subseteq V$ a closed connected subset such that for each $x \in X$ there exists a neighborhood U_x of x in V and a closed convex cone $C_x \subseteq V$ with vertex x such that $U_x \cap X = U_x \cap C_x$. Then X is convex.*

Proof. The inclusion mapping $\Psi\colon X \to V$ is proper because X is closed, and since it is injective, the assumption on X shows that Ψ is a locally fiber connected map with convexity data $(C_x)_{x\in X}$. Hence the assertion follows from Theorem 3.10. □

§ 4. Hamiltonian torus actions

The following theorem is a strengthened version of the convexity theorem proved by Atiyah (cf. [At82]) and, independently, by Guillemin and Sternberg (cf. [GS82]).

THEOREM 4.1. (The convexity theorem for Hamiltonian torus actions) *Let M be a connected symplectic manifold, T a torus, and $\mu: T \times M \to M$ a Hamiltonian action of T on M such that the corresponding moment mapping $\Phi: M \to \mathfrak{t}^*$ is proper. Then the following assertions hold:*

(i) $\Phi(M)$ is a closed locally polyhedral convex set.
(ii) $\Phi: M \to \Phi(M)$ is an open mapping.
(iii) The inverse images of points in $\Phi(M)$ are connected.
(iv) Each extreme point f of $\Phi(M)$ is of the form $\Phi(x)$ for some T-fixed point $x \in M$.
(v) If $\mathfrak{t}_0$ is the Lie algebra of the stabilizer of m in T, then $L_{\Phi(m)}(\Phi(M)) = \mathfrak{t}_0^{\perp} + \operatorname{cone} \mathcal{P}_V$, where $\mathcal{P}_V$ is the set of all $\mathfrak{t}$-weights on the vector space $V := T_m(M)^{\perp}/T_m(M)$ as described in Lemma 2.2.

Proof. This follows immediately from Theorem 2.3 and Theorem 3.10. □

COROLLARY 4.2. *Let T be a torus and $j: A \to T$ a homomorphism of Lie groups. Let further μ be a Hamiltonian action of T on the connected symplectic manifold M and μ_A the corresponding Hamiltonian action of A on M. Suppose that the moment mapping $\Phi_A: M \to \mathfrak{a}^*$ is proper. Then*

(i) $\Phi_A(M)$ is a locally polyhedral convex set.
(ii) The inverse images of points in $\Phi_A(M)$ are connected.
(iii) Each extreme point f of $\Phi_A(M)$ is of the form $\Phi_A(x)$ for some A-fixed point $x \in M$.

Proof. Let $q: \mathfrak{t}^* \to \mathfrak{a}^*, f \mapsto f \circ dj(\mathbf{1})$. Then $\Phi_A = q \circ \Phi_T$. Then the properness of Φ_T follows from Lemma 1.14. Hence Theorem 4.1 applies to the action of T on M. Moreover Lemma 1.14 shows that the restriction of q to $\Phi_T(M)$ is a proper mapping. Hence $\Phi_A(M) = q(\Phi_T(M))$ is a locally polyhedral convex set by Proposition 1.15. This proves (i).

To see that (ii) also holds for $\Phi_A(M)$, let $\alpha \in \Phi_A(M)$. Then $q^{-1}(\alpha) \cap \Phi_T(M)$ is a closed convex set, hence connected. To see that the set

$$\Phi_A^{-1}(\alpha) = \Phi_T^{-1}(q^{-1}(\alpha))$$

is also connected, suppose that $\Phi_A^{-1}(\alpha) = F_1 \cup F_2$ is a disjoint union of closed sets. Then the fact that Φ_T has connected fibers yields that the sets F_1 and F_2 are Φ_T-saturated. Thus $\Phi_T(F_1)$ and $\Phi_T(F_2)$ are disjoint closed subsets of the connected set $q^{-1}(\alpha)$. We conclude that one of these sets is empty, hence either F_1 or F_2 is empty and consequently $\Phi_A^{-1}(\alpha)$ is connected.

Finally, we use Proposition 1.15 to see that

$$q\Big(\operatorname{Ext}(\Phi_T(M))\Big) = \operatorname{Ext}(\Phi_A(M)).$$

Thus, if α is an extreme point of $\Phi_A(M)$, there exists an extreme point $\widetilde{\alpha}$ of $\Phi_T(M)$ with $\widetilde{\alpha}\,|_{\mathfrak{a}} = \alpha$. Using (iii) for $\Phi_T(M)$, we find a T-fixed point $x \in M$ with $\Phi_T(x) = \widetilde{\alpha}$. Then $\alpha = \Phi_A(x)$ and x is also fixed by A. □

In some cases one only has a Hamiltonian action of A which extends to an action of the torus T on M which is symplectic. In this case the following proposition is helpful because it shows that the action of T is automatically Hamiltonian in such a case.

PROPOSITION 4.3. *Let T be a torus and $j: A \to T$ a homomorphism of Lie groups. Suppose that $\sigma: T \times M \to M$ is a symplectic action of T on the connected simply connected symplectic manifold M such that the action $\sigma \circ (j \times \mathrm{id}_M)$ of A is Hamiltonian. Then σ is Hamiltonian.*

If, in addition, $j(A)$ is dense in T and τ is an antisymplectic involution of M such that all Hamiltonian functions associated to the action of A on M are invariant under τ and τ has at least one fixed point, then the Hamiltonian functions associated to the Hamiltonian action of T are also invariant under τ.

Proof. Let $\varphi: \mathfrak{a} \to C^\infty(M)$ be the homomorphism of Lie algebras defining the Hamiltonian action of A.

Let $X \in \mathfrak{t}$ and $\dot{\sigma}(X)$ the corresponding vector field on M given by

$$\dot{\sigma}(X)(p) = \frac{d}{dt}\Big|_{t=0} \exp(-tX).p.$$

Further let ω denote the symplectic 2-form on M. Then we have for the Lie derivative $\mathcal{L}_{\dot{\sigma}(X)}\omega = 0$, so that the 1-form $i_{\dot{\sigma}(X)}\omega$ is closed. Since M is simply connected, it is exact and there exists a function $f \in C^\infty(M)$ with $df = -i_{\dot{\sigma}(X)}\omega$.

Therefore we find a linear mapping $\varphi_T: \mathfrak{t} \to C^\infty(M)$ extending φ, i.e., $\varphi_T \circ dj(\mathbf{1}) = \varphi$. Let $X, Y \in \mathfrak{t}$ and set $C(X,Y) := \{\varphi_T(X), \varphi_T(Y)\}$. Then C defines a cocycle $\mathfrak{t} \times \mathfrak{t} \to \mathbf{R}$, hence a central extension

$$\{0\} \to \mathbf{R} \to \widetilde{\mathfrak{t}} \to \mathfrak{t} \to \{0\}.$$

More precisely $\widetilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathbf{R}$ with the bracket

$$[(X,t),(X',t')] = (0, C(X,X')).$$

Let $\widetilde{T}$ denote the simply connected Lie group with $\mathbf{L}(\widetilde{T}) = \widetilde{\mathfrak{t}}$. Then we have a homomorphism $\beta: \widetilde{T} \to T$ so that we obtain an action of $\widetilde{T}$ on M which is Hamiltonian with respect to the mapping $\widetilde{\varphi}(X,t) := \varphi_T(X) + t\mathbf{1}$, where $\mathbf{1}$ denotes the constant function with value 1 on M. Therefore we have a moment mapping $\Phi_{\widetilde{T}}: M \to \widetilde{\mathfrak{t}}^*$ which maps the $\widetilde{T}$-orbits which are also T-orbits onto compact coadjoint orbits in $\widetilde{\mathfrak{t}}^*$. A direct calculation shows that the only compact coadjoint orbits in $\widetilde{\mathfrak{t}}^*$ are the one-point orbits, i.e., those which correspond to elements vanishing on $[\widetilde{\mathfrak{t}}, \widetilde{\mathfrak{t}}]$. We conclude that $\{\varphi_T(X), \varphi_T(Y)\} = \{\widetilde{\varphi}(X,0), \widetilde{\varphi}(Y,0)\} = \widetilde{\varphi}([(X,0),(Y,0)]) = 0$.

This shows that φ_T is a homomorphism of Lie algebras and therefore the action of T is Hamiltonian.

Next we assume in addition that τ is an antisymplectic involution on M which has a fixed point m_0 and that all Hamiltonian functions associated to the action of A are invariant under τ. Let $f = \varphi(X') = \varphi_T(X)$ be such a function. Then $\tau^* df = d(f \circ \tau) = df$ and therefore $\tau^*\omega = -\omega$ yields that $\tau_* \dot{\sigma}(X) = -\dot{\sigma}(X)$. We conclude that $\tau(t.m) = t^{-1}\tau(m)$ for all $t \in j(A)$ and $m \in M$. Now the density of $j(A)$ in T yields the same for all elements in T and from this we obtain $\tau_*\dot{\sigma}(X) = -\dot{\sigma}(X)$ for all $X \in \mathfrak{t}$. Hence $\tau^* d\varphi_T(X) = d\varphi_T(X)$ for all $X \in \mathfrak{t}$. It follows that the function $\varphi_T(X) - \varphi_T(X) \circ \tau$ is constant. In the point m_0 it vanishes, thus it vanishes on M and therefore the functions $\varphi_T(X)$ are invariant under τ. □

We note that the preceding proposition applies also with $A = \{\mathbf{1}\}$, where it shows that a symplectic torus action on a simply connected manifold is Hamiltonian. The fact that Theorem 4.1 allows us to treat not only compact manifolds M has some interesting consequences for moment maps on coadjoint orbits. These will be spelled out in Section 5.

Next we obtain a version of Duistermaat's Convexity Theorem (cf. [Dui83]) for non-compact manifolds.

THEOREM 4.4. (Duistermaat's convexity theorem for non-compact manifolds) *Let (M, ω) be a connected symplectic manifold, $\sigma : T \times M \to M$ a Hamiltonian action of a torus T on M which is given by the Lie algebra homomorphism $\lambda: \mathfrak{t} \to C^\infty(M)$. Let further τ be an antisymplectic involution on M such that all Hamiltonian functions $\lambda(X)$, $X \in \mathfrak{t}$ are invariant under τ and Q the manifold of τ-fixed points in M. Suppose that the moment mapping $\Phi: M \to \mathfrak{t}^*$ is proper. Then, for every connected component Q' of Q, we have $\Phi(M) = \Phi(Q')$.*

Proof. Let Q' be a connected component of the closed submanifold Q of M. Then Q' is closed in M and therefore $\Phi(Q')$ is a closed connected subset of $\mathfrak{t}^*$. Let $q \in Q'$ and set $x := \Phi(q)$. Then it follows from the second part of Theorem 2.3 that there exists a neighborhood V_q of q in Q' and a neighborhood U_x of x in $\mathfrak{t}^*$ such that $\Phi(V_q) = C_q \cap U_x$. On the other hand $\Phi(Q')$ lies in the convex set $\Phi(M)$, hence is contained in $C_q = x + L_x(\Phi(M))$ (Theorem 3.10). Thus $C_q \cap U_x = \Phi(Q') \cap U_x$, so that we see that $\Phi(Q')$ satisfies the assumptions of Corollary 3.11. We conclude that $\Phi(Q')$ is convex. Now

$$\Phi(Q') = \bigcap_{x \in \widetilde{\mathrm{Ext}}(\Phi(Q'))} \Big(x + L_x(\Phi(Q'))\Big)$$

by Proposition 1.8.

Suppose that $x \in \widetilde{\mathrm{Ext}}(\Phi(Q'))$. Then it follows from the second part of Theorem 2.3 that

$$x + L_x(\Phi(Q')) = C_q = x + L_x(\Phi(M)).$$

Therefore

$$\Phi(M) = \bigcap_{x\in\Phi(M)} \Big(x+L_x(\Phi(M))\Big) \subseteq \bigcap_{x\in\widetilde{\mathrm{Ext}}(\Phi(Q'))} \Big(x+L_x(\Phi(Q'))\Big) = \Phi(Q').$$

From this we conclude that $\Phi(Q') = \Phi(M)$. □

COROLLARY 4.5. *The assertion of* Theorem 4.4 *remains valid in the situation of* Corollary 4.2, *i.e., if* $j: A \to T$ *is a homomorphism of connected Lie groups,* μ_A *the corresponding Hamiltonian action of* A *on* M, *and the moment mapping* $\Phi_A: M \to \mathfrak{a}^*$ *is proper. Then* $\Phi_A(M) = \Phi_A(Q)$.

Proof. This is an immediate consequence of Corollary 4.2 and Theorem 4.4. □

Note that Proposition 4.3 shows how this corollary can be applied if M is simply connected.

§ 5. Applications to coadjoint orbits

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. We consider the *Poisson structure* defined on $\mathfrak{g}^*$ by

$$\{G,H\}(f) := \langle f, [dG(f), dH(f)]\rangle \qquad \forall G,H \in C^\infty(\mathfrak{g}^*), f \in \mathfrak{g}^*$$

(cf. [LM87, p.108]).

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then the coadjoint action $\mathrm{Ad}^*(g).f := f\circ \mathrm{Ad}(g)^{-1}$ is Hamiltonian with the identity as moment mapping (cf. [LM87, p.213]), i.e., for $X \in \mathfrak{g}$ the function $H_X(f) := f(X)$ corresponds to the Hamiltonian vector field $\mathcal{X}_X(f) := f\circ \mathrm{ad}\, X$ on $\mathfrak{g}^*$. Let $f \in \mathfrak{g}^*$. Then the coadjoint orbit $\mathcal{O}_f := \mathrm{Ad}^*(G).f$ is a symplectic leaf in the Poisson manifold $\mathfrak{g}^*$ (cf. [LM87, p.212]).

In this section we are concerned with the properness of the moment mapping $\mathcal{O}_f \to \mathfrak{t}^*$ for coadjoint orbits. Here we consider the set $\mathcal{O}_f$ as endowed with its manifold topology which might be different from the topology inherited from $\mathfrak{g}^*$. We start with a simple observation showing that this cannot be the case.

LEMMA 5.1. *Let* $\mathfrak{h} \subseteq \mathfrak{g}$ *be a subalgebra. If the moment mapping* $p_{\mathfrak{h}}: \mathcal{O}_f \to \mathfrak{h}^*$ *is proper, then the orbit* $\mathcal{O}_f$ *is a closed subset of* $\mathfrak{g}^*$.

Proof. We apply Lemma 1.14 with $X = \mathcal{O}_f$, $Y = \mathfrak{g}^*$ and $Z = \mathfrak{h}^*$. □

LEMMA 5.2. *If* $\mathfrak{a} \subseteq \mathfrak{b}$ *are subalgebras of* $\mathfrak{g}$ *such that the moment mapping* $p_{\mathfrak{a}}: \mathcal{O}_f \to \mathfrak{a}^*$ *is proper, then the moment mapping* $p_{\mathfrak{b}}: \mathcal{O}_f \to \mathfrak{b}^*$ *is proper.*

Proof. We apply Lemma 1.14 with $X = \mathcal{O}_f$, $Y = \mathfrak{b}^*$ and $Z = \mathfrak{a}^*$. □

So far these formulas hold for arbitrary elements $f \in \mathfrak{g}^*$. Now we apply these to the case where $\mathfrak{t} \subseteq \mathfrak{g}$ is a compactly embedded abelian subalgebra, i.e., the group $\overline{e^{\mathrm{ad}\,\mathfrak{t}}} \subseteq \mathrm{Aut}(\mathfrak{g})$ is a torus.

PROPOSITION 5.3. *Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded abelian subalgebra and $\mathcal{O}_f$ a closed simply connected coadjoint orbit such that the moment mapping $p_\mathfrak{t}: \mathcal{O}_f \to \mathfrak{t}^*$ is proper. Then $p_\mathfrak{t}(\mathcal{O}_f)$ is a locally polyhedral subset of $\mathfrak{t}^*$ and if it has extreme points, then $\mathcal{O}_f$ contains points which are fixed by the group $\mathrm{Ad}^*(T)$, where $T = \exp \mathfrak{t}$.*

If the group $\mathrm{Ad}(T)$ is closed, then the assumption that $\mathcal{O}_f$ is simply connected is not needed.

Proof. Let T' denote the torus $\overline{\mathrm{Ad}^*(T)}$.

Let $\gamma \in \mathrm{Aut}(\mathfrak{g})$. We claim that γ^* is a Poisson automorphism of $\mathfrak{g}^*$. To see this let $f, g \in C^\infty(\mathfrak{g}^*)$ and note that

$$\begin{aligned}
\{(f \circ \gamma), (g \circ \gamma)\}(\alpha) &= \langle \alpha, [d(f \circ \gamma)(\alpha), d(g \circ \gamma)(\alpha)] \rangle \\
&= \langle \alpha, [df(\gamma.\alpha) \circ \gamma, dg(\gamma.\alpha) \circ \gamma] \rangle \\
&= \langle \alpha, [df(\gamma.\alpha), dg(\gamma.\alpha)] \circ \gamma \rangle \\
&= \langle \gamma.\alpha, [df(\gamma.\alpha), dg(\gamma.\alpha)] \rangle \\
&= (\{f, g\} \circ \gamma)(\alpha).
\end{aligned}$$

If $\mathcal{O}_f$ is a coadjoint orbit invariant under γ^*, then it follows that γ^* induces a symplectomorphism on $\mathcal{O}_f$.

Since the coadjoint orbit $\mathcal{O}_f$ is closed, it is also invariant under the group T and the above argument yields that this action preserves the symplectic structure. Let $j := \mathrm{Ad}^* \mid_T: T \to T'$. Then the action of T on $\mathcal{O}_f$ is Hamiltonian and in view of Proposition 4.3, the corresponding homomorphism $\varphi: \mathfrak{t} \to C^\infty(\mathcal{O}_f)$ extends to $\mathfrak{t}' := \mathbf{L}(T')$ in the sense that there exists a Lie algebra homomorphism $\varphi_{T'}: \mathfrak{t}' \to C^\infty(\mathcal{O}_f)$ with $\varphi_{T'} \circ dj(\mathbf{1}) = \varphi$. Now Corollary 4.2 applies and proves the assertion.

If, in the situation above, the group $\mathrm{Ad}(T)$ is closed, i.e., a torus, then we can apply Corollary 4.2 directly and therefore we do not need the assumption that $\mathcal{O}_f$ is simply connected in this case. □

To see how to compute the image of the moment mapping, we first have to compute the local convexity data. The tangent space in f is given by

$$T_f(\mathcal{O}_f) = f \circ \mathrm{ad}\, \mathfrak{g} := \{f \circ \mathrm{ad}\, X : X \in \mathfrak{g}\}.$$

For the symplectic structure Ω on $\mathcal{O}_f$ we have

$$\Omega(f)(f \circ \mathrm{ad}\, X, f \circ \mathrm{ad}\, Y) = \{H_X, H_Y\}(f) = f([X, Y]) \qquad \forall X, Y \in \mathfrak{g}.$$

If (V, J) is a symplectic vector space, then the action of the symplectic group $\mathrm{Sp}(V)$ on V is a Hamiltonian action with moment mapping

$$\Psi_V(v)(A) = \frac{1}{2} J(A.v, v) \qquad \forall v \in V, A \in \mathfrak{sp}(V).$$

Now let $V := T_f(\mathcal{O}_f)$. This is a symplectic vector space and the stabilizer group $G_f := \{g \in G: g.f = f\}$ acts on V by symplectic isomorphisms. We write $\pi: G_f \to \mathrm{Sp}(V)$ for the corresponding homomorphism of Lie groups. To find the local convexity data for the local convexity theorem, we need the moment mapping for the action of G_f on V.

Before we compute it, we note that

$$\pi(\exp Y)(f\circ \mathrm{ad}\, X) = f\circ \mathrm{ad}\, X \circ e^{-\mathrm{ad}\, Y} = f\circ e^{\mathrm{ad}\, Y}\circ \mathrm{ad}\, X\circ e^{-\mathrm{ad}\, Y} = f\circ \mathrm{ad}\left(e^{\mathrm{ad}\, Y}X\right)$$

holds for all $Y \in \mathfrak{g}_f = \mathbf{L}(G_f) = \{X \in \mathfrak{g}: f \circ \mathrm{ad}\, X = 0\}$ and $X \in \mathfrak{g}$. Hence

$$d\pi(Y).(f \circ \mathrm{ad}\, X) = f \circ \mathrm{ad}\,(\mathrm{ad}\, Y(X)) = f \circ \mathrm{ad}[Y, X].$$

This leads to the following formula for the moment mapping:

$$\begin{aligned}\Psi_V(f \circ \mathrm{ad}\, X)(Y) &= \frac{1}{2}\Omega(f)(d\pi(Y)(f \circ \mathrm{ad}\, X), f \circ \mathrm{ad}\, X)\\ &= \frac{1}{2}\Omega(f)(f \circ \mathrm{ad}[Y, X], f \circ \mathrm{ad}\, X)\\ &= \frac{1}{2}f(\,[[Y, X], X]\,).\end{aligned}$$

Hence

$$\Psi_V(f \circ \mathrm{ad}\, X) = \frac{1}{2}f \circ (\mathrm{ad}\, X)^2|_{\mathfrak{g}_f}.$$

The complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ decomposes according to the action of $\mathrm{ad}\, \mathfrak{t}$ into a direct sum of the subspaces

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} := \{Y \in \mathfrak{g}_{\mathbb{C}} : (\forall X \in \mathfrak{t}_{\mathbb{C}})\, [X, Y] = \alpha(X)Y\},$$

where $\mathfrak{g}_{\mathbb{C}}^0 = Z_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{t})$ is the centralizer of $\mathfrak{t}$. We denote the set of roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ by

$$\Lambda := \Lambda(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) := \{\alpha \in \mathfrak{t}_{\mathbb{C}}^* \setminus \{0\}: \mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}\}.$$

If $Z \mapsto \overline{Z}$ denotes complex conjugation on $\mathfrak{g}_{\mathbb{C}}$, then $\overline{\mathfrak{g}_{\mathbb{C}}^{\alpha}} = \mathfrak{g}_{\mathbb{C}}^{-\alpha}$.

Let $f \in \mathfrak{g}^*$ such that $\mathfrak{t} \subseteq \mathfrak{g}_f$. Then $\mathfrak{g}_f$ is invariant under $\mathfrak{t}$ and therefore

$$\mathfrak{g}_f = \mathfrak{t} \oplus (\mathfrak{g}^0 \cap \mathfrak{g}_f) \oplus \bigoplus_{\alpha\in\Lambda} \left((\mathfrak{g}_{\mathbb{C}}^{\alpha} + \mathfrak{g}_{\mathbb{C}}^{-\alpha}) \cap \mathfrak{g}_f\right).$$

Let $\alpha \in \Lambda$ and $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$. We consider $X := X_\alpha + \overline{X}_\alpha \in \mathfrak{g}$. Then $f \circ \mathrm{ad}\, X \in T_f(\mathcal{O}_f)$. For $Y \in \mathfrak{t}$ we find that

$$\begin{aligned}(\mathrm{ad}\, X)^2 Y &= \left[X, [X_\alpha + \overline{X}_\alpha, Y]\right] = [X, -\alpha(Y)(X_\alpha - \overline{X}_\alpha)]\\ &= \alpha(Y)[X_\alpha + \overline{X}_\alpha, \overline{X}_\alpha - X_\alpha] = 2\alpha(Y)[X_\alpha, \overline{X}_\alpha].\end{aligned}$$

It follows that

$$\Psi_V(f \circ \operatorname{ad} X) = f([X_\alpha, \overline{X}_\alpha])\alpha = -f(i[X_\alpha, \overline{X}_\alpha])i\alpha.$$

Note that the linear functional $i\alpha$ is real on $\mathfrak{t}$ and that $i[X_\alpha, \overline{X}_\alpha] \in \mathfrak{g}^0$.

This leads to the following description of the local convexity data (cf. Lemma 5.16). Here we recall that cone(M) for a subset M of a vector space means the smallest closed convex cone containing M.

REMARK 5.4. Let $f \in \mathfrak{g}^*$ be such that $\mathfrak{t} \subseteq \mathfrak{g}_f$ and

$$C_f = \operatorname{cone}\{-i\alpha f(i[X_\alpha, \overline{X}_\alpha]) : X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha, \alpha \in \Lambda^+\}. \tag{5.1}$$

Then $f(i[X_\alpha, \overline{X}_\alpha])$ vanishes for $X_\alpha \in (\mathfrak{g}_f)_{\mathbb{C}}$ and the stabilizer $\mathfrak{g}_f$ of f does not contribute to the cone C_f. If $g \in N_G(\mathfrak{t})$, then $p_\mathfrak{t}$ commutes with $\operatorname{Ad}^*(g)$ and therefore $C_{\operatorname{Ad}^*(g).f} = \operatorname{Ad}^*(g)C_f$. □

In the following we say that a subalgebra $\mathfrak{t}$ of a Lie algebra is a *Cartan subalgebra* if it is nilpotent and self-normalizing. A *compactly embedded Cartan algebra* is a Cartan algebra for which the closure of $e^{\operatorname{ad}\mathfrak{t}}$ is a compact group. In this case $\mathfrak{t}$ is abelian, and the operators $\operatorname{ad} X$, $X \in \mathfrak{t}$ are semisimple with purely imaginary spectrum.

REMARK 5.5. Let $f \in \mathfrak{t}^*$ and define $B_\alpha(X) := f(i[X, \overline{X}])$ for $X \in \mathfrak{g}_{\mathbb{C}}^\alpha$. Then, for $\alpha \in \Lambda$ we have four possibilities:

(1) B_α vanishes. Then $\mathfrak{g}_{\mathbb{C}}^\alpha \subseteq (\mathfrak{g}_f)_{\mathbb{C}}$ and α does not contribute to C_f because $i\alpha f([X_\alpha, \overline{X}_\alpha]) = 0$ for all $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$.
(2) B_α is non-zero and positive semidefinite. Then $-i\alpha \in C_f$.
(3) B_α is non-zero and negative semidefinite. Then $i\alpha \in C_f$.
(4) B_α is indefinite. Then $\pm i\alpha \in C_f$.

For an example where case (4) occurs, we refer to [Ne93c, Ex. IV.2]. Example IV.14 in [Ne93c] shows also that it may happen that $\mathfrak{g}_f = \mathfrak{t}$, $\mathfrak{t}$ is a Cartan algebra, only cases (2) and (3) occur, and that C_f is a plane which is the sum of three rays. □

PROPOSITION 5.6. *Let $f \in \mathfrak{g}^*$ be such that $\mathfrak{t} \subseteq \mathfrak{g}_f$ and that $\mathcal{O}_f$ is simply connected if the group $e^{\operatorname{ad}\mathfrak{t}}$ is not closed. Suppose further that the restriction map $p_\mathfrak{t}\colon \mathcal{O}_f \to \mathfrak{t}^*$ is proper and that the convex hull of its image contains no lines.*

For $T = \exp X$ we denote the T-fixed points in $\mathcal{O}_f$ by $\mathcal{O}_{f,\mathrm{fix}}$. Then

$$C = \operatorname{conv} p_\mathfrak{t}(\mathcal{O}_{f,\mathrm{fix}}) + \bigcap_{f' \in \mathcal{O}_{f,\mathrm{fix}}} C_{f'}.$$

Proof. First we note that according to Corollary 4.2 and Proposition 4.3, the set $C := p_\mathfrak{t}(\mathcal{O}_f)$ is closed and convex, hence has extremal points because it contains no lines (Lemma 1.4). Further each extremal point of C is the image of a T-fixed point in $\mathcal{O}_f$ under $p_{\mathfrak{t}}$. Now the claim follows from Remark 5.4 and Proposition 1.8. $\square$

If, in addition, the subalgebra $\mathfrak{t}$ is a Cartan algebra, then the following observation simplifies the computation of the set C considerably.

PROPOSITION 5.7. *Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan algebra and $f \in \mathfrak{t}^*$. Then $\mathcal{O}_{f,\text{fix}} = \mathcal{W}.f$, where $\mathcal{W} = N_G(\mathfrak{t})/Z_G(\mathfrak{t})$.*

Proof. Suppose that $f' = \mathrm{Ad}^*(g).f \in \mathcal{O}_{f,\text{fix}}$. Then $\mathfrak{t} \subseteq \mathfrak{g}_{f'} = \mathrm{Ad}(g).\mathfrak{g}_f$ and hence there exists an inner automorphism $\mathrm{Ad}(h)$ of $\mathfrak{g}_f$ such that $\mathrm{Ad}(hg^{-1}).\mathfrak{t} = \mathfrak{t}$ (cf. [HiNe91, Satz III.7.10]). Thus $hg^{-1} \in N_G(\mathfrak{t})$ and

$$f' = \mathrm{Ad}^*(g).f = \mathrm{Ad}^*(gh^{-1}).f = \mathrm{Ad}^*\left((hg^{-1})^{-1}\right).f$$

proves the claim. $\square$

Admissible coadjoint orbits

In this section $\mathfrak{g}$ denotes a finite dimensional real Lie algebra which contains a compactly embedded Cartan algebra $\mathfrak{t}$.

DEFINITION 5.8. (a) We say that an element $f \in \mathfrak{g}^*$ is *admissible* if the coadjoint orbit $\mathcal{O}_f$ is closed and its convex hull contains no lines. We call $f \in \mathfrak{g}^*$ *strictly admissible* if there exists a closed invariant convex set $C \subseteq \mathfrak{g}^*$ which contains no lines and which contains the coadjoint orbit $\mathcal{O}_f$ in its algebraic interior. We say that $\mathcal{O}_f$ is (*strictly*) *admissible* if f is (strictly) admissible. We will see later that strict admissibility implies admissibility. That this property implies that the convex hull of $\mathcal{O}_f$ contains no lines is clear.
An element $f \in \mathfrak{g}^*$ is said to be of *convexity type* if the coadjoint orbit $\mathcal{O}_f$ lies in a closed pointed convex cone and of *strict convexity type* if $\mathcal{O}_f$ lies in the algebraic interior of a pointed convex cone in $\mathfrak{g}^*$ which is invariant under the coadjoint action.
$\square$

The following lemma provides the link between the notions of admissibility and convexity type.

LEMMA 5.9.
(i) $\operatorname{conv} \mathcal{O}_f$ *contains no lines if and only if* $(f, 1) \in \mathfrak{g}^* \times \mathbf{R}$ *is of convexity type.*
(ii) f *is strictly admissible if and only if* $(f, 1) \in \mathfrak{g}^* \times \mathbf{R}$ *is of strict convexity type.*

Proof. (i) If $(f, 1)$ is of convexity type, then $\operatorname{conv} \mathcal{O}_{(f,1)} = (\operatorname{conv} \mathcal{O}_f) \times \{1\}$ contains no lines and therefore the same holds for $\operatorname{conv} \mathcal{O}_f$.

If, conversely, $\operatorname{conv} \mathcal{O}_f$ contains no lines, then we set $C := \overline{\operatorname{conv} \mathcal{O}_f}$. Then C is a closed convex set in $\mathfrak{g}^*$ and $\operatorname{conv} \mathcal{O}_f$ is a dense convex subset. Hence $\operatorname{conv} \mathcal{O}_f$ contains the algebraic interior of C. Since this set contains no lines, it follows

immediately that C contains no lines. Now Proposition 1.11 shows that $C \times \{1\}$ lies in a pointed closed convex cone and therefore $(f, 1)$ is of convexity type.
(ii) Suppose first that f is strictly admissible and that $\mathcal{O}_f \subseteq \operatorname{algint} C$, where C is invariant and contains no lines. Then, according to Proposition 1.11, $C \times \{1\}$ is containd in a pointed closed convex cone $W \subseteq \mathfrak{g}^* \times \mathbf{R}$. Taking the smallest cone with this property, the invariance of C implies the invariance of W.

If E is the affine subspace generated by C, then we may w.l.o.g. assume that $W \subseteq E \times \mathbf{R}$. Then $\mathbf{R}^*(\operatorname{algint} C \times \{1\}) = \mathbf{R}^*(\operatorname{int}_E C \times \{1\})$ is an open subset of $E \times \mathbf{R}$ and therefore $(\operatorname{algint} C) \times \{1\}$ is contained in $\operatorname{algint} W$. Hence $\mathcal{O}_f \times \{1\}$ is contained in $\operatorname{algint} W$ which means that $(f, 1)$ is of strict convexity type.

If, conversely, $\mathcal{O}_{(f,1)} \in \operatorname{algint} W$, where W is a closed pointed invariant convex cone in $\mathfrak{g}^* \times \mathbf{R}$, we set $C := \{\alpha \in \mathfrak{g}^* \colon (\alpha, 1) \in W\}$. Then C is a closed convex set containing no lines and $(\operatorname{algint} C) \times \{1\} \supseteq (\operatorname{algint} W) \cap (\mathfrak{g}^* \times \{1\})$. It follows in particular that $\mathcal{O}_f \subseteq \operatorname{algint} C$, hence that f is admissible. □

For the following we recall that if $\mathfrak{t}$ is a compactly embedded Cartan algebra, then $\mathfrak{g} = \mathfrak{t} \oplus [\mathfrak{t}, \mathfrak{g}]$ so that we can identify $\mathfrak{t}^*$ with the subspace $[\mathfrak{t}, \mathfrak{g}]^\perp$ in $\mathfrak{g}^*$. Note that this is exactly the set of T-fixed points in $\mathfrak{g}^*$.

LEMMA 5.10. *Let $f \in \mathfrak{g}^*$ be admissible and $\mathfrak{t} \subseteq \mathfrak{g}$ a compactly embedded Cartan subalgebra. Then the restriction mapping $p_\mathfrak{t} \colon \mathcal{O}_f \to \mathfrak{t}^*$ is proper, $\mathcal{O}_f \cap \mathfrak{t}^* \neq \emptyset$, and the closed convex hull of $\overline{p_\mathfrak{t}(\mathcal{O}_f)}$ contains no lines.*

Proof. Let $C := \overline{\operatorname{conv} \mathcal{O}_f}$. Then the closedness of $\mathcal{O}_f$ implies that the inclusion mapping $\mathcal{O}_f \to C$ is proper. Hence, to prove the properness of $p_\mathfrak{t}$, it suffices to show that $p_\mathfrak{t} \colon C \to \mathfrak{t}^*$ is proper.

To see this, we first note that C is a closed convex set which contains no lines and which is invariant under the coadjoint action. Let $W := \lim(C)^\star$. Then, according to Proposition 1.11, this a generating invariant cone in the Lie algebra $\mathfrak{g}$. Hence $\operatorname{int} W \cap \mathfrak{t} \neq \emptyset$ by [Ne93a, Lemma III.5]. Pick a regular element $X \in \mathfrak{t} \cap \operatorname{int} W$. Then the function $H_X \colon \alpha \mapsto \alpha(X)$ is proper on C (Proposition 1.12). Therefore the mapping

$$C \to \mathfrak{t}^* \to \mathbf{R}, \alpha \mapsto \alpha|_\mathfrak{t} \mapsto \alpha(X)$$

is proper and the properness of $p_\mathfrak{t}$ on C follows from Lemma 1.14. Moreover, we also get from Lemma 1.14 that H_X is proper on the closed convex set $p_\mathfrak{t}(C)$. Hence $\operatorname{int} B(p_\mathfrak{t}(C)) \neq \emptyset$ which in turn implies that $p_\mathfrak{t}(C)$ contains no lines (Proposition 1.11).

Since H_X is proper and bounded from below on $\mathcal{O}_f$ (Proposition 1.12), there exists an element $f' \in \mathcal{O}_f$ such that H_X takes a minimal value in f'. Let $Y \in \mathfrak{g}$. Then it follows that

$$\left.\frac{d}{dt}\right|_{t=0} H_X(\operatorname{Ad}^*(\exp tY).f') = \operatorname{ad}^* Y.f'(X) = f'([X, Y]) = 0.$$

Hence $f' \in [X, \mathfrak{g}]^\perp = [\mathfrak{t}, \mathfrak{g}]^\perp = \mathfrak{t}^*$. □

For the following theorem we recall that a convex set contains no affine lines if and only if its closure has this property (cf. [Le80, Satz 1.8 and Lemma 1.1]).

THEOREM 5.11. *For $f \in \mathfrak{g}^*$ the following are equivalent:*
(1) f is admissible.
(2) $\mathcal{O}_f \cap \mathfrak{t}^* \neq \emptyset$ and conv $\mathcal{O}_f$ *contains no lines.*
(3) $p_\mathfrak{t}\colon \mathcal{O}_f \to \mathfrak{t}^*$ is proper and the convex hull of $p_\mathfrak{t}(\mathcal{O}_f)$ contains no lines.
(4) $\mathcal{O}_f$ is closed and the convex hull of $p_\mathfrak{t}(\mathcal{O}_f)$ contains no lines.
If these conditions are satisfied, then $\mathcal{O}_f$ is simply connected.

Proof. (1) $\Rightarrow$ (2): This follows from Lemma 5.10.
(2) $\Rightarrow$ (1): According to Theorem I.18 in [Ne93b] the condition that $\mathcal{O}_f$ intersects $\mathfrak{t}^*$ implies that this set is closed and simply connected. Hence f is admissible if $\mathcal{O}_f$ intersects $\mathfrak{t}^*$ and the convex hull contains no lines.
(1) $\Rightarrow$ (3): Lemma 5.10.
(3) $\Rightarrow$ (4): It follows from Lemma 5.1 that $\mathcal{O}_f$ is closed if $p_\mathfrak{t}\colon \mathcal{O}_f \to \mathfrak{t}^*$ is proper.
(4) $\Rightarrow$ (1): Let $C :=$ conv $\mathcal{O}_f$. According to our assumption, the convex set

$$p_\mathfrak{t}(C) \subseteq \operatorname{conv} p_\mathfrak{t}(\mathcal{O}_f) \subseteq \mathfrak{t}^*$$

contains no lines. Hence $H(C) := H(\overline{C}) \subseteq \ker p_\mathfrak{t}$. Since C is invariant under the coadjoint action, the subspace $\mathfrak{a} := H(C)^\perp$ of $\mathfrak{g}$ is an ideal. Since $H(C) \subseteq \ker p_\mathfrak{t} = \mathfrak{t}^\perp$, it follows that $\mathfrak{t} \subseteq \mathfrak{a}$. Then $\mathfrak{a} = \mathfrak{g}$ is a consequence of $\mathfrak{g} = \mathfrak{t} + [\mathfrak{t}, \mathfrak{g}]$. Finally $H(C) = \mathfrak{a}^\perp = \{0\}$ follows, i.e., C contains no lines. □

COROLLARY 5.12. *If f is strictly admissible, then it is admissible.*

Proof. In view of Theorem 5.11, we only have to show that for a strictly admissible element f the coadjoint orbit $\mathcal{O}_f$ intersects $\mathfrak{t}^*$. Using Lemma 5.9, we see that we even may assume that f is of strict convexity type. Now the assertion follows from Theorem II.4 in [Ne93b]. □

REMARK 5.13. Let $\mathfrak{g} = \mathbf{R}^2 \rtimes \mathbf{R}$ denote the Lie algebra of the group of motions of the euclidean plane. Then $\mathfrak{t} = \{0\} \times \mathbf{R}$ is a compactly embedded Cartan algebra and the coadjoint orbits in $\mathfrak{g}^*$ are cylinders centered about the $\mathfrak{t}^*$-axis. Hence the moment mappings $p_\mathfrak{t}\colon \mathcal{O}_f \to \mathfrak{t}^*$ are proper but the convex hulls of these orbits are cylinders, hence contain the line $\mathfrak{t}^*$. □

To prepare the proof of the Convexity Theorem for coadjoint orbits we recall some notions concerning compactly embedded Cartan algebras and root decompositions.

DEFINITION 5.14. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan algebra and $\Lambda \subseteq i\mathfrak{t}^*$ the corresponding set of roots.

Let $\mathfrak{k} \supseteq \mathfrak{t}$ denote a maximal compactly embedded subalgebra. Then a root is said to be *compact* if $\mathfrak{g}_\mathbb{C}^\lambda \subseteq \mathfrak{k}_\mathbb{C}$. We write Λ_k for the set of compact roots and Λ_p for the set of non-compact roots.

The Lie algebra $\mathfrak{g}$ is said to have *cone potential* if $[X_\alpha, \overline{X_\alpha}] \neq 0$ holds for all non-zero elements $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$.

The finite group $\mathcal{W} := N_K(\mathfrak{t})/Z_K(\mathfrak{t})$ is called the *Weyl group* of $\mathfrak{t}$. It coincides with the Weyl group for the system of *compact roots* (cf. [Ne93a, Prop. III.1]).

A subset $\Lambda^+ \subseteq \Lambda$ is called a *positive system* if there exists $X_0 \in i\mathfrak{t}$ such that

$$\Lambda^+ = \{\lambda \in \Lambda : \lambda(X_0) > 0\}.$$

A positive system is said to be $\mathfrak{k}$-*adapted* if the set Λ_p^+ of positive non-compact roots is invariant under the Weyl group $\mathcal{W}$.

Let $\Lambda^+ \subseteq \Lambda$ be a $\mathfrak{k}$-adapted positive system of roots. We define the *maximal cone* and the *minimal cone*

$$C_{\max} := C_{\max}(\Lambda^+) := (i\Lambda_p^+)^\star \subseteq \mathfrak{t},$$

$$C_{\min} := C_{\min}(\Lambda^+) := \operatorname{cone}\{i[\overline{X}, X] : X \in \mathfrak{g}_{\mathbb{C}}^\lambda, \lambda \in \Lambda_p^+\} \subseteq \mathfrak{t}.$$

We note that these cones are invariant under the Weyl group because Λ_p^+ is invariant.

We recall that a Lie algebra $\mathfrak{g}$ is called *quasihermitean* if the centralizer of $\mathfrak{z} = \mathfrak{z}(\mathfrak{k})$ equals $\mathfrak{k}$. Note that this is equivalent to the existence of a $\mathfrak{k}$-adapted positive system of roots (cf. [Ne93d, Prop. II.7]). □

To illuminate the assumptions we will make in the following, we recall that the stabilizer $\mathfrak{g}_f$ of $f \in \mathfrak{g}^*$ is said to be *reduced* if the largest ideal contained in $\mathfrak{g}_f$ is central in $\mathfrak{g}$ and *strictly reduced* if $\ker f$ contains no non-zero ideal.

If $\mathfrak{g} \to C^\infty(\mathcal{O}_f)$ is the Lie algebra homomorphism corresponding to the Hamiltonian coadjoint action, then $\mathfrak{g}_f$ is strictly reduced if this homomorphism is injective. The largest ideal in $\mathfrak{g}_f$ contains those elements of $\mathfrak{g}$ which correspond to constant Hamiltonian functions on $\mathcal{O}_f$. Note that this assumption is only for technical convenience since one can always mod out the largest ideal $\mathfrak{a}$ in $\ker f$ without affecting the coadjoint orbit which lies in $\mathfrak{a}^\perp \cong (\mathfrak{g}/\mathfrak{a})^* \subseteq \mathfrak{g}^*$. Then $\mathfrak{g}_f$ is strictly reduced in $\mathfrak{g}/\mathfrak{a}$ and therefore isomorphic to a Lie algebra of Hamitonian functions on $\mathcal{O}_f$. It follows in particular that the largest ideal in $\mathfrak{g}_f$ is central because it consists of constant functions on $\mathcal{O}_f$.

LEMMA 5.15. *If there exists an admissible element $f \in \mathfrak{g}^*$ such that $\mathfrak{g}_f$ is strictly reduced, then the Lie algebra $\mathfrak{g}$ is admissible, i.e., $\mathfrak{g} \oplus \mathbf{R}$ contains pointed generating invariant cones. It follows in particular that $\mathfrak{g}$ is quasihermitean.*

Proof. Let $C := \overline{\operatorname{conv} \mathcal{O}_f}$. Since $\mathfrak{g}_f$ is strictly reduced, the subspace $C^\perp \subseteq \mathfrak{g}$ is trivial, i.e., C generates $\mathfrak{g}^*$ as a vector space. Furthermore, since f is admissible, C contains no lines. Hence $W := (C \times \{1\})^\star \subseteq \mathfrak{g}^* \times \mathbf{R}$ is a generating invariant cone (Proposition 1.11). An element (X, t) is contained in $H(W)$ if and only if $H_X(C) + t = \{0\}$, i.e., if the Hamiltonian function H_X is constant on C. In view of the remark above, this is only possible if $X = 0$ or if $t \neq 0$. So there are two cases. If $H(W) = \{0\}$, then W is pointed and generating, and $\mathfrak{g}$ is clearly admissible. If

$H(W) \neq \{0\}$, then $H(W)$ is a one-dimensional ideal which intersects $\mathfrak{g}$ trivially. Hence $W \cap \mathfrak{g}$ is a pointed generating invariant cone in $\mathfrak{g}$ and therefore $\mathfrak{g}$ is admissible because $(W \cap \mathfrak{g}) \times \mathbf{R}^+$ is a pointed generating invariant cone in $\mathfrak{g} \times \mathbf{R}$.

That an admissible Lie algebra is quasihermitean and has cone potential follows from [HHL89, III.2.14, III.6.18] (cf. also [Ne93a, Prop. III.15]). □

LEMMA 5.16. *Let λ be a non-compact root, $Z \in \mathfrak{g}_{\mathbb{C}}^{\lambda}$, $X = \frac{1}{2}(Z + \overline{Z})$, and $f \in \mathfrak{t}^*$. Then*

$$p_{\mathfrak{t}}(f \circ e^{\mathbf{R}^+ \operatorname{ad} X}) = f + \mathbf{R}^+ f([Z, \overline{Z}])\lambda = f + \mathbf{R}^+ f(i[\overline{Z}, Z])i\lambda.$$

Proof. We have to calculate $\langle f, e^{\operatorname{ad} X} E\rangle$ for $E \in \mathfrak{t}$. Using the formulas from [HiNe93a, 7.8], we find that

$$\begin{aligned}\langle f, e^{\operatorname{ad} X} E\rangle &= \langle f, \cosh(\operatorname{ad} X)E\rangle \\ &= f(E) + i\lambda(E)\frac{\cosh\sqrt{\frac{1}{2}\lambda([Z,\overline{Z}])} - 1}{\frac{1}{2}\lambda([Z,\overline{Z}])}\frac{-i}{2} f([Z,\overline{Z}]) \\ &= f(E) + \frac{\cosh\sqrt{\frac{1}{2}\lambda([Z,\overline{Z}])} - 1}{\lambda([Z,\overline{Z}])} f([Z,\overline{Z}])\lambda(E).\end{aligned}$$

Note that this formula yields $\langle f, e^{\operatorname{ad} X} E\rangle = f(E) + \frac{1}{4} f([Z,\overline{Z}])\lambda(E)$ for $\lambda([Z,\overline{Z}]) = 0$. In both cases we use the fact that $[zZ, \overline{zZ}] = |z|^2[Z,\overline{Z}]$ to complete the proof of the assertion. □

THEOREM 5.17. (The Convexity Theorem for coadjoint orbits) *Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan algebra, and $f \in \mathfrak{t}^*$ an admissible element such that $\mathfrak{g}_f$ is strictly reduced. Then there exists a $\mathfrak{k}$-adapted positive system Λ^+ with $f \in C_{\min}^{\star}$. For every admissible element $f \in C_{\min}^{\star}$ we have*

$$p_{\mathfrak{t}}(\mathcal{O}_f) = \operatorname{conv} \mathcal{W}.f + \operatorname{cone}(i\Lambda_f^+),$$

where

$$\Lambda_f^+ = \{\alpha \in \Lambda_p^+ : (\exists \gamma \in \mathcal{W})(\exists X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha})\ \langle f, \gamma.i[X_\alpha, \overline{X}_\alpha]\rangle < 0\}.$$

Proof. According to Theorem 5.11, the coadjoint orbit $\mathcal{O}_f$ is simply connected, the moment mapping $p_{\mathfrak{t}} : \mathcal{O}_f \to \mathfrak{t}^*$ is proper, and the convex hull of the image contains no lines. Using Proposition 5.7, we see that the set of T-fixed points in $\mathcal{O}_f$ is given by $\mathcal{W}.f$ and by Proposition 5.6 we therefore have

$$p_{\mathfrak{t}}(\mathcal{O}_f) = \operatorname{conv} \mathcal{W}.f + \bigcap_{\gamma \in \mathcal{W}} \gamma.C_f, \tag{5.2}$$

where $C_f = \overline{\text{cone}\{-i\alpha f(i[X_\alpha, \overline{X}_\alpha]) : X_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha\}}$ (cf. Remark 5.4).

Let $C := \overline{\text{conv}\, \mathcal{O}_f}$. This is a closed convex invariant subset of $\mathfrak{g}^*$ which contains no lines. Hence $W := \lim(C)^\star$ is a generating invariant cone in $\mathfrak{g}^*$. Let $\mathfrak{z}$ denote the center of $\mathfrak{k}$. Then $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{z}, \mathfrak{g}]$ since $\mathfrak{g}$ is quasihermitean (Lemma 5.15) and therefore $\text{int}\, W \cap \mathfrak{z} \neq \emptyset$ follows from [Ne93a, Th. I.10].

Pick $X \in \text{int}\, W \cap \mathfrak{z}$ such that $\ker \text{ad}\, X = \mathfrak{k}$ (the set of all such elements is dense in $\mathfrak{z}$). Then $\alpha(X) \neq 0$ holds for all non-compact roots α (Lemma 5.15, [Ne93a, Prop. II.20]). We conclude that there exists a positive system Λ^+ such that $\Lambda_p^+ = \{\alpha \in \Lambda_p : i\alpha(X) > 0\}$ (cf. [Ne93d, Lemma II.2]). Moreover, by Lemma II.4 in [Ne93d] we may even choose Λ^+ in such a way that

$$f(i[X_\alpha, \overline{X}_\alpha]) \geq 0$$

holds for all $\alpha \in \Lambda_k^+$. Now the fact that X is fixed by $\mathcal{W}$ yields that Λ^+ is $\mathfrak{k}$-adapted.

According to the choice of X, the function H_X on $p_\mathfrak{t}(\mathcal{O}_f)$ and therefore also on C takes its minimal value in f because H_X is proper on C by Proposition 1.12, it is constant on the Weyl group orbits in $\mathfrak{t}^*$, and $\mathcal{W}.f$ is the set of extreme points of $p_\mathfrak{t}(\mathcal{O}_f)$ (Theorem 4.1, Proposition 5.7). It follows that even $X \in C_f^\star$. Hence $-i\alpha(X) f(i[X_\alpha, \overline{X}_\alpha]) \geq 0$ for all $\alpha \in \Lambda_p^+$, $X_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha$. Since $i\alpha(X) > 0$, it follows that $f(i[X_\alpha, \overline{X}_\alpha]) \leq 0$, i.e., $f \in C_{\min}^\star$.

Now let $f \in C_{\min}^\star$ be an admissible element, where $\mathfrak{g}_f$ need not necessarily be reduced. In view of what has already been shown above, it remains to calculate the cone which occurs in (5.2). We claim that $\bigcap_{\gamma \in \mathcal{W}} \gamma.C_f = \text{cone}(i\Lambda_f^+)$. According to the special choice of Λ^+,

$$C_f \subseteq \text{cone}(i\Lambda_f^+) - \text{cone}(i\Lambda_k^+).$$

Moreover the set Λ_f^+ is by definition invariant under the Weyl group. Hence

$$\lim\,(p_\mathfrak{t}(\mathcal{O}_f)) = \bigcap_{\gamma \in \mathcal{W}} \gamma.C_f \subseteq \bigcap_{\gamma \in \mathcal{W}} \gamma.(\text{cone}(i\Lambda_f^+) - \text{cone}(i\Lambda_k^+)) = \text{cone}(i\Lambda_f^+)$$

follows from Proposition II.12 in [Ne92].

On the other hand let $\alpha \in \Lambda_f^+$ and $\gamma \in \mathcal{W}$ with $\langle \gamma.f, [X_\alpha, \overline{X}_\alpha] \rangle < 0$. Then Lemma 5.16 yields that $i\alpha \in \lim p_\mathfrak{t}(\mathcal{O}_{\gamma.f}) = \lim p_\mathfrak{t}(\mathcal{O}_f)$. Therefore

$$\lim\,(p_\mathfrak{t}(\mathcal{O}_f)) = \text{cone}(i\Lambda_f^+).$$

□

REMARK 5.18. Suppose that $\mathfrak{g}$ has cone potential. If, in addition to the assumptions of Theorem 5.17, $f \in \text{int}\, C_{\min}^\star$, then $f(i[X_\alpha, \overline{X}_\alpha]) < 0$ holds whenever $X_\alpha \neq 0$. Hence $\Lambda_f^+ = \Lambda_p^+$ and therefore

$$p_\mathfrak{t}(\mathcal{O}_f) = \text{conv}\, \mathcal{W}.f + \text{cone}(i\Lambda_p^+).$$

□

The preceding theorem generalizes the Convexity Theorem of Paneitz for Hermitean simple Lie algebras (cf. [Pa84]) and Kostant's Convexity Theorem for compact Lie algebras to arbitrary Lie algebras. Moreover it even generalizes Paneitz's Theorem for Hermitean Lie algebras because our theorem applies also to elements lying on the boundary of the cone $C^\star_{\min}$ and not necessarily on the interior. If $\mathfrak{g}$ is a compact Lie algebra, then $\Lambda_p = \emptyset$ and therefore $p_\mathfrak{t}(\mathcal{O}_f) = \operatorname{conv} \mathcal{W}.f$ for every $f \in \mathfrak{t}^*$.

So far we have an explicit description of the image of the moment mapping $p_\mathfrak{t}: \mathcal{O}_f \to \mathfrak{t}^*$ if f is admissible. Next we will characterize the admissible elements. The preceding theorem tells us that we have to search among those which are lying in a cone $C^\star_{\min}$ for a $\mathfrak{k}$-adapted positive system. We will see below that this condition describes precisely the set of admissible elements.

LEMMA 5.19. *Let $\mathfrak{g}$ be a Hermitean simple Lie algebra, $\mathfrak{t} \subseteq \mathfrak{g}$ a compactly embedded Cartan algebra, Λ^+ a $\mathfrak{k}$-adapted positive system of roots and $f \in C^\star_{\min} \subseteq \mathfrak{t}^*$. Then f is admissible.*

Proof. Let $Z \in \mathfrak{z}(\mathfrak{k})$ be a non-zero element such that $i\alpha(Z) < 0$ holds for all non-compact positive roots (this fixes the choice of sign), $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}_0$ a Cartan decomposition of $\mathfrak{g}$, θ the corresponding involution, and $\kappa_\theta(X, Y) := -\kappa(X, \theta Y)$ the associated scalar product, where κ is the Cartan Killing form of $\mathfrak{g}$.

For $g = \exp Y k$, $k \in K$, and $Y \in \mathfrak{p}_0$, we find that $\mathrm{Ad}(g).Z = e^{\operatorname{ad} Y} Z$ and

$$\kappa_\theta(e^{\operatorname{ad} Y} Z, Z) = -\kappa(e^{\operatorname{ad} Y} Z, Z) \geq 0$$

holds for all $Y \in \mathfrak{p}_0$ because $\operatorname{ad} Y$ is symmetric with respect to κ_θ, hence $e^{\operatorname{ad} Y}$ is positive definite.

Let $C := \overline{\operatorname{conv} \mathrm{Ad}(G).Z}$. Then the preceding paragraph shows that $C \neq \mathfrak{g}$. Hence $H(C)$ is an ideal of $\mathfrak{g}$ which is different from $\mathfrak{g}$, whence $H(C) = \{0\}$ because $\mathfrak{g}$ is simple. We conclude that C contains no lines.

Next we use κ to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by assigning to $X \in \mathfrak{g}$ the functional $\widehat{X}(Y) := \kappa(X, Y)$. Then it follows that $\widehat{Z}$ is admissible and the Convexity Theorem 5.17 yields that

$$p_\mathfrak{t}(\mathcal{O}_{\widehat{Z}}) \subseteq \widehat{Z} + \operatorname{cone}(i\Lambda_p^+)$$

whenever $\widehat{Z} \in C^\star_{\min}$.

On the other hand we have $i\alpha(i[\overline{X}_\alpha, X_\alpha]) > 0$ for $X_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha \setminus \{0\}$ and $\alpha \in \Lambda_p^+$ (cf. [HiNe93a, Th. 7.4]) and therefore

$$\kappa(i[\overline{X}_\alpha, X_\alpha], Y) = \kappa(i\overline{X}_\alpha, [X_\alpha, Y]) = -i\alpha(Y)\kappa(\overline{X}_\alpha, X_\alpha)$$

together with the negative definiteness of κ on $\mathfrak{t}$ yields that $i[\overline{X}_\alpha, X_\alpha]\widehat{\ }$ is a positive multiple of $-i\alpha \in \mathfrak{t}^* \subseteq \mathfrak{g}^*$. Thus $\widehat{Z} \in C^\star_{\min}$ and the projection of $\mathrm{Ad}(G)Z$ onto $\mathfrak{t}$ is contained in the set $Z - C_{\min}$.

Now we turn to the functional $f \in C_{\min}^{\star}$. Since $f(Z - C_{\min}) \subseteq]-\infty, f(Z)]$, the functional $-f$ is contained in $B(C)$. This set is invariant under the coadjoint action. Hence $-\mathcal{O}_f \subseteq \overline{B(C)} = \lim(C)^{\star}$ and since the cone $\lim(C)$ is generating, the cone $\lim(C)^{\star}$ is pointed. We conclude that the convex hull of $\mathcal{O}_f$ contains no lines. Consequently f is admissible by Theorem 5.11(2). □

THEOREM 5.20. *Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan algebra and $f \in \mathfrak{t}^*$ such that $\mathfrak{g}_f$ is strictly reduced. Then f is admissible if and only if $\mathfrak{g}$ is admissible and there exists a $\mathfrak{k}$-adapted positive system such that $f \in C_{\min}^{\star}$.*

Proof. In view of Theorem 5.17, we only have to show that f is admissible if $f \in C_{\min}^{\star}$ holds for a $\mathfrak{k}$-adapted positive system.

To see this, we first pick a $\mathfrak{t}$-invariant Levi decomposition $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$, where $\mathfrak{r}$ denotes the radical of $\mathfrak{g}$ ([HiNe93a, Prop. 7.3]) and $\mathfrak{t} = (\mathfrak{t} \cap \mathfrak{r}) \oplus (\mathfrak{t} \cap \mathfrak{s})$. We set $\mathfrak{t}_{\mathfrak{r}} := \mathfrak{t} \cap \mathfrak{r}$ and $\mathfrak{t}_{\mathfrak{s}} := \mathfrak{t} \cap \mathfrak{s}$ and write S and R for the subgroups of G corresponding to $\mathfrak{r}$ and $\mathfrak{s}$. Then we use Lemma I.17 in [Ne93b] to see that

$$\mathcal{O}_f = \mathcal{O}_{f_r} + \mathcal{O}_{f_s},$$

where $f_r := f\mid_{\mathfrak{t}_{\mathfrak{r}}} \in \mathfrak{t}_{\mathfrak{r}}^* \subseteq \mathfrak{g}^*$, $f_{\mathfrak{s}} := f\mid_{\mathfrak{t}_{\mathfrak{s}}} \in \mathfrak{t}_{\mathfrak{s}}^* \subseteq \mathfrak{g}^*$, $\mathcal{O}_{f_r} = \mathrm{Ad}^*(R).f_r$, and $\mathcal{O}_{f_s} = \mathrm{Ad}^*(S).f_s$.

In view of Theorem 5.11, it only remains to show that the closed convex hull of $p_{\mathfrak{t}}(\mathcal{O}_f) = p_{\mathfrak{t}}(\mathcal{O}_{f_r}) + p_{\mathfrak{t}}(\mathcal{O}_{f_s})$ contains no lines. This follows from the claim that

$$p_{\mathfrak{t}}(\mathcal{O}_f) \subseteq \mathrm{conv}(\mathcal{W}.f) + \mathrm{cone}(i\Lambda_p^+).$$

We first deal with the semisimple part. Since $\mathfrak{g}$ permits a $\mathfrak{k}$-adapted positive system, it is a quasihermitean Lie algebra. Therefore it contains only Hermitean and compact simple ideals. Accordingly we write f_s as $f_1 + \ldots + f_k$, where $\mathfrak{s} = \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_k$ is the decomposition into simple ideals. If $\mathfrak{s}_j$ is compact, then $\mathcal{O}_{f_j}$ is compact and $p_{\mathfrak{t}}(\mathcal{O}_{f_j}) = \mathrm{conv}\, \mathcal{W}.f$ by Theorem 5.17. If $\mathfrak{s}_j$ is Hermitean, then Lemma 5.19 shows that f_j is admissible, and since there exist only 2 possibilities for a $\mathcal{W}$-invariant set Λ_p^+ in $\mathfrak{s}_j$, it follows from Theorem 5.17 that $p_{\mathfrak{t}}(\mathcal{O}_{f_j}) \subseteq \mathrm{conv}(\mathcal{W}.f_j) + \mathrm{cone}(i\Lambda_p^+)$. Finally we use that

$$\mathcal{O}_{f_s} = \mathcal{O}_{f_1} \oplus \ldots \oplus \mathcal{O}_{f_k}$$

and $\mathcal{W}.f_s = \mathcal{W}.f_1 \oplus \ldots \oplus \mathcal{W}.f_k$ to see that

$$p_{\mathfrak{t}}(\mathcal{O}_{f_s}) \subseteq \mathrm{conv}\, \mathcal{W}.f_s + \mathrm{cone}(i\Lambda_p^+).$$

Next we consider $\mathcal{O}_{f_r} = \mathrm{Ad}^*(R).f_r$. We write $\mathfrak{r} = \mathfrak{n} + \mathfrak{t}_{\mathfrak{r}}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. Then $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}(\mathfrak{g})$ by [HiNe93a, Th. 7.15] because $\mathfrak{g}$ is admissible. Since $e^{\mathrm{ad}^* \mathfrak{t}_{\mathfrak{r}}}.f_r = f_r$, we conclude that

$$\mathcal{O}_{f_r} = e^{\mathrm{ad}^* \mathfrak{n}}.f_r,$$

where $\mathrm{ad}^* \mathfrak{n} \cong \mathfrak{n}/\mathfrak{z}(\mathfrak{g})$ is abelian. Hence Lemma 5.16 applies and, using that $f \in C^*_{\min}$, we obtain that $p_{\mathfrak{t}}(\mathcal{O}_{f_r}) \subseteq f_r + \mathrm{cone}(i\Lambda_p^+)$.

Putting these results together, we find that

$$p_{\mathfrak{t}}(\mathcal{O}_f) \subseteq \mathrm{conv}(\mathcal{W}.f_s)+\mathrm{cone}(i\Lambda_p^+)+f_r+\mathrm{cone}(i\Lambda_p^+) = \mathrm{conv}(\mathcal{W}.f)+\mathrm{cone}(i\Lambda_p^+).$$

□

OLAFSSON'S CONVEXITY THEOREM

In this subsection we will show how Olafsson's Convexity Theorem ([Ola90, Th. 5.5.1]) which is a version of Paneitz's Convexity Theorem for symmetric spaces can be obtained by combining the non-compact version of Duistermaat's Convexity Theorem (Section 4) with the Convexity Theorem for coadjoint orbits (Theorem 5.17). In the compact case we obtain a proof of Kostant's Linear Convexity Theorem as a consequence of Duistermaat's Theorem and the Convexity Theorem for coadjoint orbits in compact Lie algebras (cf. [Dui83]).

LEMMA 5.21. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and σ an involutive automorphism of $\mathfrak{g}$. Further let $f \in \mathfrak{g}^*$. Then the mapping $\tau := -\sigma^*: \mathfrak{g}^* \to \mathfrak{g}^*$ induces an antisymplectic mapping $\mathcal{O}_f \to \mathcal{O}_{\tau.f}$.*

Proof. Since conjugation by σ preserves the group of inner automorphisms of $\mathfrak{g}$, the mapping τ preserves the set of coadjoint orbits, hence maps $\mathcal{O}_f$ onto $\mathcal{O}_{\tau.f}$. That this mapping is antisymplectic follows from

$$\begin{aligned}
(\tau^*\Omega)(f)(f\circ \mathrm{ad}\,X, f\circ \mathrm{ad}\,Y) &= \Omega(\tau.f)(\tau(f\circ \mathrm{ad}\,X), \tau(f\circ \mathrm{ad}\,Y)) \\
&= \Omega(\tau.f)(f\circ \mathrm{ad}\,X\circ\sigma, f\circ \mathrm{ad}\,Y\circ\sigma) \\
&= \Omega(\tau.f)(f\circ\sigma\circ \mathrm{ad}\,\sigma(X), f\circ\sigma\circ \mathrm{ad}\,\sigma(Y)) \\
&= \Omega(\tau.f)(\tau(f)\circ \mathrm{ad}\,\sigma(X), \tau(f)\circ \mathrm{ad}\,\sigma(Y)) \\
&= \tau(f)([\sigma(X),\sigma(Y)]) \\
&= \tau(f)\circ\sigma([X,Y]) = -f([X,Y]) \\
&= -\Omega(f)(f\circ \mathrm{ad}\,X, f\circ \mathrm{ad}\,Y).
\end{aligned}$$

□

Let σ be an involutive automorphism of the Lie algebra $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$, where

$$\mathfrak{h} = \{X \in \mathfrak{g}: \sigma(X) = X\} \quad \text{and} \quad \mathfrak{q} = \{X \in \mathfrak{g}: \sigma(X) = -X\}.$$

Accordingly we have a direct decomposition of $\mathfrak{g}^*$ as $\mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{q}^* = \mathfrak{q}^\perp \oplus \mathfrak{h}^\perp$.

Suppose that $\mathfrak{t} \subseteq \mathfrak{g}$ is a compactly embedded Cartan algebra which is invariant under σ. Then $\mathfrak{t} = \mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{q}}$, where $\mathfrak{t}_{\mathfrak{h}} := \mathfrak{t} \cap \mathfrak{h}$ and $\mathfrak{t}_{\mathfrak{q}} := \mathfrak{t} \cap \mathfrak{q}$ and $\mathfrak{t}^* = \mathfrak{t}^*_{\mathfrak{h}} \oplus \mathfrak{t}^*_{\mathfrak{q}}$. We consider an element $f \in \mathfrak{t}_\mathfrak{q}^*$. Then $\tau(f) = f$ and therefore τ is an antisymplectic involution of the coadjoint orbit $\mathcal{O}_f$ (Lemma 5.21).

LEMMA 5.22. *If $f \in \mathfrak{t}_\mathfrak{q}^*$ is admissible, then the projection $p_{\mathfrak{t}_\mathfrak{q}}: \mathcal{O}_f \to \mathfrak{t}_\mathfrak{q}^*$ is proper.*

Proof. First we note that the coadjoint orbit $\mathcal{O}_f$ is invariant under τ. Let $C := \overline{\operatorname{conv} \mathcal{O}_f}$. Then C is also invariant under τ. As in the proof of Lemma 5.10, since $\mathcal{O}_f$ is closed, it suffices to show that $p_{\mathfrak{t}_\mathfrak{q}}: C \to \mathfrak{t}_\mathfrak{q}^*$ is a proper mapping. Let $W := \lim(C)^\star$. Then $\operatorname{int} W \cap \mathfrak{t} \neq \emptyset$ (cf. Lemma 5.10) is invariant under $\tau^* = -\sigma$, hence there exists $X \in \operatorname{int} W \cap \mathfrak{t}_\mathfrak{q}$. Now, according to Proposition 1.17, the function $H_X: \alpha \mapsto \alpha(X)$ is proper on C. Moreover, $H_X \circ \tau = H_X$, i.e., $\ker(\tau + \mathrm{id}_{\mathfrak{g}^*}) \subseteq \ker H_X$, so that Lemma 1.15 implies that $p_{\mathfrak{t}_\mathfrak{q}} = \frac{1}{2}(\tau + \mathrm{id}_{\mathfrak{g}^*}) \circ p_\mathfrak{t}$ is a proper mapping on C. □

For the following we fix a simply connected Lie group G with $\mathbf{L}(G) = \mathfrak{g}$ and lift the automorphism σ of $\mathfrak{g}$ to an involutive automorphism σ_G of the group G. Then $H := \langle \exp \mathfrak{h} \rangle$ is the connected component of the group of fixed points of σ_G in G.

LEMMA 5.23. *Let $f \in \mathfrak{q}^*$, $\mathcal{O}_f = \mathrm{Ad}^*(G).f$ be the coadjoint orbit of f and $Q = \mathcal{O}_f \cap \mathfrak{q}^*$ the fixed point set of the antisymplectic involution τ of $\mathcal{O}_f$. Then Q is a manifold invariant under H and every H-orbit is a connected component of Q.*

Proof. Since we know already that Q is a Lagrangian submanifold of $\mathcal{O}_f$ (cf. Section 2), it suffices to show that every H-orbit in Q is open. Since f was arbitrary, it even suffices to show that $\mathrm{Ad}^*(H).f$ is open in Q.

To see this, we first note that

$$\begin{aligned} T_f(Q) &= \{f \circ \operatorname{ad} X : X \in \mathfrak{g}, f \circ \operatorname{ad} X \in \mathfrak{q}^*\} \\ &= \{f \circ \operatorname{ad} X : X \in \mathfrak{g}, f([X, \mathfrak{h}]) = \{0\}\} \\ &= (f \circ \operatorname{ad} \mathfrak{h}) + \{f \circ \operatorname{ad} X : X \in \mathfrak{q}, f([X, \mathfrak{h}]) = \{0\}\}. \end{aligned}$$

Using the fact that $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}$, it follows that $X \in \mathfrak{q}$ implies that $f([X, \mathfrak{q}]) \subseteq f(\mathfrak{h}) = \{0\}$. Hence $X \in \mathfrak{q}$ and $f([X, \mathfrak{h}]) = \{0\}$ yields $f \circ \operatorname{ad} X = 0$. Therefore $T_f(Q) = f \circ \operatorname{ad} \mathfrak{h}$ and consequently the H-orbit of f is open in Q. □

THEOREM 5.24. *Let $\mathfrak{g}$ be a Lie algebra, σ an involutive isomorphism of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ the corresponding decomposition. Further let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan algebra, Λ^+ a $\mathfrak{k}$-adapted positive system such that the cone $C^\star_{\min}$ is pointed, and $f \in \mathfrak{t}_\mathfrak{q}^* \cap C^\star_{\min}$. Then*

$$p_{\mathfrak{t}_\mathfrak{q}}(\mathrm{Ad}^*(H).f) = p_{\mathfrak{t}_\mathfrak{q}}(\mathcal{O}_f) = \operatorname{conv} p_{\mathfrak{t}_\mathfrak{q}}(\mathcal{W}.f) + \operatorname{cone}(i\Lambda_f^+|_{\mathfrak{t}_\mathfrak{q}}),$$

where

$$\Lambda_f^+ = \{\alpha \in \Lambda_p^+ : (\exists \gamma \in \mathcal{W})(\exists X_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha)\, \langle f, \gamma.[X_\alpha, \overline{X_\alpha}] \rangle < 0\}.$$

Proof. First we note that f is admissible by Theorem 5.20. We consider the symplectic manifold $M = \mathcal{O}_f$. Then $\tau = -\sigma^*$ induces an antisymplectic isomorphism of M (Lemma 5.21) and the moment mapping for the action of the group $T_q := e^{\mathrm{ad}\, \mathfrak{t}_\mathfrak{q}}$, namely the projection $p_{\mathfrak{t}_\mathfrak{q}}\colon M \to \mathfrak{t}_\mathfrak{q}^*$ is a proper mapping (Lemma 5.22).

Moreover M is simply connected by Theorem 5.11 and therefore Corollary 4.2 applies in view of Proposition 4.3. To apply Corollary 4.5, we note that, according to Lemma 5.23, the H-orbit of f is a connected component of Q. Now Corollary 4.5 shows that $p_{\mathfrak{t}_\mathfrak{q}}(\mathrm{Ad}^*(H).f) = p_{\mathfrak{t}_\mathfrak{q}}(\mathcal{O}_f)$. The second equality follows from Theorem 5.17. □

COROLLARY 5.25. *We keep the assumptions from* Theorem 5.24 *and assume in addition that* $\mathfrak{t} \subseteq \mathfrak{g}$ *is a compactly embedded Cartan algebra such that* $\dim \mathfrak{t}_\mathfrak{q}$ *is maximal. Then*

$$p_{\mathfrak{t}_\mathfrak{q}}(\mathrm{Ad}^*(H).f) = \mathrm{conv}(\widetilde{\mathcal{W}}.f) + \mathrm{cone}(i\Lambda_f^+ |_{\mathfrak{t}_\mathfrak{q}}),$$

where

$$\Lambda_f^+ = \{\alpha \in \Lambda_p^+ \colon (\exists \gamma \in \mathcal{W})(\exists X_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha)\, \langle f, \gamma.[X_\alpha, \overline{X}_\alpha]\rangle < 0\}$$

and $\widetilde{\mathcal{W}}$ *is the subgroup of* $\mathrm{Gl}(\mathfrak{t}_\mathfrak{q})$ *generated by the reflections in the hyperplanes which are non-zero restrictions of compact roots to* $\mathfrak{t}_\mathfrak{q}$.

Proof. In view of Theorem 5.24, it only remains to show that $p_{\mathfrak{t}_\mathfrak{q}}(\mathcal{W}.f) = \widetilde{\mathcal{W}}.f$. According to [Ne92, Th. II.15], it suffices to check that for every compact root α with $\alpha\,|_{\mathfrak{t}_\mathfrak{q}} \neq 0$, there exists $\gamma \in \mathcal{W}$ such that $\gamma(\mathfrak{t}_\mathfrak{q}) \subseteq \mathfrak{t}_\mathfrak{q}$ and $\gamma\,|_{\mathfrak{t}_\mathfrak{q}}$ is the associated reflection in $\ker(\alpha\,|_{\mathfrak{t}_\mathfrak{q}})$.

To do so, we first note that the invariance of $\mathfrak{t}$ under σ and the fact that $\mathfrak{k}$ is the unique maximal compactly embedded subalgebra containing $\mathfrak{t}$ ([HHL89, A.2.40]) imply that $\sigma(\mathfrak{k}) = \mathfrak{k}$. Moreover the automorphism σ leaves the center and the commutator algebra of $\mathfrak{k}$ invariant. Write $\mathfrak{k} = \mathfrak{k}_\mathfrak{h} + \mathfrak{k}_\mathfrak{q}$ and $\mathfrak{z} = \mathfrak{z}_\mathfrak{h} + \mathfrak{z}_\mathfrak{q}$ for $\mathfrak{z} = \mathfrak{z}(\mathfrak{k})$. Then we also have $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}_\mathfrak{h}' + \mathfrak{k}_\mathfrak{q}'$ and according to our assumption on $\mathfrak{t}_\mathfrak{q}$, this subalgebra is a maximal abelian subspace of $\mathfrak{k}_\mathfrak{q}$. Since every Cartan algebra contains the center, it even follows that $\mathfrak{t}_\mathfrak{q}' := \mathfrak{t}_\mathfrak{q} \cap \mathfrak{k}_\mathfrak{q}'$ is maximal abelian in $\mathfrak{k}_\mathfrak{q}'$.

Now $(\mathfrak{k}', \mathfrak{k}_\mathfrak{h}')$ is a semisimple symmetric pair of compact type and $\widetilde{\mathcal{W}}$ is the Weyl group corresponding to the system of restricted roots on $\mathfrak{t}_\mathfrak{q}'$. Using [Hel78, p.289] we see that every element of $\widetilde{\mathcal{W}}$ is induced by an element in $e^{\mathrm{ad}\, \mathfrak{k}_\mathfrak{h}}$, hence by an element in $\mathcal{W}$ which leaves $\mathfrak{t}_\mathfrak{q}$ invariant. This proves the assertion. □

REMARK 5.26. So far we have simplified the formula for the set of extreme points of the image of an H-orbit. If $f \in \mathrm{int}\, C_{\mathrm{min}}^*$, then $\Lambda_f^+ = \Lambda_p^+$ and therefore the limit cone of the image is given by $\mathrm{cone}(\Lambda_p^+ |_{\mathfrak{t}_\mathfrak{q}})$, the cone generated by the restrictions of the non-compact roots to $\mathfrak{t}_\mathfrak{q}$. Thus we have

$$p_{\mathfrak{t}_\mathfrak{q}}(\mathrm{Ad}^*(H).f) = \mathrm{conv}(\widetilde{\mathcal{W}}.f) + \mathrm{cone}(i\Lambda_p^+ |_{\mathfrak{t}_\mathfrak{q}}).$$

If, in addition, $\mathfrak{g} = \mathfrak{k}$ is a compact Lie algebra, then $\Lambda_p^+ = \emptyset$ and therefore

$$p_{\mathfrak{t}_{\mathfrak{q}}}\big(\mathrm{Ad}^*(H).f\big) = \mathrm{conv}(\widetilde{\mathcal{W}}.f)$$

is the Linear Convexity Theorem of Kostant in the version for symmetric spaces of compact type. □

In the theory of symmetric spaces it is often convenient to pass from the symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ to the c-dual Lie algebra $(\mathfrak{g}^c, \mathfrak{h}, \sigma^c)$, where $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ and $\sigma^c(X + iY) = X - iY$. To explain how Olafsson's Convexity Theorem follows from our results, we assume from now on that $\mathfrak{g}$ is a semisimple Lie algebra endowed with an involutive automorphism σ. We choose a Cartan involution θ commuting with τ ([Lo69, p.153]) so that we obtain a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and a direct decomposition

$$\mathfrak{g} = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{h}_{\mathfrak{p}} + \mathfrak{q}_{\mathfrak{k}} + \mathfrak{q}_{\mathfrak{p}}.$$

We choose a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{q}_{\mathfrak{p}}$. Then $\mathfrak{g}^c = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{h}_{\mathfrak{p}} + i\mathfrak{q}_{\mathfrak{k}} + i\mathfrak{q}_{\mathfrak{p}}$ and $\mathfrak{k}^c = \mathfrak{h}_{\mathfrak{k}} + i\mathfrak{q}_{\mathfrak{p}}$ is a maximal compactly embedded subalgebra of $\mathfrak{g}^c$. Therefore $i\mathfrak{a}$ is contained in a Cartan algebra $\mathfrak{t}$ of $\mathfrak{k}^c$ which is invariant under σ^c and which satisfies $\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}} \oplus i\mathfrak{a}$, where $\mathfrak{t}_{\mathfrak{k}} = \mathfrak{t} \cap \mathfrak{k}$.

Let $X \in \mathfrak{q}_{\mathfrak{p}}$. For many applications in representation theory it is important to have some information on the projection of the H-orbit $\mathrm{Ad}(H).X \subseteq \mathfrak{q}$ onto $\mathfrak{a}$ along the orthogonal complement of $\mathfrak{a}$ with respect to the Cartan Killing form. We explain how this can be turned into a problem on a coadjoint orbit endowed with an antisyplectic involution.

We consider $\mathfrak{g}$ and $\mathfrak{g}^c$ as subalgebras of the complexification $\mathfrak{g}_{\mathbb{C}}$. Let B denote the Cartan Killing form of $\mathfrak{g}_{\mathbb{C}}$ and $\kappa = \mathrm{Im}\, B$ its imaginary part. Then κ is a real symmetric invariant bilinear form on $\mathfrak{g}_{\mathbb{C}}$. The restriction to the real form $\mathfrak{g}^c$ is therefore 0, so that we can identify $(\mathfrak{g}^c)^*$ with the subspace $i\mathfrak{h} + \mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$. More explicitly, we have an isomorphism $\psi\colon i\mathfrak{h} + \mathfrak{q} \to (\mathfrak{g}^c)^*$ satisfying $\psi(Y)(Z) = \kappa(Y, Z)$ which is equivariant with respect to $\mathrm{Ad}(G^c)$. Let $X \in \mathfrak{q}$. Then

$$\psi\big(\mathrm{Ad}(H).X\big) = \mathrm{Ad}^*(H).\psi(X) \subseteq (i\mathfrak{q})^*$$

and the above problem reduces to the determination of the image of the set $\mathrm{Ad}^*(H).\psi(X)$ under the restriction mapping to $(i\mathfrak{a})^* = \psi(\mathfrak{a})$.

To meet the assumptions we have made above, one needs that $\mathfrak{g}^c$ is a quasihermitean Lie algebra, i.e., a sum of compact and Hermitean simple ideals. This is clearly satisfied if either $\sigma = \theta$ is a Cartan involution (in this case $\mathfrak{g}^c$ is compact) and in the setting of Olafsson's theorem. In this case every maximal compactly embedded subalgebra $\mathfrak{u}$ of $\mathfrak{g}^c$ has full rank so that the Cartan subalgebras of $\mathfrak{u}$ are compactly embedded Cartan algebras in $\mathfrak{g}^c$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition compatible with the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$, $\mathfrak{a} \subseteq \mathfrak{q}_{\mathfrak{p}}$ a maximal abelian subspace, $\mathfrak{t} \subseteq \mathfrak{k} + i\mathfrak{p}$ a Cartan subalgebra containing $i\mathfrak{a}$, and Λ^+ a $\mathfrak{k}$-adapted positive system of roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$. We set $\Sigma := \{\alpha\,|_{\mathfrak{a}}: \alpha \in \Lambda\} \setminus \{0\}$ and define Σ_p and Σ_k accordingly. Let $\mathcal{W}_{\mathfrak{a}}$ denote the Weyl group of the restricted root system Σ_k and Σ_p^+ the set of non-zero restrictions of positive non-compact roots. We define

$$c_{\max} := -(\Sigma_p^+)^\star = \{X \in \mathfrak{a}: (\forall \alpha \in \Sigma_p^+)\; \alpha(X) \leq 0\}$$

and

$$c_{\min} := \{X \in \mathfrak{a}: (\forall Y \in c_{\max})\; B(X,Y) \geq 0\}.$$

Let $\alpha \in \Sigma_p^+$ and $\alpha' \in \Lambda_p^+$ with $\alpha'\,|_{\mathfrak{a}} = \alpha$. Pick a non-zero element $X_{\alpha'} \in \mathfrak{g}_{\mathbb{C}}^{\alpha'}$ and $Y \in \mathfrak{a}$. Then

$$\begin{aligned}
\langle \psi(Y), i[\overline{X_{\alpha'}}, X_{\alpha'}]\rangle &= \kappa(Y, i[\overline{X_{\alpha'}}, X_{\alpha'}]) \\
&= \kappa([Y, i\overline{X_{\alpha'}}], X_{\alpha'}) \\
&= -\alpha'(Y)\kappa(i\overline{X_{\alpha'}}, X_{\alpha'}) \\
&= -\alpha'(Y)\operatorname{Re} B(\overline{X_{\alpha'}}, X_{\alpha'}) \\
&= -\frac{1}{2}\alpha'(Y)\operatorname{Re} B(X_{\alpha'} + \overline{X_{\alpha'}}, X_{\alpha'} + \overline{X_{\alpha'}}) \\
&= -\frac{1}{2}\alpha'(Y) B(X_{\alpha'} + \overline{X_{\alpha'}}, X_{\alpha'} + \overline{X_{\alpha'}})
\end{aligned}$$

and $B(X_{\alpha'} + \overline{X_{\alpha'}}, X_{\alpha'} + \overline{X_{\alpha'}}) > 0$ because $X_{\alpha'} + \overline{X_{\alpha'}} \in \mathfrak{p}$ and B is positive definite on $\mathfrak{p}$.

We conclude that $\psi(Y) \in C_{\min}^\star$ if and only if $\alpha(Y) \leq 0$ for all $\alpha \in \Sigma_p^+$. Therefore $\psi(c_{\max}) = C_{\min}^\star \cap \psi(\mathfrak{a})$. Moreover, for $\alpha \in \Sigma_p^+$ and $Y \in \mathfrak{a}$ we have

$$B(\psi^{-1}(i\alpha), Y) = \kappa(\psi^{-1}(i\alpha), iY) = -\alpha(Y)$$

so that the above calculation yields $\mathbf{R}^+\psi^{-1}(i\alpha) = \mathbf{R}^+[\overline{X_{\alpha'}}, X_{\alpha'}]$. Consequently

$$\psi^{-1}(\operatorname{cone} i\Sigma_p^+) = p_{\mathfrak{a}}(\operatorname{cone}\{[\overline{X_{\alpha'}}, X_{\alpha'}]: \alpha \in \Sigma_p^+\}).$$

This can also be expressed as follows. Choose $A_{\alpha'} \in \mathfrak{a}$ such that $B(A_{\alpha'}, Y) = \alpha'(Y)$ for all $Y \in \mathfrak{a}$. Then $A_{\alpha'} \in \mathbf{R}^+[X_{\alpha'}, \overline{X_{\alpha'}}]$ and

$$\psi^{-1}(\operatorname{cone} i\Sigma_p^+) = -\operatorname{cone}\{A_{\alpha'}: \alpha' \in \Sigma_p^+\} = c_{\min}.$$

THEOREM 5.27. *Let $\mathfrak{g}$ be a semisimple Lie algebra, σ an involutive isomorphism of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ the corresponding decomposition and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a compatible Cartan decomposition. Further let $\mathfrak{a} \subseteq \mathfrak{q}_{\mathfrak{p}}$ be a maximal abelian subspace, $\mathfrak{t} \subseteq \mathfrak{k} + i\mathfrak{p}$ a Cartan subalgebra containing $i\mathfrak{a}$, and Λ^+ a $\mathfrak{k}$-adapted positive system of roots of*

$\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ such that $X \in \operatorname{int} c_{\max}$ holds for the associated system Σ^+ of positive restricted roots. Then

$$p_{\mathfrak{a}}(\operatorname{Ad}(H).X) = \operatorname{conv}\left(\mathcal{W}_{\mathfrak{a}}.X\right) + c_{\min},$$

where $\mathcal{W}_{\mathfrak{a}}$ is the subgroup of $\mathrm{Gl}(\mathfrak{a})$ *generated by the reflections in the hyperplanes which are non-zero restrictions of compact roots to $\mathfrak{a}$.*

Proof. In view of above considerations, this is just a restatement of Corollary 5.25 in the above context. □

REMARK 5.28. (a) As Corollary 5.25 shows, the above result can be generalized to the case where $X \in \partial c_{\max}$, but in this case the image is harder to describe.
(b) If $(\mathfrak{g}, \mathfrak{h})$ is an irreducible symmetric Lie algebra of regular type, i.e., $\mathfrak{g}^c$ is a Hermitean simple Lie algebra or the complexification of a simple hermitean Lie algebra, then Theorem 5.27 is Olafsson's Convexity Theorem.
(c) If $\mathfrak{g}$ is a semisimple Lie algebra and $\theta = \sigma$ a Cartan involution, then $\Lambda_p = \emptyset$ and therefore Theorem 5.27 simply states that

$$p_{\mathfrak{a}}(\operatorname{Ad}(K).X) = \operatorname{conv}(\mathcal{W}_{\mathfrak{a}}.X) \qquad \forall X \in \mathfrak{a}$$

which is Kostant's Linear Convexity Theorem. □

§ 6. Hamiltonian actions of compact Lie groups

The main result of this section is a strengthened version of Kirwan's convexity theorem (cf. [Ki84]).

REMARK 6.1. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and consider the root system $\Lambda(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \subseteq i\mathfrak{t}^*$. We choose a set of positive roots $\Lambda^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and denote the corresponding set of simple roots by Δ. We can view $\mathfrak{t}^*$ as a subset of $\mathfrak{g}^*$ extending linear forms on $\mathfrak{t}^*$ by zero on the orthogonal complement with respect to the Killing form. Moreover we denote the Euclidean scalar product defined on $i\mathfrak{t}^*$ by duality and the Killing form which we denote by $(\cdot \mid \cdot)$. □

Let $f \in \mathfrak{t}^*$. Then the Lie algebra of G_f is given by

$$\mathfrak{g}_f = \mathfrak{t} + \sum_{\beta \in \Lambda(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}), (if \mid \beta) = 0} \mathfrak{g} \cap (\mathfrak{g}_{\mathbb{C}}^{\beta} + \mathfrak{g}_{\mathbb{C}}^{-\beta}). \tag{6.1}$$

In fact, if $0 \neq X_\beta \in \mathfrak{g}_{\mathbb{C}}^{\beta}$ are chosen in such a way that

$$\langle \nu, [X_\beta, X_{-\beta}] \rangle = (\nu \mid \beta)$$

for all $\nu \in i\mathfrak{t}^*$, this follows from the following calculation

$$\langle f, [X_{\mathfrak{t}} + \sum_{\beta \in \Lambda(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} c_\beta X_\beta, X_{-\beta_o}] \rangle = \langle f, c_{\beta_o}[X_{\beta_o}, X_{-\beta_o}] \rangle = -ic_{\beta_o}(if \mid \beta_o).$$

PROPOSITION 6.2. *Let $f \in \mathfrak{g}^*$. Then the stabilizer G_f of f in G is connected.*

Proof. If we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via a G-invariant form. Then we have

$$G_f = Z_G(f) = Z_G(\overline{\exp \mathbf{R} f})$$

and this centralizer is connected according to [Hel78, Cor. VII.2.8] since $\overline{\exp \mathbf{R} f}$ is a toral subgroup of G. □

An element $f \in \mathfrak{t}^*$ is *regular* if $\mathfrak{g}_f = \mathfrak{t}$, which is the same as saying $(if \mid \beta) \neq 0$ for all $\beta \in \Lambda(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ or $G_f = T$, where T is the maximal torus in G corresponding to $\mathfrak{t}$. We choose a Weyl chamber $\mathfrak{t}_o^*$ in $\mathfrak{t}^*$ via

$$\mathfrak{t}_o^* := \{f \in \mathfrak{t}^* \colon (\forall \beta \in \Delta)\, (if \mid \beta) > 0\}.$$

We denote its closure by $\mathfrak{t}_+^*$.

Below we will have to deal with faces of $\mathfrak{t}_+^*$, so we note right away that these correspond to the subsets of Δ via

$$F \mapsto \Delta_F := \{\alpha \in \Delta : (\forall f \in F)\, (if \mid \alpha) = 0\}$$

and

$$\Sigma \mapsto F_\Sigma := \{f \in \mathfrak{t}_+^* \colon (\forall \alpha \in \Sigma)\, (if \mid \alpha) = 0\}.$$

The algebraic interior of F is then given by

$$\operatorname{algint} F = \{f \in F \colon (\forall \alpha \in \Delta \setminus \Delta_F)\ (if \mid \alpha) > 0\}.$$

LEMMA 6.3. *Let F be a face of $\mathfrak{t}_+^*$ and $f, f' \in \operatorname{algint} F$. Then the stabilizers G_f and $G_{f'}$ are equal.*

Proof. According to (6.1) and Proposition 6.2 it suffices to show that for $\beta \in \Lambda(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ we have $(if \mid \beta) = 0$ if and only if $(if' \mid \beta) = 0$. But that is true since for $f \in \operatorname{algint} F$ the equation $(if \mid \beta) = 0$ is satisfied precisely for the roots which lie in the span of Δ_F. □

For any face F of $\mathfrak{t}_+^*$ we set $G_F := G_f$, where f is any element of $\operatorname{algint} F$. Moreover we set

$$\Sigma_j := \{f \in \mathfrak{g}^* \colon \dim G.f = j\}.$$

REMARK 6.4. The Σ_j are submanifolds of $\mathfrak{g}^*$. This is a consequence of the slice theorem (cf. [GS84, p.201]) because for every slice S through $f \in \mathfrak{g}^*$ the set of all elements with orbits of the same dimension is given by those elements in S which are fixed by G_f, and for a linear action the set of fixed points is a subspace. The intersections of the Σ_j with $\mathfrak{t}_+^*$ are algebraic interiors of certain faces which have the right amount of singularity. More precisely, we have

$$\Sigma_j \cap \mathfrak{t}_+^* = \bigcup_{\dim G_F = \dim G - j} \operatorname{algint} F.$$

Let now F be a face with $\dim G - \dim G_F = j$ and $f \in \operatorname{algint} F$. Then we have

$$T_f(G.f) + T_f(\operatorname{algint} F) = T_f \Sigma_j. \tag{6.2}$$

□

We collect a few facts from [GS84]: Note first (cf. [GS84, p.184]) that

$$r := \max_{x \in M}\{\operatorname{rk}(d\Phi_x)\} = \max_{x \in M}\{\dim G.x\} = \dim \mathfrak{g} - \min_{x \in M}\{\dim G_x\}.$$

Then [GS84, Prop. 27.3] implies that the set

$$M_{\text{reg}} := \{x \in M \colon \operatorname{rk}(d\Phi_x) = r\}$$

is open dense and connected in M. Note that $\Phi\colon M_{\text{reg}} \to \mathfrak{g}^*$ has constant rank r. For $s = \max_{x \in M}\{\dim G.\Phi(x)\}$ we set

$$M_s := \{x \in M \colon \dim(G.\Phi(x)) = s\}.$$

and $M_{\text{reg},s} := M_{\text{reg}} \cap M_s$. Note that M_s is also open in M.

LEMMA 6.5. *The subset $M_{\text{reg},s}$ of M is open, dense and connected. Moreover $\Phi(\dot{M}_s) \subseteq \Sigma_s$.*

Proof. Consider $x \in M_{\text{reg}}$. Then because of the constant rank of Φ on M_{reg} we find a neighborhood U of x such that $\Phi\colon U \to \Phi(U)$ is a submersion onto the locally closed submanifold $\Phi(U)$ of $\mathfrak{g}^*$. Moreover we may assume that the fibers of this submersion are all connected. Since Φ is G-equivariant, we may assume that $\Phi(U)$ is the union of connected open pieces of the symplectic foliation of $\mathfrak{g}^*$ given by the coadjoint orbits. Then a dense open connected subset Ω_x of $\Phi(U)$ consists of points whose coadjoint orbits have dimension

$$s_x := \max_{f \in \Phi(U)}\{\dim G.f\}.$$

Since the fibers of the submersion are connected, $\Phi^{-1}(\Omega_x) \cap U$ is an open dense connected subset of U.

If we now consider two points $x, x' \in M_{\text{reg}}$ and the corresponding neighborhoods U and U', then $s_x = s_{x'}$ if the two neighborhoods intersect. Thus the connectedness of M_{reg} shows that $s_x = s$ for all $x \in M_{\text{reg}}$. This in turn shows

$$\Phi^{-1}(\Omega_x) \cap U \subseteq M_{\text{reg},s}$$

and hence proves that $M_{\mathrm{reg},s}$ is open dense and connected. The last claim is obvious. $\square$

LEMMA 6.6. *Let M be a symplectic manifold and $G \times M \to M$ a Hamiltonian action with moment map $\Phi: M \to \mathfrak{g}^*$. Suppose that F is a face of $\mathfrak{t}_+^*$ such that* $\operatorname{algint} F \subseteq \Sigma_s$ *and the set* $M_F := M_s \cap \Phi^{-1}(\operatorname{algint} F)$ *is non-empty.*

(i) M_F is a G_F-invariant symplectic submanifold of M.

(ii) The action $T \times M_F \to M_F$ is Hamiltonian with moment map $\Phi|_{M_F}$.

(iii) For each open subset U_F of M_F the set $G.U_F := \{g.x \in M: g \in G, x \in U_F\}$ is open in M.

Proof. (i) For $x \in M_s$ and $f = \Phi(x) \in \operatorname{algint} F$ we have

$$T_f(G.f) + T_f(\operatorname{algint} F) = T_f(\Sigma_s) \tag{6.3}$$

and $d\Phi_x\left(T_x(G.x)\right) = T_f(G.f)$ so that the mapping $\Phi: M_s \to \Sigma_s$ is transversal to $\operatorname{algint} F$ which is a submanifold of Σ_s. This implies that M_F is a manifold (cf. [LM87, p.345]). In order to show that M_F with the restriction of the symplectic form ω of M is actually a symplectic manifold, we note first that for each $x \in M_F$ we have

$$\ker d\Phi_x \subseteq T_x(M_F) \tag{6.4}$$

and

$$T_x(M) = T_x(M_s) = d\Phi_x^{-1}\left(T_f(\Sigma_s)\right). \tag{6.5}$$

From (6.3) and (6.5) we deduce

$$d\Phi_x(T_x(G.x) + T_x(M_F)) = d\Phi_x\left(T_x(M)\right) \tag{6.6}$$

and hence, using (6.4), we find

$$T_x(G.x) + T_x(M_F) = T_xM. \tag{6.7}$$

From (6.4) together with [GS84, p.184] we find

$$T_x(M_F)^\perp \subseteq (\ker d\Phi_x)^\perp = T_x(G.x),$$

where $\perp$ denotes the orthogonal complement with respect to the symplectic form ω. Moreover,

$$d\Phi_x(T_x(G.x)) \cap T_f(\operatorname{algint} F) = T_f(G.f) \cap T_f(\operatorname{algint} F) = \{0\}$$

implies

$$\begin{aligned} T_x(M_F) \cap T_x(G.x) &= \{\sigma(X)_x: d\Phi_x(\sigma(X)_x) \in T_f(\operatorname{algint} F)\} \\ &= \{\sigma(X)_x: X \in \mathfrak{g}_f\} \subseteq \ker d\Phi_x. \end{aligned}$$

If now $w \in T_x(M_F) \cap T_x(M_F)^\perp$ we can write $w = \sigma(X)_x$ with $X \in \mathfrak{g}_f$ and have

$$w \in \ker d\Phi_x = T_x(G.x)^\perp$$

as well as $\omega_x(w, v) = 0$ for $v \in T_x(M_F)$. Thus (6.7) shows that w is perpendicular to all of $T_x(M)$, i.e., $w = 0$. But this shows that M_F is symplectic. The G_F-invariance is clear since Φ is G-equivariant and algint F consists of G_F-fixed points.

(ii) This is obvious, since the relevant homomorphism $\mathfrak{t} \to C^\infty(M_F)$ simply is the composition of the homomorphism $\mathfrak{g} \to C^\infty(M)$ corresponding to Φ and the restriction to M_F.

(iii) The equality (6.7) shows that $G.U_F$ contains an open neighborhood of U_F. Since G acts by homeomorphisms, this implies that $G.U_F$, which is the union of translates of this open neighborhood, is itself open. □

Recall from Remark 6.4 that

$$\Sigma_s \cap \mathfrak{t}_+^* = \operatorname{algint} F^{(1)} \cup \ldots \cup \operatorname{algint} F^{(m)},$$

where the $F^{(j)}$ are certain faces of $\mathfrak{t}_+^*$.

LEMMA 6.7. $\Phi(M_s) \cap \mathfrak{t}_+^*$ *is contained in one of the* algint $F^{(i)}$.

Proof. We know from Lemma 6.5 that

$$\Phi(M_s) \cap \mathfrak{t}_+^* \subseteq \Sigma_s \cap \mathfrak{t}_+^* \subseteq \operatorname{algint} F^{(1)} \cup \ldots \cup \operatorname{algint} F^{(m)}.$$

Moreover we have seen that for each $i \in \{1, \ldots, m\}$ the manifold $M_{F^{(i)}}$ satisfies

$$\Phi(M_{F^{(i)}}) = \Phi(M_s) \cap F^{(i)}.$$

Each element of M_s is G-conjugate to an element of precisely one of the $M_{F^{(i)}}$ since elements of different faces of $\mathfrak{t}_+^*$ are not G-conjugate ([Bou82, Ch. 9, § 5, no. 2, Prop. 2]). This means that we have a disjoint finite union

$$M_s = \bigcup_{i=1}^m G.M_{F^{(i)}}$$

and hence also

$$M_{\text{reg},s} = \bigcup_{i=1}^m (G.M_{F^{(i)}}) \cap M_{\text{reg}}.$$

But $M_{\text{reg},s}$ is connected and each of the $(G.M_{F^{(i)}}) \cap M_{\text{reg}}$ is open. Moreover, whenever $G.M_{F^{(i)}}$ is non-empty, it intersects the open set M_{reg} (Lemma 6.6(iii)), so only one of the sets $G.M_{F^{(i)}}$ can be non-empty and that proves the claim. □

We denote the face of $\mathfrak{t}^*$ containing $\Phi(M_s) \cap \mathfrak{t}_+^*$ by F and its span in $\mathfrak{t}^*$ by $\mathfrak{t}^*(F)$.

LEMMA 6.8.
(i) $x, g.x \in M_F$ *for* $g \in G$ *implies* $g \in G_F$.
(ii) $G.M_F = M_s$.
(iii) The manifold M_F *is connected.*

Proof. (i) We have $\Phi(x) \in \operatorname{algint} F$ and $g.\Phi(x) = \Phi(g.x) \in \operatorname{algint} F$ which shows $g.\Phi(x) = \Phi(x)$ and hence $g \in G_F$.
(ii) This follows immediately from Lemma 6.7 since each element of M_s is G-conjugate to some element in $\Psi^{-1}(\mathfrak{t}_+^*)$.
(iii) Now suppose that M_F is the disjoint union of two sets $M_F^{(1)}$ and $M_F^{(2)}$ each of which is a union of connected components in M_F. It suffices to show that one of these sets has to be empty. We claim that $G.M_F^{(1)}$ and $G.M_F^{(2)}$ are disjoint. In fact, if they are not disjoint, we find $g_1, g_2 \in G$ and $x_i \in M_F^{(i)}$ such that

$$g_2^{-1} g_1.x_1 = x_2.$$

But then (i) implies that $g_2^{-1} g_1 \in G_F$ and since G_F is connected, we see that x_1 and x_2 belong to the same connected component of M_F which is a contradiction. Lemma 6.6(iii) shows that the $G.M_F^{(i)}$ are open. Now (ii) implies that $(G.M_F) \cap M_{\mathrm{reg}} = M_{\mathrm{reg},s}$ and since $M_{\mathrm{reg},s}$ is connected, this shows that one of the $(G.M_F^{(i)}) \cap M_{\mathrm{reg}}$ has to be empty. Thus one of the $M_F^{(i)}$ has to be empty since M_{reg} is dense. □

LEMMA 6.9. $\Phi(M) \cap \mathfrak{t}_+^* \subseteq \overline{\Phi(M_F)}$. *If* $\Phi(M)$ *is closed, then equality holds.*

Proof. Let $x \in M$ and $(x_j)_{j \in \mathbb{N}}$ a sequence in $M_{\mathrm{reg},s}$ converging to x. Then we can find a sequence $(g_j)_{j \in \mathbb{N}}$ in G such that $g_j.x_j \in M_F$. Since G is compact, we may asumme that $g := \lim_j g_j$ exists, so that $\Phi(g.x) \in F$. If $\Phi(x) \in \mathfrak{t}_+^*$, then this implies $g.\Phi(x) = \Phi(x)$ and hence

$$\Phi(M) \cap \mathfrak{t}_+^* \subseteq \overline{\Phi(M_F)}.$$

The converse inclusion in the case of $\Phi(M)$ closed is trivial. □

REMARK 6.10. The map $\Phi\colon M_F \to \mathfrak{t}^*(F)$ gives rise to local convexity data $(C_x)_{x \in M_F}$ by Lemma 6.6(ii) and Theorem 2.3. On the other hand there is no reason to believe that $\Phi|_{M_F}$ is proper even if Φ is. In fact, examples show that this is not so since if $\Phi|_{M_F}$ is proper, then $\Phi(M_F)$ is a *closed* subset of $\operatorname{algint} F$ and hence also $\Phi(M) \cap \mathfrak{t}_+^*$ has to be a closed subset of $\operatorname{algint} F$ which in general is not true. □

PROPOSITION 6.11. *Let* $\Psi\colon X \to V$ *be a locally fiber connected map and* $(C_x)_{x \in X}$ *a local convexity data for* Ψ. *Suppose* D *is any closed convex subset of* V *which is locally polyhedral and set* $Y := \Psi^{-1}(D)$. *Then* $\Psi|_Y\colon Y \to V$ *together with* $y \mapsto C_y \cap L_{\Psi(y)}(D)$ *is a local convexity data.*

Proof. If $(U_x)_{x\in X}$ is a collection of (suitably small) neighborhoods of $x \in X$ satisfying (LC) and (O) for $\Psi: X \to V$, then their intersections $U_x \cap Y$ satisfy (LC) and (O) for $\Psi|_Y: Y \to V$. In fact, let $x \in Y$ and $U' \subseteq U_x \cap Y$ be open in Y. Then there exists an open set $U \subseteq U_x$ in X with $U \cap Y = U'$ and

$$\Psi|_Y(U') = \Psi(U \cap Y) = \Psi(U \cap \Psi^{-1}(D)) = \Psi(U) \cap D$$

is open in $C_x \cap L_{\Psi(x)}(D)$, where we assume U_x so small that

$$\Psi(U_x) \cap C_x \cap D = \Psi(U_x) \cap C_x \cap L_{\Psi(x)}(D).$$

This proves (O) and (LC) is trivial since Y is a union of fibers of Ψ. □

LEMMA 6.12. *The set $\Phi(M_F)$ is convex.*

Proof. Recall that the map $\Phi|_{M_F}: M_F \to \mathfrak{t}^*(F)$ provides us with the local convexity data $(C_x)_{x\in M_F}$. Choose an ascending sequence of closed convex cones $D_j \subseteq \operatorname{algint} F$ with

$$\bigcup_{j\in\mathbb{N}} D_j = \operatorname{algint} F.$$

To see that such a sequence exists, take a sequence $f_j \in \operatorname{algint} F$ with $f_j \to 0$ and $f_j \in f_{j+1} + \operatorname{algint} F$. Then set $D_j := f_j + F$.

Then we can apply Proposition 6.11 to the closed sets $Y_j := \Phi^{-1}(D_j) \subseteq M_s$ and then apply Theorem 3.10 to the connected components of the Y_j. This works since Y_j is closed in M and hence the restriction of Φ to Y_j is proper. Since M_F is connected, we can find an ascending sequence of connected components Y_j' of the Y_j's such that

$$M_F = \bigcup_{j\in\mathbb{N}} Y_j'.$$

Now Theorem 3.10 shows that the $\Phi(Y_j')$ form an ascending sequence of closed convex subsets of $\operatorname{algint} F$. This implies that

$$\Phi(M_F) = \bigcup_{j\in\mathbb{N}} \Phi(Y_j')$$

is convex. □

We now have our strengthened version of Kirwan's convexity theorem.

THEOREM 6.13. *Let $G \times M \to M$ be a Hamiltonian action of a compact group G on a connected symplectic manifold M such that the corresponding moment map $\Phi: M \to \mathfrak{g}^*$ is proper. Then for any Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and any choice of a closed Weyl chamber $\mathfrak{t}_+^*$ the set $\Phi(M) \cap \mathfrak{t}_+^*$ is convex.*

Proof. We only need to apply Lemma 6.9 to prove the convexity of $\Phi(M)\cap\mathfrak{t}^*_+$ from the convexity of $\Phi(M_F)$. □

For the next result we assume that our symplectic manifold M is compact. Let $\Phi_T\colon M \to \mathfrak{t}^*$ be the moment map for the (restricted) action of the maximal torus T of G on M. If $p\colon \mathfrak{k}^* \to \mathfrak{t}^*$ is the canonical projection this means that $\Phi_T = p \circ \Phi$.

LEMMA 6.14. *Let* $\mathrm{Ext}\Phi_T(M)$ *be the set of extreme points of* $\Phi_T(M)$. *Then*

$$\mathrm{Ext}\Phi_T(M) \subseteq \Phi(M) \cap \mathfrak{t}^*.$$

Proof. The local convexity theorem (more precisely, the local normal form for Hamiltonian torus actions) implies that for $f \in \mathrm{Ext}\Phi_T(M)$ the set $\Phi_T^{-1}(f)$ consists of T-fixed points. But then

$$\Phi(M) \cap p^{-1}(f) = \Phi(\Phi_T^{-1}(f))$$

consists of T-fixed points in $\mathfrak{k}^*$, i.e., is contained in $\mathfrak{t}^*$. This says that for $x \in M$ with $\Phi_T(x) = f$ we have $\Phi(x) = \Phi_T(x) = f$ which clearly implies the claim. □

REMARK 6.15. For each $x \in M$ we have $K.\Phi(x) \cap \mathfrak{t}^* \neq \emptyset$. Moreover $\Phi(M) \cap \mathfrak{t}^* \subseteq \Phi_T(M)$. □

The following proposition has been proved in a special case by Arnal and Ludwig (cf. [AL92, Prop. 17] and [Ne93f]).

PROPOSITION 6.16. *The following statements are equivalent.*

(1) $\Phi(M)$ *is convex.*

(2) $\Phi(M) \cap \mathfrak{t}^*$ *is convex.*

(3) $\Phi_T(M) = \Phi(M) \cap \mathfrak{t}^*$.

Proof. Clearly (1) implies (2). Also (2) implies (3) because of Lemma 6.14. Thus it only remains to show that (3) implies (1). So we assume (3) and note that Remark 6.15 shows that in this case $\Phi(M) = K.\Phi_T(M)$. Now let $x \in \operatorname{conv}(\Phi(M)) \subseteq \mathfrak{k}^*$. Then there exists a $k \in K$ such that

$$k.x \in \operatorname{conv}(\Phi(M)) \cap \mathfrak{t}^*.$$

Now we calculate

$$\begin{aligned} k.x &= p(k.x) \in p(\operatorname{conv}\Phi(M)) \\ &= \operatorname{conv}(p(\Phi(M))) = \operatorname{conv}(\Phi_T(M)) = \Phi_T(M) \end{aligned}$$

which proves $x \in K.\Phi_T(M) = \Phi(M)$ and hence the claim. □

References

[AL92] Arnal, D., and J. Ludwig, La convexité de l'application moment d'un groupe de Lie, J. Funct. Anal. **105** (1992), 256–300

[At82] Atiyah, M., Convexity and commuting hamiltonians, Bull. London Math. Soc. **14** (1982), 1–15

[vdB86] van den Ban, E., *A Convexity Theorem for Semisimple Symmetric Spaces*, Pacific Journal of Math. **124**(1986), 21–55

[Bou71] Bourbaki, N., "Topologie Générale", Chap. 1–10, Hermann, Paris, 1971

[Bou82] Bourbaki, N., *Groupes et algèbres de Lie, Chap. 9*, Masson, Paris, 1982

[BJ73] Bröcker, T., and K. Jänich, "Einführung in die Differentialtopologie", Springer Verlag, Berlin, Heidelberg, 1973

[CDM88] Condevaux, M., P. Dazord and P. Molino, Geometrie du moment, in Sem. Sud-Rhodanien 1988

[tD91] tom Dieck, T., "Topologie", de Gruyter, Berlin, New York, 1991

[Dui83] Duistermaat, J., Convexity and tightness for restrictions of hamiltonian functions to fixed point sets of antisymplectic involution, Trans AMS. **275** (1983), 417–429.

[GS82] Guillemin, V., and S. Sternberg, Convexity properties of the moment mapping I, Invent. Math. **67** (1982), 491–513

[GS84] Guillemin, V., and S. Sternberg, *Symplectic techniques in physics*, Cambridge Univ. Press, 1984

[Hel78] Helgason, S., *Differential geometry, Lie groups, and symmetric spaces*, Acad. Press, London, 1978

[Hil91] Hilgert, J., *Vorlesung über symplektische Geometrie*, Erlangen, 1991

[HHL89] Hilgert, J., K. H. Hofmann, and J. D. Lawson, *Lie Groups, Convex Cones, and Semigroups*, Oxford University Press, 1989

[HiNe93a] Hilgert, J., and K.-H. Neeb, *Lie semigroups and their applications*, Lecture Notes in Math. **1552**, Springer Verlag, 1993

[HiNe93b] Hilgert, J., and K.-H. Neeb, Non-linear Convexity Theorems and Poisson Lie groups, in preparation

[Ki84] Kirwan, F., Convexity properties of the moment mapping III, Invent. Math. **77** (1984), 547–552

[Ko73] Kostant, B., *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. Sci. Ecole Norm. Sup. **6**(1973), 413-455

[Le80] Leichtweiß, K., *Konvexe Mengen*, Springer Verlag, Heidelberg, 1980

[LM87] Libermann, P., and C. Marle, *Symplectic geometry and analytical mechanics*, Reidel, Dordrecht, 1987

[Lo69] Loos, O., "Symmetric Spaces I : General Theory", Benjamin, New York, Amsterdam, 1969

[LR91] Lu, J., and T. Ratiu, *On the nonlinear convexity theorem of Kostant*, Journal of the AMS **4**(1991), 349–363

[Me81] Meyer, K. R., Hamiltonian systems with a discrete symmetry, J. Diff. Eq. **41**(1981), 228–238

[Ne92] Neeb, K.-H., A convexity theorem for semisimple symmetric spaces, Pac. J. Math., **162** (1994), 305–349.

[Ne93a] Neeb, K.-H., Invariant subsemigroups of Lie groups, Mem. of the AMS **499**, 1993

[Ne93b] Neeb, K.-H., On closedness and simple connectedness of coadjoint orbits, Manuscripta Math., **82** (1994), 51–56.

[Ne93c] Neeb, K.-H., Kähler structures and convexity properties of coadjoint orbits, Forum Math., to appear

[Ne93d] Neeb, K.-H., Holomorphic representation theory II, *Acta Math.*, to appear.

[Ne93e] Neeb, K.-H., On the convexity of the moment mapping for a unitary highest weight representation, Journal of Funct. Anal., to appear

[Ne93f] Neeb, K.-H., Locally polyhedral sets, unpublished note

[Ola90] Olafsson, G., Causal Symmetric Spaces, Mathematica Gottingensis **15**, Preprint, 1990

[Pa84] Paneitz, S., Determination of invariant convex cones in simple Lie algebras, Arkif för Mat.**21**(1984), 217–228

[Pl92] Plank, W., *Konvexität in der symplektischen Geometrie*, Diplomarbeit, Erlangen, 1992