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Equivariant short exact sequences of vector bundles and their analytic torsion forms

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Abstract. In this paper, we construct analytic torsion forms associated to a short exact sequence of holomorphic Hermitian vector bundles equipped with a holomorphic unitary endomorphism g, thus extending results of a previous paper where the case $g = 1$ was considered. We calculate these forms explicitly, in terms of an additive equivariant genus $D(\theta, x)$. We introduce a related additive equivariant genus $R(\theta, x)$, which for $\theta = 0$, coincides with the genus $R(x)$ of Gillet and Soulé. By comparison with explicit computations by Gillet-Soulé and Köhler of the Ray-Singer analytic torsion of projective spaces, we conjecture a formula of Riemann-Roch in equivariant Arakelov geometry, in which the genus $R(\theta, x)$ appears explicitly.

The purpose of this paper is to construct and identify certain characteristic classes associated to a short exact sequence of holomorphic Hermitian vector bundles

$$E: 0 \to L \underset{i}{\to} M \underset{j}{\to} N \to 0 \tag{0.1}$$

on a complex manifold B, equipped with a holomorphic unitary chain map g. This extends our previous work [B1], where the case $g = 1$ was considered.

The constructions of [B1] were motivated by a program to calculate the behaviour of Quillen metrics by complex immersions. In fact, if $i: Y \to X$ is a complex immersion of compact complex manifolds, and if $\xi \to i_* \eta \to 0$ is a resolution of the sheaf of holomorphic sections of a holomorphic vector bundle η on Y by a complex of holomorphic vector bundles ξ on X, the determinants of the cohomology $\lambda(\xi)$ and $\lambda(\eta)$ are canonically isomorphic. If metrics are introduced on TX, TY, ξ and η, then $\lambda(\xi)$ and $\lambda(\eta)$ carry natural metrics, the Quillen metrics [Q2], [BGS3], which are constructed using the Ray-Singer analytic torsion [RS] of the corresponding Dolbeault complexes. It is then natural to calculate the Quillen norm of the canonical section identifying $\lambda(\eta)$ and $\lambda(\xi)$. Using the results of [B1], Bismut and

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Lebeau [BL] gave an explicit local formula for this norm, where a genus R introduced by Gillet and Soulé [GS1] appears explicitly.

In fact, in [GS1], Gillet and Soulé had given a conjectural Riemann-Roch formula in Arakelov geometry. By a difficult calculation (together with Zagier) of the Ray-Singer analytic torsion [RS] of the trivial bundle on $\mathbf{P}^n$ equipped with the Fubini-Study metric, Gillet and Soulé exhibited an additive genus R which appears in their conjectural formula. If ζ is the Riemann zeta function, the generating series $R(x)$ of [GS1] is given by

$$R(x) = \sum_{\substack{n \geqslant 1 \\ n \text{ odd}}} \left(\sum_1^n \frac{1}{j} + 2\,\frac{\zeta'(-n)}{\zeta(-n)} \right) \zeta(-n)\,\frac{x^n}{n!}. \tag{0.2}$$

Using [BGS1,2,3], the main result of [BL], and also [GS2,3], Gillet and Soulé [GS4] were able to prove their conjectural formula for the first Chern class. Their result was later extended by Faltings [F] to higher Chern classes.

The characteristic classes of [B1] can be explicitly calculated in terms of the additive genus associated to the series $D(x)$ given by

$$D(x) = \sum_{\substack{n \geqslant 1 \\ n \text{ odd}}} \left(\Gamma'(1) + \sum_1^n \frac{1}{j} + 2\,\frac{\zeta'(-n)}{\zeta(-n)} \right) \zeta(-n)\,\frac{x^n}{n!}. \tag{0.3}$$

The striking similarity of formulas (0.2) and (0.3) was unexpected. In fact the calculations of [GS1] were made on $\mathbf{P}^n$, while the computations of [B1] use essentially harmonic oscillators techniques. Still this similarity explains why the genus R of [GS1] ultimately appears in [BL].

Here, we start a program to extend the results of [B1], [BL] to an equivariant situation. Namely, consider again the immersion problem. Assume that G is a compact group, and that $i: Y \to X$ is a G-equivariant embedding of compact complex manifolds. Take ξ, η as before, and assume that the action of G extends to ξ and η. Finally suppose that G preserves the given metrics on TX, TY, ξ, η. By a suitable extension of the formalism of [RS] to the equivariant context, one can define "equivariant Quillen metrics" associated to the cohomology groups of η and ξ, which depend on $g \in G$. This construction reduces to usual Quillen metrics on the determinants of the cohomology for $g = 1$. Still the problem of comparing these "metrics" for arbitrary $g \in G$ makes sense. This paper is the first step in this direction.

Now, we will describe our main results.

Let h^M be a Hermitian metric on M, and let h^L, h^N be the induced metrics on L, N. As in [B1], for $u > 0$, we consider the family of operators

$\bar{\partial} + \sqrt{-u} i_{j(y)}$ acting on smooth sections of $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$ along the fibres of $M_{\mathbf{R}}$, and we introduce the associated Levi-Civita superconnection $\mathscr{B}_u$ of the family [B4]. The curvature $\mathscr{B}_u^2$ of $\mathscr{B}_u$ in the sense of Quillen [Q1] is a non trivial coupling of a harmonic oscillator on $N_{\mathbf{R}}$ and a Laplacian on $L_{\mathbf{R}}$.

As in [B1], except when $L = \{0\}$, the operator $\exp(-\mathscr{B}_u^2)$ is not trace class. Still we form a generalized supertrace $\mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2)]$ of $g \exp(-\mathscr{B}_u^2)$, by integrating the supertrace of the kernel of $g \exp(-\mathscr{B}_u^2)$ evaluated on the diagonal on a subbundle of $M_{\mathbf{R}}$. The most exotic case, $g = 1$, was considered in detail in [B1].

Let $\mathrm{Td}_g(L, h^L)$, $\mathrm{Td}_g(M, h^M)$, $\mathrm{Td}_g(N, g^N)$ be the Chern-Weil representatives of the corresponding equivariant Todd characteristic classes (which appear as the contribution of the fixed point set in Lefschetz fixed point formulas). Let $\varphi: \Lambda^{\mathrm{even}}(T_{\mathbf{R}}^* B) \to \Lambda^{\mathrm{even}}(T_{\mathbf{R}}^* B)$ be such that $\varphi(\alpha) = (2i\pi)^{-(\deg \alpha)/2} \alpha$. Then we prove in Theorems 2.5, 3.2 and 5.3 that the forms $\varphi \mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2)]$ are sums of form of type (p, p), are closed, that their cohomology class not depend on $u > 0$, and that moreover,

$$\lim_{u \to 0} \varphi \, \mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2)] = \frac{\mathrm{Td}_g(M, h^M)}{\mathrm{Td}_g(N, h^N)},$$

$$\lim_{u \to +\infty} \varphi \, \mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2)] = \mathrm{Td}_g(L, h^L). \tag{0.4}$$

As in [B1], we prove a double transgression formula formally identical to similar formulas in [BGS1, Theorem 1.15], [BGS2, Theorem 2.9] for usual finite or infinite dimensional supertraces. Namely, if N_H is the number operator of $\Lambda(N^*)$, we show in Theorem 2.5 that

$$\frac{\partial}{\partial u} \varphi \, \mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2)] = \frac{\bar{\partial}\partial}{2i\pi} \mathrm{Tr}_s\left[\frac{N_H}{u} g \exp(-\mathscr{B}_u^2)\right]. \tag{0.5}$$

As in [B1], it is possible to integrate (0.5) by a zeta function technique inspired from Ray-Singer [RS]. Thus in Section 6, we construct generalized analytic torsion forms $\mathbf{B}_g(L, M, h^M)$ on B, which by Theorem 6.3, verify the equation

$$\frac{\bar{\partial}\partial}{2i\pi} \mathbf{B}_g(L, M, h^M) = \mathrm{Td}_g(L, h^L) - \frac{\mathrm{Td}_g(M, h^M)}{\mathrm{Td}_g(N, h^N)}. \tag{0.6}$$

The forms $\mathbf{B}(L, M, h^M)$ of [B1] are exactly the forms $\mathbf{B}_g(L, M, h^M)$ for $g = 1$.

The main purpose of this paper is to calculate the form $\mathbf{B}_g(L, M, h^M)$ (modulo ∂ and $\bar{\partial}$ coboundaries). By a construction of Bott-Chern [BoC], Donaldson [D], Bismut-Gillet-Soulé [BGS1], we know how to solve in a natural way the equation

$$\frac{\bar{\partial}\partial}{2i\pi}\widetilde{\mathrm{Td}}_g(L, M, h^M) = \mathrm{Td}_g(M, h^M) - \mathrm{Td}_g(L, h^L)\,\mathrm{Td}_g(N, h^N), \tag{0.7}$$

so that if the complex E in (0.1) splits, then $\widetilde{\mathrm{Td}}_g(L, M, h^M) = 0$. In view of (0.6), (0.7), the question arises to calculate

$$\mathbf{B}_g(L, M, h^M) + \mathrm{Td}_g^{-1}(N, h^N)\,\widetilde{\mathrm{Td}}_g(L, M, h^M).$$

Set

$$\sigma(u, \eta, x) = 4\sinh\left(\frac{x-2\eta+\sqrt{x^2+4u}}{4}\right)\sinh\left(\frac{-x+2\eta+\sqrt{x^2+4u}}{4}\right). \tag{0.8}$$

In Proposition 4.2, we show that

$$\sigma(u, i\theta, x) = (\theta^2 + i\theta x + u)\prod_{k\in\mathbf{Z}^*}\left(\frac{(\theta+2k\pi)^2+i(\theta+2k\pi)x+u}{4k^2\pi^2}\right). \tag{0.9}$$

Let $C(s, \theta, x)$ be the Mellin transform of $\frac{\partial\sigma/\partial x}{\sigma}(u, i\theta, -x)$, i.e. for $0 < \mathrm{Re}(s) < 1/2$,

$$C(s, \theta, x) = \frac{1}{\Gamma(s)}\int_0^{+\infty} u^{s-1}\,\frac{\partial\sigma/\partial x}{\sigma}(u, i\theta, -x)\,du. \tag{0.10}$$

Then $C(s, \theta, x)$ extends holomorphically near $s = 0$. Set

$$D(\theta, x) = \frac{\partial}{\partial s}C(0, \theta, x). \tag{0.11}$$

Then $D(x) = D(0, x)$.

Let $e^{i\theta_1}, \dots, e^{i\theta_q}$ be the eigenvalues of g on N. One can construct the obvious additive genus $D_g(N)$ attached to $D(\theta, x)$. Then $\mathrm{Td}_g(L)D_g(N)$ is a well-defined class of sums of (p, p) forms modulo ∂ and $\bar{\partial}$ coboundaries.

In Theorem 6.8, we prove the following extension of [B1, Theorem 8.5],

$$\mathbf{B}_g(L, M, h^M) = -\mathrm{Td}_g^{-1}(N, h^N)\,\widetilde{\mathrm{Td}}_g(L, M, h^M) + \mathrm{Td}_g(L)D_g(N) \quad \text{modulo } \partial \text{ and } \bar{\partial} \text{ coboundaries.} \tag{0.12}$$

Set

$$\zeta(y,s)=\sum_{n=1}^{+\infty}\frac{\cos(ny)}{n^s},$$
$$\eta(y,s)=\sum_{n=1}^{+\infty}\frac{\sin(ny)}{n^s}. \tag{0.13}$$

Then $\zeta(y, s)$ and $\eta(y, s)$ are the real and imaginary parts of the Lerch series [L]. In Theorem 7.2, and extending a result of Bismut-Soulé [B1, Appendix], which is valid for $\theta = 0$, we prove the formula

$$D(\theta,x)=\sum_{\substack{n\geqslant 0\\ n\ \text{even}}} i\left\{\left(\Gamma'(1)+\sum_1^n\frac{1}{j}\right)\eta(\theta,-n)+\frac{2\partial\eta}{\partial s}(\theta,-n)\right\}\frac{x^n}{n!}$$
$$+\sum_{\substack{n\geqslant 1\\ n\ \text{odd}}}\left\{\left(\Gamma'(1)+\sum_1^n\frac{1}{j}\right)\zeta(\theta,-n)+\frac{2\partial\zeta}{\partial s}(\theta,-n)\right\}\frac{x^n}{n!}. \tag{0.14}$$

By comparison with (0.2), we introduce the power series

$$R(\theta,x)=\sum_{\substack{n\geqslant 0\\ n\ \text{even}}} i\left\{\sum_1^n\frac{1}{j}\eta(\theta,-n)+\frac{2\partial\eta}{\partial s}(\theta,-n)\right\}\frac{x^n}{n!}$$
$$+\sum_{\substack{n\geqslant 1\\ n\ \text{odd}}}\left\{\sum_1^n\frac{1}{j}\zeta(\theta,-n)+\frac{2\partial\zeta}{\partial s}(\theta,-n)\right\}\frac{x^n}{n!}, \tag{0.15}$$

such that $R(0, x) = R(x)$.

In Sections 7(d) and 7(e), we conjecture that the additive equivariant genus associated to $R(\theta, x)$ should play the role of the genus R in an equivariant Riemann-Roch formula in Arakelov geometry. In fact in [K], Köhler has calculated the equivariant analytic torsion of $\mathbf{P}^n$ equipped with the Fubini-Study metric, for $g \in U(n + 1)$, whose action on $\mathbf{P}^n$ has only isolated fixed points. Because the fixed points are isolated, the calculation of Köhler is much simpler than the corresponding calculation of Gillet, Soulé and Zagier [GS1], where the case $g = 1$ was considered.

In Section 7(d), we verify that the formula of [K] can be very simply expressed in terms of $R(\theta, 0)$. The similarity between the calculations of [K] and our own construction of $R(\theta, x)$ is strictly parallel to the resemblance of the result of [GS1] to the ones in [B1] for the case $g = 1$. It gives more weight to the possibility of proving an equivariant Riemann-Roch formula in Arakelov geometry.

More recently [B9, 10], using the results of the present paper, we have extended the formula of Bismut-Lebeau [BL] to the equivariant case, thus completing the analytic part of the program just outlined.

This paper is organized as follows. In Section 1, we recall the results of [B1] on the superconnection $\mathscr{B}_u$ and its curvature $\mathscr{B}_u^2$. In Section 2, we introduce the generalized supertraces $\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2)]$ and we prove the double transgression formula in (0.5). In Section 3, we prove the first half of (0.4). In Section 4, we introduce the function $\sigma(u, \eta, x)$ and the associated multiplicative genus. In Section 5, by extending the results of [B1, Section 7], we calculate the forms $\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2)]$ explicitly, and we prove the second half of (0.4). In Section 6, we construct the forms $\mathbf{B}_g(L, M, h^M)$, the function $D(\theta, x)$, and we establish (0.12). Finally in Section 7, we give various formulas for $D(\theta, x)$ including (0.14), we introduce the function $R(\theta, x)$ and we give a conjectural Riemann-Roch formula in equivariant Arakelov theory.

In the whole paper, we use the superconnection formalism of Quillen [Q1]. Let us here briefly recall that if A is a $\mathbf{Z}_2$-graded algebra, if $a, a' \in A$, the supercommutator $[a, a'] \in A$ is defined by

$$[a, a'] = aa' - (-1)^{\deg(a)\deg(a')}a'a. \tag{0.16}$$

Also, we use probabilistic arguments and techniques of Fermionic integration taken from [B1], in particular in our explicit computations of Section 5. Also, if Y is a Brownian motion, $\mathrm{d}Y$ denotes its Stratonovitch differential [IkW, Chapters II and III].

For an introduction to the probabilistic techniques used in this paper, we refer to our survey [B7]. A suitable modification of the analytic arguments of [BL, Section 14] could be used as a substitute.

The results of this paper were announced in [B8].

1. The Levi-Civita superconnection associated to a short exact sequence

In this section, we recall the construction of the Levi-Civita superconnection $\mathscr{B}_u$ associated to a short exact sequence of holomorphic vector bundles which was given in [B1, Section 3]. Also we establish elementary estimates on the heat kernel along the fibres for $\exp(-\mathscr{B}_u^2)$.

This section is organized as follows. In (a), we construct $\mathscr{B}_u$, and in (b), we estimate the corresponding heat kernels.

(a) *The Levi-Civita superconnection*

Let B be a compact complex manifold. Let

$$E: 0 \to L \to M \to N \to 0$$

be a holomorphic acyclic complex of vector bundles on B. Let l, m, n be the complex dimensions of L, M, N. We identify L with a holomorphic subbundle of M, and N with M/L. Let π be the projection $M \to B$.

Let h^M be a Hermitian metric on M. Let h^L be the induced metric on L. By identifying N to the orthogonal bundle to L in M, N inherits a Hermitian metric h^N. Let ∇^L, ∇^M, ∇^N be the holomorphic Hermitian connections on L, M, N, and let R^L, R^M, R^N be their curvatures.

Let P^L, P^N be the orthogonal projection operators from M on L, N. Classically [B1, Proposition 2.4], we know that

$$\nabla^L = P^L\nabla^M, \quad \nabla^N = P^N\nabla^M.$$

Let ${}^0\nabla^M$ be the connection on $M \simeq L \oplus N$,

$${}^0\nabla^M = \nabla^L \oplus \nabla^N.$$

Set

$$A = \nabla^M - {}^0\nabla^M.$$

Then A is a 1-form on M with values in skew-adjoint endomorphisms of M which exchange L and N.

The Hermitian metrics h^M, h^N induce corresponding Hermitian metrics on $\Lambda(\bar{M}^*)$, $\Lambda(N^*)$. Let $\langle\, ,\rangle$ be the corresponding Hermitian product on $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$. Similarly let $\nabla^{\Lambda(\bar{M}^*)}$, $\nabla^{\Lambda(N^*)}$ be the connections on $\Lambda(\bar{M}^*)$, $\Lambda(N^*)$ induced by ∇^M, ∇^N. Let $\nabla^{\Lambda(\bar{M}^*)\hat{\otimes}\Lambda(N^*)}$ be the connection on $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$

$$\nabla^{\Lambda(\bar{M}^*)\hat{\otimes}\Lambda(N^*)} = \nabla^{\Lambda(\bar{M}^*)} \hat{\otimes} 1 + 1 \hat{\otimes} \nabla^{\Lambda(N^*)}.$$

Then $\nabla^{(\bar{M}^*)\hat{\otimes}\Lambda(N^*)\bar{M}^*}$ preserves the Hermitian product $\langle\, ,\rangle$.

Let $R^{\Lambda(\bar{M}^*)\hat{\otimes}\Lambda(N^*)}$ be the curvature of $\nabla^{\Lambda(\bar{M}^*)\hat{\otimes}\Lambda(N^*)}$. If $R^{\Lambda(\bar{M}^*)}$, $R^{\Lambda(N^*)}$ are the natural actions of R^M, R^N on $\Lambda(\bar{M}^*)$, $\Lambda(N^*)$, we have the obvious

$$R^{\Lambda(\bar{M}^*)\hat{\otimes}\Lambda(N^*)} = R^{\Lambda(\bar{M}^*)} \hat{\otimes} 1 + 1 \hat{\otimes} R^{\Lambda(N^*)}.$$

If $X \in M$, let $X^* \in \bar{M}^*$ correspond to X be the metric h^M. If $X \in M$, $X' \in \bar{M}$, set

$$c(X) = \sqrt{2}\, X^* \wedge, \quad c(X') = -\sqrt{2}\, i_{X'}. \tag{1.1}$$

We extend the definition of $c(X)$ by $\mathbf{C}$-linearity to $X \in M \oplus \bar{M}$. Then for $X \in M_{\mathbf{R}}$, $c(X)$ acts on $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$. Moreover if X, $X' \in M_{\mathbf{R}}$,

$$c(X)c(X') + c(X')c(X) = -2\langle X, X'\rangle_{h^M}. \tag{1.2}$$

By (1.1), (1.2), we see that $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$ is a $M_{\mathbf{R}}$-Clifford module.

Similarly, if $Y' \in \bar{N}$, let $Y'^* \in N^*$ correspond to Y' be the metric h^N. If $Y \in N$, $Y' \in \bar{N}$, set

$$\hat{c}(Y) = \sqrt{2}\, i_Y, \quad \hat{c}(Y') = -\sqrt{2}\, Y'^* \wedge. \tag{1.3}$$

Again we extend the definitions of $c(Y)$ by $\mathbf{C}$-linearity to $Y \in N \oplus \bar{N}$. Then for $Y \in N_{\mathbf{R}}$, $\hat{c}(Y)$ acts on $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$. Also if $Y, Y' \in N_{\mathbf{R}}$

$$\hat{c}(Y)\hat{c}(Y') + \hat{c}(Y')\hat{c}(Y) = -2\langle Y, Y'\rangle_{h^N}. \tag{1.4}$$

So (1.3), (1.4) show that $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$ is a $N_{\mathbf{R}}$-Clifford module. Finally if $X \in M_{\mathbf{R}}$, $Y \in N_{\mathbf{R}}$,

$$c(X)\hat{c}(Y) + \hat{c}(Y)c(X) = 0. \tag{1.5}$$

Note that our conventions in (1.3) differ from those in [B1, eq. (1.17)].

Let dv_M, dv_N be the volume forms on the fibres of $M_{\mathbf{R}}$, $N_{\mathbf{R}}$, which are associated to the metrics h^M, h^N.

DEFINITION 1.1. For $x \in B$, let I_x (resp. I_x^0) be the vector space of smooth (resp. square integrable) sections of $(\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*))_x$ over $M_{\mathbf{R},x}$.

In the sequel, the I_x's will be considered as the fibres of a smooth infinite dimensional vector bundle I on B. The set of smooth sections of I over B will be identified with the set of smooth sections of $\pi^*(\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*))$ over $M_{\mathbf{R}}$.

We equip I_x^0 with the Hermitian product

$$f, g \in I_x^0 \to \langle f, g\rangle = \left(\frac{1}{2\pi}\right)^{\dim M} \int_{M_{\mathbf{R},x}} \langle f, g\rangle\, dv_M. \tag{1.6}$$

Let $\bar{\partial}^{M_x}$ be the Dolbeault operator acting on I_x, and let $\bar{\partial}^{M_x^*}$ be the formal adjoint of $\bar{\partial}^{M_x}$ with respect to the Hermitian product (1.6). Set

$$D_x^M = \bar{\partial}^{M_x} + \bar{\partial}^{M_x^*}.$$

Let $e_1, \ldots, e_{2m}$ be an orthonormal base of $M_{\mathbf{R}}$. Then, we have the obvious

$$D^M = \frac{1}{\sqrt{2}} \sum_1^{2m} c(e_i)\nabla_{e_i}. \tag{1.7}$$

The connection ∇^{TM} defines a C^∞ splitting of TM into

$$TM = \pi^*(M) \oplus T^H M.$$

If $U \in T_{\mathbf{R}}B$, let $U^H \in T^H_{\mathbf{R}}M$ be the lift of U, so that $\pi_* U^H = U$.

DEFINITION 1.2. If s is a smooth section of the vector bundle I on B, if $U \in T_{\mathbf{R}}B$, set

$$\nabla^I_U s = \nabla^{\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)}_{U^H} s. \tag{1.8}$$

Then ∇^I is a connection on I, which preserves the Hermitian product (1.6).
Let $(\nabla^I)^2$ be the curvature of ∇^I. Then if $U, V \in T_{\mathbf{R}}B$, by [B1, Proposition 3.3],

$$(\nabla^I)^2(U, V) = R^{\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)}(U, V) - \nabla_{R^M(U,V)Y}$$

Let $f_1, \ldots, f_{2k}$ be a base of $T_{\mathbf{R}}B$, and let $f^1, \ldots, f^{2k}$ be the corresponding dual base of $T^*_{\mathbf{R}}B$. If $Y \in M_{\mathbf{R}}$, set

$$c(R^M Y) = \tfrac{1}{2} \sum f^\alpha f^\beta c(R^M(f_\alpha, f_\beta) Y). \tag{1.9}$$

Then $c(R^M Y)$ lies in $\Lambda^2(T^*_{\mathbf{R}}B) \hat{\otimes} c(M_{\mathbf{R}})$.
If $y' \in N$, set $Y' = y' + \bar{y}' \in N_{\mathbf{R}}$. Put

$$V(Y') = \sqrt{-1}\,(i_{y'} - i^*_{y'}). \tag{1.10}$$

Equivalently,

$$V(Y') = \frac{\sqrt{-1}}{\sqrt{2}}\,\hat{c}(Y'). \tag{1.11}$$

If $Y \in M_{\mathbf{R}}$, $V(P^N Y)$ acts naturally as an odd operator on $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$.

DEFINITION 1.3. For $u > 0$, set

$$\mathscr{B}_u = \nabla^I + D^M + \sqrt{u}\, V(P^N Y) - \frac{c(R^M Y)}{2\sqrt{2}}. \tag{1.12}$$

The superconnection $\mathscr{B}_u$ on the $\mathbf{Z}$-graded vector bundle I was constructed first in [B1, Definition 3.8], and was called the Levi-Civita superconnection of parameter $u > 0$, by reference to our work in [B4] on the local families index theorem.

Set

$$S = \frac{\sqrt{-1}}{2} \sum_1^{2m} c(P^N e_i)\hat{c}(P^N e_i). \tag{1.13}$$

Observe that S is exactly the operator considered in [B1, Definition 1.3 and Eq. (3.35)]. Set

$$\hat{c}(AP^L Y) = -\sum f^\alpha \hat{c}(A(f_\alpha)P^L Y) \in (\Lambda(T_{\mathbf{R}}^* B) \hat{\otimes} \operatorname{End}(\Lambda(N^*))^{\text{even}}. \tag{1.14}$$

The following formula was proved in [B1, Theorem 3.10].

THEOREM 1.4. *The following identity holds*

$$\mathscr{B}_u^2 = -\frac{1}{2}\sum_1^{2m} (\nabla_{e_i} + \tfrac{1}{2}\langle R^M Y, e_i\rangle)^2 + \frac{u|P^N Y|^2}{2}$$
$$+ \sqrt{u}\, S + \frac{\sqrt{-u}}{\sqrt{2}} \hat{c}(AP^L Y) + \tfrac{1}{2}\operatorname{Tr}[R^M] + R^{\Lambda(N^*)}. \tag{1.15}$$

DEFINITION 1.5. For $x \in B$, let $\mathscr{J}_x$ (resp. $\mathscr{J}_x^0$) be the vector space of smooth (resp. square integrable) sections of $\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)$ over $M_{\mathbf{R},x}$.

As explained in [B1, Section 3], $\mathscr{B}_{u,x}^2$ lies in $(\Lambda(T_{\mathbf{R}}^* B) \hat{\otimes} \operatorname{End}(\mathscr{J}))_x^{\text{even}}$.

Let N_H be the number operator of $\Lambda(N^*)$, i.e. N_H act on $\Lambda^p(N^*)$ by multiplication by p.

(b) *The heat kernel associated to* $\exp(-\mathscr{B}_u^2 + bN_H)$

For $b \in \mathbf{C}$, let $P_u^{x,b}(Y, Y')$ ($x \in B$, $Y, Y' \in M_{\mathbf{R},x}$) be the smooth kernel associated to the operator $\exp(-\mathscr{B}_{u,x}^2 + bN_H)$, calculated with respect to the volume element $\frac{dv_{M_x}}{(2\pi)^{\dim M}}$. For the existence and uniqueness of $P_u^{x,b}(Y, Y')$, we refer to [B1, Section 4(a)].

If s is a bounded smooth section of $(\Lambda(T_{\mathbf{R}}^* B) \hat{\otimes} \Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*))_x$ over $M_{\mathbf{R},x}$,

$$\exp(-\mathscr{B}_{u,x}^2 + bN_H)s(Y) = \int_{M_{\mathbf{R},x}} P_u^{x,b}(Y, Y')s(Y') \frac{dv_M(Y')}{(2\pi)^{\dim M}}. \tag{1.16}$$

If $Y, Y' \in M_{\mathbf{R},x}$, then $P_u^{x,b}(Y, Y') \in (\Lambda(T_{\mathbf{R}}^* B) \hat{\otimes} \operatorname{End}(\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)))_x^{\text{even}}$.

THEOREM 1.6. *For any* $u > 0$, $A > 0$, *and any multiindices* α, α', *there exist* $C > 0$, $C' > 0$, $C'' > 0$ *such that for* $x \in B$, $|b| \leqslant A$, $Y, Y' \in M_{\mathbf{R},x}$,

$$\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Y^\alpha \partial Y'^{\alpha'}} P_u^{x,b}(Y, Y')\right|$$
$$\leqslant C \exp\left\{-\frac{|P^L(Y-Y')|^2}{2} + C'(|P^L Y|+|P^L Y'|) - C''(|P^N Y|^2+|P^N Y'|^2)\right\}. \tag{1.17}$$

Proof. In the sequel, the constants $C > 0$, $C' > 0$ may vary from line to line. For $Y, Y' \in M_{\mathbf{R},x}$, let $Q_{(Y,Y')}$ be the probability law on $\mathscr{C}([0,1]; M_{\mathbf{R},x})$ of the Brownian bridge Y_t, with $Y_0 = Y$, $Y_1 = Y'$, [Sil, p. 40]. Let $E^{Q_{(Y,Y')}}$ be the corresponding expectation operator. Under $Q_{(Y,Y')}$, consider the differential equation

$$\frac{dU_t}{dt} = U_t\left[-\sqrt{u}S - \frac{\sqrt{-u}}{\sqrt{2}}\hat{c}(AP^L Y) - R^{\Lambda(N^*)} + bN_H\right],$$
$$U_0 = 1. \tag{1.18}$$

Clearly, for a given $u > 0$, there exist $C > 0$, $C' > 0$ such that for $|b| \leqslant A$,

$$|U_t| \leqslant C \exp\left\{C' \int_0^1 |P^L Y_t|\, dt\right\}. \tag{1.19}$$

Then by using Itô's formula and by proceeding as in [B1, proof of Theorem 4.1], we get the formula in [B1, Eq. (4.11)],

$$P_u^{x,b}(Y, Y') = \exp\left(-\frac{|Y-Y'|^2}{2}\right) E^{Q_{(Y,Y')}}\left[\exp\left\{\frac{1}{2}\int_0^1 \langle R^M Y, dY\rangle - \frac{u}{2}\int_0^1 |P^N Y_t|^2\, dt - \frac{1}{2}\operatorname{Tr}[R^M]\right\} U_1\right] \tag{1.20}$$

By (1.20) and by Cauchy-Schwarz's inequality, we obtain,

$$|P_u^{x,b}(Y, Y')| \leqslant C \exp\left(-\frac{|Y-Y'|^2}{2}\right)$$
$$\left\{E^{Q_{(Y,Y')}}\left[\left|\exp\left\{\frac{1}{2}\int_0^1 \langle R^M Y, dY\rangle\right\}\right|^2\right]\right.$$
$$\left. E^{Q_{(Y,Y')}}\left[\exp\left(-u\int_0^1 |P^N Y|^2\, dt + 2C'\int_0^1 |P^L Y|\, dt\right)\right]\right\}^{1/2}. \tag{1.21}$$

Now using the fact that R^M is a 2-form, so that the power series for

$$\exp\left\{\frac{1}{2}\int_0^1 \langle R^M Y, \mathrm{d}Y\rangle\right\}$$

contains only a finite number of terms, following [B1, Eq. (4.17)], we find that there exist $p \in \mathbf{N}$, $C > 0$ such that,

$$E^{Q_{(Y,Y')}}\left[\left|\exp\left\{\frac{1}{2}\int_0^1 \langle R^M Y, \mathrm{d}Y\rangle\right\}\right|^2\right] \leqslant C(1+|Y|^p+|Y'|^p). \tag{1.22}$$

Under $Q_{(Y,Y')}$, $P^L Y$ and $P^N Y$ are independent. Therefore,

$$E^{Q_{(Y,Y')}}\left[\exp\left\{-u\int_0^1 |P^N Y|^2\,\mathrm{d}t + 2C'\int_0^1 |P^L Y|\,\mathrm{d}t\right\}\right]$$
$$= E^{Q_{(Y,Y')}}\left[\exp\left\{-u\int_0^1 |P^N Y|^2\,\mathrm{d}t\right\}\right]$$
$$E^{Q_{(Y,Y')}}\left[\exp\left\{2C'\int_0^1 |P^L Y|\,\mathrm{d}t\right\}\right]. \tag{1.23}$$

By [GlJ, Theorem 1.5.10], we know that

$$E^{Q_{(Y,Y')}}\left[\exp\left\{-u\int_0^1 |P^N Y|^2\,\mathrm{d}t\right\}\right] = \left(\frac{(2u)^{1/2}}{\sinh(2u^{1/2})}\right)^n \exp\left(\frac{|P^N(Y-Y')|^2}{2}\right)$$
$$\exp\left\{-\frac{(2u)^{1/2}}{2\sinh(2u^{1/2})}(\cosh(2u^{1/2})(|P^N Y|^2+|P^N Y'|^2)-2\langle P^N Y, P^N Y'\rangle)\right\}. \tag{1.24}$$

Using (1.24), we get,

$$E^{Q_{(Y,Y')}}\left[\exp\left\{-u\int_0^1 |P^N Y|^2\right\}\right] \leqslant C\exp\left(\frac{|P^N(Y-Y')|^2}{2}\right)$$
$$\exp\left\{-\frac{(2u)^{1/2}}{2}\tanh((2u)^{1/2}/2)(|P^N Y|^2+|P^N Y'|^2)\right\}. \tag{1.25}$$

By [IMK, p. 27], we know that

$$E^{Q_{(P^N Y, P^N Y')}}\left[\exp\left\{2C'\int_0^1 |P^L Y|\,\mathrm{d}t\right\}\right] \leqslant C'''. \tag{1.26}$$

By [Si1, p. 41], under $Q_{(P^N Y, P^N Y')}$, the probability law of $t \in [0,1] \to Y'_t = (1-t)P^L Y + tP^L Y' + Y_t$ is exactly $Q_{(Y,Y')}$. From (1.26), we deduce,

$$E^{Q_{(Y,Y')}}\left[\exp\left\{2C'\int_0^1 |P^L Y|\,\mathrm{d}t\right\}\right] \leqslant C''' \exp\{2C'(|P^L Y| + |P^L Y'|)\}. \tag{1.27}$$

By (1.21)–(1.27), we find that there exist $C > 0$, $C' > 0$, $C'' > 0$ such that

$$|P_u^{x,b}(Y, Y')| \leqslant C \exp\left\{-\frac{|P^L(Y-Y')|^2}{2} + C'(|P^L Y| + |P^L Y'|) - C''(|P^N Y|^2 + |P^N Y'|^2)\right\}. \tag{1.28}$$

Then (1.28) coincides with (1.17) for $\alpha = 0$, $\alpha' = 0$.

For general α, α', as in [B1, proof of Theorem 4.1], one can use the Malliavin calculus [Ma], [B2] to obtain the bound (1.17) in full generality. This is especially easy here, because we are dealing with a flat Brownian motion. □

In the sequel, we use the notation

$$P_u^x(Y, Y') = P_u^{x,b}(Y, Y')|_{b=0}. \tag{1.29}$$

2. Equivariant short exact sequences and generalized supertraces

In this section, we consider an equivariant short exact sequence of holomorphic Hermitian vector bundles, i.e. a short exact sequence equipped with a parallel unitary chain map g. Then, by extending our work in [B1, Section 4], we define a generalized supertrace $\mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2)]$, which is a smooth closed form on B. The form $\mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2)]$ is called a generalized supertrace because, in general, $g \exp(-\mathscr{B}_u^2)$ is not fibrewise trace class. The case where $g = 1$, which is from a certain point of view the most exotic, was considered in detail in [B1].

We prove the double transgression formulas (0.5), which are the obvious analogues of [BGS1, Theorem 1.15], [BGS2, Theorem 2.9], [B1, Theorem 4.6].

This section is organized as follows. In (a), we introduce the action of g on the short exact sequence. In (b), we construct the generalized supertraces. In (c), we establish our double transgression formulas.

We make the same assumptions and we use the same notation as in Section 1.

(a) *Equivariant short exact sequences and the Levi-Civita superconnection*

Let g be a smooth section of End(M), which preserves L. Then g acts naturally on L and N.

We assume that g is an isometry of M, which is parallel with respect to the connection ∇^M. Then g also acts as an isometry of L and N, which is parallel with respect to the connections ∇^L and ∇^N.

Let $e^{i\theta_1},\dots,e^{i\theta_q}$ $(0 \leqslant \theta_1,\dots,\theta_q < 2\pi)$ be the distinct eigenvalues of g acting on L, M, N. Since g is parallel, these eigenvalues are locally constant. Clearly E splits holomorphically as an orthogonal sum of complexes E^{θ_j}, with

$$E^{\theta_j}: 0 \to L^{\theta_j} \to M^{\theta_j} \to N^{\theta_j} \to 0, \tag{2.1}$$

and g acts on E^{θ_j} by multiplication by $e^{i\theta_j}$. Moreover M^{θ_j} inherits the metric $h^{M^{\theta_j}}$ induced by h^M on M^{θ_j}.

For $1 \leqslant j \leqslant q$, let Q^{θ_j} be the projection operator $E \to E^{\theta_j}$.

Let $E^{0,\perp}$ be the direct sum of the complexes E^{θ_j}'s associated to the nonzero θ_j's. The complex $E^{0,\perp}$ can be written in the form

$$E^{0,\perp}: 0 \to L^{0,\perp} \to M^{0,\perp} \to N^{0,\perp} \to 0. \tag{2.2}$$

Moreover

$$E = E^0 \oplus E^{0,\perp}. \tag{2.3}$$

Then to each E^{θ_j}, we can associate the objects which we associated before to E itself. They will be denoted with the superscript θ_j.

In particular, for $u > 0$,

$$\begin{aligned}\mathscr{B}_u &= \sum_1^q Q^{\theta_j *}\mathscr{B}_u^{\theta_j},\\ \mathscr{B}_u^2 &= \sum_1^q Q^{\theta_j *}\mathscr{B}_u^{\theta_j,2}.\end{aligned} \tag{2.4}$$

If $Y \in M_{\mathbf{R}}$, we write $Y = \Sigma_1^q Y^{\theta_j}$, $Y^{\theta_j} = Q^{\theta_j} Y$. Then using (1.15), we obtain

$$P_u^{x,b}(Y, Y') = \bigoplus_{j=1}^q P_u^{\theta_j,x,b}(Y^{\theta_j}, Y'^{\theta_j}). \tag{2.5}$$

Of course, the $P_x^{\theta_j,x,b}$'s verify bounds similar to (1.17).

When necessary, we will also introduce the objects naturally associated to

E^0 or $E^{0,\perp}$, which will be denoted with the superscript 0 or 0, $\perp$. In particular,

$$P_u^{0,\perp,x,b}(Y,Y') = \bigotimes_{\theta_j\neq 0} P_u^{\theta_j,x,b}(Y^{\theta_j},Y'^{\theta_j}),\ Y,Y'\in M^{0,\perp}_{\mathbf{R},x}. \tag{2.6}$$

Let N_V, N_H be the number operators of $\Lambda(\bar{M}^*)$, $\Lambda(N^*)$. Then N_V, N_H act on $\Lambda^p(\bar{M}^*)$, $\Lambda^q(N^*)$ by multiplication by p, q. Similarly for $1 \leqslant j \leqslant q$, we introduce the number operators $N_V^{\theta_j}$, $N_H^{\theta_j}$ acting on $\Lambda(\bar{M}^{\theta_j *})$, $\Lambda(N^{\theta_j *})$.

Then g acts naturally as an algebra homomorphism on $\Lambda(\bar{M}^*)$ and $\Lambda(N^*)$. Clearly g acts on $\Lambda(\bar{M}^{\theta_j *})$, $\Lambda(N^{\theta_j *})$ as the operator $e^{i\theta_j N_V^{\theta_j}}$, $e^{-i\theta_j N_H^{\theta_j}}$. Then g acts on $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$ as the operator $e^{i\sum_{j=1}^{q}\theta_j(N_V^{\theta_j}-N_H^{\theta_j})}$.

Clearly g acts on $\Lambda(T^*_{\mathbf{R}}B) \hat{\otimes} \Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$ as $1 \hat{\otimes} g$. If $h \in I$, let $gh \in I$ be given by

$$(gh)(Y) = gh(g^{-1}Y). \tag{2.7}$$

Also $\exp(-\mathscr{B}_u^2 + bN_H)$ acts on

$$\Lambda(T^*_{\mathbf{R}}B) \hat{\otimes} I = \Lambda(\bar{L}^*) \hat{\otimes} \Lambda(T^*_{\mathbf{R}}B) \hat{\otimes} J \quad \text{as } 1 \hat{\otimes} \exp(-\mathscr{B}_u^2 + bN_H),$$

and the corresponding kernel is still given by $P_u^{x,b}(Y,Y')$ $(Y, Y' \in M_{\mathbf{R},x})$.

Then the operator $g\exp(-\mathscr{B}_u^2 + bN_H)$ acts on $\Lambda(T^*_{\mathbf{R}}B) \hat{\otimes} I$, and its kernel $(gP_u^{x,b})(Y,Y')$ is given by,

$$(gP_u^{x,b})(Y,Y') = gP_u^{x,b}(g^{-1}Y,Y'). \tag{2.8}$$

By Theorem 1.6, we get,

$$|P_u^{x,b}(g^{-1}Y,Y)| \leqslant C\exp\{-\tfrac{1}{2}|(g^{-1}-1)P^LY|^2 + 2C'|P^LY| - 2C''|P^NY|^2\}. \tag{2.9}$$

Also $M^{0,\perp}$ and N^0 are mutually orthogonal in M. It follows from (2.9) that there exists $C>0$, $C'>0$ such that if $x \in B$, $Y \in M^{0,\perp}_{\mathbf{R},x} \oplus N^0_{\mathbf{R},x}$, then

$$|P_u^{x,b}(g^{-1}Y,Y)| \leqslant C\exp(-C'|Y|^2). \tag{2.10}$$

By applying (2.10) to the $P_u^{\theta_j,x,b}$'s, we find that

$$\begin{aligned} &|P_u^{0,x,b}(g^{-1}Y,Y)| \leqslant C\exp(-C'|Y|^2),\ Y \in N^0_{\mathbf{R},x},\\ &|P_u^{\theta_j,x,b}(g^{-1}Y,Y)| \leqslant C\exp(-C'|Y|^2),\ \theta_j \neq 0,\ Y \in M^{\theta_j}_{\mathbf{R},x}. \end{aligned} \tag{2.11}$$

Now g acts on $\Lambda(\bar{L}^{0,\perp,*}) \hat{\otimes} \Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)$ and so g acts on

$$\Lambda(T^*_{\mathbf{R}}B) \hat{\otimes} \Lambda(\bar{L}^{0,\perp,*}) \hat{\otimes} \Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*).$$

Similarly $P_u^{x,b}(Y, Y')$ acts on the same bundle (in fact it acts trivially on $\Lambda(\bar{L}^{0,\perp,*})$). Therefore

$$gP_u^{x,b}(g^{-1}Y, Y) \in (\Lambda(T_{\mathbf{R}}^*B) \hat{\otimes} \mathrm{End}(\Lambda(\bar{L}^{0,\perp,*}) \hat{\otimes} \Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)))_x^{\mathrm{even}}.$$

Let $\mathrm{Tr}_s[gP_u^{x,b}(g^{-1}Y, Y)] \in \Lambda(T_{\mathbf{R}}^*B)$ be the corresponding supertrace. We define in the same way $\mathrm{Tr}_s[N_H gP_u^x(g^{-1}Y, Y)]$.

Let us apply the previous considerations to the E^{θ_j}'s. First, if $\theta_j = 0$, then $P_u^{0,x,b}(Y, Y')$ lies in $(\Lambda(T_{\mathbf{R}}^*B) \hat{\otimes} \mathrm{End}(\Lambda(\bar{N}^{0*}) \hat{\otimes} \Lambda(N^{0*})))_x^{\mathrm{even}}$. So we can define $\mathrm{Tr}_s[P_u^{0,x,b}(Y, Y)]$ and $\mathrm{Tr}_s[N_H^0 P_u^{0,x}(Y, Y)]$, $(Y \in M_{\mathbf{R},x}^0)$, which lie in $\Lambda(T_{\mathbf{R}}^*B)_x$. For $\theta_j \neq 0$, $gP_u^{\theta_j,x,b}(g^{-1}Y, Y) \in (\Lambda(T_{\mathbf{R}}^*B) \hat{\otimes} \mathrm{End}(\Lambda(\bar{M}^{\theta_j*}) \hat{\otimes} \Lambda(N^{\theta_j*})))_x^{\mathrm{even}}$. Therefore, we can define the supertraces $\mathrm{Tr}_s[gP_u^{\theta_j,x,b}(g^{-1}Y, Y)]$ and

$$\mathrm{Tr}_s[N_H^{\theta_j} gP_u^{\theta_j,x,b}(g^{-1}Y, Y)] \quad (Y \in M_{\mathbf{R},x}^{\theta_j}),$$

which lie in $\Lambda^{\mathrm{even}}(T_{\mathbf{R}}^*B)$.

Using (2.5), we see that,

$$\mathrm{Tr}_s[gP_u^{x,b}(g^{-1}Y, Y)] = \prod_{j=1}^{q} \mathrm{Tr}_s[gP_u^{\theta_j,x,b}(g^{-1}Y^{\theta_j}, Y^{\theta_j})], \tag{2.12}$$

$$\begin{aligned}\mathrm{Tr}_s[N_H P_u^x(g^{-1}Y, Y)] = \sum_{j=1}^{q} & \mathrm{Tr}_s[gP_u^{\theta_1,x}(g^{-1}Y^{\theta_1}, Y^{\theta_1})] \\ & \ldots \mathrm{Tr}_s[gP_u^{\theta_{j-1},x}(g^{-1}Y^{\theta_{j-1}}, Y^{\theta_{j-1}})]\, \mathrm{Tr}_s[N_H^{\theta_j} gP_u^{\theta_j,x}(g^{-1}Y^{\theta_j}, Y^{\theta_j})] \\ & \ldots \mathrm{Tr}_s[gP_u^{\theta_q,x}(g^{-1}Y^{\theta_q}, Y^{\theta_q})].\end{aligned}$$

(b) *Equivariant generalized supertraces*

Let $dv_{M_0^\perp \oplus N_0}$ be the obvious volume element on the fibres of $M_0^\perp \oplus N_0$. In view of (2.10), we are now entitled to set the following definition, which extends [B1, Definition 4.4].

DEFINITION 2.1. For $u > 0$, set

$$\mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2 + bN_H)]_x = \int_{M_{\mathbf{R},x}^{0,\perp} \oplus N_{\mathbf{R},x}^0} \mathrm{Tr}_s[gP_u^{x,b}(g^{-1}Y, Y)] \frac{dv_{M^{0,\perp} \oplus N^0}(Y)}{(2\pi)^{\dim M^{0,\perp} + \dim N^0}},$$

$$\mathrm{Tr}_s[N_H g \exp(-\mathscr{B}_u^2)]_x = \int_{M_{\mathbf{R},x}^{0,\perp} \oplus N_{\mathbf{R},x}^0} \mathrm{Tr}_s[N_H gP_u^x(g^{-1}Y, Y)] \frac{dv_{M^{0,\perp} \oplus N^0}(Y)}{(2\pi)^{\dim M^{0,\perp} + \dim N^0}}. \tag{2.13}$$

Then $\mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2 + bN_H)]$ and $\mathrm{Tr}_s[N_H g \exp(-\mathscr{B}_u^2)]$ are smooth even forms on B. Of course, we define in the same way the forms

$$\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^{\theta_j,2}+bN_H^{\theta_j})] \quad \text{and} \quad \mathrm{Tr}_s[N_H^{\theta_j}g\exp(-\mathscr{B}_u^{\theta_j,2})].$$

REMARK 2.2. As in [B1, Section 4(d)], we observe that the operators $g\exp[-\mathscr{B}_u^2+bN_H)]$ are not trace class, except when $L=\{0\}$. So the supertraces in (2.12) are only generalized supertraces.

PROPOSITION 2.3. *The following identities hold,*

$$\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2+bN_H)] = \prod_{j=1}^{q} \mathrm{Tr}_s[g\exp(-\mathscr{B}_u^{\theta_j,2}+bN_H^{\theta_j})],$$

$$\mathrm{Tr}_s[N_Hg\exp(-\mathscr{B}_u^2)] = \sum_{j=1}^{q}\prod_{k\neq j} \mathrm{Tr}_s[g\exp(-\mathscr{B}_u^{\theta_k,2})]\,\mathrm{Tr}_s[N_H^{\theta_j}g\exp(-\mathscr{B}_u^{\theta_j,2})]. \tag{2.14}$$

Proof. Proposition 2.3 follows from (2.12). □

DEFINITION 2.4. Let P^B be the vector space of smooth forms on B which are sums of forms of type (p,p). Let $P^{B,0}$ be the vector space of forms $\alpha\in P^B$ such that there exist smooth forms β,γ with $\gamma=\partial\beta+\bar\partial\gamma$.

(c) *Double transgression formulas for generalized supertraces*

Now we prove an extension of [B1, Theorem 4.6].

THEOREM 2.5. *For $b\in\mathbf{C}$ and $u>0$, the forms $\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2+bN_H)]$ and $\mathrm{Tr}_s[N_Hg\exp(-\mathscr{B}_u^2)]$ lie in P^B. The forms $\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2)]$ are closed, and their cohomology class does not depend on $u>0$. More precisely,*

$$\frac{\partial}{\partial u}\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2)] = \frac{\bar\partial\partial}{u}\left\{\frac{\partial}{\partial b}\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2+bN_H)]_{b=0}\right\}. \tag{2.15}$$

Also,

$$\mathrm{Tr}_s[N_Hg\exp(-\mathscr{B}_u^2)] = \frac{\partial}{\partial b}\mathrm{Tr}_s[\exp(-\mathscr{B}_u^2+bN_H)]_{b=0}. \tag{2.16}$$

Proof. By Proposition 2.3, we only need to prove our theorem when $E=E^\theta$, i.e. when g acts on M by multiplication by $e^{i\theta}$ $(0\leqslant\theta<2\pi)$. When $\theta=0$, our theorem was already proved in [B1, Theorem 4.6]. So we may and we will assume that $\theta\in]0,2\pi[$.

First, we proved as in [BGS1, proof of Theorem 1.9]. We define the total grading on $\Lambda(\bar M^*)$, $\bar\Lambda(N^*)$ by the operator N_V-N_H.

If $Y = y + \bar{y}$, $Y \in M_{\mathbf{R}}$, $y \in M$, then

$$\frac{\sqrt{-1}}{\sqrt{2}} \hat{c}(AP^L Y) = \sqrt{-1}\,(i_{AP^L y} - i^*_{AP^L y}). \tag{2.17}$$

As an element of $\Lambda(T^*_{\mathbf{R}}B) \hat{\otimes} \mathrm{Hom}(L, N)$, AP^L is of complex type $(1,0)$. So $i_{AP^L Y} \in \Lambda^1(T^*_{\mathbf{R}}B) \hat{\otimes} \mathrm{End}(\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*))$ is of complex type $(1,0)$ and increases the total degree in $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$ by 1. Similarly $i^*_{AP^L Y}$ is a 1 form of complex type $(0,1)$, which decreases the total degree in $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$ by 1. Also [B1, Eq. (1.26)] or a direct computation show that S preserves the total degree in $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$.

Let N_B, $\bar{N}_B$ be the number operators of $\Lambda(T^{*(1,0)}B)$, $\Lambda(T^{*(0,1)}B)$. Using Theorem 1.4, the previous consideration show that for $\alpha \in \mathbf{R}$,

$$e^{\alpha(N_V - N_H)}(\mathscr{B}_u^2 - bN_H)e^{-\alpha(N_V - N_H)} = e^{\alpha(N_B - \bar{N}_B)}(\mathscr{B}_u^2 - bN_H)e^{-\alpha(N_B - \bar{N}_B)}. \tag{2.18}$$

Also, if we identify g with its action on $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$, then

$$e^{\alpha(N_V - N_H)} g e^{-\alpha(N_V - N_H)} = g. \tag{2.19}$$

From (2.18), (2.19), we deduce that

$$e^{\alpha(N_V - N_H)} g P_u^{x,b}(g^{-1}Y, Y) e^{-\alpha(N_V - N_H)} = e^{\alpha(N_B - \bar{N}_B)} g P_u^{x,b}(g^{-1}Y, Y) e^{-\alpha(N_B - \bar{N}_B)}. \tag{2.20}$$

Since by [Q1], Tr_s vanishes on supercommutators, we deduce from (2.20) that

$$\mathrm{Tr}_s[g P_u^{x,b}(g^{-1}Y, Y)] = e^{\alpha(N_B - \bar{N}_B)} \mathrm{Tr}_s[g P_u^{x,b}(g^{-1}Y, Y)]. \tag{2.21}$$

From (2.21), we see that $\mathrm{Tr}_s[g P_u^{x,b}(g^{-1}Y, Y)]$ is a sum of forms of type (p, p). It is now clear that $\mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2 + bN_H)]$ lies in P^B.

Take $x \in B$, and let $\mathscr{V}$ be an open neighborhood of x, which is holomorphically equivalent to an open ball in $\mathbf{C}^k$, with 0 representing x.

Then the exterior algebra $\Lambda(T^*_{\mathbf{R}}B)$ is canonically trivialized on $\mathscr{V}$. Let $l \in \mathbf{R} \to x_l \in \mathbf{C}^k$ be a straight line, with $x_0 = x$. We trivialize the vector bundle M along the line x_l by parallel transport with respect to the connection ∇^M. Of course this trivialization preserves the volume element dv_M. Similarly, we trivialize the vector bundle I along the line x_l using the connection ∇^I, so that I_{x_l} is identified with I_x. The operator $\mathscr{B}^2_{u,x_l}$ now acts on the fixed vector space $(\Lambda(T^*_{\mathbf{R}}B) \hat{\otimes} I)_x$. Under this trivialization, g acts as a constant operator.

For $s > 0$, let $P^x_{u,s}(Y, Y')$ $(Y, Y' \in M_{\mathbf{R},x})$ be the smooth kernel associated to the operator $\exp(-s\mathscr{B}^2_{u,x})$. Of course $P^x_{u,1}(Y, Y') = P^x_u(Y, Y')$. For the existence and uniqueness of $P^x_{u,s}(Y, Y')$, we refer to [B1, Section 4(a)]. Then by using

Duhamel's formula as in [B1, Eq. (4.38)], we see that if $Y, Y' \in M_{\mathbf{R},x}$,

$$\frac{\partial}{\partial l} P_u^{x_l}(Y, Y') = -\int_0^1 ds \int_{M_{\mathbf{R},x}} P_{u,s}^{x_l}(Y, Y'') \frac{\partial \mathscr{B}_{u,x_l}^2}{\partial l} P_{u,1-s}^{x_l}(Y'', Y') \frac{dv_M(Y'')}{(2\pi)^{\dim M}}. \tag{2.22}$$

From (2.22), we deduce,

$$\mathrm{d}\,\mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2)]_x = -\int_0^1 ds \int_{M_{\mathbf{R},x} \times M_{\mathbf{R},x}} \mathrm{Tr}_s[g P_{u,s}^x(g^{-1}Y, Y') \times [\nabla^I, \mathscr{B}_u^2] P_{u,1-s}^x(Y', Y)] \frac{dv_M(Y')}{(2\pi)^{\dim M}} \frac{dv_M(Y)}{(2\pi)^{\dim M}}. \tag{2.23}$$

Of course, obvious analogues of the bounds (1.17) for the kernels $P_{u,s}^x(Y, Y')$ and the fact that $\theta \in]0, 2\pi[$ guarantee that the integral in the right-hand side of (2.23) makes sense.

Set

$$\mathscr{A}_u = D^M + \sqrt{u}\, V(P^N Y) - \frac{c(R^M Y)}{2\sqrt{2}}.$$

Then one has the trivial relation

$$[\mathscr{A}_u, g] = 0. \tag{2.24}$$

Also by the same method as in (2.22), we find that

$$[\mathscr{A}_u, P_u^x](Y, Y') = -\int_0^1 ds \int_{M_{\mathbf{R},x}} P_{u,s}^x(Y, Y'')[\mathscr{A}_u, \mathscr{B}_u^2] P_{u,1-s}^x(Y'', Y') \frac{dv_M(Y'')}{(2\pi)^{\dim M}}. \tag{2.25}$$

So if $(gP_u^x)(Y, Y') = gP_u^x(g^{-1}Y, Y')$ is the kernel of $g \exp(-\mathscr{B}_u^2)$, we deduce from (2.24), (2.25),

$$[\mathscr{A}_u, (gP_u^x)](Y, Y) = -\int_0^1 ds \int_{M_{\mathbf{R},x}} gP_{u,s}^x(g^{-1}Y, Y')[\mathscr{A}_u, \mathscr{B}_u^2] P_{u,1-s}^x(Y', Y) \frac{dv_M(Y')}{(2\pi)^{\dim M}}. \tag{2.26}$$

Take $\varepsilon > 0$. Let $Q(Y, Y')(Y, Y' \in M_{\mathbf{R},x})$ be a bounded smooth kernel acting on $(\Lambda(T_{\mathbf{R}}^* B) \hat{\otimes} \Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*))_x$, which vanishes for $|Y - Y'| \geq \varepsilon$. Then using the bound (1.17), we see that there exists $C > 0$ such that

$$|Q(Y, Y')gP_u^x(g^{-1}Y', Y)| \leqslant C1_{|Y'-Y|\leqslant\varepsilon} \exp\left(-\frac{|Y|^2}{2}\right),$$

$$|gP_u^x(g^{-1}Y, Y')Q(Y', Y)| \leqslant C1_{|Y'-Y|\leqslant\varepsilon} \exp\left(-\frac{|Y|^2}{2}\right). \tag{2.27}$$

From (2.27) and from the fact that Tr_s vanishes on supercommutators [Q1], we get

$$\int_{M_{\mathbf{R},x}} \mathrm{Tr}_s[[Q, (gP_u^x)](Y, Y)] \frac{dv_M(Y)}{(2\pi)^{\dim M}} = 0. \tag{2.28}$$

Using the bounds (1.17) for the derivatives of $P_u^x(Y, Y')$ and approximating $\mathscr{A}_u$ by a sequence of smooth kernels having the support property described before, we get

$$\int_{M_{\mathbf{R},x}} \mathrm{Tr}_s[[\mathscr{A}_u, (gP_u^x)](Y, Y)] \frac{dv_M(Y)}{(2\pi)^{\dim M}} = 0. \tag{2.29}$$

Clearly,

$$\mathscr{B}_u = \nabla^I + \mathscr{A}_u. \tag{2.30}$$

Since $[\mathscr{B}_u, \mathscr{B}_u^2] = 0$, we deduce from (2.30),

$$[\nabla^I, \mathscr{B}_u^2] = -[\mathscr{A}_u, \mathscr{B}_u^2]. \tag{2.31}$$

Using (2.23), (2.26), (2.29), (2.31), we get,

$$d\,\mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2)] = 0. \tag{2.32}$$

Now we replace the manifold B by $B \times \mathbf{R}_+^*$. The exact sequence E lifts to $B \times \mathbf{R}_+^*$, and g also lifts. The superconnection $\mathscr{B}_u$ is replaced by $\mathscr{B}'_u$ given by

$$\mathscr{B}'_u = \mathscr{B}_u + du \frac{\partial}{\partial u}.$$

Clearly

$$\mathscr{B}'^2_u = \mathscr{B}_u^2 + du \frac{\partial}{\partial u} \mathscr{B}_u. \tag{2.33}$$

As in [B1, proof of Theorem 4.6], although $B\times\mathbf{R}_+^*$ is not a complex manifold, we can reproduce for $B\times\mathbf{R}_+^*$ what has been done for $\mathscr{B}_u$. By (2.32), (2.33), we get,

$$\mathrm{d}\,\mathrm{Tr}_s\left[g\exp\left(-\mathscr{B}_u^2-\mathrm{d}u\,\frac{\partial\mathscr{B}_u}{\partial u}\right)\right]=0. \tag{2.34}$$

The form

$$\mathrm{Tr}_s\left[g\exp\left(-\mathscr{B}_u^2-\mathrm{d}u\,\frac{\partial}{\partial u}\mathscr{B}_u\right)\right]$$

can be written in the form,

$$\mathrm{Tr}_s\left[g\exp\left(-\mathscr{B}_u^2-\mathrm{d}u\,\frac{\partial}{\partial u}\mathscr{B}_u\right)\right]=\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2)]+\mathrm{d}u\,\alpha_u. \tag{2.35}$$

Using (2.34), (2.35), it is clear that the cohomology class of the forms $\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2)]$ does not vary with $u>0$. More precisely,

$$\frac{\partial}{\partial u}\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2)]=\mathrm{d}\alpha_u. \tag{2.36}$$

The form α_u can be calculated using Duhamel's formula. We get,

$$\alpha_{u,x}=-\int_0^1\mathrm{d}s\int_{M_{\mathbf{R},x}\times M_{\mathbf{R},x}}\mathrm{Tr}_s\left[gP_{u,s}^x(g^{-1}Y,Y')\,\frac{\partial\mathscr{B}_u}{\partial u}\,P_{u,1-s}^x(Y',Y)\right]\times\frac{\mathrm{d}v_M(Y)}{(2\pi)^{\dim M}}\,\frac{\mathrm{d}v_M(Y')}{(2\pi)^{\dim M}}.$$

Equivalently,

$$\alpha_{u,x}=\frac{-1}{2\sqrt{u}}\int_0^1\mathrm{d}s\int_{M_{\mathbf{R},x}\times M_{\mathbf{R},x}}\mathrm{Tr}_s[gP_{u,s}^x(g^{-1}Y,Y')\sqrt{-1}\,(i_{P^NY'}-i^*_{P^NY'})P_{u,1-s}^x(Y',Y)]\times\frac{\mathrm{d}v_M(Y)}{(2\pi)^{\dim M}}\,\frac{\mathrm{d}v_M(Y')}{(2\pi)^{\dim M}}. \tag{2.37}$$

Now we will prove (2.15). Let $P_{u,s}^{x,b}(Y,Y')$ be the smooth kernel associated to

$\exp(-\mathscr{B}_{u,x}^2+bN_H)$. The analogue of (2.23) is,

$$\mathrm{d}\,\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2+bN_H)]_x$$
$$= -\int_0^1 \mathrm{d}s \int_{M_{\mathbf{R},x}\times M_{\mathbf{R},x}} \mathrm{Tr}_s[gP_{u,s}^{x,b}(g^{-1}Y, Y')[\nabla^I, \mathscr{B}_u^2]P_{u,1-s}^{x,b}(Y', Y)]$$
$$\times \frac{\mathrm{d}v_M(Y)}{(2\pi)^{\dim M}} \frac{\mathrm{d}v_M(Y')}{(2\pi)^{\dim M}}. \tag{2.38}$$

We rewrite (2.31) in the form,

$$[\nabla^I, \mathscr{B}_u^2] = -[\mathscr{A}_u, \mathscr{B}_u^2 - bN_H] - b\sqrt{u}[V(P^N Y), N_H]. \tag{2.39}$$

By proceeding as in (2.23)–(2.32) and using (2.38), (2.39), we get,

$$\mathrm{d}\,\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2+bN_H)]_x$$
$$= \int_0^1 \mathrm{d}s \int_{M_{\mathbf{R},x}\times M_{\mathbf{R},x}} \mathrm{Tr}_s[gP_{u,s}^{x,b}(g^{-1}Y, Y')b\sqrt{u}\,[V(P^N Y), N_H]P_{u,1-s}^{x,b}(Y', Y)]$$
$$\times \frac{\mathrm{d}v_M(Y)}{(2\pi)^{\dim M}} \frac{\mathrm{d}v_M(Y')}{(2\pi)^{\dim M}}. \tag{2.40}$$

Also,

$$[V(P^N Y'), N_H] = \sqrt{-1}\,(i_{P^N y'} + i^*_{P^N y'}). \tag{2.41}$$

By (2.40), (2.41), we obtain,

$$\mathrm{d}\frac{\partial}{\partial b}\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2+bN_H)]_{b=0}$$
$$= \sqrt{u}\int_0^1 \mathrm{d}s \int_{M_{\mathbf{R},x}\times M_{\mathbf{R},x}} \mathrm{Tr}_s[gP_{u,s}^{x}(g^{-1}Y, Y')\sqrt{-1}\,(i_{P^N y'} + i^*_{P^N y'})P_{u,1-s}^{x}(Y', Y)]$$
$$\times \frac{\mathrm{d}v_M(Y)}{(2\pi)^{\dim M}} \frac{\mathrm{d}v_M(Y')}{(2\pi)^{\dim M}}. \tag{2.42}$$

By using the degree counting arguments of [BGS1, proof of Theorem 1.9], or the arguments in (2.17), (2.21), we deduce from (2.42),

$$\partial \frac{\partial}{\partial b} \mathrm{Tr}_s[\, g \exp(-\mathscr{B}_u^2 + bN_H)]_{b=0}$$

$$= \sqrt{u} \int_0^1 \mathrm{d}s \,\mathrm{Tr}_s[g P^x_{u,s}(g^{-1}Y, Y')\sqrt{-1}\, i^*_{P^N y'} P^x_{u,1-s}(Y', Y)] \frac{\mathrm{d}v_M(Y)}{(2\pi)^{\dim M}} \frac{\mathrm{d}v_M(Y')}{(2\pi)^{\dim M}},$$

$$\bar{\partial} \frac{\partial}{\partial b} \mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2 + bN_H)]_{b=0}$$

$$= \sqrt{u} \int_0^1 \mathrm{d}s \,\mathrm{Tr}_s[g P^x_{u,s}(g^{-1}Y, Y')\sqrt{-1}\, i_{P^N y'} P^x_{u,1-s}(Y', Y)] \frac{\mathrm{d}v_M(Y)}{(2\pi)^{\dim M}} \frac{dv_M(Y')}{(2\pi)^{\dim M}}. \tag{2.43}$$

From (2.37), (2.43), we get,

$$\alpha_u = \frac{1}{2u} (\partial - \bar{\partial}) \frac{\partial}{\partial b} \mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2 + bN_H)]_{b=0}. \tag{2.44}$$

Using (2.36), (2.44), we obtain (2.15).

Finally (2.16) can be proved easily by the same method as before. Since $\theta \in]0, 2\pi[$, the proof is in fact much simpler as in [B1, proof of Theorem 4.6], where the case $\theta = 0$ was considered.

The proof of Theorem 2.5 is completed. □

3. The asymptotics as $u \to 0$ of the generalized supertraces

We make the same assumptions and we use the same notation as in Section 2. The purpose of this section is to calculate the asymptotics as $u \to 0$ of the generalized supertraces.

For $1 \leqslant j \leqslant q$, let $h^{M^{\theta_j}}$ be the restriction of h^M to M^{θ_j}. Let $\nabla^{M^{\theta_j}}$ be the holomorphic Hermitian connection on $(M^{\theta_j}, h^{M^{\theta_j}})$ and let $R^{M^{\theta_j}}$ be its curvature. Of course, $\nabla^{M^{\theta_j}}, R^{M^{\theta_j}}$ are the restrictions of ∇^M, R^M to M^{θ_j}.

Recall that if A is a (q, q) matrix,

$$\mathrm{Td}(A) = \det\left(\frac{A}{1 - e^{-A}}\right). \tag{3.1}$$

Set

$$\mathrm{e}(A) = \det(A). \tag{3.2}$$

The genera associated to Td and e are the Todd genus and the maximal Chern class.

DEFINITION 3.1. Set

$$\mathrm{Td}_g(M, h^M) = \mathrm{Td}\left(-\frac{R^{M0}}{2i\pi}\right) \prod_{\theta_j \neq 0} \left(\frac{\mathrm{Td}}{\mathrm{e}}\right)\left(-\frac{R^{M^{\theta_j}}}{2i\pi} + i\,\theta_j\right),$$

$$\mathrm{Td}_g(M, h^M, b) = \mathrm{Td}\left(-\frac{R^{M0}}{2i\pi} + b\right) \prod_{\theta_j \neq 0} \left(\frac{\mathrm{Td}}{\mathrm{e}}\right)\left(-\frac{R^{M^{\theta_j}}}{2i\pi} + i\theta_j + b\right),$$

$$(\mathrm{Td}_g^{-1})'(M, h^M) = \frac{\partial}{\partial b} \mathrm{Td}_g^{-1}(M, h^M, b)_{b=0}. \tag{3.3}$$

We define the corresponding objects attached to (L, h^L), (N, g^N) in the same way.

Clearly, the forms in (3.3) are closed and lie in P^B.

If $(\omega_u)_{u \geqslant 0}$ is a family of smooth forms on B, we will write that as $u \to 0$,

$$\omega_u = \omega_0 + \mathcal{O}(u),$$

if for any $k \in \mathbb{N}$, the sup of the norm of $\omega_u - \omega_0$ and its derivatives of order $\leqslant k$ can be dominated by Cu for $u \leqslant 1$.

Let φ be the map from $\Lambda^{\mathrm{even}}(T^*_{\mathbb{R}}B)$ into itself: $\alpha \to ((2i\pi)^{-\deg \alpha/2}\alpha$.

Now, we prove an extension of [B1, Theorem 4.8].

THEOREM 3.2. *As* $u \to 0$,

$$\varphi\,\mathrm{Tr}_s[\exp(-\mathscr{B}_u^2 + bN_H)] = \mathrm{Td}_g(M, h^M)\,\mathrm{Td}_g^{-1}(N, h^N, -b) + \mathcal{O}(u). \tag{3.4}$$

In particular, as $u \to 0$,

$$\varphi\,\mathrm{Tr}_s[\exp(-\mathscr{B}_u^2)] = \mathrm{Td}_g(M, h^M)\,\mathrm{Td}_g^{-1}(N, h^N) + \mathcal{O}(u),$$

$$\varphi\,\frac{\partial}{\partial b}\,\mathrm{Tr}_s[\exp(-\mathscr{B}_u^2 + bN_H)]_{b=0} = -\mathrm{Td}_g(M, h^M)(\mathrm{Td}_g^{-1})'(N, h^N) + \mathcal{O}(u). \tag{3.5}$$

Proof. By Proposition 2.3, it is clear that to prove (3.4), we may as well assume that $E = E^\theta$, i.e. g acts on E by multiplication by $e^{i\theta}$ $(0 \leqslant \theta < 2\pi)$. If $\theta = 0$, our theorem was already proved in [B1, Theorem 4.8]. So, we may and we will assume that $0 < \theta < 2\pi$.

Observe first that $\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2)]$, $\mathrm{Tr}_s[N_H g\exp(-\mathscr{B}_u^2)]$ also make sense for $u = 0$. In fact from (1.21), we deduce that there exists $C > 0$ such that if $0 \leqslant u \leqslant 1$, $|b| \leqslant A$, $Y, Y' \in M_{\mathbf{R},x}$,

$$|P_u^{x,b}(Y, Y')| \leqslant C \exp\left(-\frac{|Y-Y'|^2}{2}\right)\left\{E^{Q(Y,Y')}\left[\left|\exp\left\{\frac{1}{2}\int_0^1 \langle R^M Y, \mathrm{d}Y\rangle\right\}\right|^2\right]\right.$$
$$\left.\times E^{Q(Y,Y')}\left[\exp\left(2C'\int_0^1 |P^L Y|\,\mathrm{d}t\right)\right]\right\}^{1/2}. \tag{3.6}$$

Using (1.22), (1.27), (3.6), we see that there exist $C > 0$, $C' > 0$, $p \in \mathbb{N}$ such that for $0 \leqslant u \leqslant 1$, $Y, Y' \in M_{\mathbf{R},x}$,

$$|P_u^x(Y, Y')| \leqslant C \exp\left(-\frac{|Y-Y'|^2}{2} + C'(|P^L Y| + |P^L Y'|)\right)(1+|Y|^p+|Y'|^p). \tag{3.7}$$

From (3.7) we deduce that for $0 \leqslant u \leqslant 1$, $Y \in M_{\mathbf{R},x}$,

$$|P_u^{x,b}(g^{-1}Y, Y)| \leqslant C \exp\left(-\frac{|(g^{-1}-1)Y|^2}{2} + 2C'|P^L Y|\right)(1+2|Y|^p). \tag{3.8}$$

So by (3.8), we see that $\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2+bN_H)]$ can be defined as in (2.13) for $u=0$. Also from (3.8), we deduce easily that as $u \to 0$,

$$\varphi\,\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2+bN_H)] = \varphi\,\mathrm{Tr}_s[g\exp(-\mathscr{B}_0^2+bN_H)] + \mathscr{O}(u). \tag{3.9}$$

Set

$$\mathscr{A} = -\frac{1}{2}\sum_1^{2m}(\nabla_{e_i} + \tfrac{1}{2}\langle R^M Y, e_i\rangle)^2 + \tfrac{1}{2}\mathrm{Tr}[R^M]. \tag{3.10}$$

Let $p^x(Y, Y')(Y, Y' \in M_{\mathbf{R},x})$ be the smooth kernel associated to $\exp(-\mathscr{A}_x)$.
By Theorem 1.4, we get,

$$\mathscr{B}_0^2 = \mathscr{A} + R^{\Lambda(N^*)}. \tag{3.11}$$

From (3.11), we deduce

$$P_0^{x,b}(Y, Y') = \exp(-R^{\Lambda(N^*)} + bN_H)p^x(Y, Y'). \tag{3.12}$$

So using (3.12), we get,

$$\operatorname{Tr}_s[g \exp(-\mathscr{B}_0^2+bN_H)]$$
$$= \operatorname{Tr}_s[g \exp(-R^{\Lambda(N^*)} + bN_H)] \int_{M_{\mathbf{R},x}} p^x(g^{-1}Y, Y) \frac{dv_M(Y)}{(2\pi)^{\dim M}}. \tag{3.13}$$

Clearly

$$\operatorname{Tr}_s[g \exp(-R^{\Lambda(N^*)}+bN_H)] = \operatorname{Tr}_s^{\Lambda(\bar{M}^*)}[g] \operatorname{Tr}_s^{\Lambda(N^*)}[g \exp(-R^{\Lambda(N^*)}+bN_H)]. \tag{3.14}$$

From (3.14), we obtain,

$$\operatorname{Tr}_s[g \exp(-R^{\Lambda(N^*)}+bN_H)] = (1-e^{i\theta})^{\dim M} \det(1-\exp(R^N-i\theta+b)). \tag{3.15}$$

Let $C \in \operatorname{End}(M)$ be skew-adjoint. We identify C with the corresponding antisymmetric element in $\operatorname{End}(M_{\mathbf{R}})$. Set

$$A = -\frac{1}{2} \sum_1^{2m} (\nabla_{e_i} + \tfrac{1}{2}\langle CY, e_i\rangle)^2 + \tfrac{1}{2}\operatorname{Tr}[C]. \tag{3.16}$$

Let $x_1, \ldots, x_m$ be the eigenvalues of $C \in \operatorname{End}(M)$. Assume that $\sup |x_i| < 2\pi$. Then by [B1, Eq. (6.37)], which itself relies on Mehler's formula, the smooth kernel $q(Y, Y')$ associated to $\exp(-A)$ is given by

$$q(Y, Y') = \operatorname{Td}(-C) \exp\left\{-\frac{1}{2}\left\langle\frac{C/2}{\tanh(C/2)} Y, Y\right\rangle \right.$$
$$\left. -\frac{1}{2}\left\langle\frac{C/2}{\tanh(C/2)} Y', Y'\right\rangle + \left\langle\frac{C/2}{\sinh(C/2)} e^{C/2}Y, Y'\right\rangle\right\}. \tag{3.17}$$

From (3.17), we deduce,

$$q(g^{-1}Y, Y) = \operatorname{Td}(-C) \exp\left\{-\left\langle\frac{C/2}{\sinh(C/2)} (\cosh(C/2)-e^{C/2}g^{-1})Y, Y\right\rangle\right\}. \tag{3.18}$$

As before we identify the skew-adjoint $i\theta \in \operatorname{End}(M)$ to the corresponding antisymmetic element of $\operatorname{End}(M_{\mathbf{R}})$. Then one has the obvious,

$$\left\langle\frac{C/2}{\sinh(C/2)} e^{C/2}g^{-1}Y, Y\right\rangle = \left\langle\frac{C/2}{\sinh(C/2)} \cosh(C/2-i\theta)Y, Y\right\rangle. \tag{3.19}$$

Also

$$\cosh(C/2) - \cosh(C/2 - i\theta) = 2\sin\left(\frac{\theta}{2}\right) i \sinh\left(\frac{C - i\theta}{2}\right). \tag{3.20}$$

From (3.17)–(3.20), if $\sup |x_i| < \theta$, we get

$$\int_{M_{\mathbf{R},x}} q(g^{-1}Y, Y) \frac{dv_M(Y)}{(2\pi)^{\dim M}} = \frac{e^{-1/2\,\mathrm{Tr}[C]}}{(2i\sinh(\theta/2))^{\dim M} \det_M\left(2\sinh\left(\frac{C - i\theta}{2}\right)\right)}. \tag{3.21}$$

From (3.13), (3.15), (3.21), replacing formally C by R^M, we obtain

$$\varphi\,\mathrm{Tr_s}[\exp(-\mathscr{B}_0^2 + bN_H)] = \mathrm{Td}_g(M, h^M)\,\mathrm{Td}_g^{-1}(N, h^N, -b). \tag{3.22}$$

Then (3.4) follows from (3.9), (3.22). Equation (3.5) follows from (3.4). The proof of Theorem 3.2 is completed. □

4. The function $\sigma(u, \eta, x)$ and the associated traces in finite and infinite dimension

The main result of [B1] was expressed in terms of certain additive genera attached to the function

$$\varphi(u, x) = \frac{4}{u}\sinh\left(\frac{x + \sqrt{x^2 + 4u}}{4}\right)\sinh\left(\frac{-x + \sqrt{x^2 + 4u}}{4}\right).$$

In our more general equivariant situation, our main result will be expressed in terms of the more complicated function $\sigma(u, \eta, x)$,

$$\sigma(u, \eta, x) = 4\sinh\left(\frac{x - 2\eta + \sqrt{x^2 + 4u}}{4}\right)\sinh\left(\frac{-x + 2\eta + \sqrt{x^2 + 4u}}{4}\right).$$

In this section, we show that:

1. $\sigma(u, i0, x)$ can be expressed as an infinite product, or more precisely as the determinant of a differential operator over S_1.
2. $\sigma(u, \eta, x)$ appears naturally in the explicit computation of certain supertraces on Clifford algebras, or in the evaluation of the trace of a harmonic oscilator.

The results of the section extend the result of [B1, Section 6].

This section is organized as follows. In (a), we give various properties of the function $\sigma(u, \eta, x)$. In (b), we introduce the corresponding multiplicative genus Σ. In (c), we show that Σ appears explicitly in the evaluation of certain finite dimensional supertraces. In (d), we obtain a similar result for the trace of the heat kernel of an harmonic oscillator. In Remark 4.8, we give a group theoretic interpretation of the similarity of these two calculations. This uses results of [B1, Section 1] and also a simple extension to our approach [B3] of the infinitesimal Lefschetz formulas.

Except for occasional reference to [B1], this section is self-contained.

(a) *The function* $\sigma(u, \eta, x)$

If $z \in \mathbf{C}$, $\sqrt{z}$ denotes an arbitrary (but fixed) square root of z. Our results do not depend on the choice of $\sqrt{z}$.

Recall that for $u \in \mathbf{C}$, $x \in \mathbf{C}$, the function $\varphi(u, x)$ was defined in [B1, Definition 6.1] by the formula

$$\varphi(u, x) = \frac{4}{u} \sinh\left(\frac{x + \sqrt{x^2 + 4u}}{4}\right) \sinh\left(\frac{-x + \sqrt{x^2 + 4u}}{4}\right). \tag{4.1}$$

DEFINITION 4.1. For $u \in \mathbf{C}$, $\eta \in \mathbf{C}$, $x \in \mathbf{C}$, set

$$\sigma(u, \eta, x) = 4 \sinh\left(\frac{x - 2\eta + \sqrt{x^2 + 4u}}{4}\right) \sinh\left(\frac{-x + 2\eta + \sqrt{x^2 + 4u}}{4}\right). \tag{4.2}$$

Clearly

$$\varphi(u, x) = \frac{\sigma(u, 0, x)}{u}. \tag{4.3}$$

Also $\sigma(u, i\theta, x)$ is a periodic function of θ with period 2π.

In the sequel, we will often meet infinite product $\Pi_{k \in \mathbf{Z}} a_k$, or infinite sums $\Sigma_{k \in \mathbf{Z}} b_k$, with the following conventions,

$$\prod_{k \in \mathbf{Z}} a_k = \lim_{n \to +\infty} \prod_{k=-n}^{+n} a_k,$$
$$\sum_{k \in \mathbf{Z}} b_k = \lim_{n \to +\infty} \sum_{k=-n}^{n} b_k. \tag{4.4}$$

The right-hand sides of (4.4) will in fact converge unambiguously.

PROPOSITION 4.2. *For $u \in \mathbf{C}$, $\theta \in \mathbf{C}$, $x \in \mathbf{C}$, the following identity holds*

$$\sigma(u, i\theta, x) = (\theta^2 + i\theta x + u) \prod_{k \in \mathbf{Z}^*} \left(\frac{(\theta + 2k\pi)^2 + i(\theta + 2k\pi)x + u}{4k^2\pi^2} \right). \tag{4.5}$$

If $\theta \notin 2\pi\mathbf{Z}$, (4.5) can be written in the form

$$\sigma(u, i\theta, x) = 4\sin^2\left(\frac{\theta}{2}\right) \prod_{k \in \mathbf{Z}} \left(1 + \frac{ix}{\theta + 2k\pi} + \frac{u}{(\theta + 2k\pi)^2}\right). \tag{4.6}$$

Proof. We use the formula

$$\sinh(y) = y \prod_{k=1}^{+\infty} \left(1 + \frac{iy}{k\pi}\right)\left(1 - \frac{iy}{k\pi}\right). \tag{4.7}$$

Set $z = x - 2i\theta$. Using (4.7), we get

$$\begin{aligned}
\sigma(u, i\theta, x) &= \frac{x^2 - z^2 + 4u}{4} \prod_{k=1}^{+\infty} \left(1 + i\left(\frac{z + \sqrt{x^2 + 4u}}{4k\pi}\right)\right) \\
&\quad \times \left(1 + i\left(\frac{z - \sqrt{x^2 + 4u}}{4k\pi}\right)\right)\left(1 - i\left(\frac{z + \sqrt{x^2 + 4u}}{4k\pi}\right)\right) \\
&\quad \times \left(1 - i\left(\frac{z - \sqrt{x^2 + 4u}}{4k\pi}\right)\right) \\
&= \frac{x^2 - z^2 + 4u}{4} \prod_{k=1}^{+\infty} \left(\left(1 + \frac{iz}{4k\pi}\right)^2 + \frac{x^2 + 4u}{16k^2\pi^2}\right) \\
&\quad \times \left(\left(1 - \frac{iz}{4k\pi}\right)^2 + \frac{x^2 + 4u}{4k^2\pi^2}\right) \\
&= \frac{x^2 - z^2 + 4u}{4} \prod_{k=1}^{+\infty} \left(1 + \frac{iz}{2k\pi} + \frac{x^2 - z^2 + 4u}{16k^2\pi^2}\right) \\
&\quad \times \left(1 - \frac{iz}{2k\pi} + \frac{x^2 - z^2 + 4u}{16k^2\pi^2}\right).
\end{aligned} \tag{4.8}$$

Now for $z = x - 2i\theta$,

$$\frac{x^2 - z^2 + 4u}{4} = \theta^2 + i\theta x + u \tag{4.9}$$

and for $k \in \mathbf{Z}^*$,

$$1 + \frac{iz}{2k\pi} + \frac{x^2 - z^2 + 4u}{16k^2\pi^2} = \frac{(\theta + 2k\pi)^2 + i(\theta + 2k\pi)x + u}{4k^2\pi^2}. \tag{4.10}$$

Using (4.8)–(4.10), we get (4.5). From (4.5), (4.7), we obtain (4.6). □

PROPOSITION 4.3. *If* $u \in \mathbf{R}$, $\theta \in \mathbf{R}$, $y \in \mathbf{R}$, *if* $-y^2 + 4u > 0$, *then*

$$\sigma(u, i\theta, iy) > 0. \tag{4.11}$$

Proof. By (4.2),

$$\sigma(u, i\theta, iy) = 4 \sinh\left(\frac{iy - 2i\theta + \sqrt{-y^2 + 4u}}{4}\right) \sinh\left(\frac{-iy + 2i\theta + \sqrt{-y^2 + 4u}}{4}\right). \tag{4.12}$$

Using (4.12), it follows that if $-y^2 + 4u > 0$, then (4.11) holds. □

(b) *The multiplicative genus associated to* σ

Let E be a complex Hermitian vector space of dimension n.

Let $B \in \mathrm{End}(E)$ be skew-adjoint. Let $i\theta_1, \dots, i\theta_n$ $(\theta_1, \dots, \theta_n \in \mathbf{R})$ be the eigenvalues of B.

As we saw in Proposition 4.2, $\sigma(u, \eta, x)$ is a holomorphic function of u, θ, x. Therefore if $C \in \mathrm{End}(E)$ commutes with B, for $u \in \mathbf{C}$, $\sigma(u, B, C) \in \mathrm{End}(E)$ is unambiguously defined.

DEFINITION 4.4. If $u \in \mathbf{C}$, if $C \in \mathrm{End}(E)$ commutes with B, set

$$\textstyle\sum(u, B, C) = \det_E(\sigma(u, B, C)). \tag{4.13}$$

If C is diagonalizable, if $x_1, \dots, x_n$ are the eigenvalues of C corresponding to $i\theta_1, \dots, i\theta_n$, then

$$\sum(u, B, C) = \prod_{j=1}^{n} \sigma(u, i\theta_j, x_j). \tag{4.14}$$

Set

$$K_E = L_2([0, 1]; E). \tag{4.15}$$

Equivalently,

$$K_E = L_2([0, 1]; \mathbf{C}) \otimes_{\mathbf{C}} E.$$

We identify $[0, 1[$ with $S_1 = \mathbf{R}/\mathbf{Z}$. In particular the operator $J = d/dt$ acts naturally on distributions on S_1. The eigenspace decomposition of K_E with respect to J is given by

$$K_E = \bigoplus_{k \in \mathbf{Z}} \{e^{2i\pi kt}\} \otimes E.$$

If $C \in \mathrm{End}(E)$, C acts naturally on K_E. Set

$$J_B = J + B. \tag{4.16}$$

Then J_B acts as a skew-adjoint operator on K_E.

Assume that $\theta_1, \ldots, \theta_n$, do not lie in $2\pi\mathbf{Z}$. Then the operator J_B is invertible.

Take $C \in \mathrm{End}(E)$. Since, in general, $CJ_B^{-1} + uJ_B^{-2}$ is not trace class, the operator $1 - CJ_B^{-1} - uJ_B^{-2}$ does not possess a determinant in the sense of [Si2]. Still it has a normalized determinant. Namely set

$$\det_{K_E}(1 - CJ_B^{-1} - uJ_B^{-2}) = \prod_{k \in \mathbf{Z}} \det_E(1 - C(2i\pi k + B)^{-1} - u(2i\pi k + B)^2). \tag{4.17}$$

Due to our conventions in (4.4), the product in the right-hand side of (4.17) converges unambiguously.

By (4.6), (4.13), (4.17), if $\theta_1, \ldots, \theta_n$ do not lie in $2\pi\mathbf{Z}$, if $C \in \mathrm{End}(E)$ commutes with B,

$$\sum(u, B, C) = \det_E\left(4 \sinh\left(\frac{B}{2}\right) \sinh\left(\frac{-B}{2}\right)\right) \det_{K_E}(1 - CJ_B^{-1} - uJ_B^{-2}). \tag{4.18}$$

By Proposition 4.3, we see that if C is skew-adjoint, and if $C^2 + 4u > 0$, then

$$\sum(u, B, C) > 0. \tag{4.19}$$

(c) *A computation of certain finite dimensional supertraces*

Let $e_1, \ldots, e_{2n}$ be an orthonormal base of $E_{\mathbf{R}}$. Set

$$S = \frac{\sqrt{-1}}{2} \sum_{1}^{2n} c(e_i)\hat{c}(e_i). \tag{4.20}$$

Then S acts on $\Lambda(\bar{E}^*) \hat{\otimes} \Lambda(E^*)$.

Let $B \in \operatorname{End}(E)$ be skew-adjoint. Let $B^{\Lambda(\bar{E}^*) \hat{\otimes} \Lambda(E^*)}$ be the natural action of B on $\Lambda(\bar{E}^*) \otimes \Lambda(E^*)$. Then one verifies easily that

$$[B^{\Lambda(\bar{E}^*) \hat{\otimes} \Lambda(E^*)}, S] = 0. \tag{4.21}$$

Let $C \in \operatorname{End}(E)$ commuting with B. Let $C^{\Lambda(E^*)}$ be the action of C on $\Lambda(E^*)$. Then $C^{\Lambda(E^*)}$ acts like $1 \hat{\otimes} C^{\Lambda(E^*)}$ on $\Lambda(\bar{E}^*) \otimes \Lambda(E^*)$.

For $u \geqslant 0$, $\exp(B^{\Lambda(\bar{E}^*) \hat{\otimes} \Lambda(E^*)} - \sqrt{u}\, S - C^{\Lambda(E^*)})$ acts on $\Lambda(\bar{E}^*) \otimes \Lambda(E^*)$. Then one can calculate the corresponding supertrace.

We prove the following extension of [B1, Theorem 6.4].

THEOREM 4.5. *The following identity holds,*

$$\operatorname{Tr}_s[\exp(B^{\Lambda(\bar{E}^*) \hat{\otimes} \Lambda(E^*)} - \sqrt{u}\, S - C^{\Lambda(E^*)})] = \sum(u, B, C) \exp\{\tfrac{1}{2}\operatorname{Tr}_E[C]\}. \tag{4.22}$$

Proof. Let $\mathscr{G}$ be the Lie algebra of the skew-adjoint endomorphisms of E commuting with B. We may and we will assume that $C \in \mathscr{G}$, the extension of (4.22) to a general C commuting with B being obvious by analyticity. Set

$$\hat{c}(C) = \tfrac{1}{4}\sum \langle Ce_i, e_j\rangle \hat{c}(e_i)\hat{c}(e_j). \tag{4.23}$$

Then one has the trivial,

$$C^{\Lambda(E^*)} = \hat{c}(C) - \tfrac{1}{2}\operatorname{Tr}_E[C]. \tag{4.24}$$

Consider the oriented vector space $E_{\mathbf{R}} \oplus E_{\mathbf{R}}$ equipped with the scalar product which is the direct sum of the obvious scalar products of $E_{\mathbf{R}}$. Let $\tilde{c}(E_{\mathbf{R}} \oplus E_{\mathbf{R}})$ be the Clifford algebra of $E_{\mathbf{R}} \oplus E_{\mathbf{R}}$. Let $F = F_+ \oplus F_-$ be the spinors of $E_{\mathbf{R}} \oplus E_{\mathbf{R}}$. Then F is a $\tilde{c}(E_{\mathbf{R}} \oplus E_{\mathbf{R}})$ Clifford module.

Let $\mathscr{A}$ be the algebra of antisymmetric elements in $\operatorname{End}(E_{\mathbf{R}} \oplus E_{\mathbf{R}}) \otimes_{\mathbf{R}} \mathbf{C}$. Let $f_1, \dots, f_{4n}$ be an orthonormal base of $E_{\mathbf{R}} \oplus E_{\mathbf{R}}$. If $\mathbf{D} \in \mathscr{A}$, the action $\tilde{c}(\mathbf{D})$ of $\mathbf{D}$ on F is given by,

$$\tilde{c}(\mathbf{D}) = \tfrac{1}{4}\sum \langle Cf_i, f_j\rangle \tilde{c}(f_i)\tilde{c}(f_j). \tag{4.25}$$

Let $\mathbf{D} \in \operatorname{End}(E_{\mathbf{R}} \oplus E_{\mathbf{R}})$ given in matrix form by

$$\mathbf{D} = \begin{bmatrix} B & \sqrt{-u} \\ -\sqrt{-u} & B - C \end{bmatrix}. \tag{4.26}$$

Clearly $\mathbf{D} \in \mathscr{A}$. By the same argument as in [B1, proof of Theorem 6.4, Eq. (6.21), (6.24)], we have the identity,

$$\operatorname{Tr}_s[\exp(B^{\Lambda(\bar{E}^*)\hat{\otimes}\Lambda(E^*)} - \sqrt{u}S - C^{\Lambda(E^*)})] \\ = (-1)^{\dim E}\exp\{\tfrac{1}{2}\operatorname{Tr}_E[C]\}\operatorname{Tr}_s[\exp(\tilde{c}(\mathbf{D}))]. \tag{4.27}$$

Let λ be an eigenvalue of $\mathbf{D}$ associated to a nonzero eigenvector $(X, X') \in E_{\mathbf{R}} \oplus E_{\mathbf{R}}$. Since B and C commute, we may as well assume that

$$BX = i\theta X, \quad BX' = i\theta X', \quad CX' = xX'. \tag{4.28}$$

Then

$$(B-\lambda)X + \sqrt{-u}\,X' = 0, \qquad -\sqrt{-u}\,X + (B - C - \lambda)X' = 0. \tag{4.29}$$

Assume that $u \neq 0$. Then X' is nonzero. From (4.28), (4.29), we get

$$((i\theta - \lambda)(i\theta - \lambda - x) - u)X' = 0. \tag{4.30}$$

By (4.30), we obtain,

$$\lambda^2 + (x - 2i\theta)\lambda - \theta^2 - xi\theta - u = 0. \tag{4.31}$$

Conversely if (4.28), (4.29), (4.31) hold, then

$$\left(\frac{(B - C - \lambda)X'}{\sqrt{-u}}, X'\right)$$

is an eigenvector of $\mathbf{D}$ associated to the eigenvalue λ. By (4.31), we obtain

$$\lambda = \frac{-x + 2i\theta \pm \sqrt{(x - 2i\theta)^2 + 4(xi\theta + \theta^2 + u)}}{2}. \tag{4.32}$$

Equivalently

$$\lambda = \frac{-x + 2i\theta \pm \sqrt{x^2 + 4u}}{2}. \tag{4.33}$$

Let Λ be the set of eigenvalues λ in (4.33).

Classically (see e.g. [B1, Eq. (5.6)]),

$$\operatorname{Tr}_s[\exp(\tilde{c}(\mathbf{D}))] = \prod_{\lambda \in \Lambda} 2\sinh(-\lambda/2). \tag{4.34}$$

So from (4.33), (4.34), we obtain,

$$\operatorname{Tr}_s[\exp(\tilde{c}(\mathbf{D}))] = \prod_{j=1}^{n} 4 \sinh\left(\frac{x_j - 2i\theta_j + \sqrt{x_j^2 + 4u}}{4}\right) \sinh\left(\frac{x_j - 2i\theta_j - \sqrt{x_j^2 + 4u}}{4}\right). \tag{4.35}$$

Using (4.27), (4.35), for $u \neq 0$, we get

$$\operatorname{Tr}_s[\exp(B^{\Lambda(\bar{E}^*)\hat{\otimes}\Lambda(E^*)} - \sqrt{u}\,S - C^{\Lambda(E^*)})] = \prod_{j=1}^{n} 4 \sinh\left(\frac{x_j - 2i\theta_j + \sqrt{x_j^2 + 4u}}{4}\right)$$
$$\times \sinh\left(\frac{-x_j + 2i\theta_j + \sqrt{x_j^2 + 4u}}{4}\right) \exp\{\tfrac{1}{2}\operatorname{Tr}_E[C]\}, \tag{4.36}$$

which coincides with (4.22). By continuity, we see that (4.32) is also valid for $u = 0$.

The proof of Theorem 4.5 is completed. □

(d) *A computation of the trace of a harmonic oscillator with a magnetic field*

Let $C \in \operatorname{End}(E)$ be skew-adjoint. Let $x_1, \ldots, x_n$ be the eigenvalues of C. Set $|C| = \sup |x_i|$. We identify C with the corresponding element in $\operatorname{End}(E_{\mathbf{R}})$.

Let $e_1, \ldots, e_{2n}$ be an orthonormal base of $E_{\mathbf{R}}$. For $u \geqslant 0$, set

$$\mathscr{L}_u = -\frac{1}{2}\sum_1^{2n} (\nabla_{e_i} + \tfrac{1}{2}\langle CY, e_i\rangle)^2 + \tfrac{1}{2}u|Y|^2 + \tfrac{1}{2}\operatorname{Tr}_E[C]. \tag{4.37}$$

In [B1, Theorem 6.6], we stated incorrectly that for $u > 0$, if $|C| < 2\pi$, then $\exp(-\mathscr{L}_u)$ is trace class. First, we correct this statement.

THEOREM 4.6. *If the self-adjoint element of* $\operatorname{End}(E)$, $C^2 + 4u$, *is positive, then* $\exp(-\mathscr{L}_u)$ *is trace class.*

Proof. We proceed as in [B1, proof of Theorem 6.6]. Clearly

$$\mathscr{L}_u = -\frac{1}{2}\Delta - \frac{1}{2}\nabla_{CY} + \frac{1}{2}\left\langle\left(\frac{C^2}{4} + u\right)Y, Y\right\rangle + \frac{1}{2}\operatorname{Tr}_E[C]. \tag{4.38}$$

Set

$$\mathscr{M}_u = -\frac{1}{2}\Delta + \frac{1}{2}\left\langle\left(\frac{C^2}{4} + u\right)Y, Y\right\rangle. \tag{4.39}$$

Then

$$\mathscr{L}_u = \mathscr{M}_u - \tfrac{1}{2}\nabla_{CY} + \tfrac{1}{2}\mathrm{Tr}_E[C]. \tag{4.40}$$

Also since C is skew-adjoint,

$$[\nabla_{CY}, \mathscr{M}_u] = 0. \tag{4.41}$$

The operator $e^{(1/2)\nabla_{CY}}$ is a unitary operator acting on $L_2(E_{\mathbf{R}};\mathbf{C})$ by the formula

$$h \to (e^{(1/2)\nabla_{CY}}h)(Y) = h(e^{C/2}Y). \tag{4.42}$$

By (4.42), we deduce that

$$e^{-\mathscr{L}_u} = e^{(1/2)\nabla_{CY}}e^{-\mathscr{M}_u}. \tag{4.43}$$

Also if $C^2 + 4u > 0$, $\mathscr{M}_u$ is a harmonic oscillator, and so $e^{-\mathscr{M}_u}$ is trace class. From (4.43), we deduce that if $C^2 + 4u > 0$, $e^{-\mathscr{L}_u}$ is trace class. □

Let now B be skew-adjoint in End(E). Set

$$g = e^B. \tag{4.44}$$

Then g acts as a unitary operator on $L_2(E_{\mathbf{R}};\mathbf{C})$ by the formula

$$h \in L_2(E_{\mathbf{R}};\mathbf{C}) \to gh(Y) = h(g^{-1}Y) \in L_2(E_{\mathbf{R}};\mathbf{C}). \tag{4.45}$$

By Theorem 4.6, if $C^2 + 4u$ is positive, $g\exp(-\mathscr{L}_u)$ is trace class.

Now we establish a result, which was proved in [B1, Theorem 6.6] when $B = 0$.

THEOREM 4.7. *Assume that B and C commute. Then if $C^2 + 4u$ is positive,*

$$\mathrm{Tr}[g\exp(-\mathscr{L}_u)] = \frac{\exp\{-\frac{1}{2}\mathrm{Tr}_E[C]\}}{\Sigma(u, B, C)}. \tag{4.46}$$

Proof. We proceed as in [B1, proof of Theorem 6.6]. Let Q be the positive square root of $C^2/4 + u$. Let $q_u(Y, Y')$ be the smooth kernel associated to the operator $\exp(-\mathscr{L}_u)$ with respect to the measure $dv_E/(2\pi)^{\dim E}$. Then

$$\mathrm{Tr}[g\exp(-\mathscr{L}_u)] = \int_{E_{\mathbf{R}}} q_u(g^{-1}Y, Y)\frac{dv_E(Y)}{(2\pi)^{\dim E}}. \tag{4.47}$$

By using Mehler's formula as in [B1, Eq. (6.37), (6.38)], we get

$$q_u(Y, Y') = \det_E\left(\frac{Q}{\sinh(Q)}\right)\exp\left\{-\frac{1}{2}\left\langle\frac{Q}{\tanh(Q)}Y, Y\right\rangle - \frac{1}{2}\left\langle\frac{Q}{\tanh(Q)}Y', Y'\right\rangle + \left\langle\frac{Q}{\sinh(Q)}e^{C/2}Y, Y'\right\rangle - \frac{1}{2}\operatorname{Tr}[C]\right\}. \quad (4.48)$$

Since $[B, C] = 0$, $[B, Q] = 0$. From (4.48), we obtain

$$q_u(g^{-1}Y, Y) = \det_E\left(\frac{Q}{\sinh(Q)}\right)\exp\left\{-\left\langle\frac{Q}{\sinh(Q)}(\cosh(Q) - e^{C/2-B})Y, Y\right\rangle - \frac{1}{2}\operatorname{Tr}[C]\right\}. \quad (4.49)$$

Since $C/2 - B$ is skew-adjoint, as in [B1, Eq. (6.40)], we get,

$$\begin{aligned}\left\langle\frac{Q}{\sinh(Q)}(\cosh(Q) - e^{C/2-B})Y, Y\right\rangle &= \left\langle\frac{Q}{\sinh(Q)}(\cosh(Q) - \cosh(C/2 - B))Y, Y\right\rangle \\ &= 2\left\langle\frac{Q}{\sinh(Q)}\sinh\left(\frac{C - 2B + 2Q}{4}\right)\sinh\left(\frac{-C + 2B + 2Q}{4}\right)Y, Y\right\rangle. \quad (4.50)\end{aligned}$$

By Proposition 4.3, if $C^2 + 4u > 0$, the self-adjoint operator

$$\sinh\left(\frac{C - 2B + \sqrt{C^2 + 4u}}{4}\right)\sinh\left(\frac{-C + 2B + \sqrt{C^2 + 4u}}{4}\right)$$

is positive. Using (4.50), we obtain,

$$\int_{E_{\mathbf{R}}} q_u(g^{-1}Y, Y)\frac{dv_E(Y)}{(2\pi)^{\dim E}} = \frac{\exp\{-\frac{1}{2}\operatorname{Tr}_E[C]\}}{\det_E\left[4\sinh\left(\frac{C - 2B + \sqrt{C^2 + 4u}}{4}\right)\sinh\left(\frac{-C + 2B + \sqrt{C^2 + 4u}}{4}\right)\right]}. \quad (4.51)$$

By (4.2), (4.13), (4.47), (4.51), we get (4.46). The proof of Theorem 4.7 is completed. □

REMARK 4.8. As in [B1, Remark 6.7], we observe that the similarity of formulas (4.22) and (4.46) is no accident. In effect by Theorems 4.5 and 4.7, we obtain, for C skew-adjoint commuting with B and $C^2 + 4u > 0$,

$$\mathrm{Tr}_s\left[g \exp\left\{\frac{1}{2}\sum_1^{2n}(\nabla_{e_i}+\tfrac{1}{2}\langle CY, e_i\rangle)^2 - \frac{u|Y|^2}{2} - \sqrt{u}\,S - C^{\Lambda(E^*)} - \tfrac{1}{2}\mathrm{Tr}_E[C]\right\}\right] = 1. \tag{4.52}$$

Equation (4.52) has a clear geometric interpretation. In fact, the Dolbeault operator $\bar\partial^E$ and the interior multiplication operator i_y ($y \in E$) act on the vector space $\Omega(E_{\mathbf{R}}, \Lambda(\bar E^*) \hat\otimes \Lambda(E^*))$ of smooth sections of $\Lambda(\bar E^*) \otimes \Lambda(E^*)$ on $E_{\mathbf{R}}$. Let $\bar\partial^{E*}$ and i_y^* be the formal adjoints of $\bar\partial^E$ and i_y with respect to the obvious Hermitian product on $\Omega(E_{\mathbf{R}}, \Lambda(\bar E^*) \hat\otimes \Lambda(E^*))$. The Lie derivative operator L_{CY} also acts on $\Omega(E_{\mathbf{R}}, \Lambda(\bar E^*) \hat\otimes \Lambda(E^*))$.

An obvious extension of a formula proved in [B3, Theorem 1.6] shows that for $u > 0$,

$$\begin{aligned} &-L_{CY} + \left(\bar\partial^E + \sqrt{-u}\,i_y + \bar\partial^{E*} - \sqrt{-u}\,i_y^* - \frac{c(CY)}{2\sqrt{2}}\right)^2 \\ &= \frac{1}{2}\sum_1^{2n}(\nabla_{e_i}+\tfrac{1}{2}\langle CY, e_i\rangle)^2 - \frac{u|Y|^2}{2} - \sqrt{u}\,S - C^{\Lambda(E^*)} - \tfrac{1}{2}\mathrm{Tr}_E[C]. \end{aligned} \tag{4.53}$$

In fact (1.15) and (4.53) are directly related.

Now since g and C commute, inspection of the proof of [B3, Theorem 1.8] shows that for $C^2 + 4u > 0$, the right-hand side of (4.53) is a non trivial analytic expression for the Lefschetz numbers of ge^{-C} acting on the L^2 cohomology of the operator $\bar\partial + \sqrt{-u}\,i_y$. By [B1, Theorem 1.8], this L^2 cohomology is canonically isomorphic to the cohomology of $\{0\}$, which is 1 dimensional, concentrated in degree 0, on which ge^{-C} acts trivially. Tautologically, the Lefschetz numbers of ge^{-C} is equal to 1. This fits with (4.52)–(4.53).

5. An explicit evaluation of the generalized supertraces, and their asymptotics as $u \to +\infty$

In this section, we extend the results of [B1, Section 7]. Namely in (a), we give an explicit expression for the generalized supertraces in terms of infinite determinants. In (b) we show that as $u \to +\infty$,

$$\varphi \operatorname{Tr}_s[g \exp(-\mathscr{B}_u^2)] = \operatorname{Td}_g^{-1}(L, h^L) + \mathscr{O}\left(\frac{1}{\sqrt{u}}\right).$$

We make the same assumptions and we use the same notation as in Section 2. Also the methods and results of [B1, Sections 5 and 7] will be of constant use in this section.

(a) *Generalized supertraces and infinite determinants*

Let $B \in \operatorname{End}(F)$ act on E^{θ_j} by multiplication by $i\theta_j$. Then

$$g = e^B. \tag{5.1}$$

For $x \in B$, set

$$K_{E_x} = L_2([0, 1]; E_x).$$

Equivalently,

$$K_{M_x} = L_2([0, 1], \mathbf{C}) \otimes_{\mathbf{C}} E_x. \tag{5.2}$$

We define K_{L_x}, $K_{M_x}, \ldots$, in the same way.

Set $J = \mathrm{d}/\mathrm{d}t$. As in Section 4, J acts as an unbounded skew-adjoint operator on K_L, K_M, K_N. Set

$$J_B = J + B. \tag{5.3}$$

If $e \in K_E$ is such that $J_B e = 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} e + Be = 0, \quad e_0 = e_1, \tag{5.4}$$

and so $e_t = e^{-tB} e_0$. From (5.4), we deduce that $e^B e_0 = e_0$, i.e. $e_0 \in E^0$. So we find that

$$E^0 \simeq \operatorname{Ker} J_B.$$

Let $K_{E_x}^{\perp}$ be the orthogonal space to E_x^0 in K_{E_x}.

In the sequel, if $C \in \operatorname{End}(M)$ is skew-adjoint and commutes with B, we will consider quantities like the determinant over K_M of $1 - CJ_B^{-1} - uJ_B^{-2}$. It will always be understood that this determinant is evaluated on the eigenspaces where J_B is invertible. This determinant will be denoted

$$\det'_{K_M}(1 - CJ_B^{-1} - uJ_B^{-2}).$$

In view of (4.17), we get

$$\det'_{K_M}(1 - CJ_B^{-1} - uJ_B^{-2}) = \prod_{k\in \mathbf{Z}^*} \det_{M^0}(1 - C(2i\pi k)^{-1} - u(2i\pi k)^{-2}) \\ \times \prod_{k\in\mathbf{Z}} \det_{M^{0,\perp}}(1 - C(2i\pi k + B)^{-1} - u(2i\pi k + B)^{-2}). \tag{5.5}$$

Let B^L, B^M, $B^N \dots$ be the restriction of B to L, M, $N \dots$.

In the sequel, we use the conventions of [MQ], [B1, Section 7(a))]. Namely for $u > 0$, $b \in \mathbf{C}$, if $|b|$ is small enough, $J_B^2 - (R^N + b)J_B - u$ is invertible. Still because the expressions which follow contain $\det'_{K_N}(1 - (R^N + b)J_B^{-1} - uJ_B^{-2})$ as a factor, we may formally invert $J_B^2 - (R^N + b)J - u$ for arbitrary $b \in \mathbf{C}$. The reader not interested with these subtleties may as well assume that $|b|$ is small enough.

Now we prove an extension of [B1, Theorem 7.3].

THEOREM 5.1. *For $u > 0$, $b \in \mathbf{C}$, the following identity holds,*

$$\mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2 + bN_H)] = \exp\left\{-\frac{1}{2}\mathrm{Tr}[R^L - B^L] + \frac{b}{2}\dim N\right\} \\ \times \frac{\det_{N^{0,\perp}}(2\sinh(B^{N^{0,\perp}}))}{\det_{M^{0,\perp}}(2\sinh(B^{M^{0,\perp}}))} \det'_{K_N}(1 - (R^N + b)J_B^{-1} - uJ_B^{-2}) \\ \times [\det'_{K_M}(1 - R^M J_B^{-1} - uP^L A(J_B^2 - (R^N + b)J_B - u)^{-1} AP^L J_B^{-1} - uP^N J_B^{-2})]^{-1} \tag{5.6}$$

Proof. By Proposition 2.3, it is clear that we only need to prove (5.6) when $E = E^\theta$, i.e. when $B = i\theta$, with $0 \leqslant \theta < 2\pi$.

If $\theta = 0$, (5.6) was already established in [B1, Theorem 7.3]. So now, we assume that $0 < \theta < 2\pi$.

Let $B^{\Lambda(\bar{M}^*)\otimes\Lambda(N^*)}$ be the obvious action of B on $\Lambda(\bar{M}^*) \hat{\otimes} \Lambda(N^*)$. With the notation of Section 2,

$$B^{\Lambda(\bar{M}^*)\otimes\Lambda(N^*)} = i\theta(N_V - N_H).$$

Set

$$L_{BY} = \nabla_{BY} - B^{\Lambda(\bar{M}^*)\otimes\Lambda(N^*)}. \tag{5.7}$$

Then one verifies easily that

$$[\mathscr{B}_u^2, B^{\Lambda(\bar{M}^*)\hat{\otimes}\Lambda(N^*)}] = 0. \tag{5.8}$$

From (5.8), we get

$$g \exp(-\mathscr{B}_u^2 + bN_H) = \exp(-L_{BY} - \mathscr{B}_u^2 + bN_H). \tag{5.9}$$

As in the proof of Theorem 2.5, our calculations take place in a given fibre $M_{\mathbf{R},x}$. To simplify our notation, we will not write x explicitly. Let $\tilde{P}_u^b(Y, Y')$ be the smooth kernel associated to the operator $\exp(-L_{BY} - \mathscr{B}_u^2 + bN_H)$. Then by (5.9),

$$gP_u^b(g^{-1}Y, Y) = \tilde{P}_u^b(Y, Y). \tag{5.10}$$

Therefore

$$\mathrm{Tr}_s[g \exp(-\mathscr{B}_u^2 + bN_H)] = \int_{M_\mathbf{R}} \mathrm{Tr}_s[\tilde{P}_u^b(Y, Y)] \frac{dv_M(Y)}{(2\pi)^{\dim M}}. \tag{5.11}$$

Using Theorem 1.4, we see that

$$\begin{aligned}\mathscr{B}_u^2 + L_{BY} - bN_H = &-\frac{1}{2}\sum_1^{2m} (\nabla_{e_i} + \tfrac{1}{2}\langle (R^M - 2B)Y, e_i\rangle)^2 \\ &+ \tfrac{1}{2}\langle (R^M - B)BY, Y\rangle + \frac{u|P^N Y|^2}{2} - B^{\Lambda(\bar{M}^*)\hat{\otimes}\Lambda(N^*)} \\ &+ \sqrt{u}\, S + \frac{\sqrt{u}}{\sqrt{2}}\, \hat{c}(AP^L Y) + \tfrac{1}{2}\mathrm{Tr}[R^M] + R^{\Lambda(N^*)} - bN_H.\end{aligned} \tag{5.12}$$

Take $Y \in M_\mathbf{R}$. Let $Q_{(Y,Y)}$ be the probability law of the Brownian bridge $t \to Y_t \in M_\mathbf{R}$, with $Y_0 = Y_1 = Y$. Consider the differential equation

$$\begin{aligned}\frac{dV}{dt} &= V_t\left[B^{\Lambda(\bar{N}^*)\otimes\Lambda(N^*)} - \sqrt{u}\, S - R^{\Lambda(N^*)} + bN_H - \frac{\sqrt{-u}}{\sqrt{2}}\, \hat{c}(AP^L Y_t)\right], \\ V_0 &= 1.\end{aligned} \tag{5.13}$$

Using (5.12) and Itô's formula as in [B1, Proof of Theorem 4.1], we get

$$\tilde{P}_u^b(Y, Y) = E^{Q(Y,Y)}\left[\exp\left\{\frac{1}{2}\int_0^1 \langle (R^M - 2B)Y, \mathrm{d}Y\rangle \right.\right.$$
$$\left.\left. -\frac{1}{2}\int_0^1 \langle (R^M B - B^2 + uP^N)Y, Y\rangle dt - \tfrac{1}{2}\mathrm{Tr}[R^M]\right\} e^{B^{\Lambda(\bar{L}^*)}} V_1\right]. \quad (5.14)$$

From (5.14), we obtain

$$\mathrm{Tr}_s[\tilde{P}_u^b(Y, Y)] = e^{-(1/2)\mathrm{Tr}[R^M]}(1 - e^{i\theta})^{\dim L} E^{Q(Y,Y)}$$
$$\left[\exp\left\{\frac{1}{2}\int_0^1 \langle (R^M - 2B)Y, \mathrm{d}Y\rangle \right.\right.$$
$$\left.\left. -\frac{1}{2}\int_0^1 \langle (R^M B - B^2 + uP^N)Y, Y\rangle dt\right\} \mathrm{Tr}_s[V_1]\right]. \quad (5.15)$$

Now we use the formalism of the proof of Theorem 4.5, with $E = N$. In the sequel, we identify b with the element of $\mathrm{End}(N_{\mathbf{R}}) \otimes_{\mathbf{R}} \mathbf{C}$ which acts on N by multiplication by b, on $\bar{N}$ by multiplication by $-b$. Set

$$C = R^N + b. \quad (5.16)$$

Let $\mathbf{D} \in \Lambda(T_{\mathbf{R}}^* B) \hat{\otimes} \mathrm{End}(N_{\mathbf{R}} \oplus N_{\mathbf{R}})$ be given in matrix form by

$$\mathbf{D} = \begin{bmatrix} B & \sqrt{-u} \\ -\sqrt{u} & B - C \end{bmatrix}.$$

Temporarily, we will consider $AP^L Y$ as a 1-form with values in the second copy of $N_{\mathbf{R}}$ in $N_{\mathbf{R}} \oplus N_{\mathbf{R}}$. By (5.13) and by [B1, Theorem 5.1], we get

$$\mathrm{Tr}_s[V_1] = \mathrm{Tr}_s[\exp(B^{\Lambda(\bar{M}^*)\otimes\Lambda(N^*)} - \sqrt{u}\, S - C^{\Lambda(N^*)})]$$
$$\times \exp\left\{-\frac{u}{2}\left\langle \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{D}\right)^{-1} AP^L Y, AP^L Y\right\rangle\right\}. \quad (5.17)$$

Set $K_t = AP^L Y_t$. To calculate $(\mathrm{d}/\mathrm{d}t + \mathbf{D})^{-1}K$, we proceed as in [B1, proof of Theorem 7.2]. Namely, we solve the differential equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + B\right)\varphi + \sqrt{-u}\,\psi = 0$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + B\right)\psi - \sqrt{-u}\,\varphi - C\psi = K \quad (5.18)$$

with periodic boundary conditions. If $b \in \mathbf{R}$, $J_B^2 - CJ_B - u$ is invertible. From

(5.18), we get,

$$\psi = (J_B^2 - CJ_B - u)^{-1} J_B K. \tag{5.19}$$

Using (4.18), Theorem 4.5, (5.17), (5.19), we obtain,

$$\begin{aligned}\operatorname{Tr}_s[V_1] &= (4\sin^2(\theta/2))^{\dim N} \det_{K_N}(1-(R^N+b)J_B^{-1} - uJ_B^{-2}) \\ &\times \exp\left\{-\frac{u}{2}\langle P^L A(J_B^2 - CJ_B - u)^{-1} J_B A P^L Y, Y\rangle + \tfrac{1}{2}\operatorname{Tr}[R^N] + \frac{b}{2}\dim N\right\}.\end{aligned} \tag{5.20}$$

Moreover,

$$\operatorname{Tr}[R^M] = \operatorname{Tr}[R^L] + \operatorname{Tr}[R^N]. \tag{5.21}$$

By (5.11), (5.15), (5.20), (5.21), we get

$$\begin{aligned}\operatorname{Tr}_s[g\exp(-\mathscr{B}_u^2)] &= \exp\left\{-\frac{1}{2}\operatorname{Tr}[R^L] + \frac{b}{2}\dim N\right\}(1-e^{i\theta})^{\dim L} \\ &\times (4\sin^2(\theta/2))^{\dim N} \det_{K_N}(1-(R^N+b)J_B^{-1} - uJ_B^{-2}) \\ &\times \int_{M_{\mathbf{R}}} E^{Q(Y,Y)}\Bigg[\exp\Bigg\{\frac{1}{2}\int_0^1 \langle (R^M - 2B)Y, \mathrm{d}Y\rangle \\ &- \frac{1}{2}\int_0^1 \langle (R^M B - B^2 + uP^N)Y, Y\rangle\, dt \\ &- \frac{u}{2}\langle P^L A(J_B^2 - (R^N+b)J_B - u)^{-1} J_B A P^L Y, Y\rangle\Bigg\}\Bigg]\frac{\mathrm{d}v_M(Y)}{(2\pi)^{\dim M}}.\end{aligned} \tag{5.22}$$

Let $F \in \operatorname{End}(M)$ be self-adjoint and positive. Let $f \in L^2([0,1]; M_{\mathbf{R}})$. Set

$$I = \int_{M_{\mathbf{R}}} E^{Q(Y,Y)}\left[\exp\left\{\int_0^1 \langle f, \mathrm{d}Y - \sqrt{F}\, Y\,\mathrm{d}t\rangle - \frac{1}{2}\int_0^1 \langle FY, Y\,\mathrm{d}t\right\}\right]\frac{\mathrm{d}v_M(Y)}{(2\pi)^{\dim M}}. \tag{5.23}$$

By [B1, Eq. (7.34)],

$$I = \exp\left\{\frac{1}{2}\int_0^1 |f|^2\,\mathrm{d}t\right\}\frac{1}{\det_M(F)\det'_{K_M}(1-FJ^{-2})}. \tag{5.24}$$

Let Q_F be the probability law on $\mathscr{C}(S_1; M_{\mathbf{R}})$ of the gaussian process whose covariance is given by the operator

$$K_F = (-J^2 + F)^{-1}.$$

As in [B1, Eq. (7.36)], we deduce from (5.23), (5.24) the identification of positive measures on $\mathscr{C}(S_1; M_{\mathbf{R}})$,

$$\int_{M_{\mathbf{R}}} \exp\left\{-\frac{1}{2}\int_0^1 \langle FY, Y\rangle \, dt\right\} Q_{(Y,Y)} \frac{dv_M(Y)}{(2\pi)^{\dim M}} = \frac{Q_F}{\det_M(F)\det'_{K_M}(1 - FJ^{-2})}. \tag{5.25}$$

Now by proceeding as in [B1, Eq. (7.45)], we find,

$$\begin{aligned} E^{Q_F}\Big[\exp\Big\{\frac{1}{2}\int_0^1 &\langle (R^M - 2B)Y, dY\rangle \\ &- \frac{u}{2}\langle P^L A(J_B^2 - (R^N + b)^{-1}J_B - u)^{-1} J_B A P^L Y, Y\rangle\Big\}\Big] \\ = [\det_{K_M}((-J^2 &+ F)^{-1}(-J^2 + F + (R^M - 2B)J \\ &+ uP^L A(J_B^2 - (R^N + b)J_B - u)^{-1} J^B A P^L)]^{-1}. \end{aligned} \tag{5.26}$$

Set,

$$F = R^M B - B^2 + uP^N. \tag{5.27}$$

Then,

$$\begin{aligned} -J^2 + F + (R^M - 2B)J + uP^L A &\left(J_B^2 - (R^N + b)J_B - u\right)^{-1} J^B A P^L \\ = -J_B^2 + R^M J_B + uP^N &+ uP^L A(J_B^2 - (R^N + b)J_B - u)^{-1} J_B A P^L. \end{aligned} \tag{5.28}$$

So in view of (5.22), (5.25)–(5.28), we get,

$$\begin{aligned} \operatorname{Tr}_s[g &\exp(-\mathscr{B}_u^2 + bN_H)] \\ &= \exp\left\{-\frac{1}{2}\operatorname{Tr}[R^L] + \frac{b}{2}\dim N\right\} \frac{(1 - e^{i\theta})^{\dim L}(4\sin^2(\theta/2))^{\dim N}}{\det_M(-B^2)\det'^2_{K_M}(1 + J^{-1}B)} \\ &\times \det_{K_N}(1 - (R^N + b)J_B^{-1} - uJ_B^{-2})[\det_{K_M}(1 - R^M J_B^{-1} \\ &- uP^L A(J_B^2 - (R^N + b)J_B - u)^{-1} A P^L J_B^{-1} - uP^N J_B^{-2})]^{-1}. \end{aligned} \tag{5.29}$$

Moreover,

$$\det_M(-B^2)\,\det{}'^{2}_{K_M}(1+J^{-1}B)=\left\{\theta\prod_{k=1}^{+\infty}\left(1-\frac{\theta^2}{4k^2\pi^2}\right)\right\}^{2\dim M}$$
$$=(4\sin^2(\theta/2))^{\dim M}, \tag{5.30}$$

and so,

$$\frac{(1-e^{i\theta})^{\dim L}(4\ \sin^2(\theta/2))^{\dim N}}{\det_M(-B^2)\det{}'^{2}_{K_M}(1+J^{-1}B)}=\left(\frac{e^{i\theta/2}}{2\ \sinh(i\theta/2)}\right)^{\dim L}. \tag{5.31}$$

From (5.29), (5.31), we get,

$$\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2+bN_H)]=\exp\left\{-\frac{1}{2}\mathrm{Tr}[R^L-B^L]+\frac{b}{2}\dim N\right\} \tag{5.32}$$
$$\times\frac{\det_N(2\ \sinh(B^N))}{\det_M(2\ \sinh(B^M))}\det_{K_N}(1-(R^N+b)J_B^{-1}-uJ_B^{-2})$$
$$\times[\det_{K_M}(1-R^MJ_B^{-1}-uP^LA(J_B^{-2}-(R^N+b)J_B-u)^{-1}AP_LJ_B^{-1}-uP^NJ_B^{-2})]^{-1}.$$

Clearly (5.32) coincides with (5.6) when $E=E^0$. The proof of Theorem 5.1 is completed. □

REMARK 5.2. Let us verify Theorem 3.2 directly. We only need to check (3.4) when $E=E^0$. If $\theta=0$, this was already done in [B1, Remark 7.4]. For $0<\theta<2\pi$, by (5.6), as $u\to 0$,

$$\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2+bN_H)]=\exp\left\{-\frac{1}{2}\mathrm{Tr}[R^L-B^L]+\frac{b}{2}\dim N\right\}$$
$$\times\frac{(2i\ \sin(\theta/2))^{\dim N}\ \det_{K_N}(1-(R^N+b)J_B^{-1})}{(2i\ \sin(\theta/2))^{\dim M}\ \det_{K_M}(1-R^MJ_B^{-1})}+\mathscr{O}(u). \tag{5.33}$$

Moreover,

$$\det_{K_M}(1-R^MJ_B^{-1})=\frac{\det_M(B-R^M)\det{}'_{K_M}(1+(B-R^M)J^{-1})}{\det_M(B)\det{}'_{K_M}(1+BJ^{-1})} \tag{5.34}$$

By proceeding as in (5.30), we deduce from (5.34),

$$\det_{K_M}(1 - R^M J_B^{-1}) = \frac{\det_M\left(2\sinh\left(\frac{i\theta - R^M}{2}\right)\right)}{(2i\sin(\theta/2))^{\dim M}}. \tag{5.35}$$

A similar formula holds for $\det_{K_N}(1 - (R^N + b)J_B^{-1})$. So by using (5.21), (5.33), (5.35), we find that as $u \to 0$,

$$\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2 + bN_H)] = \frac{\det_N(1 - e^{R^N + b - i\theta})}{\det_M(1 - e^{R^M - i\theta})} + \mathcal{O}(u), \tag{5.36}$$

which coincides with (3.4).

It should also be pointed out that with respect to the formulas of [B1, Theorem 7.3], the formulas of Theorem 5.1 differ by the fact that J has been replaced by J_B.

This fits with the interpretation of the generalized supertraces as being related to the equivariant cohomology of the loop space developed in [B1, Section 9] and also in [B5, B6].

(b) *The asymptotics as $u \to +\infty$ of the generalized supertraces*

Now, we establish an extension of [B1, Theorem 7.7].

THEOREM 5.3. *As $u \to +\infty$, for $b \in \mathbf{C}$,*

$$\varphi\,\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2 + bN_H)] = \exp\left(\frac{b}{2}\dim N\right)\mathrm{Td}_g(L, g^L) + \mathcal{O}\left(\frac{1}{\sqrt{u}}\right). \tag{5.37}$$

In particular, as $u \to +\infty$,

$$\varphi\,\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2)] = \mathrm{Td}_g(L, g^L) + \mathcal{O}\left(\frac{1}{\sqrt{u}}\right),$$
$$\varphi\frac{\partial}{\partial b}\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2 + bN_H)]_{b=0} = \frac{\dim N}{2}\mathrm{Td}_g(L, g^L) + \mathcal{O}\left(\frac{1}{\sqrt{u}}\right). \tag{5.38}$$

Proof. By Proposition 2.3, we only need to establish our theorem when $E = E^0$. If $\theta = 0$, this was done in [B1, Theorem 7.7].

So assume that $0 < \theta < 2\pi$. By proceeding as in [B1, Theorem 7.6], we see that as $u \to +\infty$,

$$\det_{K_N}(1 - (R^N + b)J_B^{-1} - uJ_B^{-2}) = \det_{K_N}(1 - uJ_B^{-2})\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{u}}\right)\right),$$

$$\det_{K_M}(1-R^M J_B^{-1}-uP^L A(J_B^2-(R^N+b)J_B-u)^{-1}AP_L J_B^{-1}-uP^N J_B^{-2})$$

$$=\det_{K_N}(1-uJ_B^{-2})\det_{K_L}(1-R^L J_B^{-1})\left(1+\mathcal{O}\left(\frac{1}{\sqrt{u}}\right)\right). \tag{5.39}$$

So from (5.6), (5.39), we see that as $u\to+\infty$,

$$\mathrm{Tr}_s[g\exp(-\mathscr{B}_u^2+bN_H)]$$
$$=\frac{\exp\{-\frac{1}{2}\mathrm{Tr}[R^L-B^L]+(b/2)\dim N\}}{(2i\sin(\theta/2))^{\dim L}\det_{K_L}(1-R^L J_B^{-1})}+\mathcal{O}\left(\frac{1}{\sqrt{u}}\right). \tag{5.40}$$

By (5.35), (5.40), we get (5.37). □

REMARK 5.4. To establish (5.37), we can use the method of Bismut-Lebeau [BL, Section 14], where this result was established for $\theta=0$, $E=E^0$. To apply the method of [BL] to the case $0<\theta<2\pi$, the estimates of [BL, Sections 13 and 14] have to be adequately modified.

6. Generalized equivariant supertraces, and their analytic torsion forms

The purpose of this section is to construct generalized analytic torsion forms associated to the equivariant exact sequence E of holomorphic Hermitian vector bundles. If h^M is a g-invariant Hermitian metric on M, we thus obtain a form $\mathbf{B}_g(L, M, h^M)$ on B.

Extending [B1], where only the case $g=1$ has been considered, the main result of this section is an evalution of $\mathbf{B}_g(L, M, h^M)$ in $P^B/P^{B,0}$ in terms of a standard Bott-Chern class [BoC], [D], [BGS1] and of an additive genus $D(0,x)$ naturally associated to the derivative at 0 of the Mellin transform in u of

$$\frac{\partial\sigma/\partial x}{\sigma}(u, i\theta, -x).$$

This section is organized as follows. In (a), we construct the analytic forms $\mathbf{B}_g(L, M, h^M)$, and in (b), we calculate $\mathbf{B}_g(L, M, h^M)$.

We make the same assumptions, and we use the same notation as in Sections 2–5.

(a) *A construction of the generalized analytic torsion forms*

DEFINITION 6.1. For $s \in \mathbf{C}$, $0 < \mathrm{Re}(s) < 1/2$, set

$$A(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \left(\varphi \operatorname{Tr}_s[N_H g \exp(-\mathscr{B}_u^2)] - \frac{\dim N}{2} \operatorname{Td}_g(L, h^L) \right) du. \tag{6.1}$$

By Theorems 2.5, 3.2 and 5.3, it is clear that $A(s)$ extends to a function which is holomorphic near $s = 0$.

DEFINITION 6.2. Set

$$\mathbb{B}_g(L, M, h^M) = \frac{\partial A}{\partial s}(0). \tag{6.2}$$

By Theorem 3.2 and (5.38), we have the identity

$$\begin{aligned}\mathbf{B}_g(L, M, h^M) = & \int_0^1 \left\{ \varphi \operatorname{Tr}_s[N_H g \exp(-\mathscr{B}_u^2)] + \operatorname{Td}_g(M, h^M)(\operatorname{Td}_g^{-1})'(N, h^N) \right\} \frac{du}{u} \\ & + \int_1^{+\infty} \left\{ \varphi \operatorname{Tr}_s[N_H g \exp(-\mathscr{B}_u^2)] - \frac{\dim N}{2} \operatorname{Td}_g(L, h^L) \right\} \frac{du}{u} \\ & + \Gamma'(1) \left\{ \operatorname{Td}_g(M, h^M)(\operatorname{Td}_g^{-1})'(N, h^N) + \frac{\dim N}{2} \operatorname{Td}_g(L, h^L) \right\}.\end{aligned} \tag{6.3}$$

Now, we extend [B1, Theorem 8.3].

THEOREM 6.3. *The form $\mathbf{B}_g(L, M, h^M)$ lies in P^B. Moreover*

$$\frac{\bar{\partial}\partial}{2i\pi} \mathbf{B}_g(L, M, h^M) = \operatorname{Td}_g(L, h^L) - \frac{\operatorname{Td}_g(M, h^M)}{\operatorname{Td}_g(N, h^N)}. \tag{6.4}$$

Proof. Using Theorems 2.5, 3.2 and 5.3 and also (6.3), (6.4) follows. □

By (4.5), it is clear that for $\theta \in 2\pi\mathbf{Z}$, $|x| < 2\pi$, or for $\theta \notin 2\pi\mathbf{Z}$, $|x| < \inf_{k \in \mathbf{Z}} |\theta + 2k\pi|$, and $u \geqslant 0$, $\frac{\partial\sigma/\partial x}{\sigma}(u, i\theta, x)$ is well-defined. As a function of θ, $\frac{\partial\sigma/\partial x}{\sigma}(u, i\theta, x)$ is periodic, with period 2π.

By (4.2), we find that as $u \to +\infty$,

$$\sigma(u, i\theta, x) \simeq e^{\sqrt{u}}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{u}}\right)\right). \tag{6.5}$$

From (6.5), we see that as $u \to +\infty$,

$$\frac{\partial\sigma/\partial x}{\sigma}(u, i\theta, x) = \mathcal{O}\left(\frac{1}{\sqrt{u}}\right). \tag{6.6}$$

DEFINITION 6.4. For $s \in \mathbf{C}$, $0 < \mathrm{Re}(s) < 1/2$, $\theta \in \mathbf{R}$, $x \in \mathbf{C}$, and $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, set

$$C(s, \theta, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \frac{\partial\sigma/\partial x}{\sigma}(u, i\theta, -x)\,\mathrm{d}u. \tag{6.7}$$

Using (6.5), (6.6), one verifies that $C(s, \theta, x)$ extends to a holomorphic function of $s \in \mathbf{C}$ near $s = 0$.

DEFINITION 6.5. For $\theta \in \mathbf{R}$, $x \in \mathbf{C}$, $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, set

$$D(\theta, x) = \frac{\partial C}{\partial s}(0, \theta, x). \tag{6.8}$$

Clearly $D(\theta, x)$ is a periodic function of θ with period 2π. Also for fixed θ, $x \to D(\theta, x)$ is holomorphic on its domain of definition. However, as we shall see, $D(\theta, x)$ is not a regular function of θ.

REMARK 6.6. In [B1, Definition 8.4], the functions $C(s, x)$, $D(x)$ are defined for $|x| < 2\pi$ by the formulas

$$C(s, x) = -\frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \frac{\partial\varphi/\partial x}{\varphi}(u, x)\,\mathrm{d}u,$$
$$D(x) = \frac{\partial}{\partial s} C(0, x). \tag{6.9}$$

By (4.3),

$$\frac{\partial\varphi/\partial x}{\varphi}(u, x) = \frac{\partial\sigma/\partial x}{\sigma}(u, 0, x),$$

and moreover by (4.2), this is an odd function of x. Therefore,

$$C(s, x) = C(s, 0, x),$$
$$D(x) = D(0, x). \tag{6.10}$$

For $\theta \in \mathbf{C}$, we identify $D(\theta, .)$ with the corresponding additive genus. Set

$$D(\theta_j, N^{\theta_j}, h^{N^{\theta_j}}) = \mathrm{Tr}\left[D\left(\theta_j, -\frac{R^{N^{\theta_j}}}{2i\pi}\right)\right]. \tag{6.11}$$

Then $D(\theta_j, N^{\theta_j}, h^{N^{\theta_j}})$ lies in P^B, and is closed.

DEFINITION 6.7. Set

$$D_g(N, h^N) = \sum D(\theta_j, N^{\theta_j}, h^{N^{\theta_j}}). \tag{6.12}$$

The class of $\mathrm{Td}_g(L, h^L) D_g(N, h^N)$ in $P^B/P^{B,0}$ does not depend on the metrics $h^L = \oplus h^{L^{\theta_j}}$, $h^N = \oplus h^{N^{\theta_j}}$. We denote this class by $\mathrm{Td}_g(L) D_g(N)$.

By a construction of Bismut-Gillet-Soulé [BGS1, Theorem 1.29], to the direct sum $E = \oplus_j E^{\theta_j}$ of short exact sequences of holomorphic Hermitian vector bundles E^{θ_j}, one can associate a unique class $\widetilde{\mathrm{Td}}_g(L, M, h^M) \in P^B/P^{B,0}$, such that:

(1) The following identity holds,

$$\frac{\bar{\partial}\partial}{2i\pi} \widetilde{\mathrm{Td}}_g(L, M, h^M) = \mathrm{Td}_g(M, h^M) - \mathrm{Td}_g(L, h^L)\,\mathrm{Td}_g(N, h^N). \tag{6.13}$$

(2) The class $\widetilde{\mathrm{Td}}(L, M, h^M)$ vanishes in $P^B/P^{B,0}$ if $E = \oplus_j E^{\theta_j}$ splits holomorphically and metrically, i.e. for any j, $M^{\theta_j} = L^{\theta_j} \oplus N^{\theta_j}$, $h^{M^{\theta_j}} = h^{L^{\theta_j}} \oplus h^{N^{\theta_j}}$.

Now we extend [B1, Theorem 8.5].

THEOREM 6.8. *The following identities holds*

$$\mathbf{B}_g(L, M, h^M) = -\mathrm{Td}_g^{-1}(N, h^N)\,\widetilde{\mathrm{Td}}_g(L, M, h^M) + \mathrm{Td}_g(L) D_g(N) \quad \text{in } P^B/P^{B,0}. \tag{6.14}$$

Proof. By using Theorem 6.3 and by proceeding as in [B1, proof of Theorem 8.5], we only need to establish (6.14) when the E^{θ_j}'s are split holomorphically and metrically, i.e. $M^{\theta_j} = L^{\theta_j} \oplus N^{\theta_j}$, $h^{M^{\theta_j}} = h^{L^{\theta_j}} \oplus h^{N^{\theta_j}}$. In this case $R^{M^{\theta_j}} = R^{L^{\theta_j}} \oplus R^{N^{\theta_j}}$.

By Theorem 5.1, we get

$$\operatorname{Tr}_s[g \exp(-\mathscr{B}_u^2 + bN_H)] = \exp\left\{-\frac{1}{2}\operatorname{Tr}[R^L - B^L] + \frac{b}{2}\dim N\right\}$$

$$\times \frac{1}{\det_{L^{0,\perp}}[2 \sinh(B^{L^{0,\perp}})]}[\det'_{K_L}(1 - R^L J_B^{-1})]^{-1}$$

$$\times \frac{\det'_{K_N}(1 - (R^N + b)J_B^{-1} - uJ_B^{-2})}{\det'_{K_N}(1 - R^L J_B^{-1} - uJ_B^{-2})}. \tag{6.15}$$

Using Theorem 2.5 and (6.15), we obtain,

$$\operatorname{Tr}_s[N_H g \exp(-\mathscr{B}_u^2)] = \frac{\exp\{-\frac{1}{2}\operatorname{Tr}[R^L - B^L]\}}{\det_{L^{0,\perp}}[2 \sinh(B^{L^{0,\perp}})]}[\det'_{K_L}(1 - R^L J_B^{-1})]^{-1}$$

$$\left\{\frac{\dim N}{2} + \frac{\partial/\partial b \det'_{K_N}(1 - (R^N + b)J_B^{-1} - uJ_B^{-2})_{b=0}}{\det'_{K_N}(1 - R^N J_B^{-1} - uJ_B^{-2})}\right\}. \tag{6.16}$$

By [B1, Eq. (7.47)] and by (5.35),

$$\varphi \frac{\exp\{-\frac{1}{2}\operatorname{Tr}[R^L - B^L]\}}{\det_{L^{0,\perp}}[2 \sinh(B^{L^{0,\perp}})] \det'_{K_L}(1 - R^L J_B^{-1})} = \operatorname{Td}_g^{-1}(L, h^L). \tag{6.17}$$

As explained after (6.4),

$$\frac{\frac{\partial}{\partial b}\Sigma(u, B, R^N + b)_{b=0}}{\Sigma(u, B, R^N)}$$

makes sense for any $u \geqslant 0$. Using [B1, Eq. (7.23)], (4.3) and (4.18), we get

$$\partial/\partial b \frac{\det'_{K_N}(1 - (R^N + b)J_B^{-1} - uJ_B^{-2})_{b=0}}{\det'_{K_N}(1 - R^N J_B^{-1} - uJ_B^{-2})} = \frac{\frac{\partial}{\partial b}\Sigma(u, B, R^N + b)_{b=0}}{\Sigma(u, B, R^N)}. \tag{6.18}$$

The critical fact is that (6.18) is valid for any B.

Clearly,

$$\frac{\frac{\partial}{\partial b}\Sigma(u, B, R^N + b)_{b=0}}{\Sigma(u, B, R^N)} = \operatorname{Tr}\left[\frac{\partial\sigma/\partial x}{\sigma}(u, B, R^N)\right]. \tag{6.19}$$

By (6.15), (6.17), (6.19), we obtain,

$$\varphi \operatorname{Tr}_s[N_H \exp(-\mathscr{B}_u^2)] = \operatorname{Td}_g^{-1}(L, h^L) \left\{ \frac{\dim N}{2} + \operatorname{Tr}\left[\frac{\partial\sigma/\partial x}{\sigma} \left(u, B, \frac{R^N}{2i\pi} \right) \right] \right\}. \tag{6.20}$$

Using (6.1), (6.2), (6.7), (6.20), we get (6.14) in the split case. The proof of Theorem 6.8 is completed. □

REMARK 6.9. To establish Theorem 6.8, we could as well have used Theorems 4.5 and 4.7.

7. A formula for $D(\theta, x)$ and $R(\theta, x)$

The purpose of this section is to extend the results of Bismut and Soulé in [B1, Appendix] to the equivariant situation considered in Section 6. Namely, we express $D(\theta, x)$ as a power series in the variable x. The derivatives at the odd negative integers of the Riemann zeta function which appear in [B1] are replaced here by the derivatives at the negative integers of the real and imaginary parts of the Lerch series. Of course when $\theta \in 2\pi\mathbf{Z}$, our formula coincides with the formula of [B1, Appendix] for $D(x)$.

Also, in view of the relation between the genus $D(x)$ of [B1] and the genus $R(x)$ of Gillet and Soulé in [GS1], and also from the results of [BL] on the behaviour of Quillen metrics by complex immersions, we construct a new genus $R(\theta, x)$ from $D(\theta, x)$, which coincides with $R(x)$ for $\theta \in 2\pi\mathbf{Z}$. We show that if $g \in U(n+1)$ acts with isolated fixed points on $\mathbf{P}^n$, the formula for the g-equivariant Ray-Singer analytic torsion obtained by Köhler [K] can be very simply expressed in terms of $R(\theta, 0)$.

By imitating Gillet and Soulé [GS1], we are thus led to speculate on the form of an equivariant Riemann-Roch-Arakelov formula.

This section is organized as follows. In (a), we express $D(\theta, x)$ as a power series in x. In (b), when $\theta \in 2\pi\mathbf{Z}$, we give a formula for $D(\theta, x)$ in terms of $D(x + i\theta)$. In (c), we construct the genus $R(\theta, x)$. In (d), we recall a result of Köhler [K]. Finally, in (e), we relate the previous considerations to the possibility of proving an equivariant version of a Riemann-Roch-Arakelov formula.

This section is self-contained.

(a) *The function $D(\theta, x)$ and the Lerch series*

For $a \in \mathbf{Z}$, $x \in \mathbf{R}$, $y \in \mathbf{R}$, $s \in \mathbf{C}$, let $S_a(x, y, s)$ be the Kronecker zeta function

$$S_a(x, y, s) = \sum_{n \in \mathbf{Z}}{}' (x+n)^a |x+n|^{-2s} e^{-2i\pi n y}, \tag{7.1}$$

where in (7.1) $\Sigma'_{n\in \mathbf{Z}}$ is a sum taken over $n\in\mathbf{Z}$, $n\neq x$. The series in (7.1) converges absolutely for $\mathrm{Re}(s) > \frac{a+1}{2}$, and defines a holomorphic function of s, on its domain of definition. Also it is periodic of period 1 in both variables x, y. Moreover, it is well-known [W, p. 57] that

(1) If a is odd, of if a is even, and $y\notin\mathbf{Z}$, $s\to S_a(x, y, s)$ has a holomorphic continuation to $\mathbf{C}$.
(2) If a is even and if $y\in\mathbf{Z}$, $s\to S_a(x, y, s)$ extends to a meromorphic function on $\mathbf{C}$ with a simple pole at $s=\frac{a+1}{2}$.

Clearly,

$$\text{if } a \text{ is even,}\quad S_a(x, y, s) = S_0\left(x, y, s-\frac{a}{2}\right),$$
$$\text{if } a \text{ is odd,}\quad S_a(x, y, s) = S_1\left(x, y, s+\frac{1-a}{2}\right). \tag{7.2}$$

Let $\zeta(s) = \sum_1^{+\infty} \frac{1}{n^s}$ be the Riemann zeta function. Then

$$S_0(0, 0, s) = 2\zeta(2s),$$
$$S_1(0, 0, s) = 0. \tag{7.3}$$

DEFINITION 7.1. For $y\in\mathbf{R}$, $s\in\mathbf{C}$, $\mathrm{Re}(s)>1$, set

$$\zeta(y, s) = \sum_{n=1}^{+\infty}\frac{\cos(ny)}{n^s},$$
$$\eta(y, s) = \sum_{n=1}^{+\infty}\frac{\sin(ny)}{n^s}. \tag{7.4}$$

The $\zeta(y, s)$ and $\eta(y, s)$ are the real and imaginary parts of a Lerch series [L]. Clearly

$$\zeta(y, s) = \tfrac{1}{2}S_0\left(0, \frac{y}{2\pi}, \frac{s}{2}\right),$$
$$\eta(y, s) = \frac{i}{2}S_1\left(0, \frac{y}{2\pi}, \frac{s+1}{2}\right). \tag{7.5}$$

Then if $y\notin 2\pi\mathbf{Z}$, $s\to\zeta(y, s)$ extends to a holomorphic function on $\mathbf{C}$, if

$y \in 2\pi\mathbf{Z}$, $s \to \zeta(y, s)$ extends to a meromorphic function on $\mathbf{C}$ with a simple pole at $s = 1$. Also $s \to \eta(y, s)$ extends to a holomorphic function on $\mathbf{C}$. Moreover,

$$\zeta(0, s) = \zeta(s),$$
$$\eta(0, s) = 0. \tag{7.6}$$

Now we prove an extension of a formula proved by Bismut and Soulé in [B1, Appendix].

THEOREM 7.2. *For $\theta \in \mathbf{R}$, $x \in \mathbf{C}$, $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, then $D(\theta, x)$ is given by the convergent power series*

$$D(\theta, x) = \sum_{\substack{n \geq 0 \\ n \text{ even}}} i \left\{ \left(\Gamma'(1) + \sum_1^n \frac{1}{j} \right) \eta(\theta, -n) + 2 \frac{\partial \eta}{\partial s}(\theta, -n) \right\} \frac{x^n}{n!}$$
$$+ \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \left\{ \left(\Gamma'(1) + \sum_1^n \frac{1}{j} \right) \zeta(\theta, -n) + 2 \frac{\partial \zeta}{\partial s}(\theta, -n) \right\} \frac{x^n}{n!}. \tag{7.7}$$

Proof. By Proposition 4.2, we get

$$\frac{\frac{\partial \sigma}{\partial x}}{\sigma}(u, i\theta, -x) = \sum_{k\in\mathbf{Z}} \frac{i(\theta + 2k\pi)}{(\theta + 2k\pi)^2 - i(\theta + 2k\pi)x + u}. \tag{7.8}$$

By [B1, Appendix, Eq. (7)], for $a \in \mathbf{C}$, $\mathrm{Re}(a) > 0$, $s \in \mathbf{C}$, $0 < \mathrm{Re}(s) < 1$,

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{u^{s-1}}{u + a} \, du = a^{s-1} \Gamma(1 - s). \tag{7.9}$$

From (6.7), (7.8), (7.9), we see that for $s \in \mathbf{C}$, $0 < \mathrm{Re}(s) < \frac{1}{2}$,

$$C(s, \theta, x) = i\Gamma(1 - s) \sum_{k\in\mathbf{Z}}{}' \frac{|\theta + 2k\pi|^{2s}}{\theta + 2k\pi} \left(1 - \frac{ix}{\theta + 2k\pi} \right)^{s-1}. \tag{7.10}$$

Also as $k \in \mathbf{N} \to +\infty$

$$\frac{|\theta + 2k\pi|^{2s}}{\theta + 2k\pi} \left(1 - \frac{ix}{\theta + 2k\pi} \right)^{s-1} + \frac{|\theta - 2k\pi|^{2s}}{\theta - 2k\pi} \left(1 - \frac{ix}{\theta - 2k\pi} \right)^{s-1} \simeq k^{2s-2}. \tag{7.11}$$

From (7.10), (7.11), we deduce that

$$C(s,\theta,x) = i\Gamma(1-s)\sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}}{}' \frac{|\theta+2k\pi|^{2s}}{\theta+2k\pi}\frac{(s-1)\cdots(s-n)}{n!}\left(\frac{-ix}{\theta+2k\pi}\right)^n. \quad (7.12)$$

Set $\theta' = \dfrac{\theta}{2\pi}$. From (6.8), (7.8), we get

$$C(s,\theta,x) = i\Gamma(1-s)(2\pi)^{2s-1}\sum_{n\in\mathbb{N}}\frac{(s-1)\cdots(s-n)}{n!}S_{-n-1}(\theta',0,-s)\left(\frac{-ix}{2\pi}\right)^n, \quad (7.13)$$

and the series in the right-hand side of (7.13) converges normally on its domain of definition. In particular, to calculate $\dfrac{\partial C}{\partial s}(s,\theta,x)$, we can differentiate term by term the sum in the right-hand side of (7.3). By (6.8), (7.13), we get

$$D(\theta,x) = \sum_{n\in\mathbb{N}}\frac{i}{2\pi}\left\{\left(-\Gamma'(1) + 2\log(2\pi) - \sum_1^n \frac{1}{j}\right)\right.$$
$$\left.\times S_{-n-1}(\theta',0,0) - \frac{\partial S_{-n-1}}{\partial s}(\theta',0,0)\right\}\left(\frac{ix}{2\pi}\right)^n. \quad (7.14)$$

By [W, p. 57], we have the functional equation for $S_a(x,y,s)$, $a = 0$ or 1,

$$\Gamma(s)S_a(x,y,s) = i^{-a}\pi^{2s-a-1/2}e^{2\pi ixy}\Gamma(a-s+\tfrac{1}{2})S_a(y,-x,a-s+\tfrac{1}{2}). \quad (7.15)$$

Taking logarithmic derivatives in (7.15), we get

$$\frac{\Gamma'(s)}{\Gamma(s)} + \frac{\partial S_a/\partial s}{S_a}(x,y,s) = 2\log(\pi) - \frac{\Gamma'(a-s+1/2)}{\Gamma(a-s+1/2)} - \frac{\partial S_a/\partial s}{S_a}(y,-x,a-s+\tfrac{1}{2}). \quad (7.16)$$

In the sequel, we use the well known relations [NO, p. 21],

$$\Gamma(n) = (n-1)!,$$

$$\frac{\Gamma'(n)}{\Gamma(n)} = \Gamma'(1) + \sum_1^{n-1}\frac{1}{j},$$

$$\frac{\Gamma'((1-2n)/2)}{\Gamma((1-2n)/2)} = \Gamma'(1) - 2\log(2) + 2\sum_{j=1}^n \frac{1}{2j-1},$$

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)},$$

$$2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2})=\sqrt{\pi}\,\Gamma(2z). \tag{7.17}$$

By (7.2), (7.15), (7.17), we see that for $n\in\mathbf{N}$, n even,

$$\begin{aligned} S_{-n-1}(\theta',0,0)&=S_1\left(\theta',0,\frac{n+2}{2}\right) \qquad (7.18)\\ &=\frac{1}{i\Gamma\left(\frac{n+2}{2}\right)}\pi^{n+1/2}\Gamma\left(\frac{1-n}{2}\right)S_1\left(0,-\theta',\frac{1-n}{2}\right)\\ &=\frac{(-1)^{n/2}}{i}\pi^{n+1}\frac{2^n}{n!}S_1\left(0,-\theta',\frac{1-n}{2}\right) \qquad (7.18)\end{aligned}$$

and that for $n\in\mathbf{N}$, n odd,

$$\begin{aligned} S_{-n-1}(\theta',0,0)&=S_0\left(\theta',0,\frac{n+1}{2}\right)\\ &=\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}\pi^{n+1/2}\Gamma\left(-\frac{n}{2}\right)S_0\left(0,-\theta',-\frac{n}{2}\right)\\ &=(-1)^{(n+1)/2}\pi^{n+1}\frac{2^n}{n!}S_0\left(0,-\theta',-\frac{n}{2}\right). \qquad (7.19)\end{aligned}$$

By (7.2), (7.14), (7.15), we see that if n is even, and if $S_{-n-1}(\theta',0,0)\neq 0$, then

$$\begin{aligned} &-\Gamma'(1)+2\log(2\pi)-\sum_1^n\frac{1}{j}-\frac{\partial S_{-n-1}/\partial s}{S_{-n-1}}(\theta',0,0)\\ &=-\Gamma'(1)+2\log(2\pi)-\sum_1^n\frac{1}{j}-\frac{\partial S_1/\partial s}{S_1}\left(\theta',0,\frac{n+2}{2}\right)\\ &=-\Gamma'(1)+2\log(2)-\sum_1^n\frac{1}{j}+\frac{\Gamma'}{\Gamma}\left(\frac{n+2}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-n}{2}\right)\\ &\quad+\frac{\partial S_1/\partial s}{S_1}\left(0,-\theta',\frac{1-n}{2}\right)\end{aligned}$$

$$= \Gamma'(1) - \sum_1^n \frac{1}{j} + \sum_1^{n/2} \frac{1}{j} + 2\sum_1^{n/2} \frac{1}{2j-1} + \frac{\partial S_1/\partial s}{S_1}\left(0, -\theta', \frac{1-n}{2}\right)$$

$$= \Gamma'(1) + \sum_1^n \frac{1}{j} + \frac{\partial S_1/\partial s}{S_1}\left(0, -\theta', \frac{1-n}{2}\right). \tag{7.20}$$

If n is even and if $S_{-n-1}(\theta', 0, 0) = 0$, we deduce from (7.15), (7.18)

$$\frac{\partial S_{-n-1}}{\partial s}(\theta', 0, 0) = -\frac{(-1)^{n/2}}{i}\pi^{n+1}\frac{2^n}{n!}\frac{\partial S_1}{\partial s}\left(0, -\theta', \frac{1-n}{2}\right). \tag{7.21}$$

By (7.2), (7.14), (7.15), we find that if n is odd, and if $S_{-n-1}(\theta', 0, 0) \neq 0$, then

$$-\Gamma'(1) + 2\log(2\pi) - \sum_1^n \frac{1}{j} - \frac{\partial S_{-n-1}/\partial s}{S_{-n-1}}(\theta', 0, 0)$$

$$= -\Gamma'(1) + 2\log(2\pi) - \sum_1^n \frac{1}{j} - \frac{\partial S_0/\partial s}{S_0}\left(\theta', 0, \frac{n+1}{2}\right)$$

$$= -\Gamma'(1) + 2\log(2) - \sum_1^n \frac{1}{j} + \frac{\Gamma'}{\Gamma}\left(\frac{n+1}{2}\right) + \frac{\Gamma'}{\Gamma}(-n/2)$$

$$+ \frac{\partial S_0/\partial s}{S_0}(0, -\theta', -n/2)$$

$$= \Gamma'(1) - \sum_1^n \frac{1}{j} + \sum_1^{(n-1)/2} \frac{1}{j} + 2\sum_1^{(n+1)/2} \frac{1}{2j-1} + \frac{\partial S_0/\partial s}{S_0}\left(0, -\theta', \frac{-n}{2}\right)$$

$$= \Gamma'(1) + \sum_1^n \frac{1}{j} + \frac{\partial S_0/\partial s}{S_0}\left(0, -\theta', \frac{-n}{2}\right). \tag{7.22}$$

If n is odd and if $S_{-n-1}(\theta', 0, 0) = 0$, we deduce from (7.15), (7.19),

$$\frac{\partial S_{-n-1}}{\partial s}(\theta', 0, 0) = -(-1)^{(n+1)/2}\pi^{n+1}\frac{2^n}{n!}\frac{\partial S_0}{\partial s}\left(0, -\theta', \frac{-n}{2}\right). \tag{7.23}$$

From (7.5), (7.14), (7.18)–(7.23), we get (7.7). The proof of Theorem 7.2 is completed. □

REMARK 7.3. Recall that $D(x) = D(0, x)$. By (7.6) and by Theorem 7.2, we get

$$D(x) = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \left(\Gamma'(1) + \sum_1^n \frac{1}{j} + \frac{2\zeta'(-n)}{\zeta(n)}\right)\zeta(-n)\frac{x^n}{n!}, \tag{7.24}$$

which is exactly the formula obtained by Bismut and Soulé in [B1, Appendix].

In a previous work [B5] on intersection formulas in complex equivariant cohomology in the presence of an excess normal bundle, we introduced in [B5, Definition 1.22] the function

$$D^{\mu}(x) = \frac{1}{(\mu + x)}\left(\Gamma'(1) - 2\log|2\pi\mu| - \log\left(1 + \frac{x}{\mu}\right)\right). \tag{7.25}$$

In the sequel $\Sigma'_{k\in\mathbf{Z}}$ denotes a sum over $k \in \mathbf{Z}$, $k \neq -\dfrac{\theta}{2\pi}$.

THEOREM 7.4. *For $\theta \in \mathbf{R}$, $x \in \mathbf{C}$, $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, then $D(\theta, x)$ is given by*

$$D(\theta, x) = \frac{i}{2\pi}\sum_{k\in\mathbf{Z}}{}' D^{-(k+\theta/2\pi)}\left(\frac{ix}{2\pi}\right). \tag{7.26}$$

Proof. From (7.10), we get

$$D(\theta, x) = \sum_{k\in\mathbf{Z}}{}' i\left(-\Gamma'(1) + 2\log|\theta + 2k\pi| + \log\left(1 - \frac{ix}{\theta + 2k\pi}\right)\right)\frac{1}{\theta + 2k\pi - ix}. \tag{7.27}$$

Equation (7.26) follows from (7.25), (7.27). □

(b) *A formula for $D(\theta, x)$ for $\theta \notin 2\pi\mathbf{Z}$*

Observe that if $|\theta| < 2\pi$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$, then $|x + i\theta| < 2\pi$.

THEOREM 7.5. *If $\theta \in]-2\pi, +2\pi[\backslash\{0\}$, if $x \in \mathbf{C}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$, then*

$$D(\theta, x) = D(x + i\theta) + i\sum_{k\in\mathbf{Z}^*}\frac{\log(1 + \theta/2k\pi)}{2k\pi - i(x + i\theta)} + \frac{\left(\Gamma'(1) - \log(\theta^2) - \log\left(1 - \frac{ix}{\theta}\right)\right)}{x + i\theta}. \tag{7.28}$$

Proof. By Theorem 7.4, we get

$$D(\theta, x) = \frac{i}{2\pi}\sum_{k\in\mathbf{Z}^*} D^{-(k+\theta/2\pi)}\left(\frac{ix}{2\pi}\right) + \frac{i}{2\pi} D^{-(\theta/2\pi)}. \tag{7.29}$$

For $k\in\mathbf{Z}^*$,

$$\frac{i}{2\pi}D^{-(k+\theta/2\pi)}\left(\frac{ix}{2\pi}\right)=\frac{i}{-2k\pi+i(x+i\theta)}$$
$$\times[\Gamma'(1)-\log((\theta+2k\pi)(\theta+2k\pi-ix))]$$
$$=\frac{i}{-2k\pi+i(x+i\theta)}$$
$$\times\left(\Gamma'(1)-\log(2k\pi(2k\pi-i(x+i\theta))-\log\left(1+\frac{\theta}{2k\pi}\right)\right), \tag{7.30}$$

and so for $k\in\mathbf{Z}^*$,

$$\frac{i}{2\pi}D^{-(k+\theta/2\pi)}\left(\frac{ix}{2\pi}\right)=\frac{i}{2\pi}D^{-k}(x+i\theta)+\frac{i\log\left(1+\dfrac{\theta}{2k\pi}\right)}{2k\pi-i(x+i\theta)}. \tag{7.31}$$

Moreover

$$\frac{i}{2\pi}D^{-\theta/2\pi}\left(\frac{ix}{2\pi}\right)=\left(\Gamma'(1)-\log(\theta^2)-\log\left(1-\frac{ix}{\theta}\right)\right)\frac{1}{x+i\theta}. \tag{7.32}$$

Using (6.10), (7.29), (7.31), (7.32), we get (7.28). □

(c) *The genus $R(\theta, x)$ as an extension of the genus $R(x)$ of Gillet and Soulé*

Now we recall the definition of the Gillet-Soulé genus R [GS1].

DEFINITION 7.6. For $x\in\mathbf{C}$, $|x|<2\pi$, set,

$$R(x)=\sum_{\substack{n\geq 1\\ n \text{ odd}}}\left(\sum_1^n\frac{1}{j}+2\frac{\zeta'(-n)}{\zeta(-n)}\right)\zeta(-n)\frac{x^n}{n!}. \tag{7.33}$$

Set

$$\hat{A}(x)=\frac{x/2}{\sinh(x/2)}.$$

Then one has the classical formula [B1, Eq. (8.35)],

$$\frac{\hat{A}'}{\hat{A}}(x)=\sum_{\substack{n\geq 1\\ n \text{ odd}}}\zeta(-n)\frac{x^n}{n!}. \tag{7.34}$$

So from (7.24), (7.33), (7.34), we obtain the formula of [B1, Eq. (8.39)],

$$R = D - \Gamma'(1)\,\frac{\hat{A}'}{\hat{A}}. \tag{7.35}$$

Now we will imitate (7.35) to construct a function $R(\theta,x)$ from $D(\theta, x)$. Set

$$\begin{aligned} \hat{\alpha}(\theta,\ x) &= \hat{A}(x) \quad \text{if } \theta \in 2\pi\mathbf{Z}, \\ &\frac{\hat{A}(x+i\theta)}{x+i\theta} \quad \text{if } \theta \notin 2\pi\mathbf{Z}. \end{aligned} \tag{7.36}$$

DEFINITION 7.7. For $\theta \in \mathbf{R}$, $x \in \mathbf{C}$, $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, set

$$R(\theta,\ x) = D(\theta,\ x) - \Gamma'(1)\,\frac{\partial\hat{\alpha}/\partial x}{\hat{\alpha}}\,(\theta,\ x). \tag{7.37}$$

Observe that the function $\theta \in \mathbf{R} \to R(\theta, x)$ is periodic with period 2π. By (7.35), (7.37), we get

$$R(0,\ x) = R(x). \tag{7.38}$$

THEOREM 7.8. *For* $\theta \in \mathbf{R}$, $x \in \mathbf{C}$, $|x| < 2\pi$ *if* $\theta \in 2\pi\mathbf{Z}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ *if* $\theta \notin 2\pi\mathbf{Z}$, *then*

$$\begin{aligned} R(\theta, x) = {} & \sum_{\substack{n\geqslant 0\\ n \text{ even}}} i \left\{ \sum_{1}^{n} \frac{1}{j}\,\eta(\theta,\ -n) + 2\,\frac{\partial\eta}{\partial s}\,(\theta,\ -n \right\} \frac{x^n}{n!} \\ & + \sum_{\substack{n\geqslant 1\\ n \text{ odd}}} \left\{ \sum_{1}^{n} \frac{1}{j}\,\zeta(\theta,\ -n) + 2\,\frac{\partial}{\partial s}\,\zeta(\theta,\ -n) \right\} \frac{x^n}{n!}. \end{aligned} \tag{7.39}$$

Proof. By (4.2),

$$\frac{\partial\hat{\alpha}/\partial x}{\hat{\alpha}}\,(\theta,\ x) = \lim_{u>0,u\to 0} \frac{\partial\sigma/\partial x}{\sigma}\,(u,\ i\theta,\ -x). \tag{7.40}$$

Using (6.7), (7.39), we get

$$\frac{\partial\hat{\alpha}/\partial x}{\hat{\alpha}}\,(\theta,\ x) = C(0,\ \theta,\ x). \tag{7.41}$$

By (7.5), (7.13), (7.18), (7.19), (7.41), we obtain

$$\frac{\partial\hat{\alpha}/\partial x}{\hat{\alpha}}(\theta, x) = \sum_{\substack{n\geq 0 \\ n \text{ even}}} i\eta(\theta, -n)\frac{x^n}{n!} + \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \zeta(\theta, -n)\frac{x^n}{n!}. \tag{7.42}$$

Using Theorem 7.2, (7.37) and (7.42), we get (7.39). □

As in [B5, Definition 1.22], set

$$R^{\mu}(x) = \frac{1}{(\mu+x)}\left(2\Gamma'(1) - 2\log|2\pi\mu| - \log\left(1+\frac{x}{\mu}\right)\right). \tag{7.43}$$

THEOREM 7.9. *For $\theta\in\mathbf{R}$, $x\in\mathbf{C}$, $|x|<2\pi$ if $\theta\in 2\pi\mathbf{Z}$, $|x|<\inf_{k\in\mathbf{Z}}|\theta+2k\pi|$ if $\theta\notin 2\pi\mathbf{Z}$, then $R(\theta, x)$ is given by*

$$R(\theta, x) = \frac{i}{2\pi}\sum_{k\in\mathbf{Z}}{}' R^{-(k+\theta/2\pi)}\left(\frac{ix}{2\pi}\right). \tag{7.44}$$

Proof. By (7.10), (7.41), we get

$$\frac{\partial\hat{\alpha}/\partial x}{\alpha}(\theta, x) = i\sum_{k\in\mathbf{Z}}{}' \frac{1}{\theta+2k\pi-ix}. \tag{7.45}$$

Using (7.25), (7.26), (7.37), (7.43), (7.45), we get (7.44). □

THEOREM 7.10. *If $\theta\in]-2\pi, 2\pi[\backslash\{0\}$, if $x\in\mathbf{C}$, $|x|<\inf_{k\in\mathbf{Z}}|\theta+2k\pi|$, then*

$$R(\theta, x) = R(x+i\theta) + i\sum_{k\in\mathbf{Z}^*}\frac{\log(1+\theta/2k\pi)}{2k\pi - i(x+i\theta)} + \frac{2\Gamma'(1) - \log(\theta^2) - \log\left(1-\frac{ix}{\theta}\right)}{x+i\theta}. \tag{7.46}$$

Proof. Clearly

$$\frac{\partial\hat{\alpha}/\partial x}{\hat{\alpha}}(\theta, x) = \frac{\partial\hat{A}/\partial x}{\hat{A}}(x+i\theta) - \frac{1}{x+i\theta}. \tag{7.47}$$

Using Theorem 7.5, (7.37), (7.47), we get (7.46). □

REMARK 7.11. In [B5, B6], we have showed that the results of [B1], [BL] can be viewed as formal consequences of a height pairing formula in equivariant cohomology in infinite dimensions. The expression (7.44) indicates that the analogy with [B5, B6] still holds here. This is confirmed by [B9, 10], where the results of [BL] have been extended to equivariant Quillen metrics, and where $R(\theta, x)$ appears explicitly.

DEFINITION 7.12. For $x \in \mathbf{C}$, $|x| < 2\pi$, set

$$\rho(x) = \sum_{\substack{n \geqslant 1 \\ n \text{ odd}}} 2\zeta'(-n) \frac{x^n}{n!}. \tag{7.48}$$

THEOREM 7.13. *For* $\theta \in \mathbf{R}$,

$$R(\theta, 0) = 2i \frac{\partial \eta}{\partial s} (\theta, 0). \tag{7.49}$$

Also for $\theta \in]-2\pi, 2\pi[\setminus\{0\}$,

$$R(\theta, 0) = \rho(i\theta) + \frac{2\Gamma'(1) - \log(\theta^2)}{i\theta}. \tag{7.50}$$

Proof. Equation (7.49) follows from (7.39). By [B1, Appendix, Eq. (20), (26)],

$$i \sum_{k \in \mathbf{Z}^*} \frac{\log(1 + \theta/2k\pi)}{2k\pi + \theta} = -\sum_{\substack{n \geqslant 1 \\ n \text{ odd}}} \left(\sum_1^n \frac{1}{j}\right) \zeta(-n) \frac{(i\theta)^n}{n!}. \tag{7.51}$$

Using (7.33), Theorem 7.10 and (7.51), we get (7.50). □

REMARK 7.14. The fact that (7.49) and (7.50) coincide was observed by Köhler [K, Proposition 1].

DEFINITION 7.15. Set

$$R_g(N, h^N) = \sum R(\theta_j, N^{\theta_j}, h^{N_{\theta_j}}). \tag{7.52}$$

We denote by $R_g(N)$ the class of $R_g(N, h^N)$ in $P^B/P^{B,0}$.

(d) *The genus $R(\theta, x)$ and the computation by Köhler of the equivariant torsion of $\mathbf{P}^n$ for an isometry with isolated fixed points*

Let $\mathbf{P}^n$ be the n dimensional complex projective space, equipped with the Fubini-Study metric.

Let $g \in U(n+1)$. Then g acts naturally on $\mathbf{P}^n$ as a holomorphic isometry. Let $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ be the Laplacian on $\mathbf{P}^n$. For $0 \leqslant p \leqslant n$, let $\square_p$ be the restriction of $\square$ to smooth section of $\Lambda(T^{*(0,1)}\mathbf{P}^n)$. Then for $p \geqslant 1$, $\square_p$ acts as an invertible operator.

For $s \in \mathbf{C}$, $\mathrm{Re}(s) > n$, set

$$\theta_g(s) = -\sum_{p=1}^{n} (-1)^p p \ \mathrm{Tr}[g \square_p^{-s}]. \tag{7.53}$$

Then one verifies easily that $s \to \theta_g(s)$ extends to a meromorphic function which is holomorphic at 0.

Set

$$\mathbf{P}_g^n = \{x \in \mathbf{P}^n, \ gx = x\}. \tag{7.54}$$

Assume that the $n+1$ eigenvalues of g are distinct. Then B_g consists of isolated fixed points. Also g acts naturally on $T\mathbf{P}^n|_{\mathbf{P}_g^n}$ as an isometry.

We now recall a result of Köhler [K, Theorem 7.3].

THEOREM 7.16. *The following identity holds*

$$\frac{\partial \theta_g}{\partial s}(0) = \int_{\mathbf{P}_g^n} \mathrm{Td}_g(T\mathbf{P}^n) R_g(T\mathbf{P}^n) - \log(n!). \tag{7.55}$$

Proof. Let $e^{i\varphi_1}, \ldots, e^{i\varphi_{n+1}}$ be the eigenvalues of g. Then in [K, Theorem 7.3], Köhler obtains the formula

$$\frac{\partial \theta_g}{\partial s}(0) = \sum_k \frac{1}{\prod_{j \neq k}(1 - e^{-i(\varphi_j - \varphi_k)})} \sum_{j' \neq k} R(\varphi_{j'} - \varphi_k, 0) - \log(n!), \tag{7.56}$$

which is equivalent to (7.53). $\square$

(e) *Towards an equivariant Riemann-Roch theorem in Arakelov theory*

By relating our results wih the calculation by Köhler [K] of the equivariant torsion of $\mathbf{P}^n$, at least for g having isolated fixed points, we have made a calculation formally very similar to what was done in Bismut [B1] when compared with the calculation by Gillet and Soulé [GS1] of the usual analytic torsion of $\mathbf{P}^n$.

Theorem 7.16 makes very likely that there is an equivariant form of the Riemann-Roch-Arakelov formula of Gillet and Soulé [GS4]. The genus R of Gillet and Soulé should then be replaced by the equivariant genus R_g. Recently, in [B9, 10], using the results of this paper, we have extended the formula of Bismut-Lebeau [BL] to equivariant Quillen metrics. As expected, the genus R_g appears explicitly in this new formula.

It seems very likely that if X and Y are non singular quasi-projective varieties, if G is a finite group acting on X and Y, and if $f: X \to Y$ is a G-equivariant projective morphism, if (E, h^E) is a G-equivariant Hermitian vector bundle on X, if $g \in G$, an equivariant Riemann-Roch Arakelov formula might take the form

$$\mathrm{ch}_g(\hat{f}!(E)) = f_*(\hat{\mathrm{Td}}_g(TX/Y)\hat{\mathrm{ch}}_g(E) - a(\mathrm{ch}_g(E_{\mathbf{C}})\mathrm{Td}_g(TX/Y)R_g(TX/Y|_{\mathbf{C}}))). \tag{7.57}$$

In (7.57), the characteristic classes with a ^ would be appropriate extensions of the classes of Gillet and Soulé [GS2, 3] to an equivariant situation.

Of course, this formula has been verified by Gillet-Soulé [GS4] and Faltings [F] when G is trivial.

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