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The enumeration of simultaneous higher-order contacts between plane curves

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Introduction

Two plane algebraic curves are said to have *contact of order o* at a common point P if each curve is smooth at P and if the intersection number at P is o. Thus, for example, a contact of order 1 is a transverse intersection, a contact of order 2 (sometimes called an *ordinary contact*) is a point of tangency; more generally, a contact of order o is a point at which, in appropriate local coordinates, the Taylor expansions of the curves agree up to order $o - 1$.

In [11], Fulton, Kleiman, and MacPherson consider, *inter alia*, a p-parameter family of plane curves together with p individual curves. They compute, in terms of certain "characteristic numbers", the number of members of the family which simultaneously have an ordinary contact with each of the curves. The analysis has two parts, the first of which is a formal calculation of an intersection number. The second part consists in establishing that, under stipulated hypotheses, this intersection number and the characteristic numbers have their intended geometric meanings. (The tendentious description we have just given greatly understates the scope of the contact formula of [11]. The contact formula for plane curves is an implicit special case—see further remarks in the next paragraph.)

In the present paper, by a similar two-part procedure, we derive a higher-order contact formula. We suppose that $C_1, C_2, \ldots, C_p$ are plane curves, and that $\mathscr{X}$ is an s-parameter family of plane curves of fixed degree. We suppose that $o_1, o_2, \ldots, o_p$ are specified positive integers, with sum equal to $s + p$. Our formula then counts, under stipulated hypotheses, the number of members of the family which simultaneously have a contact of order o_1 with C_1, a contact of order o_2 with $C_2, \ldots$, and a contact of order o_p with C_p. The formula is both a specialization and a generalization of the

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general contact formula of [11]: a specialization in that we consider only plane curves, a generalization in that it counts contacts of arbitrary order. Our formula, like that of [11], is stated using the formalism of "modules", a notion which we explain in Section 3.

The principal tool used in the proof, and in defining appropriate higher-order characteristic numbers, is a tower of $\mathbf{P}^1$-bundles implicitly introduced by Semple [27] and later reintroduced by Collino [7]. These bundles, which parametrize in a particularly lucid way the higher-order curvilinear data of the plane, are naturally adapted to the study of higher-order contact. They generalize the variety of second-order data of $\mathbf{P}^2$ studied by J. Roberts and R. Speiser in [23] and [24], and, in turn, are currently being generalized by E. Arrondo, I. Sols, and R. Speiser [4]. (They are developing a theory of "derived triangles" that provides a general approach for obtaining the higher-order data, both curvilinear and higher-dimensional, of any scheme.) The Semple bundles conform to the intuitive heuristic that $(n + 1)$st-order curvilinear data should fiber over the nth-order data. Moreover, since the $(n + 1)$st Semple bundle variety is a $\mathbf{P}^1$-bundle over the nth variety, the intersection rings are straightforward to calculate. Thus the Semple bundles are superior to the canonical parameter space for data, namely the Hilbert scheme $\mathrm{Hilb}_n\mathbf{P}^2$ of zero-dimensional subschemes of $\mathbf{P}^2$, in this respect. Indeed, considerable effort has been devoted recently to determining the homology and cohomology of punctual Hilbert schemes and to using these results to develop enumerative applications. (See, for example, [8], [9], [10], [20].) We have not explored the connections between Semple bundles and punctual Hilbert schemes in any detail, however. Nor have we begun to understand the apparently close connection between Semple bundles and certain subschemes of Kleiman's iterative multiple-point schemes [15].

In the first two sections of this paper we describe the Semple bundle varieties and their intersection rings. In Section 3 we obtain a "proto-contact formula", and in Section 4 we show that under stipulated hypotheses this formula counts, as intended, the number of simultaneous contacts between a generic family and specified curves. The proof involves a detailed analysis of the relationship between the universal family of plane curves of degree d and the Semple bundles which should be of independent interest. The fifth section briefly discusses the ingredients of the contact formula, i.e., the "higher-order characteristic numbers". The final section treats special cases and variants of the contact formula, compares the formula with those of de Jonquières and Fulton-Kleiman-MacPherson, and presents an example.

We assume that all our varieties and schemes are defined over an algebraically closed field of characteristic zero. However, the results presented remain true with only minor modifications if the characteristic is sufficiently large, provided one recognizes that in positive characteristic our intersection numbers only count weighted numbers of contacts.

1. Semple bundles

We briefly recall here the definition of Semple's bundles of higher-order curvilinear data; for further discussion see [5] or [6]. The inductive construction begins by declaring that $F(0)$ is the projective plane and that the first *Semple bundle variety* $F(1)$ is $\mathbf{P}TF(0)$, the total space of the projectivized tangent bundle. (We use $\mathbf{P}E$ for the variety representing rank 1 subbundles of the vector bundle E, rather than for the variety representing rank 1 quotient bundles.) Inductively suppose that $F(n)$ is the projectivization of a rank 2 subbundle of the tangent bundle $TF(n-1)$. The *focal plane* at a point $p \in F(n)$ is defined to be the preimage, via the derivative of the projection $f_n\colon F(n) \to F(n-1)$, of the line in $T_{f_n(p)}F(n-1)$ represented by p. This construction gives rise to a rank 2 *bundle of focal planes*, denoted $\mathscr{F}_n$; we define $F(n+1)$ to be the total space of the projectivization of this bundle. In this manner we obtain a tower of $\mathbf{P}^1$-bundles:

$$\begin{array}{c}\vdots \\ \downarrow{\scriptstyle f_4} \\ F(3) \\ \downarrow{\scriptstyle f_3} \\ F(2) \\ \downarrow{\scriptstyle f_2} \\ F(1) \\ \downarrow{\scriptstyle f_1} \\ F(0)\end{array}$$

Suppose that C is a reduced plane curve, and that $p \in C$ is a nonsingular point. The fiber of $F(1)$ over p parametrizes the various tangent directions at p, including the tangent direction p_1 of C. The totality of such tangent directions to C form a curve in $F(1)$; the closure of this curve is called the *first lift* of C, and denoted $C(1)$. Now observe that the tangent direction p_2 of $C(1)$ at p_1 maps, via the derivative of the projection $f_1\colon F(1) \to F(0)$, to p_1, the tangent direction of C at p. Hence the *second lift* $C(2)$, i.e., the lift of the lift of C, is in fact a curve in $F(2)$. The same argument shows that there are *higher-order lifts* $C(3) \subset F(3)$, $C(4) \subset F(4)$, etc. We call the rational map λ from

C to $F(n)$ the *lifting map*; it is regular away from the singularities of C and a birational map to $C(n)$. We note that the nth lift $C(n)$ is just the nth blow-up of C at its singular points. However, for our enumerative purposes we need to understand the embedding of $C(n)$ in the Semple bundle variety $F(n)$.

It is sometimes convenient to regard a point of $F(n)$ as an equivalence class of irreducible germs of plane curves. Thus we sometimes say that the irreducible germ of a curve C *represents* the point $p_n \in F(n)$, meaning that the lift of C passes through p_n above the closed point of the germ.

To illustrate the transparent nature of calculations in these Semple bundles, let us consider a reduced plane curve C defined, in an affine chart with coordinates x and y, by $f(x, y) = 0$. Then one can show that on $F(n)$ there is a *primary chart* isomorphic to affine $(n + 2)$-space, with coordinates x, y, y', y'', $y^{(3)}, \ldots, y^{(n)}$, and that the ideal defining the nth lift $C(n)$ includes the sequence of functions $f, f', f'', f^{(3)}, \ldots, f^{(n)}$ obtained by repeated implicit differentiation with respect to x. (There is a second primary chart in which the roles of x and y are reversed.)

In general these functions may not generate the ideal defining the lift. Consider, for example, the cuspidal cubic $y^2 = x^3$. Then the first lift is defined in the primary chart by

$$y^2 = x^3, \quad 2yy' = 3x^2, \quad x = \frac{4}{9}(y')^2, \quad \text{and} \quad y = \frac{8}{27}(y')^3. \tag{1}$$

The variety defined by just the first two equations contains a spurious component over the origin. (To obtain the third equation, we square both sides of the second equation, then use the first equation to replace y^2; since the curve is singular at the origin it is then legal to cancel x^3 from both sides. The fourth equation is obtained in a similar fashion. Clearly the third and fourth equations define an irreducible curve in $\mathbf{A}^3$, and the first two equations are redundant.)

If we continue this example one step further we see that the lift of a curve may leave the primary charts. Indeed, if we implicitly differentiate the third equation of (1), we obtain this equation for $C(2)$:

$$1 = \frac{8}{9} y' y''.$$

Since the unique point p_1 of $C(1)$ over the origin is $(x, y, y') = (0, 0, 0)$, and since $C(2)$ must be complete, the unique point p_2 of $C(2)$ over p_1 must be the one point over p_1 missed by the primary chart, the point which, intuitively, represents infinite curvature. There is one such point on each fiber of $F(2)$ over $F(1)$; taken together, these points form a section of the $\mathbf{P}^1$-bundle which we call the *divisor at infinity* and denote by I_2.

Here is another characterization of the divisor at infinity. The derivative of f_1 at p_1,

$$\mathrm{d}f_1 \colon T_{p_1}F(1) \to T_{f_1(p_1)}F(0),$$

maps a 3-dimensional vector space to a 2-dimensional vector space. The kernel is one-dimensional, so there is a unique direction annihilated by $\mathrm{d}f_1$. This direction is represented by a point on the divisor at infinity. More generally, the derivative of f_n at a point p_n,

$$\mathrm{d}f_n \colon T_{p_n}F(n) \to T_{f_n(p_n)}F(n-1),$$

likewise has a one-dimensional kernel. Hence there is a divisor at infinity I_{n+1} on $F(n+1)$; intuitively, a point on this divisor represents (in addition to certain lower-order data) infinite curvilinear data of order $n+1$. To avoid clumsy notation, we will also denote by I_{n+1} the pullback of this divisor to any Semple bundle variety above $F(n+1)$ in the tower, and continue to call the pullback a divisor at infinity. Note that the divisors at infinity have normal crossings; in particular, the intersection of two such divisors has codimension two.

The simplest sort of chart meeting a divisor at infinity is a *secondary chart.* To specify such a chart on $F(n)$ – over the affine chart of $\mathbf{P}^2$ with coordinates x and y – choose an integer j between 2 and n. Then there is a chart isomorphic to affine $(n+2)$-space, with coordinates $x, y, y', y'', \dots, y^{(j-1)}, x', x'', \dots, x^{(n-j+1)}$. For $i = 1, \dots, j-1$, the coordinate $y^{(i)}$ measures $\mathrm{d}y^{(i-1)}/\mathrm{d}x$, a ratio of differentials of coordinates on $F(i-1)$; for $i = 1, \dots, n-j+1$, the coordinate $x^{(i)}$ measures $\mathrm{d}x^{(i-1)}/\mathrm{d}y^{(j-1)}$, a ratio of differentials of coordinates on $F(j+i-1)$. (See [5] for an explanation of a full system of charts.) If C is a reduced plane curve defined by $f(x, y) = 0$, then the ideal defining the nth lift $C(n)$ includes a sequence of functions obtained by repeated implicit differentiation. In the first $j-1$ of these differentiations we treat x as the independent variable, and denote the derivative $\mathrm{d}y/\mathrm{d}x$ of y by y', the derivative $\mathrm{d}y'/\mathrm{d}x$ of y' by y'', etc. In the remaining $n-j+1$ differentiations we treat $y^{(j-1)}$ as the independent variable, and denote the derivative $\mathrm{d}x/\mathrm{d}y^{(j-1)}$ of x by x', the derivative $\mathrm{d}x'/\mathrm{d}y^{(j-1)}$ of x' by x'', etc. The derivative $\mathrm{d}y^{(i)}/\mathrm{d}y^{(j-1)}$ of $y^{(i)}$ $(0 \leqslant i < j-1)$ is obtained by the chain rule:

$$\frac{\mathrm{d}y^{(i)}}{\mathrm{d}y^{(j-1)}} = \frac{\mathrm{d}y^{(i)}}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}y^{(j-1)}} = y^{(i+1)}x'.$$

In other words, the defining ideal for $C(n)$ includes the functions

$$f,\ P(f),\ P^2(f), \ldots, P^{j-1}(f),\ QP^{j-1}(f),\ Q^2P^{j-1}(f), \ldots, Q^{n-j+1}P^{j-1}(f), \tag{2}$$

where P and Q are the differential operators

$$P = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots + y^{(j-1)}\frac{\partial}{\partial y^{(j-2)}}$$

$$Q = x'P + \frac{\partial}{\partial y^{(j-1)}} + x''\frac{\partial}{\partial x'} + x^{(3)}\frac{\partial}{\partial x''} + \cdots + x^{(n-j+1)}\frac{\partial}{\partial x^{(n-j)}}. \tag{3}$$

Away from the singularities of C, the sequence of functions in (2) generates the ideal, but additional generators will be needed to eliminate spurious components over the singularities. In this jth *secondary chart* the divisor at infinity I_j is defined by the vanishing of x'.

To illustrate the rules for calculating in secondary charts, we continue our example of the cuspidal cubic $y^2 = x^3$. Implicitly differentiating the third equation of (1) with respect to y', we obtain

$$x' = \frac{8}{9}y'.$$

Here x' is the coordinate of the secondary chart measuring dx/dy'. (In other words, x' is the reciprocal of the ordinary second derivative coordinate dy'/dx.) From the equations of (1) one easily sees that the first lift is tangent to the fiber of $F(1)$ over the origin. Note that, as expected, the second lift hits the divisor at infinity I_2 over the origin.

We note for future reference that, taken together, the primary and secondary charts cover all of the Semple bundle variety except for intersections of two or more divisors at infinity.

By definition $F(n+1)$ is a subvariety of $\mathbf{P}TF(n)$, the total space of the projectivized tangent bundle of $F(n)$. If $n \geqslant 2$, then $\mathbf{P}TI_n$, the total space of the projectivized tangent bundle of the divisor at infinity, is likewise a subvariety of $\mathbf{P}TF(n)$; its codimension is 2. Now one can easily verify, e.g., by a calculation in local coordinates, that $F(n+1)$ and $\mathbf{P}TI_n$ are transverse. Hence their intersection is a codimension 2 subvariety of $F(n+1)$ which we call the *locus of tangency* to I_n. A point of this locus represents a point of I_n together with a tangent direction belonging to the fiber of the focal plane at that point. Again to avoid clumsy terminology, we continue to speak of the "locus of tangency" when we ought to say "the pullback through the Semple bundle tower of the locus of tangency". The jth secondary chart on $F(n)$ meets only one such locus, namely the locus of tangency to I_j; this locus is defined by the vanishing of both x' and x''. (But if $j = n$, there is no such locus meeting the secondary chart at all.)

Suppose now that $\mathscr{X}$ is a family of plane curves, and that its general member is reduced. For each reduced member there is a (rational) lifting map to $F(n)$; these maps fit together to form a rational map $\lambda: \mathscr{X} \dashrightarrow F(n)$. We call the closure of $\lambda(\mathscr{X})$ the *lift* of the family, and denote it by $\mathscr{X}(n)$. By definition, $\mathscr{X}(n)$ is the union of the graph of λ and a closed subvariety $\mathscr{S}(n)$, each point of which lies over a singular or nonreduced point of the corresponding member of $\mathscr{X}$.

2. Intersection rings

Since the variety $F(n)$ is the projectivization of the rank 2 bundle $\mathscr{F}_{n-1}$ over $F(n-1)$, the intersection rings may be determined inductively from standard theory. Specifically, the Chow ring $A^*(F(n))$ is an $A^*(F(n-1))$-algebra generated by the tautological class $\phi_n := c_1(\mathscr{O}_{F(n)}(1))$, which satisfies a single quadratic relation

$$\phi_n^2 + c_1(\mathscr{F}_{n-1})\phi_n + c_2(\mathscr{F}_{n-1}) = 0. \tag{4}$$

(Note: Here and in the sequel, we omit pullbacks of classes when convenient and when no confusion should result.) The base variety $F(0)$ is $\mathbf{P}^2$, whose Chow ring is generated by the hyperplane class h, subject to the relation $h^3 = 0$. Thus $A^*(F(n))$ is generated by $h, \phi_1, \ldots, \phi_n$ subject to the relations mentioned above.

For the purpose of studying contact between plane curves, however, this description is not desirable. Instead, we provide an equivalent formulation using more geometric classes. In particular, we will eliminate the ϕ_n's in favor of $\check{h}$, the dual hyperplane class on $\check{\mathbf{P}}^2$, and the classes i_j of the divisors at infinity I_j.

THEOREM 1. *For $n \geqslant 1$, the Chow ring $A^*(F(n))$ is generated by $h, \check{h}, i_2, \ldots, i_n$ subject to the relations*

$$h^3 = 0, \quad \check{h}^3 = 0, \quad h^2 - h\check{h} + \check{h}^2 = 0,$$

and, for $k = 2, \ldots, n$,

$$i_k^2 = ((2k-1)h - (k+1)\check{h} - ki_2 - (k-1)i_3 - \cdots - 3i_{k-1})i_k.$$

Proof. To begin, $F(1)$ is the incidence correspondence of $\mathbf{P}^2$. Thus $F(1) \subset \mathbf{P}^2 \times \check{\mathbf{P}}^2$ and it is well known that

$$A^*(F(1)) \cong \frac{\mathbf{Z}[h, \check{h}]}{(h^3, \check{h}^3, h^2 - h\check{h} + \check{h}^2)}.$$

It follows from a change of basis calculation that

$$\phi_1 = \check{h} - 2h. \tag{5}$$

To finish the proof, we need to rewrite (4) without using any ϕ_k's. To do this, we note first that the focal plane bundle $\mathscr{F}_k, k \geqslant 1$, fits into the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{F(k)/F(k-1)} & \longrightarrow & \mathscr{F}_k & \longrightarrow & \mathcal{O}_{F(k)}(-1) & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_{F(k)/F(k-1)} & \longrightarrow & T_{F(k)} & \longrightarrow & f_k^* T_{F(k-1)} & \longrightarrow & 0 \end{array} \tag{6}$$

(This is the dual of diagram 1 of [6].) Let

$$\sigma\colon \mathcal{O}_{F(k+1)}(-1) \to f_{k+1}^* \mathscr{F}_k \to f_{k+1}^* \mathcal{O}_{F(k)}(-1)$$

be the composite of the tautological map and the pullback to $F(k + 1)$ of the map in the top row of diagram (6). Then σ has zero locus equal to $PT_{F(k)/F(k-1)} = I_{k+1}$. In addition, σ defines a section of $\mathrm{Hom}(\mathcal{O}_{F(k+1)}(-1), f_{k+1}^* \mathcal{O}_{F(k)}(-1))$. Hence

$$\begin{aligned} i_{k+1} &= [PT_{F(k)/F(k-1)}] \\ &= f_{k+1}^* c_1(\mathcal{O}_{F(k)}(-1)) - c_1(\mathcal{O}_{F(k+1)}(-1)) \\ &= \phi_{k+1} - \phi_k. \end{aligned} \tag{7}$$

Now the Euler sequence

$$0 \to \mathcal{O}_{F(k)} \to f_k^* \mathscr{F}_{k-1} \otimes \mathcal{O}_{F(k)}(1) \to T_{F(k)/F(k-1)} \to 0$$

and the top sequence of (6) together imply that, for $k \geqslant 1$,

$$\begin{aligned} c_1(\mathscr{F}_k) &= c_1(\mathscr{F}_{k-1}) + \phi_k \\ c_2(\mathscr{F}_k) &= 2c_2(\mathscr{F}_{k-1}) + c_1(\mathscr{F}_{k-1})\phi_k. \end{aligned} \tag{8}$$

Using (7) to substitute for ϕ_{k+1} in (4) and simplifying, one finds

$$i_{k+1}(i_{k+1} + 3\phi_k + c_1(\mathscr{F}_{k-1})) = 0. \tag{9}$$

Note that the inductive formula for $c_1(\mathscr{F}_k)$ in (8) yields the closed-form formula

$$c_1(\mathscr{F}_k) = 3h + \phi_1 + \phi_2 + \cdots + \phi_k.$$

Hence (9) can be rewritten as

$$i_{k+1}(i_{k+1} + 3h + \phi_1 + \cdots + \phi_{k-1} + 3\phi_k) = 0.$$

In view of (5) and (7), this is equivalent to, for $k \geqslant 1$,

$$i_{k+1}^2 = ((2k+1)h - (k+2)\check{h} - (k+1)i_2 - ki_3 - \cdots - 3i_k)i_{k+1}. \qquad \square$$

In [6] we studied the action of the projective general linear group $PGL(2)$ on the Semple bundles induced from the $PGL(2)$-action on $\mathbf{P}^2$. Since this action preserves incidence in $\mathbf{P}^2$, there is a special orbit Z_k on each $F(k)$, $k \geqslant 1$, that is isomorphic to $F(1)$ and is represented by the germ of a line. (The intuition is that Z_k measures "zero data" of orders 2 through k.) Since Z_k is represented by the germ of a smooth curve, it must be disjoint from the divisors at infinity I_j for all $j \leqslant k$. Thus if $z_k := [Z_k]$, we have

$$i_j \cdot z_k = 0, \quad j \leqslant k.$$

Also each Z_k is a section of the $\mathbf{P}^1$-bundle obtained by restricting the Semple bundle to Z_{k-1}. Hence the intersection of Z_k with the fiber of $F(k)$ over a point of $F(1)$ has degree 1. Thus, for $k \geqslant 1$,

$$\int_{F(k)} h^2\check{h}z_k = \int_{F(k)} h\check{h}^2 z_k = 1.$$

In a similar manner, it follows that

$$\int_{F(k)} h^2\check{h}z_{j-1}i_j \cdots i_k = \int_{F(k)} h\check{h}^2 z_{j-1}i_j \cdots i_k = 1, \quad 2 \leqslant j \leqslant k.$$

(Note that Z_1 is all of $F(1)$, so that $z_1 = 1$; in this case the claims above are vacuous or obvious.)

In view of these remarks and the calculation of the Chow ring of $F(n)$ above, it is not difficult to deduce the following matrix for the intersection pairing of $A^1(F(n))$ with $A^{n+1}(F(n))$, in which $f_0 = f_1 = 1$ and f_j denotes the jth Fibonacci number:

	$\check{h}^2 z_n$	$h^2 i_2 i_3 \cdots i_n$	$h^2 \check{h} i_3 \cdots i_n$	$h^2 \check{h} z_2 i_4 \cdots i_n$	$h^2 \check{h} z_3 i_5 \cdots i_n$	$\cdots$	$h^2 \check{h} z_{n-1}$
h	1	0	0	0	0	$\cdots$	0
$\check{h}$	0	1	0	0	0	$\cdots$	0
i_2	0	-3	1	0	0	$\cdots$	0
i_3	0	5	-3	1	0	$\cdots$	0
i_4	0	-8	5	-3	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
i_n	0	$(-1)^{n+1} f_{n+1}$	$(-1)^n f_n$	$(-1)^{n-1} f_{n-1}$	$(-1)^{n-2} f_{n-2}$	$\cdots$	1.

3. The proto-contact formula

We associate to each reduced plane curve C a sequence $d, \check{d}, \kappa_2, \kappa_3, \ldots$ of (*higher-order*) *characteristic numbers*, beginning with the degree and class. Each of the other numbers is defined by

$$\kappa_j := \int i_j \cap [C(j)],$$

i.e., as the intersection number of a lift of the curve with a divisor at infinity; this intersection is defined on the jth Semple bundle variety, or any Semple bundle variety above it in the tower. Since the lift of a curve cannot hit a divisor at infinity over a nonsingular point, these characteristic numbers count certain sorts of singularities. A point of C over which $C(j)$ meets I_j will be called a *jth-order cusp*, and the corresponding characteristic number κ_j will be called the *number of jth-order cusps* on C. For example, the singularity at the origin of $y^2 = x^{2j-1}$ is a jth-order cusp and contributes 1 to κ_j. Note that κ_j may count with multiplicities; for example, a curve may have a singularity at which distinct branches each contribute to κ_j. One can easily show that repeated lifting will desingularize a specified curve, hence that only finitely many characteristic numbers are non-zero.

The *nth contact module* of a reduced plane curve C is a certain element of the polynomial algebra over the integers in the following indeterminates:

$$\Lambda_j, \quad 0 \leqslant j$$

$$\Pi_j, \quad 1 \leqslant j$$

$$\Gamma_j^k, \quad 2 \leqslant k \leqslant j. \tag{10}$$

Specifically, it is

$$\begin{aligned} m_n(C) := {} & d\Lambda_n + \check{d}\Pi_n \\ & + (3\check{d} + \kappa_2)\Gamma_n^2 \\ & + (4\check{d} + 3\kappa_2 + \kappa_3)\Gamma_n^3 \\ & + (5\check{d} + 4\kappa_2 + 3\kappa_3 + \kappa_4)\Gamma_n^4 \\ & + \cdots \\ & + ((n+1)\check{d} + n\kappa_2 + (n-1)\kappa_3 + \cdots + 3\kappa_{n-1} + \kappa_n)\Gamma_n^n. \end{aligned}$$

In particular the 0th and 1st contact modules are

$$m_0(C) := d\Lambda_0, \quad \text{and} \quad m_1(C) := d\Lambda_1 + \check{d}\Pi_1.$$

If we assign weight n to each of Λ_n, Π_n, and Γ_n^k, then the nth module is homogeneous of weight n.

Now suppose that $C_1, C_2, \ldots, C_p$ are reduced plane curves. Suppose that $\mathscr{X}$ is a family of plane curves over S, a parameter space of dimension s; suppose that the general member of the family is reduced. Suppose that $n_1, n_2, \ldots, n_p$ are specified positive integers, with sum equal to s. Let $F(\mathbf{n})$ be the product of Semple bundle varieties $F(n_1) \times F(n_2) \times \cdots \times F(n_p)$, and let $\pi_1, \pi_2, \ldots, \pi_p$ be the various projections to the factors. The fiber product

$$\mathscr{X}_S(\mathbf{n}) := \mathscr{X}(n_1) \times_S \mathscr{X}(n_2) \times_S \cdots \times_S \mathscr{X}(n_p)$$

is a subvariety of $F(\mathbf{n}) \times S$; its fiber over a point s of S is the product of p lifts of the curve X_s in the family. Let σ denote the projection of $F(\mathbf{n}) \times S$ onto its first factor.

$$\begin{array}{c} \hspace{30em} C_j(n_j) \\ \hspace{30em} \downarrow \\ \mathscr{X}(n_1) \times_S \mathscr{X}(n_2) \times_S \cdots \times_S \mathscr{X}(n_p) = \mathscr{X}_S(\mathbf{n}) \longrightarrow F(\mathbf{n}) \times S \xrightarrow{\sigma} F(\mathbf{n}) \xrightarrow{\pi_j} F(n_j) \end{array}$$

We define the *proto-contact number* of *type* $(n_1, n_2, \ldots, n_p)$ by

$$I := \int_{F(\mathbf{n})} \pi_1^*[C_1(n_1)] \cdot \pi_2^*[C_2(n_2)] \cdot \ldots \cdot \pi_p^*[C_p(n_p)] \cap \sigma_*[\mathscr{X}_S(\mathbf{n})].$$

THEOREM 2. *The proto-contact number is obtained by multiplying the contact modules $m_{n_1}(C_1)$, $m_{n_2}(C_2), \ldots, m_{n_p}(C_p)$, evaluating each monomial in the resulting product, and performing the indicated arithmetic. Each monomial in the product is of weight s and of the form $y_1 y_2 \cdots y_p$, where each y_j is either Λ_{n_j} or Π_{n_j} or* *some* $\Gamma^k_{n_j}$; *to evaluate this monomial means to replace it by*

$$\int_{F(\mathbf{n})} \pi_1^*(\bar{y}_1)\cdot\pi_2^*(\bar{y}_2)\cdot\ldots\cdot\pi_p^*(\bar{y}_p)\cap\sigma_*[\mathscr{X}_S(\mathbf{n})], \tag{12}$$

where

$$\bar{y}_j := \begin{cases} h & \text{if } y_j = \Lambda_0, \\ \check{h}^2 z_{n_j} & \text{if } y_j = \Lambda_{n_j}, \quad n_j > 0, \\ h^2 i_2 i_3 \cdots i_{n_j} & \text{if } y_j = \Pi_{n_j}, \\ h^2 \check{h} z_{k-1} i_{k+1} i_{k+2} \cdots i_{n_j} & \text{if } y_j = \Gamma^k_{n_j}. \end{cases}$$

Proof. Our discussion in Section 2 shows that the set

$$\{\check{h}^2 z_n, h^2 i_2 i_3 \cdots i_n, h^2\check{h} i_3 i_4 \cdots i_n, h^2\check{h} z_2 i_4 i_5 \cdots i_n, h^2\check{h} z_3 i_5 i_6 \cdots i_n, \ldots, h^2\check{h} z_{n-1}\}$$

forms a basis for $A^{n+1}(F(n))$, and that the dual basis for $A^1(F(n))$ is

$$\{h, \check{h}, 3\check{h} + i_2, 4\check{h} + 3i_2 + i_3, 5\check{h} + 4i_2 + 3i_3 \\ + i_4, \ldots, (n+1)\check{h} + ni_2 + (n-1)i_3 + \cdots + 3i_{n-1} + i_n\}.$$

(For $n = 0$ the singleton $\{h\}$ is a self-dual basis.) The characteristic numbers of a reduced plane curve C are determined by

$$d = \int_{\mathbf{P}^2} h \cap [C] = \int_{F(n)} h \cap [C(n)],$$

$$\check{d} = \int_{F(1)} \check{h} \cap [C(1)] = \int_{F(n)} \check{h} \cap [C(n)],$$

$$\kappa_j = \int_{F(j)} i_j \cap [C(j)] = \int_{F(n)} i_j \cap [C(n)] \quad (j \leqslant n).$$

Hence the rational equivalence class of the nth lift of C is given by

$$\begin{aligned} [C(n)] = {} & d\check{h}^2 z_n + \check{d} h^2 i_2 i_3 \cdots i_n \\ & + (3\check{d} + \kappa_2) h^2 \check{h} i_3 i_4 \cdots i_n \\ & + (4\check{d} + 3\kappa_2 + \kappa_3) h^2 \check{h} z_2 i_4 i_5 \cdots i_n \\ & + (5\check{d} + 4\kappa_2 + 3\kappa_3 + \kappa_4) h^2 \check{h} z_3 i_5 i_6 \cdots i_n \\ & + \cdots \\ & + ((n+1)\check{d} + n\kappa_2 + (n-1)\kappa_3 + \cdots + 3\kappa_{n-1} + \kappa_n) h^2 \check{h} z_{n-1}. \end{aligned}$$

(For $n = 0$, the class of $C(0) = C$ is of course dh.) The theorem follows immediately from this formula. □

4. The simultaneous contact formula

Suppose that C and X are reduced plane curves. A *contact* (or *honest contact*) of *order* o between them is a point $x \in C(o-1) \cap X(o-1)$ whose image in $\mathbf{P}^2$ is a nonsingular point on each curve. Note that, for nonsingular curve germs, the following statements are equivalent:

- There is a contact of order o between them.
- In appropriate local coordinates, the Taylor expansions agree up to order $o-1$.
- The intersection number is at least o.

We will call a point $x \in C(o-1) \cap X(o-1)$ whose image in $\mathbf{P}^2$ is a singular point on C or X a *false contact.*

Next suppose that $C_1, C_2, \dots, C_p$ are reduced plane curves. A *simultaneous contact* of *order* $(o_1, o_2, \dots, o_p)$ between X and these p curves is a p-tuple $(x_1, x_2, \dots, x_p)$ in which the point $x_1 \in F(o_1 - 1)$ is a contact of order o_1 between X and C_1, the point $x_2 \in F(o_2 - 1)$ is a contact of order o_2 between X and C_2, etc.

We say that a plane curve C has a *profound cusp* if, for some j, the jth lift $C(j)$ meets the intersection of I_j and the pullback of another divisor at infinity; i.e., C has a profound cusp at P if some branch of C has simultaneously a jth-order cusp and a cusp of lower order. For example, $y^3 = x^5$ has a profound cusp; its lifts meet both I_2 and I_3. We say that C has a *flat cusp* if the jth lift of C meets the locus of tangency to the divisor at infinity I_{j-1}. If $C(j-1)$ happens to be nonsingular over P, then C has a flat cusp if and only if $C(j-1)$ is tangent to I_{j-1}. For example, the second lift of $y^3 = x^4$ is tangent to I_2; hence this curve has a flat cusp at the origin.

A family of plane curves of degree d over a parameter space S determines, and is determined by, a morphism from S to the projective space $\mathbf{P}^{N(d)}$ parametrizing such curves, where

$$N(d) = \binom{d+2}{2} - 1.$$

We call such a family *generic* if it is obtained from one particular specified family by composing the morphism from S to $\mathbf{P}^{N(d)}$ with a generic motion of the projective space. In other words, "proposition $\mathscr{P}$ is valid for a generic family" means that if $\mathscr{X}$ is a family of curves over S determined by $\sigma: S \to \mathbf{P}^{N(d)}$, then, for all γ in some Zariski open dense subset of the projective linear group $PGL(N(d))$, proposition $\mathscr{P}$ is valid for the family determined by $\gamma \circ \sigma$.

THEOREM 3. *Suppose that $\mathscr{X}$ is a generic family of degree d plane curves over S, a parameter space of dimension $s = o_1 + o_2 + \cdots + o_p - p$. Suppose that each $o_i > 1$, and that $d + 1 \geqslant \Sigma_{i=1}^{p} o_i$. Suppose that $C_1, C_2, \ldots, C_p$ are reduced plane curves, none of which has a profound cusp or a flat cusp. Suppose that these curves have only pairwise transverse intersections. Then the number of simultaneous contacts of order $(o_1, o_2, \ldots, o_p)$ between some reduced member of $\mathscr{X}$ and $C_1, C_2, \ldots, C_p$ is the proto-contact number of type $(o_1 - 1, o_2 - 1, \ldots, o_p - 1)$.*

We have assumed that each $o_i > 1$ only to avoid a clumsy exposition. For remarks concerning the omitted cases, see Section 6(c).

To prove Theorem 3, we begin in Lemmas A and B by analyzing a lift of the universal family $\mathscr{C}$ of degree d plane curves over $\mathbf{P}^{N(d)}$; a point of $\mathscr{C}$ is an ordered pair (P, C) consisting of a degree d plane curve C and a point $P \in C$. Over the affine chart with coordinates x and y, the hypersurface $\mathscr{C}$ is defined in $\mathbf{A}^2 \times \mathbf{P}^{N(d)}$ by

$$f(x, y) = \sum_{u+v \leqslant d} a_{uv} x^u y^v = 0. \tag{13}$$

Since we wish to study simultaneous contacts, in Lemmas C and D we analyze the fiber product of several lifts of the universal family. Finally, in order to ultimately rule out the possibility that our proto-contact formula counts false contacts, in Lemmas E and F we analyze the higher-order data carried by a singular point of a curve. Our analysis uses, *faute de mieux*, explicit and detailed calculations.

LEMMA A. *Suppose that $d \geqslant n$.*

- *Except possibly above intersections of two or more divisors at infinity and above the loci of tangency to divisors at infinity, the nth lift $\mathscr{C}(n)$ is smooth over $F(n)$.*
- *Over the primary chart with coordinates $x, y, y', y'', y^{(3)}, \ldots, y^{(n)}$, the nth lift $\mathscr{C}(n)$ is defined in $\mathbf{A}^{n+2} \times \mathbf{P}^{N(d)}$ by an ideal generated by the function f of* (13) *and the functions $f', f'', f^{(3)}, \ldots, f^{(n)}$ obtained by repeated implicit differentiation with respect to x. Over each point of the primary chart the matrix of this system of linear equations has rank $n + 1$.*
- *Over a point on the exceptional divisor of the secondary chart with coordinates $x, y, y', y'', \ldots, y^{(j-1)}, x', x'', \ldots, x^{(n-j+1)}$* (*i.e., a point at which x'*

vanishes) the nth lift $\mathscr{C}(n)$ is defined by an ideal generated by the function f of (13) *and the sequence of functions* (2) *obtained by the procedure explained in Section* 1. *Over this point the matrix of this system of linear equations has rank $n+1$.*

Proof. Recall that, taken together, the primary charts cover all of $F(n)$ except divisors at infinity, and that the primary and secondary charts cover all of $F(n)$ except intersections of two or more divisors at infinity. And note that the codimension of $\mathscr{C}(n)$ in $\mathbf{A}^{n+2} \times \mathbf{P}^{N(d)}$ is $n+1$. Hence the first claim follows from the other two.

Consider the primary chart. Let M denote the $(n+1) \times (N(d)+1)$ matrix of partial derivatives of the functions $f, f', f'', f^{(3)}, \ldots, f^{(n)}$ with respect to each of the a_{uv}. (Equivalently, since all these functions are linear in the a_{uv}'s, M is the matrix of the system of linear equations.) If the rank of M at a point of $\mathscr{C}(n)$ is $n+1$, then at this point $f, f', f'', f^{(3)}, \ldots, f^{(n)}$ generate the defining ideal, and the projection to $F(n)$ is smooth. Each column of M begins with a monomial in x and y, and the other entries are obtained by repeated implicit differentiation:

$$\begin{bmatrix} \dfrac{\partial f}{\partial a_{uv}} \\ \dfrac{\partial f'}{\partial a_{uv}} \\ \vdots \\ \dfrac{\partial f^{(n)}}{\partial a_{uv}} \end{bmatrix} = \begin{bmatrix} x^u y^v \\ \dfrac{d(x^u y^v)}{dx} \\ \vdots \\ \dfrac{d^n(x^u y^v)}{dx^n} \end{bmatrix}.$$

To see that M has rank $n+1$ at every point, consider the square submatrix consisting of the partial derivatives of $f, f', f'', \ldots, f^{(n)}$ with respect to $a_{00}, a_{10}, a_{20}, \ldots, a_{n0}$:

$$\begin{bmatrix} 1 & x & x^2 & \cdots & x^n \\ 0 & 1 & 2x & \cdots & nx^{n-1} \\ 0 & 0 & 2 & \cdots & n(n-1)x^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n! \end{bmatrix}. \tag{14}$$

Now consider the secondary chart. Let M denote the $(n+1) \times (N(d)+1)$ matrix of partial derivatives of the functions in (2) with respect to each of the a_{uv}. (Once again we may interpret M as the matrix of a system of linear equations.) Each column of M begins with a monomial in x and y, and the other entries are obtained by repeated implicit differentiation:

$$\begin{bmatrix} x^u y^v \\ P(x^u y^v) \\ \vdots \\ P^{j-1}(x^u y^v) \\ QP^{j-1}(x^u y^v) \\ \vdots \\ Q^{n-j+1}P^{j-1}(x^u y^v) \end{bmatrix}. \tag{15}$$

If the rank of M at a point of $\mathscr{C}(n)$ is $n+1$, then at this point the functions in (2) generate the defining ideal, and the projection to $F(n)$ is smooth.

To complete our analysis of this matrix, we need the following formulas concerning the polynomial ring in the variables x, y, y', $y'',\ldots,y^{(j-1)}$, x', $x'',\ldots,x^{(n-j+1)}$.

LEMMA B. (a) *For each integer* $k \geqslant j$,

$$Q^i P^{j-1}(x^k) \equiv \begin{cases} 0 & \text{if } i < 2(k-j+1) \\ \text{a positive multiple of } (x'')^{k-j+1} \\ \quad \text{if } i = 2(k-j+1) \end{cases}$$

modulo the ideal generated by x *and* x'.

(b) *For each pair of positive integers* i *and* m,

$$[Q^i(x^m)]_{x=0} = Q[Q^{i-1}(x^m)]_{x=0} + mx'[Q^{i-1}(x^{m-1})]_{x=0},$$

where $[\ ]_{x=0}$ *indicates evaluation.*

(c) *For each nonnegative integer* $h \leqslant j-1$ *and each nonnegative integer* i,

$$Q^i(y^{(h)}) = \sum_{b=h}^{j-1} \frac{1}{(b-h)!} y^{(b)}[Q^i(x^{b-h})]_{x=0} + \text{a polynomial in } x, x', \ldots, x^{(n-j+1)}.$$

(d) *For each nonnegative integer* $h \leqslant j-1$,

$$Q^i P^h(y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^i P^h(x^b) \equiv \begin{cases} 0 & \text{if } i < 2(j-1-h)+1 \\ \text{a positive multiple of } (x'')^{j-1-h} \\ \quad \text{if } i = 2(j-1-h)+1 \end{cases}$$

modulo the ideal generated by x and x'.

(e) *For each polynomial ϕ in x and y, for each $h \geqslant 1$, and for each $i \geqslant 2$,*

$$Q^i P^h(x \cdot \phi) \equiv hQ^i P^{h-1}(\phi) + \sum_{a=0}^{i-2} \binom{i}{a} Q^a P^h(\phi) \cdot x^{(i-a)}$$

modulo the ideal generated by x and x'.

(f) *For each nonnegative integer k and each nonnegative integer $h \leqslant j-1$,*

$$Q^i P^h(x^k y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^i P^h(x^{k+b})$$

$$\equiv \begin{cases} 0 & \textit{if } i < 2(k+j-1-h)+1 \\ \textit{a positive multiple of } (x'')^{k+j-1-h} & \\ \quad \textit{if } i = 2(k+j-1-h)+1 & \end{cases}$$

modulo the ideal generated by x and x'.

(g) *For each nonnegative integer k,*

$$Q^i P^{j-1}(x^k y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^i P^{j-1}(x^{k+b})$$

$$\equiv \begin{cases} 0 & \textit{if } i < 2k+1 \\ \textit{a positive multiple of } (x'')^k & \\ \quad \textit{if } i = 2k+1 & \end{cases}$$

modulo the ideal generated by x and x'.

Proof. To prove statement (a), declare the weight of $x^{(t)}$ to be $2-t$. Then Q decreases the weight of a polynomial in $x, x', \ldots, x^{(n-j+1)}$ by 1, so that

$$Q^i P^{j-1}(x^k) = \frac{k!}{(k-j+1)!} Q^i(x^{k-j+1})$$

is a polynomial of degree $k-j+1$ and weight $2(k-j+1)-i$. If $i < 2(k-j+1)$ then the weight is positive, so each term involves either x or x'. If $i = 2(k-j+1)$ then the weight is zero, so the only term not involving x or x' is a multiple of $(x'')^{k-j+1}$. Clearly the coefficient of this term is positive.

Observe that

$$Q^i(x^m) = \Sigma\, x^{(e_1)} x^{(e_2)} \cdots x^{(e_m)},$$

where the sum is over all functions s from $\{1, \ldots, i\}$ to $\{1, \ldots, m\}$, and e_k is the number of elements of $\{1,\ldots,i\}$ for which the value of s is k. (The function can be regarded as an instruction to differentiate $x\cdot x\cdot x\cdots x$ first at factor number $s(1)$, then at factor number $s(2)$, etc., thus obtaining one of the m^i terms created by i applications of the product rule.) The left side of formula (b) is the same sum, taken over the set of surjections. The first term on the right side of (b) is again this sum, now taken over the set of those s for which the restriction to $\{1,\ldots,i-1\}$ is surjective. The second term is the same sum, but taken over the set of those surjective s for which the restriction to $\{1,\ldots,i-1\}$ is not surjective. From these descriptions the equality is clear. (This is essentially a proof of the basic recurrence formula for Stirling numbers of the second kind; cf. formula (23), p. 33 of [28].)

We prove (c) by induction on i. When $i=0$ each term of the sum except the first is divisible by x; in this case the required polynomial in x, x', $x'',\ldots,x^{(n-j+1)}$ is zero. For the inductive step, apply Q to both sides of the equation to obtain

$$\begin{aligned} Q^{i+1}(y^{(h)}) &= \sum_{b=h}^{j-1} \frac{1}{(b-h)!} Q(y^{(b)})[Q^i(x^{b-h})]_{x=0} \\ &\quad + \sum_{b=h}^{j-1} \frac{1}{(b-h)!} y^{(b)} Q[Q^i(x^{b-h})]_{x=0} \\ &\quad + \text{a polynomial in } x, x',\ldots,x^{(n-j+1)}. \end{aligned}$$

(Note that $Q(x^{(n-j+1)})=0$ by definition of the differential operator Q in (3).) In the first sum on the right, replace b by $c-1$; in all terms except the last replace $Q(y^{(b)})$ by its chain rule equivalent $y^{(c)}x'$; and absorb the last term into the polynomial in $x, x',\ldots,x^{(n-j+1)}$. In each term of the second sum, use formula (b). With these manipulations, we find that

$$\begin{aligned} Q^{i+1}(y^{(h)}) &= \sum_{c=h+1}^{j-1} \frac{1}{(c-h-1)!} y^{(c)}x'[Q^i(x^{c-h-1})]_{x=0} \\ &\quad + \sum_{b=h}^{j-1} \frac{1}{(b-h)!} y^{(b)} \{[Q^{i+1}(x^{b-h})]_{x=0} - (b-h)x'[Q^i(x^{b-h-1})]_{x=0}\} \\ &\quad + \text{a polynomial in } x, x',\ldots,x^{(n-j+1)} \\ &= \sum_{b=h}^{j-1} \frac{1}{(b-h)!} y^{(b)}[Q^{i+1}(x^{b-h})]_{x=0} \\ &\quad + \text{a polynomial in } x, x',\ldots,x^{(n-j+1)}, \end{aligned}$$

as required.

To begin the proof of (d), observe that $P^h(x^b)$ vanishes if $b < h$ and equals $(b!/(b-h)!)x^{b-h}$ otherwise. Hence

$$Q^i P^h(y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^i P^h(x^b) = Q^i(y^{(h)}) - \sum_{b=h}^{j-1} \frac{1}{(b-h)!} y^{(b)} Q^i(x^{b-h}). \tag{16}$$

By statement (c), the right side of (16) is, modulo the principal ideal generated by x, a polynomial in $x', x'', \ldots, x^{(n-j+1)}$. Since each term of the summation involves some $y^{(b)}$, the polynomial in question is obtained by expanding $Q^i(y^{(h)})$ and ignoring all terms divisible by x or any $y^{(b)}$. Clearly all the coefficients of this polynomial are nonnegative.

Let us define a $\mathbf{Z} \oplus \mathbf{Z}$ grading by declaring the degree of $x^{(t)}$ to be $(2-t, 1)$ and the degree of $y^{(t)}$ to be $(2(j-t)-1, j-1-t)$. Then Q is a homomorphism of degree $(-1, 0)$. (Note in particular that $y^{(j-1)}$ has degree $(1, 0)$ and that $Q(y^{(j-1)}) = 1$.) Each term in the expansion of $Q^i(y^{(h)})$ has degree

$$(2(j-1-h)+1-i, j-1-h).$$

If $i < 2(j-1-h)+1$ then the first component is positive. Hence each monomial in the expansion of $Q^i(y^{(h)})$ involves either x or x' or some $y^{(b)}$. Hence the right side of (16) is contained in the ideal generated by x and x'. Similarly, if $i = 2(j-1-h)+1$ then the first component is zero. Hence each monomial in the expansion of $Q^i(y^{(h)})$ either involves one of the same variables or is a power of x''; the second component of the degree tells us that the relevant power is $(x'')^{j-1-h}$. Clearly the coefficient of this term is not zero. Hence the right side of (16) is, modulo the ideal generated by x and x', a positive multiple of $(x'')^{j-1-h}$.

The binomial formula for differential operators tells us that the left side of formula (e) equals

$$\sum_{a=0}^{i} \sum_{b=0}^{h} \binom{i}{a}\binom{h}{b} Q^a P^b(\phi) \cdot Q^{i-a} P^{h-b}(x).$$

Note that $P^{h-b}(x)$ vanishes unless b equals h or $h-1$; that $Q^{i-a}P(x)$ vanishes unless $a = i$; and that $Q^{i-a}(x) = x^{(i-a)}$. Hence

$$Q^i P^h(x \cdot \phi) = hQ^i P^{h-1}(\phi) + \sum_{a=0}^{i} \binom{i}{a} Q^a P^h(\phi) \cdot x^{(i-a)}.$$

Modulo the ideal generated by x and x' the last two terms of the sum vanish.

We prove (f) by induction on k. The base case is statement (d). For the inductive step, apply statement (e) to each term on the left side of (f). We find that, modulo the ideal generated by x and x',

$$\begin{aligned} &Q^iP^h(x^k y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^i P^h(x^{k+b}) \\ &\quad \equiv h\left[Q^iP^{h-1}(x^{k-1}y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^i P^{h-1}(x^{k+b-1})\right] \\ &\qquad + \sum_{a=0}^{i-2} \binom{i}{a} \left[Q^aP^h(x^{k-1}y) - \sum_{b=0}^{j-1} \frac{1}{b!} y^{(b)} Q^a P^h(x^{k+b-1})\right] \cdot x^{(i-a)}. \end{aligned}$$

Apply the inductive hypothesis to both bracketed expressions. If $i < 2(k+j-1-h)+1$ then they both vanish. If $i = 2(k+j-1-h)+1$ then the first term yields a positive multiple of $(x'')^{k+j-1-h}$, as does the last term ($a = i-2$) in the sum. Hence $Q^iP^h(x^k y)$ is likewise a positive multiple of $(x'')^{k+j-1-h}$.

Statement (g) is the special case $h = j-1$ of statement (f). □

We now return to the proof of Lemma A. Recall that the first row of the matrix M consists of all monomials of degree at most d in x and y, and that a typical column is shown in (15). We wish to show that this matrix has rank $n+1$ except possibly above the locus of tangency to the divisor at infinity, i.e., whenever $x' \neq 0$ or $x'' \neq 0$. There are compatible actions of the projective general linear group $PGL(2)$ on the universal family $\mathscr{C}$ and on $F(1)$, the incidence correspondence of points and lines in the plane. Hence we may assume that we are studying a point of $\mathscr{C}(n)$ over the point $x = y = y' = 0$ of $F(1)$. We may also assume that $x' = 0$, i.e., that the point lies over the divisor at infinity, since we have already dealt with other points when we examined the primary charts.

Consider the square submatrix of M obtained by extracting the columns headed by these $n+1$ monomials:

$$1, x, x^2, \ldots, x^{j-1},$$

$$y, x^j, xy, x^{j+1}, x^2y, x^{j+2}, x^3y, \ldots.$$

It is of the form

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix},$$

where the upper left $j \times j$ submatrix A is upper triangular, with diagonal entries 0!, 1!, 2!, 3!, 4!, etc. The lower left $(n - j + 1) \times j$ submatrix B is a zero matrix. The upper right $j \times (n - j + 1)$ submatrix C is irrelevant. We look at the lower right $(n - j + 1) \times (n - j + 1)$ matrix D modulo the ideal generated by x and x'. Statement (a) of Lemma B then says that columns 2, 4, 6, etc.—corresponding to the monomials x^j, x^{j+1}, x^{j+2}, etc.—are zero above the diagonal, and have positive multiples of powers of x'' on the diagonal. Statement (g) of the same lemma says that if one modifies columns 1, 3, 5, etc.—corresponding to the monomials y, xy, x^2y, etc.—by adding suitable linear combinations of previous columns, then one again obtains, modulo the ideal, columns which are zero above the diagonal, and have positive multiples of powers of x'' on the diagonal. Hence, modulo the ideal, the submatrix D is nonsingular whenever $x'' \neq 0$. Hence the full square matrix is likewise nonzero, and M is of maximal rank $n + 1$. $\square$

We now consider $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})$, the fiber product over $\mathbf{P}^{N(d)}$ of the lifts of $\mathscr{C}$ to $F(n_1), F(n_2), \ldots, F(n_p)$. We assume that each n_i is positive. Let $F(\mathbf{n})^\circ$ denote the open subvariety of $F(\mathbf{n}) = F(n_1) \times F(n_2) \times \cdots \times F(n_p)$ obtained by removing the following points:

- each point $(P_1, P_2, \ldots, P_p)$ for which some P_i lies on the intersection of two divisors at infinity;
- each point for which some P_i lies on a locus of tangency to a divisor at infinity;
- each point for which some P_i and some $P_j (i \neq j)$ lie over the same point of $\mathbf{P}^2$ (i.e., each point lying over a large diagonal of $\mathbf{P}^2 \times \mathbf{P}^2 \times \cdots \times \mathbf{P}^2$).

Let $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^\circ$ denote the inverse image of $F(\mathbf{n})^\circ$ in $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})$.

LEMMA C. *Suppose that $d \geqslant p - 1 + \Sigma_{i=1}^{p} n_i$. Then the morphism $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^\circ \to F(\mathbf{n})^\circ$ is smooth, with relative dimension $N(d) - (p + \Sigma_{i=1}^{p} n_i)$.*

Let $F(\mathbf{n})^+$ denote the open subvariety of $F(\mathbf{n})$ obtained by removing the following points:

- each point $(P_1, P_2, \ldots, P_p)$ for which some P_i lies on the intersection of two divisors at infinity;
- each point for which some P_i lies on a locus of tangency to a divisor at infinity;
- each point for which some P_i and some $P_j (i \neq j)$ lie over the same point of $F(1)$ (i.e., each point lying over a large diagonal of $F(1) \times F(1) \times \cdots \times F(1)$);
- each point for which some P_i, P_j, and $P_k (i, j, k$ distinct) all lie over the same point of $\mathbf{P}^2$;

- each point for which some P_i and some $P_j (i \neq j)$ lie over the same point of $\mathbf{P}^2$, and one of them lies on a divisor at infinity.

Note that $F(\mathbf{n})^+$ is larger than $F(\mathbf{n})^\circ$, since a point $(P_1, P_2, \ldots, P_p)$ of $F(\mathbf{n})^+$ may project to a point $(Q_1, Q_2, \ldots, Q_p)$ of $\mathbf{P}^2 \times \mathbf{P}^2 \times \cdots \times \mathbf{P}^2$ for which Q_1, Q_2, etc., are not all distinct. Let us stratify $F(\mathbf{n})^+$ by declaring a point $(P_1, P_2, \ldots, P_p)$ to be in $F(\mathbf{n})^q$ if it projects to a point $(Q_1, Q_2, \ldots, Q_p)$ of $\mathbf{P}^2 \times \mathbf{P}^2 \times \cdots \times \mathbf{P}^2$ for which the number of distinct points among Q_1, Q_2, etc., is exactly q. The possible values for q are $\lceil \frac{p}{2} \rceil, \lceil \frac{p}{2} \rceil + 1, \ldots, p$; the stratum $F(\mathbf{n})^p = F(\mathbf{n})^\circ$ is open and dense. Let $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q$ denote the inverse image of $F(\mathbf{n})^q$ in $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})$.

LEMMA D. *Suppose that* $d \geqslant p - 1 + \Sigma_{i=1}^p n_i$. *Then for each* q (from $\lceil \frac{p}{2} \rceil$ *to* p) *the morphism* $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q \to F(\mathbf{n})^q$ *is smooth, with relative dimension* $N(d) - (q + \Sigma_{i=1}^p n_i)$.

Proof of Lemmas C and D. Lemma C is a special case of Lemma D.

Over each point $(P_1, P_2, \ldots, P_p)$ of $F(\mathbf{n})^q$ the fiber of $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q$ is a subspace of $\mathbf{P}^{N(d)}$ of codimension at most $q + \Sigma_{i=1}^p n_i$, as can be seen by a naive count of the number of imposed conditions. Hence to prove Lemma D it suffices to show that this fiber is a linear subspace of $\mathbf{P}^{N(d)}$ of codimension at least $q + \Sigma_{i=1}^p n_i$. This fiber is the intersection of the fiber of $\mathscr{C}(n_1)$ over P_1, the fiber of $\mathscr{C}(n_2)$ over P_2, etc. Lemma A therefore provides an explicit set of generators for its defining ideal in $\mathbf{P}^{N(d)}$. These generators include, for each P_i not on a divisor at infinity, the functions

$$f_i(x_i, y_i),\ f_i'(x_i, y_i, y_i'), \ldots, f_i^{(n_i)}(x_i, y_i, y_i', \ldots, y_i^{(n_i)}) \tag{17}$$

(where x_i, y_i, etc., are coordinates on a primary chart of $F(n_i)$ containing P_i, and f_i generates the ideal of $\mathscr{C}$) together with, for each P_i on a single divisor at infinity I_{j_i}, the functions

$$\begin{aligned}
&f_i(x_i, y_i),\ Pf_i(x_i, y_i, y_i'), \ldots,\\
&P^{j_i-1}f_i(x_i, y_i, y_i', \ldots, y_i^{(j_i-1)}),\ QP^{j_i-1}f_i(x_i, y_i, y_i', \ldots, y_i^{(j_i-1)}, x_i'), \ldots,\\
&Q^{n_i-j_i+1}P^{j_i-1}f_i(x_i, y_i, y_i', \ldots, y_i^{(j_i-1)}, x_i', \ldots, x_i^{(n_i-j_i+1)})
\end{aligned}$$

(where x_i, y_i, etc., are coordinates on a secondary chart of $F(n_i)$ containing P_i). Each of these generators is a linear function in the a_{uv}'s; thus the fiber of $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})$ over $(P_1, P_2, \ldots, P_p)$ is defined by a system of $p + \Sigma n_i$ linear equations. Let M_n denote the matrix of this system.

Note that it suffices to prove the lemma in case $d = p - 1 + \Sigma_{i=1}^p n_i$, since increasing d will only enlarge $M_{\mathbf{n}}$ by adding more columns.

We will use induction on q; the case $q = 0$ is vacuous. The inductive step has two cases. Suppose first that the image of P_1 in $\mathbf{P}^2$ is distinct from the images of $P_2, P_3, \ldots, P_p$. Let $\mathbf{n} - n_1$ denote $(n_2, n_3, \ldots, n_p)$. On $\mathbf{P}^2$ choose an affine chart, with coordinates x and y, so that the chart contains the images of $P_2, P_3, \ldots, P_p$ and so that the image of P_1 is the point at infinity on the y-axis. Around each $P_i (i \neq 1)$ choose, as appropriate, a primary or secondary chart on $F(n_i)$ whose first two coordinates are $x_i = x$ and $y_i = y$. (Choose a secondary chart only if the point lies on a divisor at infinity.) Let $x_1 = x/y$ and $y_1 = 1/y$; then the image of P_1 is at the origin of the $\mathbf{A}^2$ with coordinates x_1 and y_1. Note that over this affine chart the defining equation for the universal family is

$$\sum_{u+v \leqslant d} a_{uv} x_1^u y_1^{d-(u+v)} = 0. \tag{18}$$

Around P_1 choose, as appropriate, a primary or secondary chart on $F(n_1)$ whose first two coordinates are x_1 and y_1.

We may assume that the matrix $M_\mathbf{n}$ has been arranged so that the initial columns correspond to those a_{uv}'s for which $u + v \leqslant d - 1 - n_1$. Then $M_\mathbf{n}$ takes the form

$$\begin{bmatrix} M_{\mathbf{n}-n_1} & C \\ B & D \end{bmatrix}.$$

By the inductive hypothesis the rank of the upper left submatrix is at least $q - 1 + \Sigma_{i=2}^{p} n_i$. The $(1 + n_1) \times (1 + N(d - 1 - n_1))$ submatrix B records the partial derivatives of the functions defining the fiber of $\mathscr{C}(n_1)$ over P_1 with respect to the a_{uv}'s just mentioned. In (18) the coefficient of each such a_{uv} is divisible by $y_1^{n_1+1}$. If we implicitly differentiate this monomial n_1 or fewer times, as we do when applying the differential operator P or Q, the result is still divisible by y_1. Hence all the entries of B vanish at P_1. Submatrix C is irrelevant. Matrix D contains the submatrix M_{n_1}; by Lemma A its rank is $1 + n_1$. Hence the rank of $M_\mathbf{n}$ is at least $q + \Sigma_{i=1}^{p} n_i$.

Next suppose that the image of P_1 in $\mathbf{P}^2$ coincides with the image of some $P_i (i \neq 1)$. Without loss of generality we may assume that P_1 and P_2 project to the same point of $\mathbf{P}^2$, but that this point is distinct from the images of $P_3, P_4, \ldots, P_p$. We may also assume that $n_1 \leqslant n_2$. Furthermore the images of P_1 and P_2 in $F(1)$ are distinct points, and neither P_1 nor P_2 is on a divisor at infinity. As in the previous case, we choose on $\mathbf{P}^2$ an affine chart, with coordinates x and y, so that the chart contains the images of $P_3, P_4, \ldots, P_p$ and so that the common image of P_1 and P_2 is the point at infinity on the y-axis.

Around each P_i $(i > 2)$ we choose, as appropriate, a primary or secondary chart on $F(n_i)$ whose first two coordinates are $x_i = x$ and $y_i = y$. Let $x_1 = x_2 = x/y$ and $y_1 = y_2 = 1/y$; then the common image of P_1 and P_2 is at the origin of the $\mathbf{A}^2$ with coordinates x_1 and y_1. Without loss of generality we may assume that the image of P_1 in $F(1)$ represents the tangent direction of the y_1-axis. Hence we may use the primary chart with coordinates y_1, x_1, x'_1, etc. on $F(n_1)$ and the primary chart with coordinates x_2, y_2, y'_2, etc. on $F(n_2)$.

We may assume that the matrix $M_{\mathbf{n}}$ has been arranged so that the initial columns correspond to those a_{uv}'s for which $u + v \leqslant d - 2 - (n_1 + n_2)$. Then $M_{\mathbf{n}}$ takes the form

$$\begin{bmatrix} M_{\mathbf{n}-n_1-n_2} & C \\ B & D \end{bmatrix}.$$

By the inductive hypothesis the rank of the upper left submatrix is at least $q - 1 + \Sigma_{i=3}^{p} n_i$. The $(2 + n_1 + n_2) \times (1 + N(d - 2 - n_1 - n_2))$ submatrix B records the partial derivatives of the functions defining the fiber of $\mathscr{C}(n_1)$ over P_1 and the fiber of $\mathscr{C}(n_2)$ over P_2, with respect to the a_{uv}'s just mentioned. In (18) the coefficient of each such a_{uv} is divisible by $y_1^{n_1+n_2+2}$. If we implicitly differentiate this monomial at most $n_2 = \max\{n_1, n_2\}$ times, the result is still divisible by y_1. Hence all the entries of B vanish at (P_1, P_2). Submatrix C is again irrelevant. Matrix D contains, after reordering the columns, the following submatrix:

$$\begin{bmatrix}
1 & y_1 & x_1 & y_1^2 & x_1^2 & \cdots & y_1^{n_1} & x_1^{n_1} & x_1^{n_1+1} & \cdots & x_1^{n_2} \\
0 & 1 & x'_1 & 2y_1 & 2x_1x'_1 & \cdots & n_1 y_1^{n_1-1} & \ddots & & & \vdots \\
0 & 0 & x''_1 & 2 & \ddots & & & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & & & & \\
0 & 0 & x_1^{(n_1)} & 0 & \cdots & & & & & & \\
1 & y_2 & x_2 & y_2^2 & x_2^2 & \cdots & y_2^{n_1} & x_2^{n_1} & x_2^{n_1+1} & \cdots & x_2^{n_2} \\
0 & y'_2 & 1 & 2y_2y'_2 & 2x_2 & \ddots & & & & & \vdots \\
0 & y''_2 & 0 & \ddots & & & & & & & \\
\vdots & \vdots & \vdots & & & & & & & & \\
0 & y_2^{(n_2)} & 0 & & & & & & & &
\end{bmatrix}.$$

We can then rearrange the rows:

$$\begin{bmatrix}
1 & y_1 & x_1 & y_1^2 & x_1^2 & \cdots & y_1^{n_1} & x_1^{n_1} & x_1^{n_1+1} & \cdots & x_1^{n_2} \\
1 & y_2 & x_2 & y_2^2 & x_2^2 & \cdots & y_2^{n_1} & x_2^{n_1} & x_2^{n_1+1} & \cdots & x_2^{n_2} \\
0 & 1 & x_1' & 2y_1 & 2x_1x_1' & \cdots & n_1y_1^{n_1-1} & \ddots & & & \vdots \\
0 & y_2' & 1 & 2y_2y_2' & 2x_2 & \ddots & & & & & \\
0 & 0 & x_1'' & 2 & \ddots & & & & & & \\
0 & y_2'' & 0 & \ddots & & & & & & & \\
0 & 0 & x_1^{(3)} & & & & & & & & \\
\vdots & \vdots & \vdots & & & & & & & & \\
0 & 0 & x_1^{(n_1)} & & & & & & & & \\
0 & y_2^{(n_1)} & 0 & & & & & & & & \\
0 & y_2^{(n_1+1)} & 0 & & & & & & & & \\
\vdots & \vdots & \vdots & & & & & & & & \\
0 & y_2^{(n_2)} & 0 & \cdots & & & & & & &
\end{bmatrix}.$$

Evaluating at $x_1 = y_1 = x_1' = x_2 = y_2 = 0$, we obtain

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & & & 2! & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & & & & 2! & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & & & & & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 & & & & & & n_1! & 0 & 0 & 0 & \cdots & 0 \\
 & & & & & & & n_1! & 0 & 0 & \cdots & 0 \\
 & & & & & & & & (n_1+1)! & 0 & \cdots & 0 \\
 & & & & & & & & & (n_1+2)! & \cdots & 0 \\
 & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & n_2!
\end{bmatrix}.$$

which clearly has rank $n_1 + n_2 + 1$. Hence at (P_1, P_2) the rank of D is at least $n_1 + n_2 + 1$, and the rank of M_n is at least $q + \Sigma_{i=1}^{p} n_i$. □

By definition $\mathscr{C}(n)$ is the union of $\lambda(\mathscr{C})$ (where λ is the lifting map $\mathscr{C} \dashrightarrow F(n)$) and a closed subvariety $\mathscr{S}(n)$, each point of which lies over a singular or nonreduced point of some member of $\mathscr{C}$. Since the codimension of $\mathscr{C}(n)$ in $\mathbf{A}^{n+2} \times \mathbf{P}^{N(d)}$ is $n + 1$, the codimension of $\mathscr{S}(n)$ is at least $n + 2$.

LEMMA E. *Suppose that $d \geqslant n$. Suppose that P is a point of $F(n)$ which is* *not on any divisor at infinity. Then the fiber of $\mathcal{S}(n)$ over P has codimension at least $n + 2$ in $\mathbf{P}^{N(d)}$.*

Proof. It suffices to check over the primary chart of $F(n)$ with coordinates $x, y, y', y'', y^{(3)}, \ldots, y^{(n)}$. Over this chart the defining ideal of $\mathcal{S}(n)$ contains the functions $f, f', f'', f^{(3)}, \ldots, f^{(n)}$ defining $\mathcal{C}(n)$ (obtained from the function f of (13) by repeated implicit differentiation) and the partial derivative $\partial f/\partial y$. Let M denote the $(n + 2) \times (N(d) + 1)$ matrix of partial derivatives of these functions with respect to each of the a_{uv}. (Once again we may interpret M as the matrix of a linear system.) The square submatrix consisting of the partials with respect to $a_{00}, a_{10}, a_{20}, \ldots, a_{n0}$, and a_{01} is nonsingular at each point:

$$\begin{bmatrix} 1 & x & x^2 & \cdots & x^n & y \\ 0 & 1 & 2x & \cdots & nx^{n-1} & y' \\ 0 & 0 & 2 & \cdots & n(n-1)x^{n-2} & y'' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n! & y^{(n)} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{19}$$

Hence at each point the functions $f, f', f'', f^{(3)}, \ldots, f^{(n)}$, and $\partial f/\partial y$ define a variety of codimension $n + 2$ which is smooth over the primary chart of $F(n)$. □

Let $\mathcal{S}(\mathbf{n})$ denote the fiber product

$$\mathcal{S}(n_1) \underset{\mathbf{P}^{N(d)}}{\times} \mathcal{C}(n_2) \underset{\mathbf{P}^{N(d)}}{\times} \cdots \underset{\mathbf{P}^{N(d)}}{\times} \mathcal{C}(n_p).$$

Let $F(\mathbf{n})^{\text{finite}}$ denote the open subvariety of $F(\mathbf{n})$ obtained by removing, from each factor, all divisors at infinity; let $\mathcal{S}(\mathbf{n})^{\text{finite}}$ denote its inverse image in $\mathcal{S}(\mathbf{n})$. Let

$$\mathcal{S}(\mathbf{n})^\circ = \mathcal{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^\circ \cap \mathcal{S}(\mathbf{n})^{\text{finite}}.$$

LEMMA F. *Suppose that $d \geqslant p - 1 + \Sigma_{i=1}^p n_i$. Then each fiber of the morphism $\mathcal{S}(\mathbf{n})^\circ \to F(\mathbf{n})^\circ \cap F(\mathbf{n})^{\text{finite}}$ has codimension at least $1 + p + \Sigma_{i=1}^p n_i$.*

Proof. The proof is similar to that of Lemma D. In this instance, we need only to work in primary charts. The defining ideal for the fiber of $\mathcal{S}(\mathbf{n})$ over a point $(P_1, P_2, \ldots, P_p)$ of $F(\mathbf{n})^{\text{finite}}$ includes the functions of (17) and the partial derivative $(\partial f_1/\partial y_1)(x_1, y_1)$. Each of these functions is linear in the a_{uv}'s; let M denote the matrix of this linear system. As in the proof of

Lemma D, we note that it suffices to prove the lemma in case $d = p - 1 + \Sigma_{i=1}^{p} n_i$.

If $(P_1, P_2, \ldots, P_p)$ is in $\mathscr{S}(\mathbf{n})^\circ$, then the images in $\mathbf{P}^2$ of these p points are all distinct. By Lemma D, the fiber of $\mathscr{C}_{\mathbf{P}^{N(d)}}(n_2, n_3, \ldots, n_p)$ over $(P_2, P_3, \ldots, P_p)$ has codimension $p - 1 + \Sigma_{i=2}^{p} n_i$. By Lemma E, the fiber of $\mathscr{S}(n_1)$ over P_1 has codimension at least $n_1 + 2$. Since the image of P_1 is distinct from the images of P_2 through P_p, we may argue as in the first case of the inductive step in the proof of Lemma D. By an appropriate choice of affine charts, in particular, by choosing a chart on $\mathbf{P}^2$ containing the images of $P_2, P_3, \ldots, P_p$ and so that the image of P_1 is the point at infinity on the y-axis, we see that M takes the form

$$\begin{bmatrix} M_{\mathbf{n}-n_1} & C \\ B & D \end{bmatrix},$$

where $M_{\mathbf{n}-n_1}$ has rank $p - 1 + \Sigma_{i=2}^{p} n_i$, B is a zero matrix, and D has rank at least $n_1 + 2$. Hence M has rank at least $1 + p + \Sigma_{i=1}^{p} n_i$. □

Proof of Theorem 3. Let

$$\mathbf{n} = (n_1, n_2, \ldots, n_p) = (o_1 - 1, o_2 - 1, \ldots, o_p - 1).$$

By Lemma D, each morphism $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q \to F(\mathbf{n})^q$ is smooth.

Let us assume for the moment that S is nonsingular. Let $\tau\colon S \to \mathbf{P}^{N(d)}$ be the morphism determined by the family $\mathscr{X}$. Then the morphism

$$\begin{aligned} S \times PGL(N(d)) &\to \mathbf{P}^{N(d)} \\ (s, \gamma) &\mapsto \gamma \cdot \tau(s) \end{aligned} \tag{20}$$

is smooth. (This assertion is justified in the course of the proof of Theorem 2 in [14].) Hence the projection

$$\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q \underset{\mathbf{P}^{N(d)}}{\times} (S \times PGL(N(d))) \to \mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q,$$

which is obtained from (20) by base extension, is smooth. Composing it with the smooth morphism $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q \to F(\mathbf{n})^q$, we obtain

$$\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q \underset{\mathbf{P}^{N(d)}}{\times} (S \times PGL(N(d))) \to F(\mathbf{n})^q,$$

which fits into the following diagram. (The vertical morphism on the left is the composite of two projections; the one on the right is inclusion.)

$$
\begin{tikzcd}
 & C_1(n_1) \times \cdots \times C_p(n_p) \cap F(\mathbf{n})^q \arrow[d] \\
\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q \underset{\mathbf{P}^{N(d)}}{\times} (S \times PGL(N(d))) \arrow[r] \arrow[d] & F(\mathbf{n})^q \\
PGL(N(d)) &
\end{tikzcd}
$$

The fiber of $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q \times_{\mathbf{P}^{N(d)}} (S \times PGL(N(d)))$ over a point $\gamma \in PGL(N(d))$ is the fiber product $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q \times_{\mathbf{P}^{N(d)}} S$, where S is regarded as a $\mathbf{P}^{N(d)}$-variety via the morphism $\gamma \circ \tau$, i.e., via τ followed by translation by γ. Kleiman's transversality lemma (Lemma 1 of [14]) tells us that for generic γ the map $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q \times_{\mathbf{P}^{N(d)}} S \to F(\mathbf{n})^q$ is transverse to $C_1(n_1) \times \cdots \times C_p(n_p) \cap F(\mathbf{n})^q$. Our assumption that the family $\mathscr{X}$ is generic means that we may assume γ is the identity.

We now count dimensions. By Lemma D,

$$\dim(\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q) = \dim F(\mathbf{n})^q + N(d) - \left(q + \sum_{i=1}^{p} n_i\right).$$

Since the morphism τ is generic,

$$\dim\left(\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q \underset{\mathbf{P}^{N(d)}}{\times} S\right) = \dim F(\mathbf{n})^q - \left(q + \sum_{i=1}^{p} n_i\right) + s = \dim F(\mathbf{n})^q - q.$$

To specify a point of $C_1(n_1) \times \cdots \times C_p(n_p) \cap F(\mathbf{n})^q$, one can first specify $p - q$ points lying on the intersection of two of the p curves $C_1, \ldots, C_p$, then specify $2q - p$ additional points, each of which lies on one of the curves; there may be further choices involved in specifying higher-order data, but these choices cannot contribute to the dimension count:

$$\dim(C_1(n_1) \times \cdots \times C_p(n_p) \cap F(\mathbf{n})^q) = 2q - p.$$

If $q < p$, the dimension count shows that the image of $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^q \times_{\mathbf{P}^{N(d)}} S$ is disjoint from $C_1(n_1) \times \cdots \times C_p(n_p)$. Hence the morphism $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^+ \times_{\mathbf{P}^{N(d)}} S \to F(\mathbf{n})^+$ is transverse to $C_1(n_1) \times \cdots \times C_p(n_p) \cap F(\mathbf{n})^+$, and all the intersections lie in the open dense stratum $F(\mathbf{n})^\circ$. Our assumptions on the p curves guarantee that $C_1(n_1) \times \cdots \times C_p(n_p)$ is contained in $F(\mathbf{n})^+$. Hence enlarging $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})^+$ to $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})$ creates no further intersections: the map from $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n}) \times_{\mathbf{P}^{N(d)}} S$ to $F(\mathbf{n})$ is likewise transverse to the p-fold product.

Note that $\mathscr{X}_S(\mathbf{n})$, the fiber product over S of the various lifts of $\mathscr{X}$, is a subvariety of $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n}) \times_{\mathbf{P}^{N(d)}} S$, and that it has the same dimension. Hence the morphism $\sigma: \mathscr{X}_S(\mathbf{n}) \to F(\mathbf{n})$ (the restriction of the projection of $F(\mathbf{n}) \times S$ onto its first factor) is also transverse to $C_1(n_1) \times C_2(n_2) \times \cdots \times C_p(n_p)$. Each intersection between $\sigma(\mathscr{X}_S(\mathbf{n}))$ and the p-fold product is a p-tuple $(x_1, x_2, \ldots, x_p)$ in which x_1 is a contact or a false contact between C_1 and some member of $\mathscr{X}$, and x_2 is a contact or a false contact between C_2 and the same member of $\mathscr{X}$, etc. The number of such intersections is

$$\int_{F(\mathbf{n})} [C_1(n_1) \times C_2(n_2) \times \cdots \times C_p(n_p)] \cdot \sigma_*[\mathscr{X}_S(\mathbf{n})],$$

which is equal to the proto-contact number as defined by (11) in Section 3.

We must show that each point of intersection is a p-tuple $(x_1, x_2, \ldots, x_p)$ of honest contacts rather than false ones. Consider $C_1(n_1)_{\text{sing}}$, the (zero-dimensional) subvariety of $C_1(n_1)$ lying over the singularities of C_1. The dimension of $C_1(n_1)_{\text{sing}} \times C_2(n_2) \times \cdots \times C_p(n_p)$ is $p - 1$. Hence by Kleiman's transversality lemma the image of $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n}) \times_{\mathbf{P}^{N(d)}} S$ in $F(\mathbf{n})$ is disjoint from this p-fold product. *A fortiori*, the image of $\mathscr{X}_S(\mathbf{n})$ is disjoint from this p-fold product. Hence x_1 does not lie over a singular point of C_1. Note this implies that x_1 does not lie over a divisor at infinity on $F(n_1)$. Similarly one sees that x_2 does not lie over a singular point of C_2, and hence that x_2 does not lie over a divisor at infinity on $F(n_2)$, etc. Lemma F, together with the appropriate dimension count, shows that the image of $\mathscr{S}(\mathbf{n})^\circ \times_{\mathbf{P}^{N(d)}} S$ is disjoint from $C_1(n_1) \times \cdots \times C_p(n_p)$. As we have already seen, enlarging $\mathscr{S}(\mathbf{n})^\circ$ to $\mathscr{S}(\mathbf{n})^{\text{finite}}$ creates no intersections. Hence for every point of intersection $(x_1, x_2, \ldots, x_p)$ between the image of $\mathscr{X}_S(\mathbf{n})$ and the p-fold product, x_1 lies over a nonsingular point of the relevant member of $\mathscr{X}$. Similarly one sees that x_2 lies over a nonsingular point of the relevant member of $\mathscr{X}$, etc.

If S is singular, we can apply our argument above to the singular locus S_{sing}. The morphism

$$S_{\text{sing}} \times PGL(N(d)) \to \mathbf{P}^{N(d)}$$

is now flat rather than smooth. (Again, this assertion is justified by Kleiman in the proof of his Theorem 2.) Kleiman's lemma now tells us, after the appropriate dimension count, that the image of $\mathscr{X}_{S_{\text{sing}}}(\mathbf{n})$ in $F(\mathbf{n})$ is disjoint from $C_1(n_1) \times \cdots \times C_p(n_p)$. Hence, for a generic family $\mathscr{X}$, the members over S_{sing} have no simultaneous contacts with $C_1, C_2, \ldots, C_p$. Thus we may assume, as we did, that S is nonsingular. $\square$

5. The higher-order characteristic numbers of a family of plane curves

At the beginning of Section 3 we defined the higher-order characteristic numbers of a plane curve. The degree and class are well known. Each of the other numbers counts, perhaps with multiplicities, the number of cusps of a specified order. These numbers are readily calculated, either implicitly or from a parametrization of the curve; in fact one needs only a local or even formal parametrization at each singular point. (See the algorithm presented in the proof of Proposition 3.9 of [5].)

For a family of curves, the numbers defined by (12) are likewise called *characteristic numbers*. For an s-parameter family $\mathscr{X}$, this formula associates such a number to each monomial of weight s in the indeterminates of (10). To denote this number we use the monomial itself, with lowercase Greek letters replacing their uppercase counterparts.

Note that Λ_0 has weight 0. If M is a monomial of weight s with associated characteristic number m, then it is easy to show that the characteristic number associated to $\Lambda_0 M$ is dm, where d is the degree of the general member of $\mathscr{X}$. In the remainder of this section we consider monomials not involving Λ_0.

If M is a monomial in Λ_1 and Π_1, then we call its associated characteristic number *ordinary*. It is well known that, under mild hypotheses on $\mathscr{X}$, the characteristic number $(\lambda_1)^r(\pi_1)^{s-r}$ is the number of members tangent to r specified general lines and passing through $s-r$ specified general points. These characteristic numbers, especially those of plane cubics, have been the subject of numerous investigations, both classical and contemporary. (For example, see [1], [2], [3], [16], [17], [18], [19], [22], [25], [31].) If M also involves the indeterminate Γ_2^2, then it should perhaps still be called "ordinary", since its definition uses only the notion of ordinary contact. For example if $\mathscr{X}$ is a general two-parameter family then

$$\gamma_2^2 = \int_{F(1)} \overline{\Gamma_2^2} \cap \sigma_*[\mathscr{X}(1)] = \int_{F(1)} h^2\check{h} \cap \sigma_*[\mathscr{X}(1)]$$

is the number of members tangent to a specified line at a specified point. This characteristic number does not, however, appear in classical ordinary contact formulas. It does appear in Schubert's formula for triple contacts between a two-parameter family and a single specified curve. (See [6], [24], [26], and Section 6(a) of the present paper.)

If M involves other indeterminates, then the associated number m is called a *higher-order characteristic number*. For example, a two-parameter family has six characteristic numbers. The ordinary characteristic numbers are $(\lambda_1)^2$, $\lambda_1\pi_1$, $(\pi_1)^2$, and γ_2^2. The higher-order characteristic numbers are

$$\lambda_2 = \int_{F(2)} \bar{\Lambda}_2 \cap \sigma_*[\mathscr{X}(2)] = \int_{F(2)} \check{h}^2 z_2 \cap \sigma_*[\mathscr{X}(2)];$$

and

$$\pi_2 = \int_{F(2)} \bar{\Pi}_2 \cap \sigma_*[\mathscr{X}(2)] = \int_{F(2)} h^2 i_2 \cap \sigma_*[\mathscr{X}(2)].$$

Theorem 3 tells us that, for a generic family of curves of degree $\geqslant 2$, λ_2 is the number of triple contacts between a member of $\mathscr{X}$ and a specified line. (More generally, λ_s is, for a generic s-parameter family $\mathscr{X}$ of curves of degree at least s, the number of contacts of order $s + 1$ between a member of $\mathscr{X}$ and a specified line.) One can show that π_2 is the number of members of $\mathscr{X}$ having a cusp at a specified point (See [6], Theorem A1. Note that a "dual" characteristic number involving flexes rather than cusps is considered in [26] and [24].)

The definition

$$\pi_s = \int_{F(s)} h^2 i_2 i_3 \cdots i_s \cap \sigma_*[\mathscr{X}(s)]$$

suggests that this characteristic number measures, for a generic s-parameter family $\mathscr{X}$ of curves of sufficiently large degree, the number of member curves of $\mathscr{X}$ whose lift, at a specified point, meets the divisors at infinity $I_2, I_3, \ldots, I_s$. (One might call such a point a "super-profound cusp".) To justify this interpretation would involve an appeal to Kleiman's transversality lemma, to rule out contributions to the intersection number created by the way in which curvilinear data specializes at a singular or nonreduced member of the family. But our crucial Lemma A fails in precisely this "super-profound" situation, and we cannot use Kleiman's lemma. We suspect that the naive interpretation of π_s is incorrect, and that one cannot even guarantee that the intersection in question can be made proper.

Similar difficulties plague the interpretation of other characteristic numbers. The number of characteristic numbers also grows rather quickly with the number of parameters. For example, a five-parameter family possesses 70 characteristic numbers.

6. Variations, further remarks, and an example

(a) Contacts between a family and a single curve

When $p = 1$, the special case of the contact formula says that the number of contacts of order $o = n + 1$ between a specified curve C (with no profound cusp) and a generic n-parameter family $\mathscr{X}$ is

$$\int_{F(n)} [C(n)] \cap \sigma_*[\mathscr{X}(n)]$$
$$= d\lambda_n + \check{d}\pi_n + (3\check{d} + \kappa_2)\gamma_n^2 + \cdots + ((n+1)\check{d} + n\kappa_2 + \cdots + 3\kappa_{n-1} + \kappa_n)\gamma_n^n$$

where

$$\lambda_n = \int \check{h}^2 z_n \cap \sigma_*[\mathscr{X}(n)],$$

$$\pi_n = \int h^2 i_2 \cdots i_n \cap \sigma_*[\mathscr{X}(n)],$$

$$\gamma_n^k = \int h^2 \check{h} z_{k-1} i_{k+1} i_{k+2} \cdots i_n \cap \sigma_*[\mathscr{X}(n)], \quad k = 2, \ldots, n.$$

Note that this formula is essentially implicit in the definition of the contact module $m_n(C)$.

(b) Simultaneous contact with nonsingular curves

Suppose that the curves $C_1, \ldots, C_p$ are all nonsingular. Then for each curve the characteristic numbers $\kappa_1, \kappa_2, \ldots$ vanish and (by the Plücker formula) the class is $\check{d} = d(d-1)$. In this case the contact formula tells us that the number of simultaneous contacts of order $(o_1, \ldots, o_p) = (n_1 + 1, \ldots, n_p + 1)$ between $C_1, \ldots, C_p$ and some member of a $\Sigma\, n_j$-parameter family $\mathscr{X}$ is obtained from

$$\prod_{j=1}^{p} \left[d_j \Lambda_{n_j} + d_j(d_j - 1)\left(\Pi_{n_j} + \sum_{k=2}^{n_j} (k+1)\Gamma_{n_j}^k \right) \right] \tag{21}$$

by the expansion and evaluation of monomials described in Theorem 2.

This contact formula is valid under the assumptions of Theorem 3, that is, when $\mathscr{X}$ is a generic family of plane curves of sufficiently high degree. There is, however, a certain freedom in choosing hypotheses. We could assume, for example, that each individual curve C_j is generic (hence nonsingular) of degree $d_j \geqslant o_j - 1$, but make no assumption whatever about the family (except, of course, that the generic member is a reduced curve). Then, if the base field is either of characteristic zero or of characterstic at least $\max\{o_1, \ldots, o_p\}$, formula (21) counts (again, after expansion and evaluation of monomials) the number of simultaneous contacts. Moreover, if each $d_j \geqslant o_j$, then all contacts are of order exactly $(o_1, \ldots, o_p)$ (that is, none of the contacts with any of the C_j's is of higher order). The proof is essentially that of Proposition 2.5 of [5], which treats the case $p = 1$.

(c) Bézout's Theorem

As we remarked in Section 4, Theorem 3 is valid even if we drop the requirement that each specified order of contact o_i be at least 2. Suppose, for example, that C and $C_1, C_2, \ldots, C_p$ are reduced plane curves of degrees $d, d_1, d_2, \ldots, d_p$ respectively. Then our contact formula says that the number of simultaneous intersections between C and C_1, $C_2, \ldots, C_p$ is $d^p d_1 d_2 \cdots d_p$. This is just a silly formulation of Bézout's Theorem. Indeed, a simultaneous intersection is a p-tuple $(x_1, x_2, \ldots, x_p)$ in which the point x_1 lies in the intersection of C and C_1, the point x_2 lies in the intersection of C and C_2, etc. There are dd_1 possibilities for x_1, together with dd_2 possibilities for x_2, etc. The other cases omitted from our statement of Theorem 3 are of the same ilk.

(d) The formula of Fulton, Kleiman, and MacPherson

The overlap of our contact formula and that of Fulton-Kleiman-MacPherson [11] is the following formula:

The number of simultaneous ordinary contacts between the members of a p-parameter family $\mathscr{X}$ of plane curves and p specified plane curves C_1 (of degree d_1 and class $\check{d}_1$), ..., C_p (of degree d_p and class $\check{d}_p$) is obtained from the product of modules

$$\prod_{j=1}^{p} (d_j \Lambda + \check{d}_j \Pi)$$

by expansion and evaluation of monomials. To evaluate $\Lambda^r \Pi^{p-r}$ means to replace it by the (ordinary) characteristic number $\lambda^r \pi^{p-r}$, the number of members of $\mathscr{X}$ tangent to r specified general lines and passing through $p - r$ specified general points.

Fulton *et al.* assume that the individual curves $C_1, C_2, \ldots, C_p$ are in general position, whereas our hypotheses involve genericity assumptions about the family $\mathscr{X}$ of curves, as well as different, but mild, conditions on $C_1, \ldots, C_p$. (See (f) below for further comments along these lines.)

(e) The formula of de Jonquières

The classical formula of de Jonquières [13] gives the number of plane curves of degree d making contacts of orders $o_1, o_2, \ldots, o_p$ with a given curve C and which pass through k points, where $k = d(d+3)/2 + p - \Sigma_j o_j$. It is tempting to use our set-up to obtain this formula by calculating

$$I := \int_{F(\mathbf{n})} \pi_1^*(h^2) \cdot \ldots \cdot \pi_k^*(h^2) \cdot \pi_{k+1}^*[C(n_1)] \cdot \ldots \cdot \pi_{k+p}^*[C(n_p)] \cap \sigma_*[\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n})].$$

where now

$$F(\mathbf{n}) := \underbrace{\mathbf{P}^2 \times \cdots \times \mathbf{P}^2}_{k \text{ factors}} \times F(n_1) \times \cdots \times F(n_p),$$

$$\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n}) := \underbrace{\mathscr{C} \underset{\mathbf{P}^{N(d)}}{\times} \cdots \underset{\mathbf{P}^{N(d)}}{\times} \mathscr{C}}_{k \text{ factors}} \underset{\mathbf{P}^{N(d)}}{\times} \mathscr{C}(n_1) \underset{\mathbf{P}^{N(d)}}{\times} \cdots \underset{\mathbf{P}^{N(d)}}{\times} \mathscr{C}(n_p),$$

each $n_j = o_j - 1$, and $\mathscr{C}$ denotes the universal family of degree d plane curves. However, Theorem 3 itself will never apply to the universal family. We have $\dim \mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n}) = N(d) + p + k$ and $\dim F(\mathbf{n}) = 2k + p + \Sigma_j o_j$ so that the map $\mathscr{C}_{\mathbf{P}^{N(d)}}(\mathbf{n}) \to F(\mathbf{n})$ is smooth only if $N(d) - k > \Sigma_j o_j$ while we must have $N(d) - k = \Sigma_j o_j - p$ for the proto-contact formula to be valid.

Thus our formula represents an approach to problems of higher-order contact distinct from that provided by the formula of de Jonquières. Note that de Jonquières's formula can lose its enumerative significance when, for example, members of the family have nonreduced components of sufficiently high multiplicity, whereas the hypotheses of Theorem 3 guarantee validity of our formula for suitable families of plane curves of fixed degree. (See [11] p. 184ff for further discussion of the enumerative significance of de Jonquières's formula.) Moreover, our results are in keeping with the spirit of Hilbert's 15th problem which asks "to establish rigorously and with an exact determination of the limits of their validity those geometrical numbers..." [12]. Finally, Fulton, Kleiman, and MacPherson [11] note that "[i]t was observed long ago that de Jonquières's formula yields via symbolic multiplication a formula in the case of several given curves." The proto-contact formula of Theorem 2 involves just such a general multiplication.

Other contemporary treatments of de Jonquières's formula include [21] and [29]. In these cases, the problem is formulated in terms of systems of divisors of degree d cut out on the fixed curve C by a family $\mathscr{X}$ of curves. The formula so produced counts the number of such divisors on C that have coalesced in a prescribed manner. In particular, the support of such degenerate divisors could include points that are singular either on C or on the relevant member of $\mathscr{X}$. In our language, such divisors represent "false contacts" and are not counted by our formula. (See Example 3.8, pp. 503–504 of [5] for an illustration of this phenomenon.)

(f) An example: triple contact with two curves

In this case we have two specified curves C and D, and seek the number of simultaneous contacts of order (3, 3) with members of a 4-parameter family $\mathscr{X}$. If C and D intersect transversely and have no profound or flat cusps, and if $\mathscr{X}$ is a generic family of curves of degree $d \geqslant 5$, then Theorems 2 and 3 show that the desired number

$$I = \int_{F(2) \times F(2)} \pi_1^*[C(2)] \cdot \pi_2^*[D(2)] \cap \sigma_*[\mathscr{X}_S(2, 2)]$$

is obtained from the product of modules

$$(d_C \Lambda_2 + \check{d}_C \Pi_2 + (3\check{d}_C + \kappa_{2C})\Gamma_2^2)(d_D \Lambda_2 + \check{d}_D \Pi_2 + (3\check{d}_D + \kappa_{2D})\Gamma_2^2),$$

where d_C, $\check{d}_C$, κ_{2C} are the characteristic numbers of C, etc. Explicitly, the number of simultaneous contacts is

$$\begin{aligned} & d_C d_D(\lambda_2)^2 + (d_C \check{d}_D + d_D \check{d}_C)\lambda_2 \pi_2 \\ & \quad + (d_C(3\check{d}_D + \kappa_{2D}) + d_D(3\check{d}_C + \kappa_{2C}))\lambda_2 \gamma_2^2 + \check{d}_C \check{d}_D (\pi_2)^2 \\ & \quad + (\check{d}_C(3\check{d}_D + \kappa_{2D}) + (\check{d}_C(3\check{d}_C + \kappa_{2C}))\pi_2 \gamma_2^2 + (3\check{d}_C + \kappa_{2C})(3\check{d}_D + \kappa_{2D})(\gamma_2^2)^2. \end{aligned} \tag{22}$$

Another approach can be taken to establishing the validity of this contact formula. In [6] and [24], the $PGL(2)$-orbits of $F(2)$ are identified. There are three of them: the dense orbit $\mathcal{O}(-)$, each point of which is represented by the germ of a nonsingular curve without a flex, the special orbit $\mathcal{O}(0) = Z_2$, each point of which is represented by the germ of a line, and the divisor at infinity $\mathcal{O}(\infty) = I_2$, each point of which is represented by the germ of an ordinary cusp. Thus the action of $PGL(2) \times PGL(2)$ on $F(2) \times F(2)$ has nine orbits:

Orbit		Dimension
$\mathcal{O}(-) \times \mathcal{O}(-)$		8
$\mathcal{O}(-) \times \mathcal{O}(0)$,	$\mathcal{O}(0) \times \mathcal{O}(-)$	7
$\mathcal{O}(-) \times \mathcal{O}(\infty)$,	$\mathcal{O}(\infty) \times \mathcal{O}(-)$	7
$\mathcal{O}(0) \times \mathcal{O}(0)$,	$\mathcal{O}(\infty) \times \mathcal{O}(\infty)$	6
$\mathcal{O}(0) \times \mathcal{O}(\infty)$,	$\mathcal{O}(\infty) \times \mathcal{O}(0)$	6

If C, D, and $\mathcal{X}$ are suitably transverse, i.e., if $\mathcal{X}_S(2,2)$ is mapped transversely to $C(2) \times D(2)$ by σ and the intersection $(C(2) \times D(2)) \cap \sigma(\mathcal{X}_S(2,2))$ is contained in the dense orbit $\mathcal{O}(-) \times \mathcal{O}(-)$, then we can argue, as in the proof of Theorem A2 of [6], that the proto-contact number (22) correctly counts the number of simultaneous contacts. In particular, we have the following result.

THEOREM 4. *Suppose that the curves C and D contain no lines, and that the general member of $\mathcal{X}$ contains no line. If C, D, and $\mathcal{X}$ are in general position with respect to the action of $PGL(2) \times PGL(2)$ on $F(2) \times F(2)$, then* (22) *counts the number of simultaneous contacts of order* (3, 3) *between C and D and the members of $\mathcal{X}$.*

Proof. We will show that, for each nondense orbit $\mathcal{O}$ listed above, we have

$$\dim((C(2) \times D(2)) \cap \mathcal{O}) + \dim(\mathcal{X}_S(2,2) \cap (\mathcal{O} \times S)) < \dim \mathcal{O}. \tag{23}$$

Since C, D, and $\mathcal{X}$ are in general position with respect to the action, transversality theory [14] then tells us that the intersection

$$(C(2) \times D(2)) \cap \sigma(\mathcal{X}_S(2,2)) \cap \mathcal{O}$$

is empty. Therefore the intersection of $\sigma(\mathcal{X}_S(2,2))$ and $C(2) \times D(2)$ is transverse, with all intersections occurring in the dense orbit $\mathcal{O}(-) \times \mathcal{O}(-)$. By definition $\mathcal{X}_S(2,2) = \mathcal{X}(2) \times_S \mathcal{X}(2)$, where $\mathcal{X}(2)$ is the closure of the graph of a function defined on a dense subset of $\mathcal{X}$, and $C(2) \times D(2)$ is the closure of the graph of a function defined on a dense subset of $C \times D$. By a general position argument, all intersections between $\sigma(\mathcal{X}_S(2,2))$ and $C(2) \times D(2)$ are intersections between the graphs of these functions. Hence each intersection point corresponds to a pair of points (c, d), in which c is a nonsingular point of both C and some member X_s of the family, and d is a nonsingular point of both D and X_s. The second-order data of C and X_s at c are identical, as are the second-order data of D and X_s at d. Hence each intersection point is a simultaneous triple contact.

To see that (23) holds for the nondense orbits, note that the intersection of $C(2)$ or $D(2)$ with the divisor at infinity is finite. Thus

$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(-) \times \mathcal{O}(\infty))) \leqslant 1$$
$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(\infty) \times \mathcal{O}(-))) \leqslant 1$$
$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(\infty) \times \mathcal{O}(\infty))) \leqslant 0.$$

If C and D contain no lines, then $C(2) \cap \mathcal{O}$ and $D(2) \cap \mathcal{O}$ are each finite. Hence

$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(-) \times \mathcal{O}(0))) \leqslant 1$$
$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(0) \times \mathcal{O}(-))) \leqslant 1$$
$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(0) \times \mathcal{O}(0))) \leqslant 0$$
$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(0) \times \mathcal{O}(\infty))) \leqslant 0$$
$$\dim((C(2) \times D(2)) \cap (\mathcal{O}(\infty) \times (\mathcal{O}(0))) \leqslant 0.$$

The dimension of $\mathscr{X}_S(2,2)$ is 6, and, since the general member of $\mathscr{X}$ is reduced (hence generically smooth), we obtain

$$\dim(\mathscr{X}_S(2,2) \cap (\mathcal{O}(-) \times \mathcal{O}(\infty) \times S)) \leqslant 5$$
$$\dim(\mathscr{X}_S(2,2) \cap (\mathcal{O}(\infty) \times \mathcal{O}(-) \times S)) \leqslant 5$$
$$\dim(\mathscr{X}_S(2,2) \cap (\mathcal{O}(\infty) \times \mathcal{O}(\infty) \times S)) \leqslant 5$$
$$\dim(\mathscr{X}_S(2,2)) \cap (\mathcal{O}(0) \times \mathcal{O}(\infty) \times S)) \leqslant 5$$
$$\dim(\mathscr{X}_S(2,2)) \cap (\mathcal{O}(\infty) \times \mathcal{O}(0) \times S)) \leqslant 5.$$

If the general member of $\mathscr{X}$ contains no line, then likewise

$$\dim(\mathscr{X}_S(2,2)) \cap (\mathcal{O}(-) \times \mathcal{O}(0) \times S)) \leqslant 5$$
$$\dim(\mathscr{X}_S(2,2)) \cap (\mathcal{O}(0) \times \mathcal{O}(-) \times S)) \leqslant 5$$
$$\dim(\mathscr{X}_S(2,2)) \cap (\mathcal{O}(0) \times \mathcal{O}(0) \times S)) \leqslant 5.$$

Thus (23) is true in each of the eight cases.

Theorem 4 has a clear advantage over Theorem 3 in that the hypotheses are readily verified. Thus results like Theorem 4 appear to be highly desirable. To establish such a result, however, one needs an understanding of the $PGL(2)$-orbit structure of $F(n)$, which becomes increasingly complicated as n grows. For example, there are 8 orbits on $F(3)$ and 21 orbits on $F(4)$. (For a derivation of the first number, see Theorem 2 of [6]. The second number was first obtained by Oberlin College student Dan Frankowski by *Mathematica* calculations [30] and later confirmed by us.) Even worse, since $PGL(2)$ has dimension 8, there are infinitely many orbits on $F(n)$, none of them dense, when $n \geqslant 7$. (In fact, the authors, along with Oberlin College students Ian Robertson and Susan Sierra, have found that there are infinitely many orbits on $F(6)$.) Hence theorems such as Theorem 4 cannot exist for simultaneous contacts of arbitrary order. This is why Theorem 3 is stated as it is, and appears to be the best possible result of its type.

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