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Supertempered virtual characters

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1. Introduction

Let G be a connected reductive p-adic group, and let $\mathscr{V}(G)$ denote the set of tempered virtual characters of G, that is the set of finite linear combinations of characters of irreducible, tempered representations of G. Every $\Theta \in \mathscr{V}(G)$ satisfies the following weak estimate. Given any Cartan subgroup T of G there is a positive constant r so that

$$\sup_{t \in T'} |D_G(t)|^{1/2}\, |\Theta(t)|(1 + \sigma(t))^{-r} < \infty.$$

We say that Θ is supertempered if it satisfies the following strong estimate. For every Cartan subgroup T of G and every positive constant r we have

$$\sup_{t \in T'} |D_G(t)|^{1/2}\, |\Theta(t)|(1 + \sigma_*(t))^{r} < \infty.$$

(In the above, D_G is the usual discriminant factor [S, 4.7], σ measures polynomial growth on G, and σ_* measures polynomial growth on G/Z_G [S, 4.1]). If Θ is the character of an irreducible tempered representation π, then Θ is supertempered if and only if π is a discrete series representation. In [A], Arthur singled out a set of tempered virtual characters which he conjectured spanned the space of supertempered virtual characters. In this paper we will show that his conjecture is correct.

More precisely, let $P = MN$ be a parabolic subgroup of G and let σ be a discrete series representation of M. Let $i_{G,M}(\sigma)$ denote the equivalence class of the tempered representation of G parabolically induced from σ and let R be the corresponding R-group. It is a finite group with the property

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that the commuting algebra of $i_{G,M}(\sigma)$ is naturally isomorphic to $\mathbf{C}[R]^{\eta}$, the complex group algebra of R with multiplication twisted by a cocycle η. Let $\tilde{R}$ be a central extension

$$1 \to Z \to \tilde{R} \to R \to 1$$

of the R-group as in [A]. (When η is trivial, we can take $\tilde{R} = R$.) Then there is a character χ of Z so that the irreducible constituents of $i_{G,M}(\sigma)$ are parameterized by $\Pi(\tilde{R}, \chi)$, the set of equivalence classes of irreducible representations ρ of $\tilde{R}$ with Z-character χ. For $\rho \in \Pi(\tilde{R}, \chi)$, let π_ρ denote the corresponding irreducible constituent of $i_{G,M}(\sigma)$ and let Θ_ρ denote its character. The representation π_ρ is called elliptic if Θ_ρ is not identically zero on the elliptic set of G.

Let $\underline{a}$, $\underline{z}$ denote the real Lie algebras of the split components of M, G respectively. Then R acts on $\underline{a}$ and for each $r \in R$ we set

$$\underline{a}_r = \{H \in \underline{a} \colon rH = H\}.$$

Define

$$R_{\text{reg}} = \{r \in R \colon \underline{a}_r = \underline{z}\}$$

and let $\tilde{R}_{\text{reg}}$ denote the inverse image of R_{reg} in $\tilde{R}$. Arthur proves in [A, 2.1] that for $\rho \in \Pi(\tilde{R}, \chi)$, π_ρ is elliptic if and only if the character of ρ does not vanish on $\tilde{R}_{\text{reg}}$. (In particular, for $i_{G,M}(\sigma)$ to have any elliptic constituents it is necessary that $\tilde{R}_{\text{reg}} \neq \varnothing$.) For each $r \in \tilde{R}$, define

$$\Theta(M, \sigma, r) = \sum_{\rho \in \Pi(\tilde{R}, \chi)} \overline{tr(\rho(r))}\, \Theta_\rho.$$

We will prove in Section 3 that, as predicted by Arthur in the introduction to [A], $\Theta(M, \sigma, r)$ is supertempered if $r \in \tilde{R}_{\text{reg}}$. We also prove that every supertempered virtual character is a linear combination of ones of this form.

The method of proof relies on ideas of Harish-Chandra [HC2]. For every $\Theta \in \mathscr{V}(G)$ and parabolic subgroup $P = MN$, we will define a weak constant term $\Theta_P^w \in \mathscr{V}(M)$. In the case that Θ is the character of an irreducible tempered representation (π, V) of G, then Θ_P^w is just the (normalized) character of the maximal tempered quotient of the Jacquet module $V/V(\bar{N})$. In Section 2 we show that $\Theta \in \mathscr{V}(G)$ is supertempered if and only if $\Theta_P^w = 0$ for all $P \neq G$.

In Section 3 we use the R-group machinery developed by Arthur in [A, §2] and the Geometrical Lemma of Bernstein and Zelevinsky [B-Z] to compute the weak constant terms of the elliptic virtual characters $\Theta(M, \sigma, r)$, $r \in \tilde{R}_{\text{reg}}$, and show they are zero. We also prove that any $\Theta \in \mathscr{V}(G)$ which is supertempered and zero on the elliptic set of G must be zero. This allows us to show that every supertempered virtual character is a linear combination of ones of the form $\Theta(M, \sigma, r)$, $r \in \tilde{R}_{\text{reg}}$.

2. Constant terms

Let F be a locally compact, non-discrete, nonarchimedean field of characteristic zero. Let G be the F-rational points of a connected, reductive algebraic group over F. In this section we will define constant terms and weak constant terms of tempered virtual characters and prove that a tempered virtual character is supertempered if and only if all of its weak constant terms vanish. We use Silberger's book [S] as a convenient reference for Harish-Chandra's theory of constant terms of matrix coefficients. We note however, that it must be used with care since there is an error in the definition of the weak constant term.

For any admissible representation π of G, let $\mathscr{A}(\pi)$ denote the set of all finite linear combinations of matrix coefficients of π. Set $\mathscr{A}(G) = \cup \mathscr{A}(\pi)$ where the union is over all admissible representations π of G. Given any $f \in \mathscr{A}(G)$ and parabolic subgroup $P = MN$ of G, there is a constant term $f_P \in \mathscr{A}(M)$ with the property that given $m \in M$, there is $t > 0$ so that

$$\delta_P(ma)^{1/2} f(ma) = f_P(ma)$$

for all $a \in A^+(t)$. Here δ_P is the modular function of P, A is the split component of M, and $A^+(t)$ is the set of all $a \in A$ such that $|\alpha(a)| \geqslant t$ for every simple root α of P with respect to A [S, 2.6].

Suppose that (π, V) is an admissible representation of G and that $P = MN$ is a parabolic subgroup. Write $\bar{P} = M\bar{N}$ for the opposite parabolic subgroup. As usual we let $V(\bar{N})$ be the subspace of V generated by elements of the form $\pi(\bar{n})v - v$, $\bar{n} \in \bar{N}$, $v \in V$, and define $V_{\bar{N}} = V/V(\bar{N})$. Let $p\colon V \to V/V(\bar{N})$ be the projection map, and for $m \in M$, $v \in V$, let

$$\pi_{\bar{N}}(m)p(v) = \delta_{\bar{P}}(m)^{-1/2} p(\pi(m)v).$$

Then $(\pi_{\bar{N}}, V_{\bar{N}})$ is called the normalized Jacquet module of (π, V) corresponding to $\bar{P}$. It is well known to be an admissible representation of M [S, 2.3.6].

Let $(\tilde{\pi}, \tilde{V})$ denote the contragredient of (π, V). For any $\tilde{v} \in \tilde{V}$, $v \in V$, define the matrix coefficient

$$\phi_{\tilde{v},v}(x) = \langle \tilde{v}, \pi(x)v \rangle, \ x \in G.$$

As in [Ca2, 4.2], the dual of $V_{\bar{N}}$ is $\tilde{V}_N$ with the pairing

$$\langle p(\tilde{v}), p(v)\rangle = (\phi_{\tilde{v},v})_P(1), \ \tilde{v}\in \tilde{V}, \ v\in V.$$

(Here $\phi_{\tilde{v},v}\in\mathscr{A}(G)$ and $(\phi_{\tilde{v},v})_P\in\mathscr{A}(M)$ denotes its constant term as above.) For $\tilde{v}\in\tilde{V}$, $v\in V$, define the matrix coefficient

$$\psi_{p(\tilde{v}),p(v)}(m) = \langle p(\tilde{v}), \pi_{\bar{N}}(m)p(v)\rangle, \quad m\in M.$$

Then it is easy to check using [S, 2.7.1] that

$$\psi_{p(\tilde{v}),p(v)}(m) = (\phi_{\tilde{v},v})_P(m)$$

for all $m\in M$, $\tilde{v}\in\tilde{V}$, $v\in V$. Thus we have

$$\mathscr{A}(\pi_{\bar{N}}) = \{f_P: f\in\mathscr{A}(\pi)\}.$$

Suppose that $\Theta = \Theta_\pi$ is the character of π. Then define $\Theta_P = \Theta_{\pi_{\bar{N}}}$, the character of $\pi_{\bar{N}}$, and call Θ_P the constant term of Θ along P. Let G' denote the set of regular semisimple elements of G, $M' = M\cap G'$.

LEMMA 2.1. *Given any $m\in M'$, there is $t>0$ so that*

$$\delta_P(ma)^{1/2}\Theta(ma) = \Theta_P(ma)$$

for all $a\in A^+(t)$. Further, $a\mapsto\Theta_P(ma)$ is the only A-finite function with this property.

Proof. The equality is a rephrasing of Casselman's theorem [Ca1, 5.2]. The uniqueness follows from [S, 2.6.2]. □

Let $f\in\mathscr{A}(G)$ and let P be a parabolic subgroup of G. Then as in [S, 3.1] we can write $f_P = \Sigma_\chi f_{P,\chi}$ where the χ are quasi-characters of A, and define $X_f(P, A) = \{\chi: f_{P,\chi}\neq 0\}$. Now if π is an admissible representation of G we set

$$X_\pi(P, A) = \bigcup_{f\in\mathscr{A}(\pi)} X_f(P, A).$$

As in [S, 3.3.1], we can decompose $V_{\bar{N}}$ as an M-module direct sum

$$V_{\bar{N}} = \sum_{\chi\in X_\pi(P, A)} V_{\bar{N},\chi}.$$

Let $\Theta_{P,\chi}$ denote the character of the restriction of $\pi_{\bar{N}}$ to $V_{\bar{N},\chi}$. Then

$$\Theta_P = \sum_{\chi\in X_\pi(P,A)} \Theta_{P,\chi}$$

and

$$\Theta_{P,\chi}(ma) = \chi(a)\Theta_{P,\chi}(m)$$

for all $m \in M$, $a \in A$.

Given parabolic pairs (P, A) and (P_1, A_1) we write $(P, A) \prec (P_1, A_1)$ if $P \subset P_1$ and $A_1 \subset A$. In this case, if $P_1 = M_1N_1$, we write $P^* = P \cap M_1$. Given $\chi_1 \in X_\pi(P_1, A_1)$, let $X_\pi(P, A, \chi_1) = \{\chi \in X_\pi(P, A): \chi|_{A_1} = \chi_1\}$. The following lemma is an easy consequence of the definition of the constant terms as characters of Jacquet modules.

LEMMA 2.2. *Suppose that* $(P, A) \prec (P_1, A_1)$. *Then*

(i) $\Theta_P = (\Theta_{P_1})_{P^*}$;

(ii) *For all* $\chi_1 \in X_\pi(P_1, A_1)$,

$$(\Theta_{P_1,\chi_1})_{P^*} = \sum_{\chi \in X_\pi(P, A, \chi_1)} \Theta_{P,\chi}.$$

For any continuous function f on G, we say f satisfies the weak inequality if there exists a positive constant r so that

$$\sup_{x \in G} |f(x)|\Xi(x)^{-1}(1 + \sigma(x))^{-r} < \infty.$$

Here σ is the usual polynomial growth factor defined in [S, 4.1] and Ξ is the usual spherical function defined in [S, 4.2]. Let $\mathscr{A}_T(G)$ denote the set of functions in $\mathscr{A}(G)$ which satisfy the weak inequality.

Fix a parabolic pair (P, A) and let a be a variable element of A. Following Harish-Chandra [HC1, Section 21], we say that $a \to_P \infty$ if there exists a number $\varepsilon > 0$ so that (1) $\log_q|\alpha(a)| \geqslant \varepsilon\sigma(a)$ and (2) $|\alpha(a)| \to \infty$ for every root α of (P, A). For any $f \in \mathscr{A}_T(G)$ there is a weak constant term $f_P^w \in \mathscr{A}_T(M)$ defined as in [S, 4.5.5]. It is the unique element of $\mathscr{A}_T(M)$ such that

$$\lim_{a \to_P \infty} |\delta_P(ma)^{1/2} f(ma) - f_P^w(ma)| = 0$$

for every $m \in M$. Note Silberger's definition of $a \to_P \infty$ in [S, p. 101] does not have property (1). This is needed for the validity of [S, 4.5.5]. For $f \in \mathscr{A}_T(G)$, set $X_f^w(P, A) = X_f(P, A) \cap \hat{A}$. Then by [S, 4.5.5],

$$f_P^w = \sum_{\chi \in X_f^w(P, A)} f_{P,\chi}.$$

Now let π be an irreducible tempered representation of G and let $P = MN$

be a parabolic subgroup of G. Decompose the Jacquet module

$$V_{\bar{N}} = \sum_{\chi \in X_\pi(P, A)} V_{\bar{N},\chi}.$$

Define $X_\pi^w(P, A) = X_\pi(P, A) \cap \hat{A}$. Then as in [S, 5.4.1.3], the maximal tempered quotient of $\pi_{\bar{N}}$ is

$$(V_{\bar{N}})^w = \sum_{\chi \in X_\pi^w(P, A)} V_{\bar{N},\chi}.$$

If $\Theta = \Theta_\pi$, we define $\Theta_P^w = \Theta_{(V_{\bar{N}})^w}$, the character of $(V_{\bar{N}})^w$. Thus we have

$$\Theta_P^w = \sum_{\chi \in X_\pi^w(P, A)} \Theta_{P,\chi}.$$

Let $\phi: A \to \mathbf{C}$ be an A-finite function. Then we can write $\phi = \Sigma_\chi \phi_\chi$. We say a quasi-character χ of A is an exponent of ϕ is $\phi_\chi \neq 0$, and say ϕ is a tempered A-finite function if all of its exponents are unitary. Fix $m \in M'$. Then $a \to \Theta_P^w(ma)$ is a tempered A-finite function.

LEMMA 2.3. *Let $m \in M'$. Then*

$$\lim_{a \to_P \infty} |\delta_P^{1/2}(ma)\Theta(ma) - \Theta_P^w(ma)| = 0.$$

Further, $a \to \Theta_P^w(ma)$ is the only tempered A-finite function with this property.

Proof. Using Lemma 2.1 it is enough to show that

$$\lim_{a \to_P \infty} |\Theta_P(ma) - \Theta_P^w(ma)| = 0.$$

Write $X_\pi^s(P, A) = X_\pi(P, A) \cap X_\pi^w(P, A)^c$. Then

$$\begin{aligned}|\Theta_P(ma) - \Theta_P^w(ma)| &= \left| \sum_{\chi \in X_\pi^s(P, A)} \Theta_{P,\chi}(ma) \right| \\ &\leqslant \sum_{\chi \in X_\pi^s(P, A)} |\chi(a)\Theta_{P,\chi}(m)|.\end{aligned}$$

But by [S, 4.5.3], for all $\chi \in X_\pi^s(P, A)$, we have $\lim_{a \to_P \infty} |\chi(a)| = 0$. The uniqueness follows from [S, 4.1.6]. □

Let $\mathscr{E}_t(G)$ denote the set of equivalence classes of irreducible tempered representations of G. For $\pi \in \mathscr{E}_t(G)$, write Θ_π for the character of π. We will say that Θ is a tempered virtual character of G, and write $\Theta \in \mathscr{V}(G)$, if there are

$\pi_1, \ldots, \pi_k \in \mathscr{E}_t(G)$, $c_1, \ldots, c_k \in \mathbf{C}$, such that

$$\Theta = \sum_{i=1}^{k} c_i \Theta_{\pi_i}.$$

For any $\Theta = \Sigma_i c_i \Theta_{\pi_i} \in \mathscr{V}(G)$ we can define constant terms Θ_P and Θ_P^w by

$$\Theta_P = \sum_i c_i (\Theta_{\pi_i})_P, \quad \Theta_P^w = \sum_i c_i (\Theta_{\pi_i})_P^w.$$

Define $X_\Theta(P, A)$ to be the set of all quasicharacters χ of A such that $\Theta_{P,\chi} \neq 0$.

Let (P, A) be a parabolic pair with simple roots $\alpha_1, \ldots, \alpha_l$. Let A^+ denote the positive chamber of A with respect to P and let $\underline{a}$ denote the real Lie algebra of A. Finally, let χ be a quasicharacter of A which is unitary when restricted to Z, the split component of the center of G. Then as in [S, 4.5.10], we can define $\gamma = -\Sigma_{i=1}^{l} c_i \alpha_i \in \underline{a}^*$ by setting

$$|\chi(a)| = q^{\langle \gamma, H_P(a) \rangle}, \ a \in A.$$

We will say that χ is rapidly decreasing on A^+ if $c_i > 0$ for all $1 \leqslant i \leqslant l$. For $x \in G$ define

$$\sigma_*(x) = \inf_{z \in Z} \sigma(xz).$$

Then χ is rapidly decreasing on A^+ implies that for every $t > 0$ we have

$$\sup_{a \in A^+} |\chi(a)|(1 + \sigma_*(a))^r < \infty.$$

LEMMA 2.4. *Let $\Theta \in \mathscr{V}(G)$. The following are equivalent.*

(i) $\Theta_P^w = 0$ *for all* $P \neq G$;

(ii) *For every parabolic pair (P, A), every $\chi \in X_\Theta(P, A)$ is rapidly decreasing on A^+.*

Proof. Let $\Theta \in \mathscr{V}(G)$. Using [S, 4.5.2, 4.5.3], we see that for every (P, A) and $\chi \in X_\Theta(P, A)$ we have $|\chi(a)| \leqslant 1$ for all $a \in A^+$. Thus if we define $\gamma = -\Sigma_{i=1}^{l} c_i \alpha_i$ as above associated to χ, we have $c_i \geqslant 0$ for all i. Further, Θ_P^w is the sum of the $\Theta_{P,\chi}$ where χ runs over the unitary characters $\chi \in X_\Theta(P, A)$, that is the χ for which $c_i = 0$ for all i. Thus (ii) clearly implies (i).

Now assume that $\Theta_P^w = 0$ for all $P \neq G$. Fix a parabolic pair (P, A) and $\chi \in X_\Theta(P, A)$. If $(P, A) = (G, Z)$ there is nothing to check. Assume that (P, A) is proper. Define the constants $c_i \geqslant 0$ associated to χ as above. Suppose that

$c_1 = 0$. Let (P_1, A_1) be a proper parabolic pair such that $(P, A) \prec (P_1, A_1)$ and $\langle \alpha_i, H_P(a) \rangle = 0$ for all $a \in A_1, 2 \leqslant i \leqslant l$. Note that if $l = 1$ we take $P_1 = P$. Let χ_1 be the restriction of χ to A_1. Then χ_1 is unitary since $c_1 = 0$. Since $P_1 \neq G$, we have $\Theta_{P_1}^w = 0$ so that $\Theta_{P_1, \chi_1} = 0$ for every unitary character χ_1 of A_1. Now using Lemma 2.2 we see that $\Theta_{P, \chi'} = 0$ for every quasicharacter χ' of A such that the restriction of χ' to A_1 is χ_1. But this contradicts the assumption that $\chi \in X_\Theta(P, A)$. □

Let π be an irreducible, admissible representation of G. By [Cl, 3.4] we know that π is tempered if and only if given any Cartan subgroup T of G there is a positive constant r so that

$$\sup_{t \in T'} |D_G(t)|^{1/2} |\Theta_\pi(t)| (1 + \sigma(t))^{-r} < \infty.$$

Here D_G is the standard discriminant factor defined in [S, 4.7]. Let $\Theta \in \mathscr{V}(G)$. Then we say that Θ is supertempered if and only if for every Cartan subgroup T of G and every positive constant r we have

$$\sup_{t \in T'} |D_G(t)|^{1/2} |\Theta(t)| (1 + \sigma_*(t))^r < \infty.$$

THEOREM 2.5. *Let $\Theta \in \mathscr{V}(G)$. Then Θ is supertempered if and only if $\Theta_P^w = 0$ for all $P \neq G$.*

Proof. Suppose that $\Theta_P^w = 0$ for $P \neq G$. We will use the argument of Clozel in [Cl, 3.4] to show that Θ is supertempered. Let T be a Cartan subgroup of G. As in [Cl, 3.4] we can write T as a finite union of subsets $T(M_c^+)$ where $P = MN$ runs over a set of representatives for conjugacy classes of parabolic subgroups of G. Here $T(M_c^+) = 0$ unless $T \subset M$ modulo conjugation. Thus we assume that the P are chosen so that $T(M_c^+) = 0$ unless $T \subset M$. For regular $t \in T(M_c^+)$ we have

$$|D_G(t)|^{1/2} \Theta(t) = |D_M(t)|^{1/2} \Theta_P(t).$$

If $P = M = G$, then $T(G_c^+)$ is compact modulo center so that we know that

$$\sup_{t \in T(G_c^+) \cap G'} |D_G(t)|^{1/2} |\Theta(t)| (1 + \sigma_*(t))^r < \infty.$$

Now suppose that $P \neq G$. Now $T(M_c^+)$ can be written as a finite union of sets of the form $A^+ T_c t_i$, where T_c is the maximal compact subgroup of T and $t_i \in T$. Now for $t = act_i \in A^+ T_c t_i$ we can write

$$|D_M(act_i)|^{1/2}|\Theta_P(act_i)|(1+\sigma_*(act_i))^r$$
$$\leqslant \sum_{\chi \in X_\Theta(P,A)} |\chi(a)|(1+\sigma_*(a))^r\, |D_M(ct_i)|^{1/2}|\Theta_{P,\chi}(ct_i)|(1+\sigma_*(ct_i))^r.$$

Now since T_c is compact,

$$\sup_{c\in T_c} |D_M(ct_i)|^{1/2}|\Theta_{P,\chi}(ct_i)|(1+\sigma_*(ct_i))^r < \infty$$

for all χ. But using Lemma 2.4, every $\chi \in X_\Theta(P, A)$ is rapidly decreasing on A^+ so that

$$\sup_{a\in A^+} |\chi(a)|(1+\sigma_*(a))^r < \infty$$

for any $r > 0$.

Conversely, suppose that Θ is supertempered. Let $P = MN \neq G$ be a parabolic subgroup of G and fix $m \in M'$. Let $T = Z_G(m)$. Then for all $a \in A$, $ma \in T$ so that for any $r > 0$ there is $C_r > 0$ so that

$$|D_G(ma)|^{1/2}|\Theta(ma)| < C_r(1+\sigma_*(ma))^{-r}$$

for all $a \in A$ with $ma \in T'$. Thus

$$\lim_{a \to_P \infty} \delta_P(ma)^{1/2}\Theta(ma) = 0.$$

Thus by Lemma 2.3, $\Theta_P^w(ma) = 0$ for all $a \in A$. □

COROLLARY 2.6. *Let $\pi \in \mathscr{E}_t(G)$. Then Θ_π is supertempered if and only if π is a discrete series representation.*

Proof. Let $\pi \in \mathscr{E}_t(G)$ and write $\Theta = \Theta_\pi$. Then for any parabolic pair (P, A), we have $X_\Theta(P, A) = \bigcup_{f\in \mathscr{A}(\pi)} X_f(P, A)$. Suppose that π is a discrete series representation. Then using [S, 4.5.10], for every $f \in \mathscr{A}(\pi)$, every $\chi \in X_f(P, A)$ is rapidly decreasing on A^+. Thus using Lemma 2.6 and Theorem 2.7 we see that Θ is supertempered. Conversely, if Θ is supertempered, again using Lemma 2.6, Theorem 2.7, and [S, 4.5.10], we see that every $f \in \mathscr{A}(\pi)$ is square integrable mod center so that π is discrete series. □

3. Supertempered characters

Suppose that $P = MN$ is any parabolic subgroup of G and let A be the split component of M. Let $W(G/A) = N_G(A)/M$ be the Weyl group of A. Elements of $W(G/A)$ normalize M and act on representations of M. Let $\sigma \in \mathscr{E}_2(M)$ and define $W_\sigma = \{w \in W(G/A): w\sigma \simeq \sigma\}$. As in [A, §2], corresponding to each $w \in W_\sigma$ there is an intertwining operator for the representation $\mathrm{Ind}_P^G(\sigma \otimes 1)$ of G unitarily induced from σ. Write W_σ^0 for the subgroup of all $w \in W_\sigma$ such that the corresponding intertwining operator is scalar. Let Σ_0 be the set of reduced roots α of (G, A) such that the corresponding reflection $w_\alpha \in W_\sigma^0$. Then it is known that Σ_0 is a root system. Let Δ_0 be the set of simple roots for a choice of positive roots in Σ_0 and define $R_\sigma = \{w \in W_\sigma: w\Delta^0 = \Delta^0\}$. Then W_σ is the semidirect product of W_σ^0 and R_σ and $R = R_\sigma$ is the R-group for $\mathrm{Ind}_P^G(\sigma \otimes 1)$.

R has the property that the commuting algebra $C(\sigma)$ of the induced representation is naturally isomorphic to the complex group algebra $\mathbf{C}[R]^\eta$ with multiplication twisted by a cocycle η. Fix a finite central extension

$$1 \to Z \to \tilde{R} \to R \to 1$$

over which the cocycle η splits and a character χ of Z as in [A, §2]. Then the irreducible constituents of $\mathit{Ind}_P^G(\sigma \otimes 1)$ are naturally parameterized by $\Pi(\tilde{R}, \chi)$, the set of equivalence classes of irreducible representations of $\tilde{R}$ whose central character on Z is χ. For each $\rho \in \Pi(\tilde{R}, \chi)$ we will write π_ρ for the corresponding irreducible constituent of $\mathrm{Ind}_P^G(\sigma \otimes 1)$ and Θ_ρ for its character. Corresponding to each $r \in \tilde{R}$ we can define a virtual character as in [A, 2.3] by

$$\Theta(M, \sigma, r) = \sum_{\rho \in \Pi(\tilde{R}, \chi)} \overline{tr(\rho(r))}\, \Theta_\rho.$$

Let $\underline{a}$ denote the real Lie algebra of A and let $\underline{z}$ denote the real Lie algebra of Z, the split component of G. For each $r \in R$, let

$$\underline{a}_r = \{H \in \underline{a}: rH = H\}.$$

Set

$$R_{\text{reg}} = \{r \in R: \underline{a}_r = \underline{z}\}$$

and let $\tilde{R}_{\text{reg}}$ denote the inverse image of R_{reg} in $\tilde{R}$. If $r \in \tilde{R}_{\text{reg}}$, we say (M, σ, r) is an elliptic triple. Arthur says in the introduction of [A] that when (M, σ, r) is an elliptic triple, then the virtual character $\Theta(M, \sigma, r)$ should be supertempered, and that virtual characters of this type should span the set of supertempered virtual characters. In this section we will prove the following theorems.

THEOREM 3.1. *For every elliptic triple (M, σ, r), the virtual character $\Theta(M, \sigma, r)$ is supertempered. Conversely, given $\Theta \in \mathscr{V}(G)$ supertempered, there are finitely many elliptic triples (M_i, σ_i, r_i) and complex numbers c_i so that*

$$\Theta = \sum_i c_i \Theta(M_i, \sigma_i, r_i).$$

THEOREM 3.2. *Suppose that $\Theta \in \mathscr{V}(G)$ is supertempered and that Θ is zero on the elliptic set of G. Then $\Theta = 0$.*

In order to prove these two theorems we will first need some lemmas. We first recall a result of Bernstein and Zelevinsky [B-Z] on Jacquet modules of induced representations. Fix a minimal parabolic subgroup $P_0 = M_0 N_0$ of G and let A_0 be the split component of M_0. Let $\mathscr{L}(G)$ denote the finite set of Levi subgroups of parabolic subgroups P of G containing P_0. For each $M \in \mathscr{L}(G)$, set $\mathscr{L}(M) = \{M' \in \mathscr{L}(G)\colon M' \subset M\}$ and let $W_M = N_M(A_0)/M_0$. Each $M \in \mathscr{L}(G)$ is the Levi component of a unique parabolic subgroup $P_M = P_0 M$ containing P_0.

Let $M \in \in \mathscr{L}(G)$. Given any admissible representation τ of M we write $i_{G,M}(\tau)$ for the equivalence class of the admissible representation $\mathrm{Ind}_{P_M}^G(\tau \otimes 1)$ of G. If Θ_τ is the character of τ, we also write $i_{G,M}(\Theta_\tau)$ for the character of $i_{G,M}(\tau)$. Given any admissible representation π of G, we write $r_{M,G}(\pi)$ for the equivalence class of the admissible representation π_N of M where as in Section 2, π_N denotes the normalized Jacquet module of π corresponding to $P_M = MN$. If Θ_π is the character of π, we also write $r_{M,G}(\Theta_\pi)$ for the character of $r_{M,G}(\pi)$. It is the constant term $(\Theta_\pi)_{\bar{P}_M}$ of Θ_π with respect to $\bar{P}_M$. We will also write $(\Theta_\pi)_M = (\Theta_\pi)_{\bar{P}_M}$ and $(\Theta_\pi)_M^w = (\Theta_\pi)_{\bar{P}_M}^w$.

Given M, $L \in \mathscr{L}(G)$, set

$$W^{M,L} = \{w \in W_G\colon w(M \cap P_0) \subset P_0, w^{-1}(L \cap P_0) \subset P_0\}.$$

Then $W^{M,L}$ gives a complete set of coset representatives for the double cosets $W_L \backslash W_G / W_M$ and for each $w \in W^{M,L}$, $wM \cap L \in \mathscr{L}(L)$ and $M \cap w^{-1}L \in \mathscr{L}(M)$. Now the Geometrical Lemma [B-Z, 2.12] implies the following character formula. Let $M, L \in \mathscr{L}(G)$ and let τ be an admissible representation of M. For each $w \in W^{M,L}$, write $L_w = L \cap wM$. Then as above we can define $r_{w^{-1}L_w, M}(\tau)$. It is an admissible representation of $w^{-1}L_w$. Now $wr_{w^{-1}L_w, M}(\tau)$ is an admissible representation of L_w and $i_{L,L_w}(wr_{w^{-1}L_w,M}(\tau))$ is an admissible representation of L. We will also denote this representation by $i_{L,L_w}(r_{L_w,wM}(w\tau))$. Then the character formula can be written as

$$r_{L,G}(i_{G,M}(\Theta_\tau)) = \sum_{w \in W^{M,L}} i_{L,L_w}(r_{L_w,wM}(w\Theta_\tau)). \tag{3.3}$$

Let $L \in \mathscr{L}(G)$ and suppose that $\Theta' \in \mathscr{V}(L)$, the set of tempered virtual characters of L. Then by linearity we can define $\Theta = i_{G,L}(\Theta') \in \mathscr{V}(G)$. Let L^{ell} denote the set of regular elliptic elements of L. Thus $x \in L^{ell}$ just in case x is a regular semisimple element of L and the centralizer of x in L is compact modulo the center of L.

LEMMA 3.4. *Let* $\Theta' \in \mathscr{V}(L)$, $\Theta = i_{G,L}(\Theta')$. *Then for all* $x \in L^{ell}$,

$$\Theta_L^w(x) = \sum_{s \in W_0^{L,L}} s\Theta'(x).$$

Here $W_0^{L,L} = \{s \in W^{L,L} \colon sL = L\}$.

Proof. First, using (3.3) which clearly extends by linearity to virtual characters, we have

$$r_{L,G}(i_{G,L}(\Theta')) = \sum_{s \in W^{L,L}} i_{L,L_s}(r_{L_s,sL}(s\Theta')).$$

Now suppose that $s \in W^{L,L}$ and $L_s = L \cap sL$ is a proper Levi subgroup of L. Then $i_{L,L_s}(r_{L_s,sL}(s\Theta'))$ is a properly induced character of L and hence is zero on L^{ell}. But when $L_s = L = sL$, then $i_{L,L_s}(r_{L_s,sL}(s\Theta')) = s\Theta'$. Thus for all $x \in L^{ell}$ we have

$$\Theta_L(x) = r_{L,G}(i_{G,L}(\Theta'))(x) = \sum_{s \in W_0^{L,L}} s\Theta'(x).$$

But since Θ' is tempered, so is $s\Theta'$ for all $s \in W_0^{L,L}$, and so

$$\Theta_L^w(x) = \Theta_L(x) = \sum_{s \in W_0^{L,L}} s\Theta'(x). \qquad \square$$

Proof of Theorem 3.2. Let $\Theta \in \mathscr{V}(G)$ such that the restriction of Θ to G^{ell} is zero, but $\Theta \neq 0$. Then using Theorem D and Proposition 1 of Kazhdan [K], there are proper Levi subgroups $M_i \in \mathscr{L}(G)$ and tempered virtual characters $\Theta_i \in \mathscr{V}(M_i)$ such that $\Theta = \Sigma_i i_{G,M_i}(\Theta_i)$. Let d_i be the dimension of M_i and let d be the maximum of the d_i. The expression of Θ as a sum of induced virtual characters is not unique, but we can assume that we have chosen the M_i, Θ_i, so that d is as small as possible. We can also assume that M_i is not conjugate to M_j for $i \neq j$.

Now assume that Θ is supertempered. Pick M_1 such that $d_1 = d$ is maximal and let $x \in M_1^{ell}$. Using (3.3), for all i we have

$$i_{G,M_i}(\Theta_i)_{M_1} = \sum_{s \in W^{M_i,M_1}} i_{M_1,M_{i,s}}(r_{M_{i,s},M_i}(s\Theta_i)).$$

Let $s \in W^{M_i,M_1}$ and suppose that $M_{i,s} = M_1 \cap sM_i = M_1$. Then $M_1 \subset sM_i$. But $\dim M_1 \geqslant \dim sM_i$ so that $M_1 = sM_i$. But we assumed that M_i is not conjugate to M_1 for $i \neq 1$. Thus for all $i \neq 1$ and $s \in W^{M_i, M_1}$, $M_{i,s}$ is a proper Levi subgroup of M_1, so that the induced characters are all zero on elliptic elements of M_1. Thus for $i \neq 1$ we have

$$i_{G,M_i}(\Theta_i)^w_{M_1}(x) = i_{G,M_i}(\Theta_i)_{M_1}(x) = 0.$$

But Θ is supertempered, so this implies that

$$i_{G,M_1}(\Theta_1)^w_{M_1}(x) = \Theta^w_{M_1}(x) = 0.$$

Define $\Theta'_1 \in \mathscr{V}(M_1)$ by

$$\Theta'_1 = k^{-1} \sum_{s \in W_0^{M_1, M_1}} s\Theta_1$$

where k is the cardinality of $W_0^{M_1, M_1}$. Then $i_{G,M_1}(\Theta'_1) = i_{G,M_1}(\Theta_1)$. But, by Lemma 3.4, we see that for $x \in M_1^{ell}$, $\Theta'_1(x) = k^{-1} i_{G,M_1}(\Theta_1)^w_{M_1}(x) = 0$. Thus Θ'_1 is zero on the elliptic set of M_1. Now using Kazhdan's theorem [K], there are proper Levi subgroups L_j of M_1 and $\Theta''_j \in \mathscr{V}(L_j)$ so that $\Theta'_1 = \Sigma_j i_{M_1, L_j}(\Theta''_j)$. Thus $i_{G,M_1}(\Theta_1) = \Sigma_j i_{G,L_j}(\Theta''_j)$. We have seen that for any M_1 such that $d_1 = d$ is maximal, $i_{G,M_1}(\Theta_1)$ can be written as a sum of virtual characters induced from Levi subgroups of strictly smaller dimension. This contradicts our assumption about the expression of Θ. Thus $\Theta = 0$. □

Let $M \in \mathscr{L}(G)$, $\sigma \in \mathscr{E}_2(M)$, and define $W_\sigma, R = R_\sigma$, and $\Pi(\tilde{R}, \chi)$ as in the beginning of this section. Define $\pi = i_{G,M}(\sigma)$ and for each $\rho \in \Pi(\tilde{R}, \chi)$, let π_ρ be the irreducible constituent of π corresponding to ρ and Θ_ρ its character. Let $L \in \mathscr{L}(G)$ such that $M \subset L$ and assume that L satisfies the compatibility condition in [A, §2]. Then the R-group for $i_{L,M}(\sigma)$ is $R(L) = R \cap W(L/A)$. Let $\tilde{R}(L)$ denote the inverse image of $R(L)$ in $\tilde{R}$. Then $\Pi(\tilde{R}(L), \chi)$ parameterizes the irreducible constituents of $i_{L,M}(\sigma)$. Given $\rho \in \Pi(\tilde{R}, \chi)$, $\rho' \in \Pi(\tilde{R}(L), \chi)$, write $\mathrm{Res}(\rho)$ for the restriction of ρ to $\tilde{R}(L)$ and $\mathrm{Ind}(\rho')$ for the representation of $\tilde{R}$ induced from ρ'. It follows easily from the remarks of Arthur in [A, §2] explaining the compatibility of formula [A, 2.4] with induction, that if $\tau_{\rho'}$ is the irreducible constituent of $i_{L,M}(\sigma)$ corresponding to $\rho' \in \Pi(\tilde{R}(L), \chi)$, then

$$m(\pi_\rho, i_{G,L}(\tau_{\rho'})) = m(\rho, \mathrm{Ind}(\rho')) = m(\rho', \mathrm{Res}(\rho)).$$

LEMMA 3.5. *Suppose that $R_\sigma = W_\sigma$ and let $L \in \mathscr{L}(G)$. Then given $\rho \in \Pi(\tilde{R}, \chi)$,*

$$(\Theta_\rho)^w_L = \sum_{\tau \in \mathscr{E}_t(L)} m(\pi_\rho, i_{G,L}(\tau))\Theta_\tau.$$

Proof. Using (3.3) we can compute

$$(\Theta_\pi)_L = \sum_{v \in W^{M,L}} i_{L,L_v}((\Theta_{v\sigma})_{L_v})$$

where $L_v = L \cap vM$. Fix $v \in W^{M,L}$ and write

$$(\Theta_{v\sigma})_{L_v} = \sum_{\gamma_v} (\Theta_{v\sigma})_{L_v,\gamma_v}$$

where the γ_v are quasicharacters of the split component A_v of L_v. Since $v\sigma \in \mathscr{E}_2(vM)$, $\Theta_{v\sigma}$ is supertempered by Corollary 2.6. Now as in Lemma 2.4, every exponent γ_v of $(\Theta_{v\sigma})_{L_v}$ is unitary on $A_{vM} \subset A_v$, and is rapidly decreasing in the appropriate chamber of A_v/A_{vM}. But the central character of $i_{L,L_v}((\Theta_{v\sigma})_{L_v,\gamma_v})$ is the restriction of γ_v to A_L. Thus $i_{L,L_v}((\Theta_{v\sigma})_{L_v,\gamma_v})$ can have unitary central character only if $A_L \subset A_{vM}$ so that $vM \subset L$. In this case $L_v = vM$. Thus we see that

$$(\Theta_\pi)_L^w = \sum_{v \in W_0^{M,L}} i_{L,vM}(\Theta_{v\sigma})$$

where $W_0^{M,L} = \{v \in W^{M,L}: vM \subset L\}$. Fix $v \in W_0^{M,L}$. Then $i_{L,vM}(v\sigma) = vi_{v^{-1}L,M}(\sigma)$. There is no compatibility condition since $R_\sigma = W_\sigma$, and we can write the irreducible constituents of $i_{v^{-1}L,M}(\sigma)$ as $\tau_{\rho'}, \rho' \in \Pi(\tilde{R}(v^{-1}L), \chi)$. Further, $\tau_{\rho'}$ occurs in $i_{v^{-1}L,M}(\sigma)$ with multiplicity $\deg \rho'$. Thus $i_{L,vM}(v\sigma)$ has irreducible constituents $v\tau_{\rho'}$ occuring with multiplicity $\deg \rho'$.

Given $v_1, v_2 \in W_0^{M,L}$, $i_{L,v_1M}(v_1\sigma)$ and $i_{L,v_2M}(v_2\sigma)$ have constituents in common if and only if they are equal. In this case there is $s \in W_L$ such that $v_2M = sv_1M$ and $v_2\sigma \simeq sv_1\sigma$. Thus $v_2^{-1}sv_1 \in W'_\sigma = \{w \in W_G: wM = M, w\sigma \simeq \sigma\}$ so that v_1 and v_2 are in the same double coset of $W_L \backslash W_G / W'_\sigma$. Recall $W_\sigma \cong W'_\sigma / W_M$ and write $W_\sigma(v^{-1}L) = W_\sigma \cap W(v^{-1}L/A)$. Then we see that $i_{L,vM}(\Theta_{v\sigma})$ occurs in $(\Theta_\pi)_L^w$ exactly $[W_\sigma / W_\sigma(v^{-1}L)]$ times. Thus for $v \in W_0^{M,L}$, each $\Theta_{v\tau_{\rho'}}, \rho' \in \Pi(\tilde{R}(v^{-1}L), \chi)$, occurs in $(\Theta_\pi)_L^w$ with multiplicity $\deg \rho' [W_\sigma / W_\sigma(v^{-1}L)] = \deg \rho' [R/R(v^{-1}L)]$ since $R = W_\sigma$ by hypothesis.

Let $\rho \in \Pi(\tilde{R}, \chi)$. By the standard Frobenius reciprocity result [Ca2, 3.2.4], for any $\tau \in \mathscr{E}_t(L)$,

$$m(\pi_\rho, i_{G,L}(\tau)) = \dim \operatorname{Hom}_L((\pi_\rho)_{N_L}, \tau)$$

where we use normalized induction and the normalized Jacquet module. However, since $(\pi_\rho)_{N_L}^w$ is the maximal tempered quotient of $(\pi_\rho)_{N_L}$ and we are assuming that τ is tempered,

$$\operatorname{Hom}_L((\pi_\rho)_{N_L}, \tau) = \operatorname{Hom}_L((\pi_\rho)_{N_L}^w, \tau).$$

Thus Θ_τ occurs in $(\Theta_\rho)_L^w$ with multiplicity $m(\rho,\tau) \geqslant m(\pi_\rho, i_{G,L}(\tau))$.

Fix $\tau \in \mathscr{E}_t(M)$. If τ is not of the form $v\tau_{\rho'}$ for some $v \in W_0^{M,L}$, $\rho' \in \Pi(\tilde{R}(v^{-1}M),\chi)$, then Θ_τ does not occur in $(\Theta_\pi)_L^w$ and so $m(\rho,\tau) = 0$ for all ρ. It is an easy consequence of [A, 1.1], that $m(\pi_\rho, i_{G,M}(\tau)) = 0$ for all ρ also. Thus $m(\rho,\tau) = m(\pi_\rho, i_{G,M}(\tau)) = 0$ in this case. Now let $\tau = v\tau_{\rho'}$ for some $v \in W_0^{M,L}$, $\rho' \in \Pi(\tilde{R}(v^{-1}M),\chi)$ and suppose that $m(\rho, v\tau_{\rho'}) > m(\pi_\rho, i_{G,L}(v\tau_{\rho'})) = m(\rho, \mathrm{Ind}(\rho'))$ for some ρ. Now since the multiplicity of π_ρ in π is $\deg \rho$, we see that the multiplicity of $\Theta_{v\tau_{\rho'}}$ in $(\Theta_\pi)_L^w$ is strictly greater than

$$\sum_\rho \deg \rho\, m(\rho, \mathrm{Ind}(\rho')) = \deg \mathrm{Ind}(\rho') = \deg \rho' [R/R(v^{-1}L)].$$

But this contradicts the above calculation using the Geometric Lemma. Thus $m(\rho, v\tau_{\rho'}) = m(\pi_\rho, i_{G,L}(v\tau_{\rho'}))$ for all ρ. □

LEMMA 3.6. *Suppose that* $R_{\mathrm{reg}} \neq \varnothing$. *Then* $R_\sigma = W_\sigma$.

Proof. Suppose that $R_\sigma \neq W_\sigma$. Then $\Delta_0 \neq \varnothing$. For each $\alpha \in \Delta_0$, define $H_\alpha \in \underline{a}$ dual to α. Set $H_0 = \Sigma_{\alpha \in \Delta_0} H_\alpha$. Then for any $r \in R$, $r\Delta_0 = \Delta_0$ so that $rH_0 = H_0$. Thus $H_0 \in \underline{a}_r$. But $\alpha(H_0) > 0$ for all $\alpha \in \Delta_0$ so that $H_0 \notin \underline{z}$. Thus $R_{\mathrm{reg}} = \varnothing$. □

Proof of Theorem 3.1

Let (M,σ,r) be an elliptic triple. Thus $M \in \mathscr{L}(G)$ is a Levi subgroup of $G, \sigma \in \mathscr{E}_2(M)$, and $r \in \tilde{R}_{\mathrm{reg}}$. We must show that $\Theta = \Theta(M,\sigma,r) = \Sigma_{\rho \in \Pi(\tilde{R},\chi)} \overline{tr(\rho(r))}\, \Theta_\rho$ is supertempered. Since $R_{\mathrm{reg}} \neq \varnothing$, we know from Lemma 3.6 that $R_\sigma = W_\sigma$. Now from Lemma 3.5 we know that for any $L \in \mathscr{L}(G)$,

$$\begin{aligned} \Theta_L^w &= \sum_\rho \overline{tr(\rho(r))} \sum_{\tau \in \mathscr{E}_t(L)} m(\pi_\rho, i_{G,L}(\tau))\Theta_\tau \\ &= \sum_{\tau \in \mathscr{E}_t(L)} \Theta_\tau \sum_\rho \overline{tr(\rho(r))}\, m(\pi_\rho, i_{G,L}(\tau)). \end{aligned}$$

But we know that $m(\pi_\rho, i_{G,L}(\tau)) = 0$ for all ρ unless there is $v \in W_G$ so that $M \subset v^{-1}L$ and $\tau = v\tau_{\rho'}$ for some $\rho' \in \Pi(\tilde{R}(v^{-1}L),\chi)$. Assume that this is the case. Then $m(\pi_\rho, i_{G,L}(v\tau_{\rho'})) = m(\rho, \mathrm{Ind}(\rho'))$. Thus

$$\sum_\rho \overline{tr(\rho(r))}\, m(\pi_\rho, i_{G,L}(v\tau_{\rho'})) = \sum_\rho m(\rho, \mathrm{Ind}(\rho'))\overline{tr(\rho(r))} = \overline{tr(\mathrm{Ind}(\rho'))(r)}.$$

But if $L \neq G$, $r \in \tilde{R}_{\mathrm{reg}}$ cannot be conjugate to an element of $\tilde{R}(v^{-1}L)$, and so $tr(\mathrm{Ind}(\rho')(r)) = 0$. Thus $\Theta_L^w = 0$.

We have proven that $\Theta_L^w = 0$ for every $L \in \mathscr{L}(G)$, $L \neq G$. But $\mathscr{L}(G)$ contains representatives for all conjugacy classes of Levi subgroups of G. Further, although the definition of $\Theta_L^w = \Theta_{\bar{P}_L}^w$ depends on the choice of the parabolic-subgroup $\bar{P}_L$ with Levi component L, the formula from Lemma 3.5 shows that it is independent of all choices. Thus Θ is supertempered.

Conversely, suppose that $\Theta \in \mathscr{V}(G)$ is supertempered. As in [A, §3], virtual characters of the form $\Theta(M, \sigma, r)$, $M \in \mathscr{L}(G)$, $\sigma \in \mathscr{E}_2(M)$, $r \in \tilde{R}_\sigma$, span $\mathscr{V}(G)$. Thus there are triples (M_i, σ_i, r_i), $1 \leqslant i \leqslant k$, as above and complex numbers c_i so that

$$\Theta = \sum_{i=1}^{k} c_i \Theta(M_i, \sigma_i, r_i).$$

Now by the results of [A, §3], $\Theta(M_i, \sigma_i, r_i)$ is zero on G^{ell} unless $i \in I_{\text{reg}} = \{1 \leqslant i \leqslant k : r_i \in (\tilde{R}_{\sigma_i})_{\text{reg}}\}$. But for all $i \in I_{\text{reg}}$, $\Theta(M_i, \sigma_i, r_i)$ is supertempered. Thus

$$\Theta' = \Theta - \sum_{i \in I_{\text{reg}}} c_i \Theta(M_i, \sigma_i, r_i)$$

is supertempered and zero on the elliptic set of G. Thus $\Theta' = 0$ by Theorem 3.2. □

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