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Spherical functions on a complex classical quantum group

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1. Introduction

The classical theory of spherical functions is a well established part of harmonic analysis on homogeneous spaces that studies the functions on a real reductive Lie group G that are bi-invariant under the action of the maximal compact subgroup K. In [10] Macdonald described a family of symmetric polynomials first discovered by Jack [5] and showed that they are closely related to the spherical functions for certain symmetric spaces. In [11] he generalized Jack's idea and found various families of polynomials associated to the root systems of semisimple Lie algebras (see also [12, Chapter VI]).

For the root system of type A_1 the Macdonald polynomials are the continuous q-ultra-spherical polynomials discovered by Roger in his proof of the Roger-Ramanujan identities [16]. By the work of Koornwinder [8] and Noumi, Mimachi [13] it appears that the q-ultraspherical polynomials of weight $\frac{1}{2}$ can be realized as spherical functions for the quantum group $SU_q(2)$. Furthermore in [1] a similar interpretation of the ultraspherical polynomials of weight 1 as spherical functions on $SL_q(2, \mathbb{C})$ is given. This suggests that the Macdonald polynomials have a realization connected to the representation theory of quantum groups.

Indeed, in this paper, we show that the Macdonald polynomials of weight 1 associated to root systems of classical type appear naturally as spherical functions on a complex classical quantum group regarded as a real group. We study the Hopf algebra of functions on the quantum group and derive directly an explicit description of the spherical functions (Theorem 5.2).

Our approach tries to mimic the situation at the classical limit $q = 1$; in particular our description of a complex quantum group G as a real group is given by the obvious extension from the classical to the quantum setting of the description of a semisimple complex group seen as a real reductive group. It turns out, however, that the restriction of functions from the group to its maximal compact subgroup in the quantum setting is no longer an algebra homomorphism but only a coalgebra homomorphism, so the space of bi-invariant functions is not a subalgebra of the space of functions on G; nevertheless we prove that the bi-invariant functions form a subalgebra of the functions on G if we substitute the multiplication with a sort of "opposite" multiplication (see Corollary 5.5). This somewhat puzzling feature allows us to define an algebra isomorphism between the algebra of bi-invariant functions and a polynomial algebra thus obtaining the desired result (Proposition 6.2).

In addition to the fundamental work on quantum groups [2, 15] we use extensively the work of Woronowicz on compact matrix quantum groups [20] and its application to the compact quantum groups of Soibelman (see [19]). The referee drew our attention to a recent paper of Podles [14] where a different construction of the space of functions on a complex quantum group is given. A series of lectures given by Prof. Heckman [4] were also very useful in providing motivations to our work.

2. Preliminaries

Let $\mathfrak{g}$ be a complex simple Lie algebra and set K to denote a compact connected group whose complexified Lie algebra is $\mathfrak{g}$. Set G to be the complexification of K. Let q be a positive real number.

Let T be a torus in K, $\mathfrak{t}_0$ its Lie algebra and set $\mathfrak{t} = (\mathfrak{t}_0)_{\mathbb{C}}$. We fix a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ for $(\mathfrak{g}, \mathfrak{t})$. Let $(\,,)$ be a scalar multiple of the Killing form such that, if $h_{\alpha_i} \in \mathfrak{t}$ denotes the element such that $\alpha_i(h) = (h, h_{\alpha_i})$, then $(h_{\alpha_i}, h_{\alpha i}) \in 2\mathbb{N}$. The form $(\,,)$ induces a form on $\mathfrak{t}^*$ by setting $(\alpha, \beta) = (h_\alpha, h_\beta)$. Set $h_i = 2h_{\alpha_i}/(\alpha_i, \alpha_i)$ and $\mathfrak{t}_{\mathbb{Q}}$ to be the $\mathbb{Q}$-linear span of $\{h_i\}$ so that Π is basis of $\mathfrak{t}_{\mathbb{Q}}^*$. Let $\Lambda \subset \mathfrak{t}_{\mathbb{Q}}^*$ denote the lattice of integral weights, and let P be the sublattice of weights integrable to T.

We use $\mathfrak{t}_{\mathbb{Q}}$, Π, $\{h_i\}$, $(\,,)$, and Λ as our data for constructing the quantized universal enveloping algebra $U_q(\mathfrak{g})$ of Drinfeld and Jimbo (see [2], [6]). This is the Hopf algebra over $\mathbb{C}$ generated by the symbols e_i, f_i, and q^h, with $1 \leqslant i \leqslant \operatorname{rank}(\mathfrak{g})$ and $h \in \Lambda^*$ ($\Lambda^* = \{h \in t_{\mathbb{Q}} \mid \Lambda(h) \subset \mathbb{Z}\}$). We refer to [7] p. 6 for the set of commutation relations between these symbols. We recall, for example, that

$$q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \tag{2.1}$$

and

$$[e_i, f_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q_i^2 - q_i^{-2}} \tag{2.2}$$

where $k_i = q^{h_{\alpha_i}}$ and $q_i = q^{(\alpha_i, \alpha_i)/2}$.

The comultiplication Δ, the counit ε, and the antipode S of $U = U_q(\mathfrak{g})$ are given by the following formulae:

$$\Delta(q^h) = q^h \otimes q^h$$

$$\Delta(e_i) = e_i \otimes k_i^{-1} + k_i \otimes e_i$$

$$\Delta(f_i) = f_i \otimes k_i^{-1} + k_i \otimes f_i \tag{2.3}$$

$$\varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \varepsilon(q^h) = 1 \tag{2.4}$$

$$S(q^h) = q^{-h}$$

$$S(e_i) = -q^{-2h_\rho} e_i q^{2h_\rho}$$

$$S(f_i) = -q^{-2h_\rho} f_i q^{2h_\rho} \tag{2.5}$$

where ρ is the sum of the fundamental weights. Let $*$ denote the conjugate linear antiautomorphism of U defined by

$$e_i^* = f_i, \quad f_i^* = e_i, \quad (q^h)^* = q^h. \tag{2.6}$$

We now recall the basic facts about the theory of finite dimensional representations of U. If M is a U-module and $\lambda \in \mathfrak{t}^*$, we set

$$M_\lambda = \{m \in M \mid q^h \cdot m = q^{\lambda(h)} m\}.$$

We say that a U-module M is K-admissible if

$$M = \bigoplus_{\lambda \in P} M_\lambda \tag{2.7}$$

Let $\mathcal{M}_K$ denote the category of finite dimensional K-admissible representations of U: it is a full subcategory of the category of finite dimensional representations of U. We remark that for $q = 1$, $\mathcal{M}_K$ is the category of finite dimensional representations of K.

Let $A_K \subset U^*$ denote the set of matrix coefficients for elements of $\mathcal{M}_K$. Since $\mathcal{M}_K$ is closed under tensor products, direct sums, and contains the trivial representation, the Hopf structure of U induces a structure of Hopf algebra on A_K. Let Δ_K, S_K, ε_K denote respectively the comultiplication, the antipode, and the counit in A_K.

The anti-automorphism $*$ on U induces a anti-automorphism of U^* by the formula:

$$l^*(X) = \overline{l(S(X)^*)}, \quad l \in U^*, X \in U. \tag{2.8}$$

If $(\pi, M) \in \mathscr{M}_K$ then we let $(\bar{\pi}, \bar{M})$ denote the conjugate representation given by $\bar{\pi}(u)(m) = \pi(S(u)^*)(m)$ for all $u \in U$ and $m \in \bar{M}$. Obviously, if $m \in M_\lambda$ then $m \in \bar{M}_{-\lambda}$, so that, if $M \in \mathscr{M}_K$, then $\bar{M} \in \mathscr{M}_K$.

If $(\pi, M) \in \mathscr{M}$, $m \in M$, and $\lambda \in M^*$, let $c_{\lambda,m}$ denote the corresponding matrix coefficient. Let also $\bar{\lambda} \in \bar{M}^*$ be defined by $\bar{\lambda}(m) = \overline{\lambda(m)}$. An obvious computation gives that $c^*_{\lambda,m} = c_{\bar{\lambda},m}$. From this observation it follows that A_K is also $*$-invariant, therefore A_K is a $*$-Hopf algebra. The following result is proved in [18]:

THEOREM 2.9. *If K is of classical type, then A_K is a compact matrix quantum group in the sense of Woronowicz.* □

Let P^+ be the set of dominant weights in P; for $\lambda \in P^+$, let $N_q(\lambda)$ denote the irreducible U-module with highest weight λ. It follows easily from [17], that these modules are precisely the irreducible element of $\mathscr{M}_K$.

We say that a U-module M is unitary if there is an inner product $(\,,)$ on M such that

$$(X \cdot v, w) = (v, X^* \cdot w), \quad v, w \in M, X \in U.$$

The main point of Theorem 2.9 is essentially the fact that, if $\lambda \in P^+$, then $N_q(\lambda)$ is unitary and this is proved case by case for the classical algebras in [18].

Because of Theorem 2.9, we can apply the various results of [20] to A_K; before stating them, we need to set some notations and definitions. We say that a linear functional h on A_K is invariant if the following holds for any $a \in A_K$:

$$(id \otimes h) \circ \Delta_K(a) = 1 \otimes h(a)$$

and

$$(h \otimes id) \circ \Delta_K(a) = h(a) \otimes 1.$$

Fix an invariant inner product on $N_q(\lambda)$. If $\{v_i^\lambda\}$ is an orthonormal basis for $N_q(\lambda)$, let $\{w_{ij}^\lambda\}$ be the set of the corresponding matrix coefficients.

The following theorem collects the results of [20] that we will need.

THEOREM 2.10.

(1) *There exists a unique invariant functional* h *on* A_K *such that* $h(1) = 1$

(2) *The set* $\{w^{\lambda}_{ij}\}$ *is a basis for* A_K.

(3) $\Delta_K(w^{\lambda}_{ij}) = \sum_k w^{\lambda}_{ik} \otimes w^{\lambda}_{kj}$.

(4) $(w^{\lambda}_{ij})^* = S_K(w^{\lambda}_{ji})$.

(5) *There are multiplicative functionals* f_1 *and* f_{-1} *on* A_K *such that, for* $\lambda, \mu \in P^+$,

$$h(w^{\lambda}_{ij}(w^{\mu}_{hk})^*) = \frac{1}{M_\lambda}\,\delta_{\lambda\mu}\delta_{ih}f_1(w^{\lambda}_{kj})$$

and

$$h((w^{\lambda}_{ij})^*\,w^{\mu}_{hk}) = \frac{1}{M_\lambda}\,\delta_{\lambda\mu}\delta_{jk}f_{-1}(w^{\lambda}_{hi}).$$

where $M_\lambda = \Sigma_i f_1(w^{\lambda}_{ii})$.

Proof.

(1) is Theorem 4.2 of [20].

(2) is Proposition 4.7 in [20].

(3) is (4.26) in [20].

(4) and (5) are given in Theorem 5.7 in [20]. □

3. The norm of the matrix coefficients

For $\lambda \in P^+$ and $\mu \in P$, set $d^{\lambda}_{\mu} = \dim N_q(\lambda)_\mu$. Since $(q^h)^* = q^h$, it follows that we can choose an orthonormal basis:

$$\{v^{\lambda}_{\mu i} \in N_q(\lambda)_\mu \mid \mu \in P,\ 1 \leqslant i \leqslant d^{\lambda}_{\mu}\}. \tag{3.1}$$

We let $w^{\lambda}_{\mu\nu ij}$ denote the corresponding matrix coefficients. The main result of this section is:

THEOREM 3.2.

$$h(w^{\lambda}_{\mu\nu ij}(w^{\lambda}_{\mu\gamma ik})^*) = \delta_{\nu\gamma}\delta_{jk}\,\frac{q^{\nu(-4h_\rho)}}{\Sigma_{\eta\in P}d^{\lambda}_{\eta}q^{\eta(-4h_\rho)}} \tag{3.2.1}$$

$$h((w^{\lambda}_{\nu\mu ji})^*\,w^{\lambda}_{\gamma\mu ki}) = \delta_{\nu\gamma}\delta_{jk}\,\frac{q^{\nu(4h_\rho)}}{\Sigma_{\eta\in P}d^{\lambda}_{\eta}q^{\eta(4h_\rho)}} \tag{3.2.2}$$

Proof. Set $V = N_q(\lambda)$. If $v \in V$, let v^{**} denote the functional on V^* given by evaluation on v. We define a map $F\colon V \to V^{**}$ by setting

$$F(v) = (q^{4h_\rho}v)^{**}.$$

It is easy to verify that F intertwines the action of U. In fact, if $X \in U$, $v \in V$, and $f \in V^*$ then

$$F(X \cdot v)(f) = (q^{-4h_\rho}Xv)^{**}(f) = f(q^{-4h_\rho}Xv).$$

On the other hand, it follows from the definition of the contragradient representation that

$$(X \cdot F(v))(f) = (q^{-4h_\rho}v)^{**}(S(X)f) = f(S^2(X)q^{-4h_\rho}v).$$

It follows at once from (2.5) that $S^2(X) = q^{-4h_\rho}Xq^{4h_\rho}$. Substituting above, we find that $F(X \cdot v) = X \cdot F(v)$, as we wished to show.

Clearly

$$Tr(F) = \sum_{\mu \in P} d_\mu^\lambda q^{\mu(-4h_\rho)}$$

is positive. Let w_0 be the longest element of the Weyl group. It is known (see [9]) that $d_\mu^\lambda = d_{w_0(\mu)}^\lambda$, thus

$$\begin{aligned} Tr(F) &= \sum_{\mu \in P} d_\mu^\lambda q^{w_0(\mu)(-4h_\rho)} \\ &= \sum_{\mu \in P} d_\mu^\lambda q^{\mu(4h_\rho)} \\ &= Tr(F^{-1}). \end{aligned}$$

Therefore F satisfies the requirement of [20, Theorem 5.4]. Using [20, (5.22)], we find that

$$f_{\pm 1}(w_{\mu\nu ij}^\lambda) = \delta_{\mu\nu}\delta_{ij}q^{\mu(\mp 4h_\rho)}.$$

On substituting in (2.10.5), we find our result. □

Notice that we have also proven that

$$M_\lambda = Tr(F) = \sum_{\eta \in P} d_\eta^\lambda q^{\eta(-4h_\rho)}$$

and, by [9], M_λ can be expressed in terms of the usual Weyl character formula.

As an application of 3.2 we give the following result.

COROLLARY 3.3. *For* $\gamma, \nu \in P$, $1 \leqslant j \leqslant d_\gamma^\lambda$ *and* $1 \leqslant s \leqslant d_\nu^\lambda$

$$\sum_{\substack{\mu \in P \\ 1 \leqslant i \leqslant d_\mu^\lambda}} w_{\nu\mu ji}^\lambda (w_{\gamma\mu si}^\lambda)^* = \delta_{\nu\gamma}\delta_{js} \tag{3.3.1}$$

$$\sum_{\substack{\mu \in P \\ 1 \leqslant i \leqslant d_\mu^\lambda}} q^{\mu(-4h_\rho)} w_{\mu\nu ij}^\lambda (w_{\mu\gamma is}^\lambda)^* = \delta_{\nu\gamma}\delta_{js} q^{\nu(-4h_\rho)} \tag{3.3.2}$$

Proof. By Theorem 3.2, we have that

$$h(w_{\mu\nu ij}^\lambda (w_{\eta\gamma rs}^\lambda)^*) = \delta_{\mu\eta}\delta_{\nu\gamma}\delta_{ir}\delta_{js} \frac{q^{\nu(-4h_\rho)}}{M_\lambda} \tag{3.4}$$

hence, by the left invariance of h,

$$\begin{aligned}
\delta_{\nu\gamma}\delta_{js} \frac{q^{\mu(-4h_\rho)}}{M_\lambda}(1 \otimes 1) &= (1 \otimes h) \circ \Delta_K(w_{\nu\mu ji}^\lambda (w_{\gamma\mu si}^\lambda)^*) \\
&= (1 \otimes h)\left(\sum_{\eta,\xi,t,k} w_{\nu\eta jt}^\lambda (w_{\gamma\xi sk}^\lambda)^* \otimes w_{\eta\mu ti}^\lambda (w_{\xi\mu ki}^\lambda)^*\right) \\
&= \sum_{\eta,\xi,t,k} w_{\nu\eta jt}^\lambda (w_{\gamma\xi sk}^\lambda)^* \otimes h(w_{\eta\mu ti}^\lambda (w_{\xi\mu ki}^\lambda)^*) \\
&= \sum_{\eta,\xi,t,k} w_{\nu\eta jt}^\lambda (w_{\gamma\xi sk}^\lambda)^* \otimes \delta_{tk}\delta_{\eta\xi} \frac{q^{\mu(-4h_\rho)}}{M_\lambda} \\
&= \sum_{\eta,t} w_{\nu\eta jt}^\lambda (w_{\gamma\eta st}^\lambda)^* \otimes \frac{q^{\mu(-4h_\rho)}}{M_\lambda}
\end{aligned} \tag{3.5}$$

By dividing both ends of (3.5) by $(q^{\mu(-4h_\rho)}/M_\lambda)$ we obtain (3.3.1). Equation (3.3.2) is obtained similarly using the right invariance of h. □

Equation (3.3.1) is just the unitarity of $N_q(\lambda)$, we included it for completeness.

4. Complex quantum groups: the classical versus the quantum case

In this section we give our description of a complex quantum group seen as a real group. In order to provide motivations for our settings, we first discuss the classical limit $q = 1$. We essentially follow the construction of Duflo given in [3]. We look upon $\mathfrak{g}$ as a real algebra and denote by $\mathfrak{g}_{\mathbb{C}}$ its complexification. We let i denote the multiplication by $\sqrt{-1}$ in the complex structure of $\mathfrak{g}$ while

j denotes the multiplication by $\sqrt{-1}$ in the complex structure of $\mathfrak{g}_\mathbb{C}$.

We define two maps $\Phi^L\colon \mathfrak{g} \to \mathfrak{g}_\mathbb{C}$ and $\Phi^R\colon \mathfrak{g} \to \mathfrak{g}_\mathbb{C}$ by setting

$$\Phi^L(X) = \frac{X - ijX}{2} \quad \Phi^R(X) = \frac{\bar{X} + ij\bar{X}}{2}.$$

Here $-$ denotes the complex conjugation of $\mathfrak{g}$ with respect to the Lie algebra of K. We observe that Φ^L and Φ^R are complex Lie algebra homomorphisms.

Let

$$\mathfrak{g}^L = \Phi^L(\mathfrak{g}) = \{X - jiX \mid X \in \mathfrak{g}\}$$

and

$$\mathfrak{g}^R = \Phi^R(\mathfrak{g}) = \{X + jiX \mid X \in \mathfrak{g}\}$$

then $\mathfrak{g}_\mathbb{C} = \mathfrak{g}^L \oplus \mathfrak{g}^R$ and

$$\Phi = \Phi^L \times \Phi^R\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}_\mathbb{C}$$

is a complex Lie algebra isomorphism. It follows that

$$U(\mathfrak{g}_\mathbb{C}) \simeq U(\mathfrak{g}) \otimes U(\mathfrak{g}).$$

Let σ denote the complex conjugation on $\mathfrak{g}_\mathbb{C}$ coming from $\mathfrak{g}$ seen as a real form, then it is easy to check that, identifying $\mathfrak{g}_\mathbb{C}$ with $\mathfrak{g} \times \mathfrak{g}$,

$$\sigma(X, Y) = (\bar{Y}, \bar{X}) \quad X, Y \in \mathfrak{g}.$$

We let $*$ denote the conjugate linear antihomomorphism of $U(\mathfrak{g})$ defined as the unique map such that

$$1^* = 1, \quad X^* = -\bar{X}, \quad X \in \mathfrak{g}.$$

We also let $*$ denote the conjugate linear antihomomorphism of $U(\mathfrak{g}_\mathbb{C})$ obtained by extending the conjugate linear antihomomorphism of $\mathfrak{g}_\mathbb{C}$ defined by $(X, Y)^* = -\sigma(X, Y)$. It is then clear that

$$(X \otimes Y)^* = Y^* \otimes X^*.$$

Let $\mathfrak{k}_0$ be the Lie algebra of K and set $\mathfrak{k} = (\mathfrak{k}_0)_\mathbb{C}$. This is a subalgebra of $\mathfrak{g}_\mathbb{C}$. If we extend $-$ linearly to $\mathfrak{g}_\mathbb{C}$ then we find that

$$\overline{(X, Y)} = (Y, X)$$

therefore, since $\mathfrak{k}$ is the set of fixed points for $-$, one see that

$$\mathfrak{k} = \{(X, X) \mid X \in \mathfrak{g}\}$$

We recall that $\mathfrak{k}$ is $\mathfrak{g}$ thus the map

$$\Phi^D : \mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$$

defined by $\Phi^D(X) = (X, X)$ identifies $\mathfrak{g}$ as the complexification of $\mathfrak{k}_0$ in $\mathfrak{g}_{\mathbb{C}}$. Let Δ be the extension of Φ^D to $U(\mathfrak{g})$. Then Δ is the unique algebra homomorphism from $U(\mathfrak{g})$ to $U(\mathfrak{g}_{\mathbb{C}}) = U(\mathfrak{g}) \otimes U(\mathfrak{g})$ such that

$$\Delta(X) = X \otimes 1 + 1 \otimes X \quad \forall X \in \mathfrak{g}.$$

i.e. Δ is the comultiplication that gives an Hopf algebra structure to $U(\mathfrak{g})$.

We are now ready to describe the quantum group case: as in the classical limit $q = 1$ we consider the Hopf algebra

$$U_q(\mathfrak{g}_{\mathbb{C}}) := U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}).$$

Recall that by (2.6) $U_q(\mathfrak{g})$ has an involutive antihomomorphism $*$, we can thus define an involutive antihomomorphism $*$ on $U(\mathfrak{g}_{\mathbb{C}})$ by setting

$$(X \otimes Y)^* = Y^* \otimes X^* \quad X, Y \in U(\mathfrak{g}); \tag{4.1}$$

this turns $U_q(\mathfrak{g}_{\mathbb{C}})$ into a $*$-Hopf algebra that we consider to be the quantized enveloping algebra corresponding to the complex quantum group G seen as a real group. The algebra of regular functions on G is therefore $A_G = A_K \otimes A_K$. We recall that the multiplication m and the comultiplication Δ on A_G are given by the formulas:

$$m(a \otimes b, c \otimes d) = ac \otimes bd \tag{4.2}$$

and

$$\Delta = \tau_{23} \circ (\Delta_K \otimes \Delta_K) \tag{4.3}$$

where τ_{23} is the flip homomorphism:

$$\tau_{23}(a \otimes b \otimes c \otimes d) = a \otimes c \otimes b \otimes d.$$

The involution $*$ on $U_q(\mathfrak{g}_{\mathbb{C}})$ induces an involution $*$ on A_G defined by

$$(f \otimes g)^*(X \otimes Y) = (f \otimes g)(S \otimes S(X \otimes Y)^*)$$

thus, by (4.1),

$$(f \otimes g)^* = g^* \otimes f^*.$$

In analogy with the classical case discussed above we consider the comultiplication map

$$\Delta: U_q(\mathfrak{g}) \to U_q(\mathfrak{g}_{\mathbb{C}})$$

as the embedding of the quantized enveloping algebra of the compact quantum group K into the algebra of its complexification G. We point out that Δ is not a coalgebra homomorphism, so $\Delta(U_q(\mathfrak{g}))$ is not a Hopf-subalgebra of $U_q(\mathfrak{g}_{\mathbb{C}})$.

We then let $\pi_K: A_G \to A_K$ be the coalgebra homomorphism given by

$$\pi_K(F) = F_{|\Delta(U(\mathfrak{g}))}.$$

More explicitly

$$\pi_K(f \otimes g)(X) = (f \otimes g) \circ \Delta(X) = fg(X) \quad \forall X \in U(\mathfrak{g}).$$

Notice that $\pi_K(F^*) = \pi_K(F)^*$.

We also observe that π_K is *not* an algebra homomorphism: despite this feature we insist on considering π_K as the restriction of functions to K. The map π_K gives to A_G the structure of a bicomodule for $A(K)$: the left coaction of $A(K)$ is given by $L_K = (\pi_K \otimes 1) \circ \Delta$ while $R_K = (1 \otimes \pi_K) \circ \Delta$ is the right coaction.

5. Bi-invariant functions

An element $f \in A_G$ is said to be a K-bi-invariant function, or simply a bi-invariant function, if $R_K(f) = f \otimes 1$ and $L_K(f) = 1 \otimes f$. We denote by $A(G \parallel K)$ the space of bi-invariant functions. Our aim is to determine a basis for $A(G \parallel K)$.

Set

$$F_\lambda = \sum_{\substack{\mu\nu \in P \\ i \leqslant d_\mu^\lambda\ j \leqslant d_\nu^\lambda}} \frac{q^{\mu(-4h_\rho)}}{M_\lambda} w_{\mu\nu ij}^\lambda \otimes (w_{\mu\nu ij}^\lambda)^*.$$

The next result is an easy consequence of Corollary 3.3

LEMMA 5.1.

$$F_\lambda \in A(G \parallel K)$$

Proof. This is just a simple computation, as an example we show that F_λ is left invariant:

$$\begin{aligned} L_K(F_\lambda) &= (\pi_K \otimes 1) \circ \Delta(F_\lambda) \\ &= \sum_{\nu\eta\xi,jrs} \frac{1}{M_\lambda} \left(\sum_{\mu,i} q^{\mu(-4h_\rho)} w^\lambda_{\mu\eta ir} (w^\lambda_{\mu\xi is})^* \right) \otimes w^\lambda_{\eta\nu rj} \otimes (w^\lambda_{\xi\nu sj})^* \end{aligned}$$

so, applying (3.3.2), we obtain that

$$\begin{aligned} L_K(F_\lambda) &= \sum_{\nu\eta\xi,jrs} \frac{1}{M_\lambda} \left(\sum_{\mu,i} q^{\mu(-4h_\rho)} w^\lambda_{\mu\eta ir} (w^\lambda_{\mu\xi is})^* \right) \otimes w^\lambda_{\eta\nu rj} \otimes (w^\lambda_{\xi\nu sj})^* \\ &= \sum_{\nu\eta\xi,jrs} \frac{1}{M_\lambda} \delta_{\eta\xi} \delta_{sr} q^{\eta(-4h_\rho)} \otimes w^\lambda_{\eta\nu rj} \otimes (w^\lambda_{\xi\nu sj})^* \\ &= 1 \otimes \sum_{\nu\eta,jr} \frac{q^{\eta(-4h_\rho)}}{M_\lambda} w^\lambda_{\eta\nu rj} \otimes (w^\lambda_{\eta\nu rj})^* \\ &= 1 \otimes F_\lambda. \end{aligned}$$

□

The following is the main result of this section.

THEOREM 5.2. *$\{F_\lambda\}_{\lambda\in P^+}$ is a basis of $A(G \parallel K)$.*

Proof. In order to simplify the notations we let $\{w^\lambda_{ij}\}$ denote the matrix coefficients, thus $w^\lambda_{ij} = w^\lambda_{\nu_i\nu_j r_i r_j}$ for some $\nu_i, \nu_j \in P$ and $1 \leqslant r_i \leqslant d^\lambda_{\nu_i}$, $1 \leqslant r_j \leqslant d^\lambda_{\nu_j}$.

Suppose that F is a bi-invariant function. By Theorem 2.10.2 we can write

$$F = \sum_{\lambda,\mu} \sum_{ij,rs} c^{\lambda\mu}_{ijrs} w^\lambda_{ij} \otimes (w^\mu_{rs})^*.$$

We now compute the conditions on the coefficients $c^{\lambda\mu}_{ijrs}$ for F to be in $A(G \parallel K)$: from (4.3) and (2.10.3) it follows that

$$\Delta(F) = \sum_{\lambda,\mu} \sum_{ijrsmn} c^{\lambda\mu}_{ijrs} w^\lambda_{im} \otimes (w^\mu_{rn})^* \otimes w^\lambda_{mj} \otimes (w^\mu_{ns})^*$$

therefore

$$F \otimes 1 = R_K(F)$$
$$= \sum_{\lambda,\mu} \sum_{ijrsmn} c^{\lambda\mu}_{ijrs} w^{\lambda}_{im} \otimes (w^{\mu}_{rn})^* \otimes w^{\lambda}_{hj}(w^{\mu}_{ns})^*.$$

Let us write

$$w^{\lambda}_{mj}(w^{\mu}_{ns})^* = \sum_{vtl} a^{vtl}_{mjns\lambda\mu} w^{v}_{tl}$$

so that

$$R_K(F) = \sum_{\lambda,\mu} \sum_{ijrsmn} \sum_{vtl} c^{\lambda\mu}_{ijrs} a^{vtl}_{mjns\lambda\mu} w^{\lambda}_{im} \otimes (w^{\mu}_{rn})^* \otimes w^{v}_{tl}.$$

In particular, if F is right invariant, then

$$\sum_{js} c^{\lambda\mu}_{ijrs} a^{011}_{mjns\lambda\mu} = c^{\lambda\mu}_{imrn}.$$

Let $h'\colon A_K \to \mathbb{C}$ be the map such that

$$h'(w^0_{11}) = h'(1) = 1 \quad \text{and } h'(w^{\lambda}_{ij}) = 0 \quad \text{for } \lambda \neq 0,$$

one verifies, using (2.10.3), that h' is an invariant functional on A_K, thus, by the uniqueness of h, we have $h = h'$. It follows that

$$a^{011}_{mjns\lambda\mu} = h(w^{\lambda}_{mj}(w^{\mu}_{ns})^*)$$

so, by Theorem 3.2, $a^{011}_{mjns\lambda\mu} = 0$ unless $m = n, j = s$, and $\lambda = \mu$. This implies that $c^{\lambda\mu}_{imrn} = 0$ if $\lambda \neq \mu$ or $m \neq n$. If $\lambda = \mu$ and $m = n$,

$$\sum_{j} c^{\lambda\lambda}_{ijrj} a^{011}_{mjmj\lambda\lambda} = c^{\lambda\lambda}_{imrm}. \tag{5.3}$$

Recall that

$$w^{\lambda}_{mj} = w^{\lambda}_{v_m v_j r_m r_j} \quad \text{so } a^{011}_{mjmj\lambda\lambda} = \frac{q^{v_m(-4h_\rho)}}{M_\lambda}.$$

Substituting in (5.3) we find that $c^{\lambda\lambda}_{imrm}$ does not depend on m.

Similarly the left invariance of F implies that $c^{\lambda\lambda}_{imrm} = 0$ if $i \neq r$ and, if $i = r$,

$$\sum_i c_{ijij}^{\lambda\lambda} \frac{q^{\nu_m(-4h_\rho)}}{M_\lambda} = c_{mjmj}^{\lambda\lambda}. \tag{5.4}$$

Obvious solutions to this linear system are given by $c_{ijij}^{\lambda\lambda} = c_\lambda q^{\nu_i(-4h_\rho)}$, with c_λ depending only on λ. We claim that they are the only solutions: in fact if we fix j and λ then finding the solutions to (5.4) becomes equivalent to finding the eigenspace of the eigenvalue 1 for a matrix having equal columns. Such an eigenspace has clearly dimension at most one.

We have thus shown that $F = \sum c_\lambda F_\lambda$. The linear independence of the F_λ is an obvious consequence of (2.10.2). □

We now define a new product $\times$ on A_G by setting

$$(a \otimes b) \times (c \otimes d) = ac \otimes db$$

We already pointed out that $A(G \parallel K)$ is not a subalgebra of A_G. A somewhat surprising consequence of Theorem 5.2 is that $A(G \parallel K)$ is a subalgebra of A_G with respect to the product $\times$:

COROLLARY 5.5. *$A(G \parallel K)$ is a subalgebra of $(A_G, \times)$.*

Proof. Clearly it is enough to show that $F_\lambda \times F_\mu \in A(G \parallel K)$ for each $\lambda, \mu \in P^+$; as in the proof of 5.2 we set $w_{ij}^\lambda = w_{\nu_i \nu_j r_i r_j}^\lambda$. Set

$$c_i^\lambda = \frac{q^{\nu_i(-4h_\rho)}}{M_\lambda},$$

then

$$R_K(F_\lambda \times F_\mu) = R_K\left(\sum_{ijrs} c_i^\lambda c_r^\mu w_{ij}^\lambda w_{rs}^\mu \otimes (w_{rs}^\mu)^* (w_{ij}^\lambda)^*\right)$$

$$= \sum_{ijrstluv} c_i^\lambda c_r^\mu w_{it}^\lambda w_{rl}^\mu \otimes (w_{rv}^\mu)^* (w_{iu}^\lambda)^* \otimes w_{tj}^\lambda w_{ls}^\mu (w_{vs}^\mu)^* (w_{uj}^\lambda)^*$$

by (3.3.1) this becomes

$$R_K(F_\lambda \times F_\mu) = \sum_{ijrtluv} c_i^\lambda c_r^\mu w_{it}^\lambda w_{rl}^\mu \otimes (w_{rs}^\mu)^* (w_{ij}^\lambda)^* \otimes w_{tj}^\lambda \left(\sum_s w_{ls}^\mu (w_{vs}^\mu)^*\right)(w_{uj}^\lambda)^*$$

$$= \sum_{irtlu} c_i^\lambda c_r^\mu w_{it}^\lambda w_{rl}^\mu \otimes (w_{rs}^\mu)^* (w_{ij}^\lambda)^* \otimes \left(\sum_j w_{tj}^\lambda (w_{uj}^\lambda)^*\right)$$

$$= \sum_{irtl} c_i^\lambda c_r^\mu w_{it}^\lambda w_{rl}^\mu \otimes (w_{rs}^\mu)^* (w_{ij}^\lambda)^* \otimes 1$$

$$= F_\lambda \times F_\mu \otimes 1.$$

Analogously one verifies the left invariance of $F_\lambda \times F_\mu$ using (3.3.2). □

The reason why we did not choose $\times$ as the product in A_G in the first place is the fact that the involution $*$ on A_G that defines the real form is not a antihomomorphism for $\times$.

6. Spherical functions and Macdonald polynomials

This section is devoted to the study of the algebra structure of $(A(G \parallel K), \times)$. Recall that we let $P \subset \mathfrak{t}^*$ denote the lattice of weights that are integrable to K; we denote by $\mathscr{W}$ the Weyl group of $(\mathfrak{g}, \mathfrak{t})$. Let $\pi: A_K \to \mathbb{C}[\Lambda]$ be the restriction of the matrix coefficients to $\mathbb{C}[\Lambda^*]$. Since $\mathbb{C}[\Lambda^*]$ is a Hopf-subalgebra of $U_q(\mathfrak{g})$, it follows that π is an algebra homomorphism. Moreover, it is fairly easy to check that

$$\pi(w^\lambda_{\mu\nu ij}) = \delta_{\mu\nu}\delta_{ij}e^\mu \tag{6.1}$$

(here $e^\mu(q^h) = q^{\mu(h)}$).

Let $\vee$ be the unique linear antihomomorphism on $U_q(\mathfrak{g})$ defined by

$$e_i^\vee = e_i^*, \quad f_i^\vee = e_i^*, \quad (q^h)^\vee = (q^h)^*;$$

since $U_q(\mathfrak{g})$ is defined over $\mathbb{R}$ (q is real), it follows that the good definition of $*$ implies the good definition of $\vee$.

The involution $\vee$ on $U_q(\mathfrak{g})$ induces an involution $\vee$ on $U_q(\mathfrak{g})^*$ defined by

$$l^\vee(X) = l(S(X)^\vee) \quad l \in U_q(\mathfrak{g})^* \, X \in U_q(\mathfrak{g}).$$

As for $*$ it is easy to check that $A_K^\vee = A_K$, thus we can restrict $\vee$ to A_K.

We let $\pi_A: A_G \to \mathbb{C}[\Lambda]$ be defined by

$$\pi_A(f \otimes g) = \pi(fg^\vee).$$

Since $\mathbb{C}[\Lambda]$ is a commutative algebra, it is clear that π_A is an algebra homomorphism between $(A_G, \times)$ and $\mathbb{C}[\Lambda]$. For $\lambda \in P^+$ let $Ch(\lambda)$ denote the Weyl character of $N_q(\lambda)$, that is

$$Ch(\lambda) = \sum_\mu d^\lambda_\mu e^\mu.$$

Finally let ϕ be the automorphism of $\mathbb{C}[\Lambda]$ such that

$$\phi(e^{\mu}) = q^{2\mu(-2h_{\rho})}e^{2\mu}.$$

PROPOSITION 6.2.

$$\pi_A(M_\lambda F_\lambda) = \phi(Ch(\lambda)).$$

In particular $\phi^{-1} \circ \pi_A$ *establishes an algebra isomorphism between* $(A(G \parallel K), \times)$ *and* $\mathbb{C}[P]^{\mathscr{W}}$.

Proof. First of all observe that

$$((w^{\lambda}_{\mu\nu ij})^*)^{\vee}(q^h) = (S(S(q^h)^{\vee})^* \cdot v^{\lambda}_{\mu i}, v^{\lambda}_{\nu j})$$
$$= q^{\mu(h)}\delta_{\mu\nu}\delta_{ij}$$

therefore

$$\pi(((w^{\lambda}_{\mu\nu ij})^*)^{\vee}) = \delta_{\mu\nu}\delta_{ij}e^{\mu}. \tag{6.3}$$

The first part of the statement follows now directly from the definition of F_λ, (6.1), and (6.3).

Since $\{Ch(\lambda)\}_{\lambda \in P^+}$ is a basis of $\mathbb{C}[P]^{\mathscr{W}}$, the second part of the statement follows readily from Theorem 5.2. □

The Macdonald polynomials $p_\lambda(q, k)$ associated to the root system R of $\mathfrak{g}$ were introduced in [11] (see also [12]). More precisely the polynomials $p_\lambda(q, k)$ form a basis of $\mathbb{C}[\Lambda]^{\mathscr{W}}$ as λ runs through Λ^+ and depend on a real parameter q and a function $k: R \to \mathbb{R}$ such that $k_\alpha = k_\beta$ if $\alpha = w \cdot \beta$ for some $w \in \mathscr{W}$.

Suppose now that G/K is a real symmetric space and write $G = KAN$ for an Iwasava decomposition of G. If R is the reduced root system of the symmetric pair (G, K) and one sets $k_\alpha = \frac{1}{2}m_\alpha$, where m_α is the multiplicity of the root, then, as $q \to 1$, the polynomials $p_\lambda(q, k)$ approach the value of spherical functions on G/K seen as functions on A.

As the polynomials $p_\lambda(q, k)$ are related to the spherical functions on symmetric spaces at the classical limit, we conjecture that for generic q they should be connected to spherical functions on quantum spaces, and indeed this is proven to be true for some rank one groups (compare [1, 13]). We claim that Proposition 6.2 gives a further improvement in this direction.

In fact the real roots of a reductive complex group have all multiplicity two, and, when $k_\alpha = 1$ for all $\alpha \in R$, then $p_\lambda(q, k) = Ch(\lambda)$. It follows that, if we interpret π_A as the restriction to A, then Proposition 6.2 says that the restriction to A of the spherical functions in A_G are essentially the Macdonald polynomials.

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