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TAKAKAZU SATOH

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Wiener measures on certain Banach spaces over non-Archimedean local fields

TAKAKAZU SATOH

Department of Mathematics, Faculty of Science, Saitama University, 255 Shimo-ookubo, Urawa, Saitama, 338, Japan

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Dedicated to Proffessor Hideo Shimizu on his 60th birthday

1. Introduction

The integral representation is a fundamental tool to study analytic properties of the zeta function. For example, the Euler p-factor of the Riemann zeta function is

$$(1 - p^{-s})^{-1} = \int_{\mathbf{Z}_p} |x|^{s-1} \frac{p}{p-1}\, dx,$$

where dx is a Haar measure of $\mathbf{Q}_p$ and $p/(p-1)$ is a normalizing constant so that measure of the unit group is 1. What is necessary to generalize such integrals to a completion of a ring finitely generated over $\mathbf{Z}_p$? For example, let T be an indeterminate. The two dimensional local field $\mathbf{Q}_p((T))$ is an infinite dimensional $\mathbf{Q}_p$-vector space. It is not locally compact, and no Haar measure exists over it. So, before considering integral representations, we must establish a σ-additive measure on an infinite dimensional space.

In case of real vector spaces, Wiener [10] constructed a σ-additive measure on the space of continuous functions on $[0, 1]$ vanishing at the origin. Gross [5] generalized this and constructed abstract Wiener measure. In this paper, we construct a non-Archimedean version of Wiener measure over a normed vector space with an orthonormal Schauder basis (which corresponds to notion of real separable Hilbert spaces). (Theorem 3.17).

Following Kuo [6, Chap. I] we summarize construction of abstract Wiener measure. Let $(H, \langle \cdot, \cdot \rangle)$ be a real separable Hilbert space, and

FOP(H) the set of all orthogonal projections whose image have finite dimension. A set of the form

$$\{x \in H : P(x) \in F\}$$

where $P \in \mathrm{FOP}(H)$ and F is a Borel subset of $P(H)$ is called a cylinder set in H. We denote by Cyl(H) the set of all cylinder sets in H. The Gauss measure

$$\mu(\{x \in H : P(x) \in F\}) = (2\pi)^{-\dim P(H)/2} \int_F \exp\left(-\frac{\langle x, x\rangle}{2}\right) dx,$$

(with variance 1) is a measure in H. Here, dx is the Lebesgue measure in $P(H)$. This is finitely additive but not σ-additive. The seminorm $\|\cdot\|$ in H is called measurable if, for any $\varepsilon > 0$, there is $P \in \mathrm{FOP}(H)$ satisfying

$$\mu(\{x \in H : \|Q(x)\| > \varepsilon\}) < \varepsilon \tag{1.1}$$

for all $Q \in \mathrm{FOP}(H)$ with $P(H) \perp Q(H)$. A measurable norm induces a weaker topology than the topology induced by the inner product of H. Let B be a completion of H with respect to the $\|\cdot\|$-topology. Generally, B is a Banach space but not a Hilbert space. A subset of the form

$$\{x \in B : (y_1(x), \ldots, y_n(x)) \in F\}$$

where n is a suitable integer, $y_1, \ldots, y_n \in B^*$ and F is a Borel subset in $\mathbf{R}^n$, is called a cylinder set in B. We denote by Cyl*(B) the set of all cylinder sets in B. If $T \in \mathrm{Cyl}^*(B)$, then $T \cap H \in \mathrm{Cyl}(H)$. Now put

$$\tilde{\mu}(T) = \mu(T \cap H).$$

The measure $\tilde{\mu}$ is a σ-additive measure on the Borel field of B ([6, Chap. I, Theorem 4.1, 4.2]).

Let K be a non-Archimedean local field and H a normed K-vector space. Unfortunately there is no inner product (bi-linear map) on H which gives the norm of H. There is notion of orthogonality in H, but a norm direct supplement (analogue of orthogonal complement) may not exist. Even if it exists, uniqueness does not hold. The problem what is non-Archimedean analogue of the normal distribution is another difficulty. We define admissible measure on K in Definition 3.3. For each admissible measure ν, we define the notion of cylinder sets in H and a non-Archimedean version of Gauss measure G_ν in Lemma 3.6 for some class of H. This is canonical in the sense that G_v is independent of choice of particular coordinate system. Section 2 is a preparation. For normed vector spaces with orthonormal Schauder basis (cf. Definition 2.9) we study sufficient conditions for existence of norm direct supplements, orthogonal projections, and so on. (E.g. Lemma 2.6 and Lemma 2.13). Then, we define measurability of seminorm on H in Definition 3.8. Once these are established, we can construct non-Archimedean Wiener measure G_v with parameter v by the similar method to the real case. One of the merits of our method is that we can compute some integrals associated to arithmetic objects. Two examples are explicitly calculated in Section 5. In the Archimedean case, the abstract Wiener measure is a generalization of the classical Wiener measure corresponding to stochastic process on $C([0, 1])$. We note Evans [3, 4], and Albeverio and Karkowski [1], have studied stochastic process on local fields. Finding the relation between our work and these works would be interesting.

The result of this paper for a special v is announced in the symposium "Construction of automorphic L-function and its application" held on Nov. 5–11, 1991 at RIMS, Kyoto [9].

2. Review on non-Archimedean analysis

Following Bosch, Güntzer and Remmert [2, Chap. II], we summarize some basic facts on non-Archimedean analysis. We introduce a notion of orthogonal projections and prove some lemmas which are necessary to construct Wiener measure.

Let K be a field with non-Archimedean multiplicative valuation $|\cdot|$. Let H be a normed K-vector space with a norm also denoted by $|\cdot|$. In other words, H is a normed K-vector space whose norm satisfies

$$|a|\,|x| = |ax| \qquad (a \in K,\ x \in H),$$

$$|x + y| \leqslant \max(|x|, |y|) \qquad (x,\ y \in H).$$

We denote the identity operator of H by 1_H. For $A \subset H$ and $x \in H$, we put

$$|x, A| = \inf_{a \in A} |a - x|.$$

For an integer $n \geqslant 1$, let K^n be a normed K-vector space provided with the norm

$$|(c_1, \ldots, c_n)| = \max_{1 \leqslant i \leqslant n} |c_i|.$$

DEFINITION 2.1. Let $V_1, V_2, \ldots, V_n$ be subspaces of a normed K-vector space H. The spaces $V_1, V_2, \ldots, V_n$ are *mutually orthogonal* if

$$\left|\sum_{k=1}^{n} v_k\right| = \max_{1 \leqslant k \leqslant n} |v_k|$$

holds for any $v_k \in V_k$. If this is the case, we say that the sum space $V_1 + V_2 + \cdots + V_n$ is a *norm direct sum* and denote it $V_1 \oplus V_2 \oplus \cdots \oplus V_n$. Especially, for two normed K-vector spaces V and W, we write $V \perp W$ if V and W are mutually orthogonal.

As is readily verified, norm direct sum space is a direct sum of each component as a K-vector space.

DEFINITION 2.2. A subspace V of H admits a *norm direct supplement* (in H) if there is a subspace W of H satisfying $H = V \oplus W$.

This notion corresponds to an orthogonal complement of real Hilbert spaces. Although there always exists the unique orthogonal complement of a closed subspace of a real Hilbert space, a norm direct supplement may not exist and may not be unique even if it exists for non-Archimedean cases. For example,

$$K^2 = K(1, 0) \oplus K(a, 1)$$

holds for any $a \in K$ satisfying $|a| < 1$.

DEFINITION 2.3. The subspace V of H is *strictly closed* if, for all $h \in H$, there is $v = v(h) \in V$ satisfying

$$|h - v| = |h, V|.$$

If a subspace V is strictly closed, then it is closed. Conversely, a closed subspace V is strictly closed if $|V - \{0\}|$ is discrete (in $\{x \in \mathbf{R} : x > 0\}$) ([2, Lemma 1.1.5/3, Proposition 1.1.5/4]).

DEFINITION 2.4. A normed K-vector space V is *spherically complete* if each descending chain of open balls $B(v_n, r_n) = \{x \in V : |x - v_n| \leqslant r_n\}$, where $r_n > 0$, $v_n \in V$ and $n \geqslant 1$, always has non-empty intersection.

Spherically complete spaces are complete. Conversely, if V is complete and $|V - \{0\}|$ is discrete, then V is spherically complete. If K is spherically complete, then every finite dimensional normed K-vector space is spherically complete ([2, Lemma 2.4.4/4]).

LEMMA 2.5. *Let V be a strictly closed subspace of H, and U a spherically complete subspace of H. If U is orthogonal to V, then $U \oplus V$ is a strictly closed subspace of H.*

Proof. Let $v \in H - (U \oplus V)$. Then there are vectors $a_n \in U$ and $b_n \in V$ such that the sequence $d_n = |v - (a_n + b_n)|$ is strictly monotone decreasing and that $\lim_{n\to\infty} d_n = |v, U \oplus V|$. We have

$$\begin{aligned}|(a_{n+1} + b_{n+1}) - (a_n + b_n)| &= |(v - (a_n + b_n)) - (v - (a_{n+1} + b_{n+1}))| \\ &\leqslant \max(d_n, d_{n+1}) = d_n.\end{aligned}$$

On the other hand,

$$\begin{aligned}|(a_{n+1} + b_{n+1}) - (a_n + b_n)| &= \max(|a_{n+1} - a_n|, |b_{n+1} - b_n|) \\ &\geqslant |a_{n+1} - a_n|\end{aligned}$$

since $U \perp V$. Hence, $|a_{n+1} - a_n| \leqslant d_n$. We put $B_n = \{x \in U : |x - a_n| \leqslant d_n\}$ to obtain a descending chain of open balls $\{B_n\}_{n=1}^{\infty}$. Because U is spherically complete, the set $B = \bigcap_{n=1}^{\infty} B_n$ is non-empty. Let a be any element of B. Since V is strictly closed, there is $b \in V$ satisfying $|v - a, V| = |v - a - b|$. For an integer $n \geqslant 1$, we have

$$\begin{aligned}|v - (a + b)| &\leqslant |v - a - b_n| = |v - a_n - b_n + a_n - a| \\ &\leqslant \max(|v - a_n - b_n|, |a_n - a|) \\ &\leqslant \max(d_n, d_n) \to |v, U \oplus V|.\end{aligned}$$

Therefore $|v - (a + b)| = |v, U \oplus V|$. □

LEMMA 2.6. *Let K be a spherically complete field. Let V be a strictly closed subspace of H with finite codimension. Then V has a norm direct supplement.*

Proof. We use induction on $n = \operatorname{codim} V$. For the case of $n = 1$, we pick an $a \in H - V$. Since V is strictly closed, there exists $x \in V$ satisfying $|x - a| = |a, V|$. Put $W = K(x - a)$. Then $H = V + W$ since $n = \operatorname{codim} V = 1$. Moreover, $V \perp W$ follows from [2, Observation 2.4.2/2]. Let $n > 1$. A similar argument shows the existence of $a_n \in H$ satisfying $Ka_n \perp V$. Since K is spherically complete, so is Ka_n. By Lemma 2.5, $Ka_n \oplus V$ is strictly closed and $\operatorname{codim} Ka_n \oplus V = \operatorname{codim} V - 1$. By the induction hypothesis, $Ka_n \oplus V$ has a norm direct supplement W_{n-1}. Then, $W_{n-1} \oplus Ka_n$ is a norm direct supplement of V. □

DEFINITION 2.7. Let V be a finite dimensional normed K-vector space and put $n = \dim_K V$. A basis $\{e_1, e_2, \dots, e_n\}$ of V is an *orthogonal base* if

$$\left|\sum_{i=1}^{n} c_i e_i\right| = \max_{1 \leqslant i \leqslant n} |c_i e_i|$$

holds for all $c_1, \dots, c_n \in K$. Moreover, if $|e_i| = 1$ for all i, this basis is called an *orthonormal basis.*

Such a basis may not exist. If K is spherically complete, every finite dimensional normed K-vector space has orthogonal basis. Moreover, if V also satisfies $|V| \subset |K|$, then V has an orthonormal basis ([2, Proposition 2.4.4/2, Observation 2.5.1/2]). However, if K is not spherically complete, there exists a two-dimensional K-vector space which has no orthogonal basis ([2, p. 193]).

DEFINITION 2.8. A normed K-vector space H (of arbitrary dimension) is called *K-cartesian* (resp. *strictly K-cartesian*) if every finite dimensional subspace of H has an orthogonal basis (resp. an orthonormal basis).

DEFINITION 2.9. A countable subset $\{e_i\}_{i=1}^{\infty}$ of H is called an *orthogonal Schauder base* if it satisfies the following two conditions:

(1) For any $v \in H$, there is the unique $\{c_i \in K\}_{i=1}^{\infty}$ such that $\Sigma_{i=1}^{\infty} c_i e_i$ converges to v.
(2) For any converging series $\Sigma_{i=1}^{\infty} c_i e_i$, we have

$$\left|\sum_{i=1}^{\infty} c_i e_i\right| = \max_{1 \leqslant i < \infty} |c_i e_i|.$$

In addition to above conditions, if $|e_i| = 1$ for all i, we call $\{e_i\}_{i=1}^{\infty}$ an *orthonormal Schauder basis* of H.

If H has an orthogonal Schauder basis, H is K-cartesian ([2, Proposition 2.7.2/7]), and if H has an orthonormal Schauder basis, it is strictly K-cartesian ([2, Proposition 2.7.5/1]).

DEFINITION 2.10. A linear map $P \in \mathrm{Hom}_K(H, H)$ is called an *orthogonal projection* if $P^2 = P$ and $\mathrm{Im}\, P \perp \mathrm{Ker}\, P$. We denote by FOP($H$) the set of all orthogonal projections of H with $\dim_K P(H) < \infty$. We order FOP(H) by putting $P \leqslant Q$ whenever $P(H) \subset Q(H)$ and $\mathrm{Ker}\, P \supset \mathrm{Ker}\, Q$.

An orthogonal projection P is continuous because

$$|x| = \max(|P(x)|, |x - P(x)|) \geqslant |P(x)|.$$

Let $x \in H - \mathrm{Ker}\, P$. For all $y \in \mathrm{Ker}\, P$ we have

$$|x - y| = |P(x) - y + (x - P(x))| \geqslant |P(x)|$$

where $y = x - P(x)$ attains equality. Hence $\operatorname{Ker} P$ is a strictly closed subspace of H.

LEMMA 2.11. *Let P and Q be orthogonal projections of H. Assume $\operatorname{Ker} P \subset \operatorname{Ker} Q$. Then $PQ(H) \perp \operatorname{Ker} Q$.*

Proof. We have only to show

$$|x + y| \geqslant |x| \tag{2.1}$$

for $x \in PQ(H)$ and $y \in \operatorname{Ker} Q$ satisfying $|x| = |y|$. Since Q is an orthogonal projection,

$$|x + y| \geqslant |Q(x + y)| = |Q(x)|.$$

Pick $z \in Q(H)$ so that $x = P(z)$. Since $\operatorname{Ker} P \subset \operatorname{Ker} Q$ implies $QP = Q$,

$$|Q(x)| = |QP(z)| = |z| \geqslant |P(z)| = |x|.$$

This proves (2.1). □

LEMMA 2.12. *Let K be spherically complete. Let H be a normed K-vector space with an orthogonal Schauder basis. Assume that $|H - \{0\}|$ is discrete. Then* $(\mathrm{FOP}(H), \leqslant)$ *is a directed set.*

Proof. Reflexivity and transitivity are obvious. For any $P_1, P_2 \in \mathrm{FOP}(H)$, let V be the sum space $P_1(H) + P_2(H)$. Since K is spherically complete and V is finite dimensional, V is spherically complete. Since H has orthogonal Schauder basis, V has a norm direct supplement U by [2, Proposition 2.7.2/7]. Put

$$A = U \cap \operatorname{Ker} P_1 \cap \operatorname{Ker} P_2. \tag{2.2}$$

Then A is a closed subspace of H with finite codimension because all subspaces appearing in the right side of (2.2) have the same properties. Since $|H - \{0\}|$ is discrete, A is strictly closed. Noting $V \perp A$, we see $V \oplus A$ is a strictly closed subspace with finite codimension by Lemma 2.5. Let W be a (one of) norm direct supplement of V, which surely exists by Lemma 2.6. Let $Q \in \mathrm{FOP}(H)$ be the projection to $V \oplus W$ component of the decomposition $H = V \oplus W \oplus A$. By definition, $Q \geqslant P_1$ and $Q \geqslant P_2$. □

LEMMA 2.13. *Let K be spherically complete. Let H be a normed K-vector space with an orthonormal Schauder basis and $|H - \{0\}|$ be discrete. For any* *given* $\{F_n \in \mathrm{FOP}(H)\}_{n=1}^{\infty}$, *there exists a sequence* $\{P_n \in \mathrm{FOP}(H)\}_{n=1}^{\infty}$ *satisfying the following four conditions*:

(2.3) $\operatorname{Ker} P_n \subset \operatorname{Ker} F_n$ *for all* $n \geqslant 1$,
(2.4) $\operatorname{Ker} P_n \subset \operatorname{Ker} P_{n-1}$ *for all* $n \geqslant 2$,
(2.5) $P_n(H) \supset P_{n-1}(H)$ *for all* $n \geqslant 2$,
(2.6) $\lim_{n\to\infty} P_n = 1_H$ *(strongly)*.

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal Schauder basis of H. Define $E_n \in \mathrm{FOP}(H)$ by

$$E_n\left(\sum_{i=1}^{\infty} c_i e_i\right) = \sum_{i=1}^{n} c_i e_i.$$

Let P_0 be the zero map. Assume P_{n-1} is defined. We define P_n as any element of $\mathrm{FOP}(H)$ satisfying $P_n \geqslant P_{n-1}$, $P_n \geqslant F_n$ and $P_n \geqslant E_n$, whose existene follows from Lemma 2.12. By the definition of order, (2.3), (2.4) and (2.5) hold. Using $P_n(H) \supset E_n(H)$, we have $P_n E_n = E_n$. For any $x \in H$,

$$|(1_H - P_n)(x)| = |(1_H - P_n)(1_H - E_n)(x)| \leqslant |(1_H - E_n)(x)|. \tag{2.7}$$

Since $\{e_i\}_{i=1}^{\infty}$ is an orthonormal Schauder basis of H, we see that (2.7) converges to 0 as $n \to \infty$. This proves (2.6). □

We end this section with two simple measure theoretic lemmas. Let n be a nonnegative integer. A Borel measure λ on K^n is isometry invariant if $\lambda(T(E)) = \lambda(E)$ for all linear isometries T of K^n and all Borel sets E. Note the Haar measure μ_n of K^n is isometry invariant. We normalize μ_n as

$$\mu_n(\{x \in K^n : |x| \leqslant 1\}) = 1. \tag{2.8}$$

LEMMA 2.14. *Let* $n \geqslant 1$ *be an integer and* V *an n-dimensional normed K-vector space with an orthonormal base* $\{e_i\}_{i=1}^{n}$. *Define* $\psi \in \operatorname{Hom}(K^n, V)$ *by*

$$\psi(c_1, \ldots, c_n) = \sum_{i=1}^{n} c_i e_i. \tag{2.9}$$

Then for any Borel set E *of* V, *the value* $\lambda(\psi^{-1}(E))$ *is independent of choice of an orthonormal base of* V.

Proof. Let $\{e_i'\}_{i=1}^{n}$ be another orthonormal base of V and put

$$\psi'(c_1, \ldots, c_n) = \sum_{i=1}^{n} c_i e_i'.$$

Since $\psi^{-1}\circ\psi'$ is an isometry of K^n, the lemma follows from $\psi^{-1}(E) = (\psi^{-1}\circ\psi')(\psi'^{-1}(E))$. □

In the rest of this paper, we put

$$\lambda_V(E) = \lambda(\psi^{-1}(E)) \tag{2.10}$$

and

$$\mu_V(E) = \mu_{\dim V}(\psi^{-1}(E)), \tag{2.11}$$

which are well defined by Lemma 2.14.

LEMMA 2.15. *Let H be a normed K-vector space with an orthonormal Schauder basis. Let P and Q be elements of* FOP(H) *satisfying* $\operatorname{Ker} P \subset \operatorname{Ker} Q$. *Then for a real valued function f and a Borel subset D of $Q(H)$, one has*

$$\int_D f(|x|)d\mu_{Q(H)}(x) = \int_{P(D)} f(|x|)d\mu_{PQ(H)}(x). \tag{2.12}$$

(*If one value exists, then the other exists and two values are equal.*)

Proof. The assumption $\operatorname{Ker} P \subset \operatorname{Ker} Q$ implies $QP = Q$. Since

$$|x| \geqslant |P(x)| \geqslant |QP(x)| = |Q(x)| = |x|$$

for $x \in Q(H)$, the restriction of P to $Q(H)$ is a surjective isometry from $Q(H)$ to $PQ(H)$. Especially, if $\{e_i\}_{i=1}^n$ is an orthogonal basis of $Q(H)$, then $\{P(e_i)\}_{i=1}^n$ is an orthogonal basis of $PQ(H)$. By Lemma 2.14, both hand sides of (2.12) are

$$\int_{\psi^{-1}(D)} f(|\psi(x)|)d\mu_n(x),$$

where ψ is defined by (2.9). □

3. Construction of non-Archimedean Wiener measure

In the previous section, we defined the notion of orthogonal projections of normed K-vector spaces with orthogonal Schauder basis. Using them, we construct Wiener measures on non-Archimedean local fields. In our construction, Proposition 3.7, Lemma 3.10, Lemma 3.11, Lemma 3.12, Lemma 3.13, Theorem 3.17 and Theorem 3.18 correspond to Proposition 4.1, Lemma 4.1, Lemma 4.2, Lemma 4.4, Lemma 4.5, Theorem 4.1 and Theorem 4.2 of Kuo [6, Chap. I], respectively where real Wiener measure is constructed. However our definition of a measurable seminorm (see Definition 3.8) is different from that of real case (1.1).

Throughout this section, K denotes a non-Archimedean local field. Therefore K is spherically complete. We normalize a valuation of K as $|\pi| = 1/q$, where q is a cardinality of residue class field of K and $\pi = \pi_K$ is a prime element of $|\cdot|$. We denote by H a normed K-vector space with an orthonormal Schauder basis. The existence of orthonormal Schauder basis implies that $|H - \{0\}|$ $(=|K - \{0\}|)$ is discrete.

DEFINITION 3.1. A subset E of H of the following form is called a *cylinder set*,

$$E = P^{-1}(F) = \{x \in F : P(x) \in H\},$$

where $P \in \mathrm{FOP}(H)$ and F is a measurable subset of $P(H)$. We denote by $\mathrm{Cyl}(H)$ the set of all cylinder sets in H.

LEMMA 3.2. *The set* $\mathrm{Cyl}(H)$ *is a field of sets.*

Proof. Let $E_1 = P_1^{-1}(F_1)$ and $E_2 = P_2^{-1}(F_2)$, where $P_1, P_2 \in \mathrm{FOP}(H)$ and F_1 and F_2 are measurable subsets in $P_1(H)$ and $P_2(H)$, respectively. Since both $\mathrm{Ker}\, P_1$ and $\mathrm{Ker}\, P_2$ are closed subspaces of H with finite codimension, $\mathrm{Ker}\, P_1 \cap \mathrm{Ker}\, P_2$ is also a closed subspace with finite codimension. Hence it is strictly closed because $|H - \{0\}|$ is discrete. By Lemma 2.6, there is $P \in \mathrm{FOP}(H)$ satisfying $\mathrm{Ker}\, P = \mathrm{Ker}\, P_1 \cap \mathrm{Ker}\, P_2$. Then, for $i = 1, 2$, we have

$$\begin{aligned} P^{-1}(E_i \cap P(H)) &= \{x \in H : P(x) \in E_i \cap P(H)\} \\ &= \{x \in H : P(x) \in E_i\} \\ &= \{x \in H : P_i P(x) \in F_i\}. \end{aligned}$$

Since P is a projection and $\mathrm{Ker}\, P_i \supset \mathrm{Ker}\, P$,

$$P^{-1}(E_i \cap P(H)) = \{x \in H : P_i(x) \in F_i\} = E_i. \tag{3.1}$$

By the definition, F_i is a measurable set in $P_i(H)$. Since restriction of P_i to $P(H)$ is continuous, $E_i \cap P(H) = \{x \in P(H) : P_i(x) \in F_i\}$ is a measurable subset in $P(H)$. Therefore both

$$E_1 \cup E_2 = P^{-1}((E_1 \cap P(H)) \cup (E_2 \cap P(H))) \tag{3.2}$$

and

$$E_1 \cap E_2 = P^{-1}(E_1 \cap E_2 \cap P(H))$$

belong to Cyl(H). On the other hand,

$$E_1^c = P_1(F_1^c).$$

Thus Cyl(H) is closed under taking union, intersection and complement.

□

For a nonnegative integer n and a nonnegative real number r, put

$$A_n(r) = \{x \in K^n : |x| = r\},$$
$$C_n(r) = \{x \in K^n : |x| \leqslant r\}.$$

Under our normalization (2.8) of the Haar measure of K^n, we have $\mu_n(C_n(1)) = 1$. We abbreviate μ_1 as μ for simplicity.

DEFINITION 3.3. A probabilistic measure ν on K is said to be *admissible* if ν satisfies the following two conditions:

(1) The measure ν is isometry invariant absolute continuous measure with respect to μ.
(2) The value of Radon-Nykodim derivative $d\nu/d\mu(x)$ at $x = \pi^n$ is a non-decreasing function on n.

Let ν be an admissible measure on K. For $m \in \mathbf{Z}$, put

$$\beta_m = \frac{d\nu}{d\mu}(\pi^{-m}) - \frac{d\nu}{d\mu}(\pi^{-m-1}). \tag{3.3}$$

The condition (2) implies $\beta_m \geqslant 0$. Since

$$1 = \nu(K) = \sum_{n=-\infty}^{\infty} (q^n - q^{n-1}) \frac{d\nu}{d\mu}(\pi^{-n}) = \left(1 - \frac{1}{q}\right) \sum_{n=-\infty}^{\infty} q^n \frac{d\nu}{d\mu}(\pi^{-n}),$$

we have

$$\lim_{n \to \pm\infty} q^n \frac{d\nu}{d\mu}(\pi^{-n}) = 0. \tag{3.4}$$

Noting convergence, we obtain

$$\sum_{m=N+1}^{\infty} \beta_m q^m = q^{N+1} \frac{d\nu}{d\mu}(\pi^{-N-1}) + \nu(C_1(q^{N+1})^c), \tag{3.5}$$

$$\sum_{m=-\infty}^{N} \beta_m q^m = -q^N \frac{d\nu}{d\mu}(\pi^{-N-1}) + \nu(C_1(q^N)) \tag{3.6}$$

and

$$\sum_{m=-\infty}^{\infty} \beta_m q^m = 1.$$

LEMMA 3.4. *Let ν be an admissible probabilistic measure on K and β_m as in* (3.3). *Define $\mathscr{D}_{\nu,n} \in \mathrm{Map}(|K|, \mathbf{R})$ by*

$$\mathscr{D}_{\nu,n}(t) = \begin{cases} 0 & (t = 0) \\ \sum_{i=m}^{\infty} q^{-(n-1)i}\beta_i & (|t| = q^m > 0). \end{cases}$$

Then $\mathscr{D}_{\nu,n}(|\cdot|) \in L^1(K^n, \mu_n)$. Moreover, for any Borel set E in K^n,

$$\int_E \mathscr{D}_{\nu,n}(|x|)d\mu_n(x) = \int_{E\times K} \mathscr{D}_{\nu,n+1}(\max(|x|, |y|))d\mu_{n+1}(x, y). \tag{3.7}$$

Proof. Put $a_{n,m} = \mathscr{D}_{\nu,n}(q^m)$ for simplicity. Noting $0 \leqslant a_{n,m} \leqslant q^{-(n-1)m}\sum_{i=m}^{\infty}\beta_m < \infty$, we obtain

$$q^m a_{n+1,m} + \sum_{l=m+1}^{\infty} a_{n+1,l}(q^l - q^{l-1}) = a_{n,m}.$$

Assume $E \subset A_n(q^m)$ for some $m \in \mathbf{Z}$. Then the left hand side of (3.7) is $a_{n,m}\mu_n(E)$, whereas the right hand side of (3.7) is

$$\sum_{i=-\infty}^{\infty} \mathscr{D}_{\nu,n}(\max(q^m, q^i))\mu_n(E)(q^i - q^{i-1})$$
$$= \mu_n(E)\left(a_{n+1,m}q^m + \sum_{i=m+1}^{\infty} a_{n+1,i}(q^i - q^{i-1})\right)$$

Therefore, (3.7) holds for $E \subset A_n(q^m)$. Note $\mathscr{D}_{\nu,1}(|x|) = d\nu/d\mu(x) \in L^1(K, \mu)$. We also see that $\int_K \mathscr{D}_{\nu,1}(|x|)d\mu(x) = \nu(K) = 1$. Assume $\int_{K^n} \mathscr{D}_{\nu,n}(|x|)d\mu_n(x) = 1$. Using (3.7) for $E = A_n(q^i)$ we have

$$\sum_{i=-N}^{M} \int_{A_n(q^i)\times K} \mathscr{D}_{\nu,n}(|x|)d\mu_{n+1}(x) = \sum_{i=-N}^{M} \int_{A_n(q^i)\times K} \mathscr{D}_{\nu,n}(|x|)d\mu_n(x)$$
$$= \int_{K^n} \chi_{C_n(q^M)-C_n(q^{-N})}(x)\mathscr{D}_{\nu,n}(|x|)d\mu_n(x) \leqslant 1$$

where $\chi_E(\cdot)$ is the characteristic function of E. Letting $M, N \to \infty$, we see that $\mathscr{D}_{\nu,n+1}(|\cdot|) \in L^1(K^{n+1}, \mu_{n+1})$ and $\int_{K^{n+1}} \mathscr{D}_{\nu,n+1}(|x|) d\mu_{n+1}(x) = 1$ by Levi's theorem. This and Lebesgue's convergence theorem establish (3.7) for an arbitrary Borel set E. □

We define a probabilistic measure $\nu^{\times n}$ on K^n by

$$\nu^{\times n}(E) = \int_E \mathscr{D}_{\nu,n}(|x|) d\mu_n(x),$$

which is σ-additive since $\mathscr{D}_{\nu,n} \in L^1(K^n, \mu_n)$. Then (3.7) is rephrased as

$$\nu^{\times n}(E) = \nu^{\times n+1}(E \times K). \tag{3.8}$$

For simplicity, we define $\nu^{\times V} = (\nu^{\times \dim V})_V$ (cf. (2.10)) for a finite dimensional normed K vector space V. It is straightforward to obtain

$$\begin{aligned}\nu^{\times n}(C_n(q^N)) &= \sum_{m=-\infty}^{N} \nu^{\times n}(A_n(q^m)) = (1 - q^{-n}) \sum_{m=-\infty}^{N} \left(\sum_{i=m}^{\infty} q^{-(n-1)i} \beta_i \right) q^{nm} \\ &= \sum_{i=-\infty}^{N} q^i \beta_i + \sum_{i=N+1}^{\infty} q^{-n(i-N)} \cdot q^i \beta_i,\end{aligned}$$

and hence

$$\sum_{i=-\infty}^{N} q^i \beta_i \leqslant \nu^{\times n}(C_n(q^N)) \leqslant \sum_{i=-\infty}^{N} q^i \beta_i + q^{-n} \sum_{i=N+1}^{\infty} q^i \beta_i. \tag{3.9}$$

DEFINITION 3.5. The *cylinder measure* G_ν with *parameter* ν is the function on Cyl(H) defined by the following formula:

$$G_\nu(P^{-1}(F)) = \nu^{\times P(H)}(F)$$

LEMMA 3.6. *The measure G_ν is well defined.*

Proof. For $E \in \mathrm{Cyl}(H)$, take any $P \in \mathrm{FOP}(H)$ and a measurable subset F of $P(H)$ satisfying $E = \{x \in H : P(x) \in F\}$. We set

$$U = \{x \in H : a + tx \in E \text{ for all } a \in E, t \in K\}.$$

(Note U depends on only E, not on the choice of P and F.) Then, U is a subspace of H and $U \supset \mathrm{Ker}\, P$. Let Q be an arbitrary element of FOP(H) satisfying $\mathrm{Ker}\, Q = U$. To show Q exists, we note that since $P(H)$ is K-cartesian, [2, Proposition 2.4.1/5] implies $U \cap P(H)$ has a norm direct supplement V in

$P(H)$, i.e. $P(H) = V \oplus (U \cap P(H))$. Let $Q \in \mathrm{FOP}(H)$ be an orthogonal projection to the V component of $H = V \oplus (U \cap P(H)) \oplus \operatorname{Ker} P$. Since $\operatorname{Ker} P \subset U$, we see $(U \cap P(H)) \oplus \operatorname{Ker} P \subset U$. On the other hand, for $x \in U$ we have $P(x) = x - (x - P(x)) \in U$ because U is a subspace containing $\operatorname{Ker} P$. Therefore $x = P(x) + (x - P(x)) \in (U \cap P(H)) \oplus \operatorname{Ker} P$. Thus

$$U = (U \cap P(H)) \oplus \operatorname{Ker} P = \operatorname{Ker} Q$$

and this Q is the desired one.

We note $Q = QP$ because of $\operatorname{Ker} P \subset \operatorname{Ker} Q$. We show

$$F = \{v + u : v \in PQ(F),\ u \in \operatorname{Ker} Q \cap P(H)\} \tag{3.10}$$

how we choose Q. Let $x \in F$. Using $QP = Q = Q^2$, we see $x - PQ(x) \in \operatorname{Ker} Q$. Hence decomposition $x = PQ(x) + (x - PQ(x))$ shows that the right hand side of (3.10) contains F. Conversely, for $u \in \operatorname{Ker} Q \cap P(H)$ and $v = PQ(t)$ with any $t \in F$, we have $u + PQ(t) - t \in \operatorname{Ker} Q = U$. Using $t \in F \subset E$ and the definition of U we have $u + v \in E$, namely, $P(u + v) \in F$. Hence $u + v \in F$ by $u + v \in P(H)$. Therefore F contains the right hand side of (3.10). Moreover, for $x \in F$, vectors $u \in \operatorname{Ker} Q \cap P(H)$ and $v \in PQ(F)$ satisfying $x = u + v$ are unique by Lemma 2.11. For simplicity, we put $S = \operatorname{Ker} Q \cap P(H)$. Lemma 2.11 also yields $PQ(F) \perp S$. Summarizing above results, we have

$$\begin{aligned} G_\nu(P^{-1}(F)) &= \int_F \mathscr{D}_{\nu,\dim P(H)}(|x|)\, d\mu_{P(H)}(x) \\ &= \int_{(u,v) \in PQ(F) \times S} \mathscr{D}_{\nu,\dim P(H)}(\max(|u|, |v|))\, d\mu_{PQ(H) \times S}(u, v) \end{aligned}$$

(See (2.11) for the definition of $\mu_{P(H)}$ etc.) By repeated use of (3.8),

$$G_\nu(E) = \int_{u \in PQ(F)} \mathscr{D}_{\nu,\dim PQ(H)}(|u|)\, d\mu_{PQ(H)}(u).$$

By Lemma 2.15 and $Q(E) = QP(E) = Q(F)$, we obtain

$$\begin{aligned} G_\nu(E) &= \int_{u \in Q(F)} \mathscr{D}_{\nu,\dim Q(H)}(|u|)\, d\mu_{Q(H)}(u) \\ &= \int_{u \in Q(E)} \mathscr{D}_{\nu,\dim Q(H)}(|u|)\, d\mu_{Q(H)}(u). \end{aligned}$$

The last formula is independent of choice of P and F. □

The measure G_ν is finitely additive by (3.1) and (3.2). However, we have the following result.

PROPOSITION 3.7. *The measure G_ν is not σ-additive.*

Proof. Let $U(x; r)$ be the open ball in H with center x and radius r. Suppose G_ν is σ-additive. Then G_ν extends to a σ-additive measure Γ on $\sigma[\mathrm{Cyl}(H)]$. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal Schauder basis of H and P_i the orthogonal projection to e_i-component. Observe that

$$U\left(\sum_{i=1}^\infty c_i e_i; r\right) = \bigcap_{i=1}^\infty \{x \in H : |P_i(x) - c_i e_i| < r\} \in \sigma[\mathrm{Cyl}(H)]$$

and that H is a separable Banach space. Hence $\sigma[\mathrm{Cyl}(H)]$ contains the Borel field of H. This implies Γ is a Borel measure on H satisfying $\Gamma(H) = 1$. Therefore Γ is a tight measure by Parthasarathy [7, Chap. II, Theorem 3.2]. That is, for every $\varepsilon > 0$, there exists compact set F_ε of $(H, |\cdot|)$ satisfying $\Gamma(F_\varepsilon) > 1 - \varepsilon$.

Let β_m be as in (3.3). Choose $N \in \mathbf{Z}$ so that $\Sigma_{i=-\infty}^N q^i \beta_i < \frac{1}{3}$. Since $F_{1/3}$ is compact, there are finitely many elements $x_1, \dots, x_l$ of $F_{1/3}$ satisfying $F_{1/3} \subset \bigcup_{j=1}^l U(x_j; q^N)$. Let $x_j = \Sigma_{i=1}^\infty c_{ji} e_i$. Since $\lim_{j\to\infty} |c_{ji}| = 0$, there is an $M \in \mathbf{Z}$ such that $|c_{j,i}| < q^N$ for all $1 \leqslant j \leqslant l$ and all $i > M$. Hence

$$\begin{aligned} F_{1/3} &\subset \left\{\sum_{i=1}^\infty a_i e_i : |a_i| < q^N \text{ for all } i > M\right\} \\ &\subset \left\{\sum_{i=1}^\infty a_i e_i : |a_i| \leqslant q^N \text{ for } M < i < M + m\right\}, \end{aligned}$$

where m is any integer satisfying $q^{-m} < \frac{1}{3}$. This and (3.9) imply

$$\frac{3}{2} < \Gamma(F_{1/3}) \leqslant \nu^{\times m}(C_m(q^N)) < \frac{2}{3},$$

which is contradiction. □

For $P, Q \in \mathrm{FOP}(H)$ satisfying $P \leqslant Q$, put $\xi_P^Q = P|_{Q(H)}$. It is easy to see $\xi_P^R = \xi_P^Q \circ \xi_Q^R$ for $P \leqslant Q \leqslant R$. By Lemma 3.6, $(\{\nu^{\times P(H)}\}_{P \in \mathrm{FOP}(H)}, \{\xi_P^Q\}_{P \leqslant Q})$ is a consistent family of tight Radon probabilistic measure on completely regular spaces (actually on completely separable metric spaces). Moreover, ξ_P^Q is a surjective continuous map. Put $\Omega = \mathrm{proj.lim}_{Q \in \mathrm{FOP}(H)} Q(H)$, where projective limit is taken in the category of topological spaces. For $P \in \mathrm{FOP}(H)$, let ξ_P be the projection from $\Omega \subset \Pi_{Q\in \mathrm{FOP}(H)} Q(H)$ to $P(H)$. Especially, $\{\xi_Q\}_{Q\in \mathrm{FOP}(H)}$ separates points of Ω (i.e. for each $\omega \neq \omega'$ in Ω, there is a $Q \in \mathrm{FOP}(H)$ with $\xi_Q(\omega) \neq \xi_Q(\omega')$). For all $(\omega_Q)_{Q\in \mathrm{FOP}(H)} \in \Omega$ and $P \leqslant Q$, we have $\omega_P = \xi_P^Q(\omega_Q)$, which implies $\xi_P = \xi_P^Q \circ \xi_Q$. On the other hand, let $P \in \mathrm{FOP}(H)$ be arbitrary. For each $x \in P(H) \subset H$, put $\hat{x} = (Q(x))_{Q\in \mathrm{FOP}(H)}$. We see $\hat{x} \in \Omega$ and $\xi_P(\hat{x}) = P(x) = x$. Therefore, ξ_P is surjective. By the Prokhorov's theorem on the existence of product measure on projective limit (see e.g. Rao [8, Sect. 6.4, Theorem 7]), there is a Radon probabilistic measure $\tilde{\nu}$ on Ω satisfying

$$\tilde{\nu}(\{\omega \in \Omega : \xi_P(\omega) \in F\}) = \nu^{\times P(H)}(F) = G_\nu(P^{-1}(F)) \tag{3.11}$$

for all $P \in \mathrm{FOP}(H)$ and all Borel set $F \subset P(H)$.

DEFINITION 3.8. A seminorm $\|\cdot\|$ in H is called *measurable* if for every $\varepsilon > 0$, there exists $P \in \mathrm{FOP}(H)$ satisfying $\|x\| \leqslant \varepsilon|x|$ for all $x \in \mathrm{Ker}\, P$.

EXAMPLE: Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal Schauder basis of H. The norm defined by

$$\left\| \sum_{i=1}^{\infty} c_i e_i \right\| = \max_{1 \leqslant i < \infty} a_i |c_i|$$

is a measurable norm in H.

LEMMA 3.9. *A measurable seminorm $\|\cdot\|$ on H satisfies the following condition:*

(*) *For every $\varepsilon > 0$, there exist $P \in \mathrm{FOP}(H)$ such that*
$G_\nu(\{x \in H : \|Q(x)\| > \varepsilon\}) < \varepsilon$ for all $Q \in \mathrm{FOP}(H)$
satisfying $\mathrm{Im}\, Q \subset \mathrm{Ker}\, P$.

Conversely a seminorm $\|\cdot\|$ on H satisfying (*) is measurable if there is a monotone increasing function $\varphi : (0, 1) \to (0, \infty)$ such that $\lim_{\varepsilon\to 0} \varphi(\varepsilon) = 0$ and that $\inf_{r>0} r\varphi(\nu(C_1(r)^c)) = M > 0$.

Proof. Let $\varepsilon > 0$ be arbitrary. Take $N \in \mathbf{N}$ so that $\nu^{\times n}(C_n(q^N)^c) < \varepsilon$ for all $n \in \mathbf{N}$ by (3.9). By the measurability of $\|\cdot\|$, there is $P \in \mathrm{FOP}(H)$ satisfying $\|x\| \leqslant \varepsilon q^{-N}|x|$ for all $x \in \mathrm{Ker}\, P$. Assume $Q \in \mathrm{FOP}(H)$ satisfies $\mathrm{Im}\, Q \subset \mathrm{Ker}\, P$. We have

$$\begin{aligned} G_\nu(\{x \in H : \|Q(x)\| > \varepsilon\}) &= \nu^{\times Q(H)}(\{x \in Q(H) : \|x\| > \varepsilon\}) \\ &\leqslant \nu^{\times Q(H)}(\{x \in Q(H) : |x| > q^N\}) \leqslant \varepsilon, \end{aligned}$$

i.e. (*) holds.

Conversely, let $\delta > 0$ be arbitrary. Put $\varepsilon = \min(M, \rho)$ where ρ is any real number satisfying $0 < \rho < 1$ and $\varphi(\rho) < \delta$ and take $P \in \mathrm{FOP}(H)$ in (*). It is enough to show $\|x\| \leqslant \delta$ for $x \in \operatorname{Ker} P$ satisfying $|x| = 1$ and $\|x\| \neq 0$. Let x be such an element of Ker P, Q an orthogonal projection whose image is Kx. Since $\operatorname{Im} Q \subset \operatorname{Ker} P$, we have

$$\varepsilon > \nu^{\times Kx}(\{v \in Kx : \|v\| > \varepsilon\}) = \nu(\{c \in K : |c|\,\|x\| > \varepsilon\}).$$

By the definition of M and monotone increasing property of φ,

$$\|x\| \leqslant \frac{M\|x\|}{\varepsilon} \leqslant \varphi(\nu(\{c \in K : |c|\,\|x\| > \varepsilon\})) < \varphi(\varepsilon) \leqslant \varphi(\rho) < \delta. \qquad \square$$

LEMMA 3.10. *Let* $\|\cdot\|$ *be a measurable seminorm. Then the net* $\{\|\xi_P(\cdot)\|\}_{P \in \mathrm{FOP}(H)}$ *converges in probability on* Ω. *This limit is denoted by* $\|\cdot\|^{\sim}$.

Proof. It is easy to see that $P \geqslant R$ and $Q \geqslant R$ for P, Q, $R \in \mathrm{FOP}(H)$ imply $\operatorname{Im}(P - Q) \subset \operatorname{Ker} R$. Note also, for $P \leqslant Q$,

$$\begin{aligned}\tilde{\nu}(\{\omega \in \Omega : |\,\|\xi_Q(\omega)\| - \|\xi_P(\omega)\|\,| > \varepsilon\}) &\leqslant \tilde{\nu}(\{\omega \in \Omega : \|(1_H - P)(\xi_Q(\omega))\| > \varepsilon\}) \\ &= \nu^{\times Q(H)}(\{x \in Q(H) : \|(1_H - P)x\| > \varepsilon\}) \\ &= G_\nu(\{x \in H : \|(Q - P)(x)\| > \varepsilon\})\end{aligned}$$

since $PQ = P$ and $\xi_P = P\xi_Q$. Therefore, this lemma follows from the same argument as that of [6, Lemma 4.1]. $\square$

LEMMA 3.11. *Let* $\|\cdot\|$ *be a measurable seminorm in* H. *Then there exists a positive constant* c *satisfying* $\|x\| \leqslant c|x|$ *for all* $x \in H$.

Proof. By the measurability of $\|\cdot\|$, there is $P \in \mathrm{FOP}(H)$ such that $\|x\| \leqslant |x|$ for all $x \in \operatorname{Ker} P$. Since $\dim P(H)$ is finite, there is an orthogonal basis $\{v_i\}_{i=1}^n$ of $P(H)$. Using $|y| \geqslant |c_i|\,|v_i|$ for $y = \Sigma_{i=1}^n c_i v_i \in P(H)$ we have

$$\|y\| \leqslant \max_{1 \leqslant i \leqslant n} |c_i|\,\|v_i\| \leqslant \max_{1 \leqslant i \leqslant n} \frac{\|v_i\|}{|v_i|}\,|y|.$$

Hence,

$$\begin{aligned}\|x\| &\leqslant \max(\|P(x)\|, \|x - P(x)\|) \\ &\leqslant \max\left(\max_{1 \leqslant i \leqslant n} \frac{\|v_i\|}{|v_i|}\,|P(x)|, |x - P(x)|\right)\end{aligned}$$

$$\leqslant \max\left(\max_{1\leqslant i\leqslant n}\frac{\|v_i\|}{|v_i|}, 1\right)|x|. \qquad \square$$

LEMMA 3.12. *Let* $\|\cdot\|$ *be a measurable seminorm in* H. *Let* $\{a_n \in \mathbf{R} : a_n > 0\}_{n=1}^{\infty}$ *be a sequence of positive real numbers. Then there exists* $\{Q_n \in \mathrm{FOP}(H)\}_{n=1}^{\infty}$ *satisfying the following four conditions*:

(3.12) $Q_m Q_n = \delta_{mn} Q_n$ *for all positive integers m and n.*
(3.13) $\Sigma_{n=1}^{\infty} Q_n = 1_H$ *(strongly).*
(3.14) $a_n \|Q_n(x)\| \leqslant n^{-1}|x|$ *for all* $x \in H$ *and* $n \geqslant 2$.
(3.15) $\|x\|_0 = \max_{1\leqslant n<\infty} a_n |Q_n(x)|$ *is a measurable seminorm in H. If* $\|\cdot\|$ *is a norm, so is* $\|\cdot\|_0$.

Proof. By the definition of measurable seminorn, there is $F_n \in \mathrm{FOP}(H)$ satisfying

$$\|x\| \leqslant \frac{1}{na_n}|x| \quad \text{for all } x \in \operatorname{Ker} F_n \tag{3.16}$$

for each n. Using Lemma 2.13, we find $P_n \in \mathrm{FOP}(H)$ satisfying (2.3)–(2.6). Put $Q_1 = P_2 \in \mathrm{FOP}(H)$, and $Q_n = P_{n+1} - P_n$ for $n \geqslant 2$. We note

$$P_n P_m = P_{\min(m,n)} \tag{3.17}$$

by (2.4) and (2.5). So we see Q_n satisfies (3.12). For $n \geqslant 2$,

$$Q_n(H) = \operatorname{Ker} P_n \cap P_{n+1}(H) \tag{3.18}$$

and

$$\operatorname{Ker} Q_n(H) = P_n(H) \oplus \operatorname{Ker} P_{n+1}$$

follow from (3.17). The subspaces $P_n H$, $\operatorname{Ker} P_{n+1}$ and $\operatorname{Ker} P_n \cap P_{n+1}(H)$ are mutually orthogonal by (2.4). Especially, $Q_n(H) \perp \operatorname{Ker} Q_n$. Therefore, $Q_n \in \mathrm{FOP}(H)$ for $n \geqslant 2$. Now, $\Sigma_{n=1}^{N} Q_n = P_{N+1}$ and this converges to 1_H strongly by (2.6), which proves (3.13).

For $n \geqslant 2$, (3.14) follows from (3.18), (2.3) and (3.16). This yields $\lim_{n\to\infty} a_n \|Q_n(x)\| = 0$, which proves existence of $\|x\|_0$. We prove measurability of $\|\cdot\|_0$. Let $\varepsilon > 0$ be arbitrary. Take an integer $N \geqslant \max(2, 1/\varepsilon)$. Using (3.17) and (3.14), we see

$$\|(1_H - P_N)(x)\|_0 = \max(a_1\|P_2(1_H - P_N)(x)\|, \max_{2\leqslant n<\infty} a_n \|(P_{n+1} - P_n)(1_H - P_N)(x)\|)$$

$$= \max_{N \leqslant n < \infty} a_n \|Q_n(x)\|$$

$$\leqslant \max_{N \leqslant n < \infty} |x|/n \leqslant \varepsilon|x|.$$

Hence $\|\cdot\|_0$ is measurable.

Finally, suppose $\|\cdot\|$ is a norm and $\|x\|_0 = 0$. Then, for all $n \geqslant 1$ we have $\|Q_n(x)\| = 0$, namely, $Q_n(x) = 0$. Hence $x = \Sigma_{n=1}^{\infty} Q_n(x) = 0$ by (3.13), and this implies that $\|\cdot\|_0$ is also a norm. This completes proof. □

LEMMA 3.13. *Let $\|\cdot\|$ be a measurable norm in H and B a completion of H with respect to $\|\cdot\|$-topology. Then, there exists a measurable norm $\|\cdot\|_0$ in H such that, for all $r > 0$,*

$$S_r = \{x \in H : \|x\|_0 \leqslant r\}$$

is precompact in B.

Proof. By Lemma 3.11, we can choose a positive real number a_1 satisfying $a_1\|x\| \leqslant |x|$ for all $x \in H$. Let $\{a_n \in \mathbf{R} : a_n > 0\}_{n=2}^{\infty}$ be a sequence of positive real numbers satisfying $\lim_{n\to\infty} a_n = \infty$. We show $\|\cdot\|_0$ defined in Lemma 3.12 is desired one.

To prove that S_r is precompact, it is sufficient to show that any infinite sequence $\{x_n\}_{n=1}^{\infty}$ in S_r has a Cauchy subsequence. We put $x_n^{(0)} = x_n$ and, for $k \geqslant 1$, define a subsequence $x_n^{(k)}$ of $x_n^{(k-1)}$ as follows. Note that

$$a_k \|Q_k(y)\| \leqslant \|y\|_0 \leqslant r \tag{3.19}$$

for any $y \in S_r$. Since $Q_k(H)$ is a finite dimensional space with an orthonormal basis, there is a subsequence $\{x_n^{(k)}\}_{n=1}^{\infty}$ of $\{x_n^{(k-1)}\}_{n=1}^{\infty}$ such that $\{Q_k(x_n^{(k)})\}_{n=1}^{\infty}$ is a $\|\cdot\|$-Cauchy sequence. Then $\{Q_k(x_n^{(n)})\}_{n=1}^{\infty}$ is a $\|\cdot\|$-Cauchy sequence for each $k \geqslant 1$.

Let $\varepsilon > 0$ be arbitrary. Let M be an integer such that $a_k > r/\varepsilon$ for all $k \geqslant M$. By Lemma 3.11 and (3.13),

$$\|x_n^{(n)} - x_m^{(m)}\| = \lim_{j\to\infty} \left\| \sum_{k=1}^{j} Q_k(x_n^{(n)} - x_m^{(m)}) \right\|$$

$$\leqslant \sup_{1 \leqslant k < \infty} \|Q_k(x_n^{(n)} - x_m^{(m)})\|. \tag{3.20}$$

For terms with $k \geqslant M$, (3.19) yields $\|Q_k(x_n^{(n)} - x_m^{(m)})\| \leqslant (r/a_k) < \varepsilon$. On the other hand, there is an integer N satisfying

$$\|Q_k(x_n^{(n)}) - Q_k(x_m^{(m)})\| < \varepsilon$$

for all $1 \leqslant k \leqslant M$ and m, $n > N$, since $\{Q_k(x_n^{(n)})\}_{n=1}^{\infty}$ is a $\|\cdot\|$-Cauchy sequence. Therefore (3.20) is less than ε for all $m, n \geqslant N$. Namely, $\{x_n^{(n)}\}_{n=1}^{\infty}$ is a $\|\cdot\|$-Cauchy sequence. □

DEFINITION 3.14. Let $\|\cdot\|$ be a measurable norm in H. Let B be a completion of H with respect to the $\|\cdot\|$-topology and B^* a set of all continuous linear maps from B to K. The subset T of B of the following form is called a *cylinder set* in B,

$$T = \{x \in B : (P_1(x), \ldots, P_n(x)) \in E\}$$

where $n \geqslant 1$ is an integer, E is a measurable set in K^n and $P_1, \ldots, P_n \in B^*$. We denote by $\mathrm{Cyl}^*(B)$ the set of all cylinder set in B.

LEMMA 3.15. *If* $T \in \mathrm{Cyl}^*(B)$, *then* $T \cap H \in \mathrm{Cyl}(H)$.

Proof. Let $T \in \mathrm{Cyl}^*(B)$. There is an integer $n \geqslant 1$, a measurable set E in K^n and $P_1, \ldots, P_n \in B^*$ satisfying

$$T = \{x \in B : (P_1(x), \ldots, P_n(x)) \in E\}.$$

Since restriction of P_i to H is continuous by Lemma 3.11,

$$V = \bigcap_{i=1}^{n} \{x \in H : P_i(x) = 0\}$$

is a closed subspace of H. By assumption, $|H - \{0\}|$ is discrete, so V is strictly closed. On the other hand, codim $V \leqslant n$. Hence V has a norm direct supplement W in H. Let $P \in \mathrm{FOP}(H)$ be a projection to W component of $H = W \oplus V$. Put

$$E' = \{x \in W : (P_1(x), \ldots, P_n(x)) \in E\}.$$

We see E' is a measurable subset in W and

$$T \cap H = \{x \in H : P(x) \in E'\} \in \mathrm{Cyl}(H)$$

by $P_i = P_i P$. □

DEFINITION 3.16. Let ν be an admissible probabilistic measure on K. The *Wiener measure* W_ν with *parameter* ν is the function on $\mathrm{Cyl}^*(B)$ defined by the following formula:

$$W_\nu(T) = G_\nu(T \cap H)$$

where $T \in \mathrm{Cyl}^*(B)$.

It is easy to see that $\mathrm{Cyl}^*(B)$ is a field of sets and that W_r is a finitely additive measure. We denote by $\sigma[\mathrm{Cyl}^*(B)]$ the σ-field generated by $\mathrm{Cyl}^*(B)$. The next theorem is the main result of this paper.

THEOREM 3.17. *The Wiener measure W_ν extends to a σ-additive measure on $\sigma[\mathrm{Cyl}^*(B)]$.*

Proof. First we show that for any $\varepsilon > 0$ there is a compact set C_ε in B such that $W_\nu(T) < 2\varepsilon$ for all $T \in \mathrm{Cyl}^*(B)$ and $T \cap C_\varepsilon = \varnothing$. Choose (and fix) a measurable norm $\|\cdot\|_0$ on H as in Lemma 3.13. There is an $r > 0$ satisfying

$$\tilde{\nu}(\{\omega \in \Omega : \|\cdot\|_0^{\sim}(\omega) > r - \varepsilon\}) < \varepsilon. \tag{3.21}$$

Let C_ε be the closure of $\{x \in H : \|x\|_0 \leqslant r\}$ in B, which is a compact set of B by Lemma 3.13. As we have proved in Lemma 3.10, there is $Q \in \mathrm{FOP}(H)$ satisfying

$$\tilde{\nu}(\{\omega \in \Omega : \big|\|\cdot\|_0^{\sim}(\omega) - \|\xi_P(\omega)\|_0\big| > \varepsilon) < \varepsilon \tag{3.22}$$

for all $P \geqslant Q$. On the other hand, by Lemma 3.15, there are $P \in \mathrm{FOP}(H)$ and a Borel set E of $P(H)$ such that $T \cap H = \{x \in H : P(x) \in E\}$. Using Lemma 2.12 and well definedness of the cylinder set measure Lemma 3.6, we may assume $P \geqslant Q$. Then,

$$\begin{aligned} W_\nu(T) &= G_\nu(T \cap H) = \nu^{\times P(H)}(E) \\ &\leqslant 1 - \nu^{\times P(H)}(C_\varepsilon \cap P(H)) \end{aligned}$$

since $(C_\varepsilon \cap P(H)) \cap E \subset C_\varepsilon \cap T = \varnothing$. We have

$$\begin{aligned} \nu^{\times P(H)}(C_\varepsilon \cap P(H)) &\geqslant \nu^{\times P(H)}(\{x \in P(H) : \|x\|_0 \leqslant r\}) \\ &= \tilde{\nu}(\{\omega \in \Omega : \|\xi_P(\omega)\|_0 \leqslant r\}) \quad \text{by (3.11)}, \\ &\geqslant \tilde{\nu}(\{\omega \in \Omega : \|\cdot\|_0^{\sim}(\omega) \leqslant r - \varepsilon\}) - \varepsilon \quad \text{by (3.22)}, \\ &\geqslant 1 - \tilde{\nu}(\{\omega \in \Omega : \|\cdot\|_0^{\sim}(\omega) > r - \varepsilon\}) - \varepsilon \geqslant 1 - 2\varepsilon \end{aligned}$$

by (3.21). Hence $W_\nu(T) \leqslant 1 - (1 - 2\varepsilon) = 2\varepsilon$. Once this has been established, we obtain σ-additivity of W_v by the exact same method as [6, Theorem 4.1, Step 1]. □

THEOREM 3.18. *The Borel field of B is* $\sigma[\mathrm{Cyl}^*(B)]$.

Proof. By [2, Proposition 2.7.2/8], there is a norm $\|\cdot\|'$ in B equivalent to $\|\cdot\|$ with respect to which B has an orthonormal Schauder basis $\{f_i\}_{i=1}^{\infty}$. It suffices to show

$$S = \{x \in B: \|x\|' \leqslant 1\} \in \sigma[\mathrm{Cyl}^*(B)]$$

since $(B, \|\cdot\|')$ is a separable metric space. Define $P_n \in B^*$ by

$$P_n\left(\sum_{i=1}^{\infty} c_i f_i\right) = c_n.$$

By the orthonormality of $\{f_i\}_{i=1}^{\infty}$,

$$\|x\|' = \max_{1 \leqslant n < \infty} |P_n(x)|.$$

Hence

$$S = \bigcap_{n=1}^{\infty} \{x \in B: |P_n(x)| \leqslant 1\}$$

and this shows $S \in \sigma[\mathrm{Cyl}^*(B)]$. □

4. Multiplicative measurable norms

Throughout this section, let K denote a complete field with respect to a non-trivial non-Archimedean valuation $|\cdot|$. In Definition 3.8 we introduced the notion of measurable norm on normed vector space. This notion especially makes sense for a normed K-algebra H with a multiplicative norm. The subset

$$R = \{x \in H: |x| \leqslant 1\}$$

is a subring of H. In this section, we associate a maximal ideal of R with each multiplicative measurable seminorm.

LEMMA 4.1. *Let H be a K-cartesian space with respect to the norm* $|\cdot|$. *Let* $P \in \mathrm{FOP}(H)$ *and put* $n = \dim P(H)$. *Let* $x_0, \ldots, x_n$ *be* $n+1$ *vectors in H. If*

$$|(1_H - P)(x_i)| < |x_i| \tag{4.1}$$

holds for all $0 \leqslant i \leqslant n$, *then there exist constants* $a_i \in K$ *satisfying*

$$\left|\sum_{i=0}^{n} a_i x_i\right| < \max_{0 \leqslant i \leqslant n} |a_i x_i|.$$

Proof. Since H is K-cartesian, there is a K-orthogonal basis $\{e_i\}_{i=1}^{n}$ of $P(H)$. Thus, there are constants $c_{ij} \in K$ satisfying $P(x_i) = \Sigma_{j=1}^{n} c_{ij} e_j$. Since $\{e_i\}_{i=1}^{n}$ is orthogonal, $|c_{i,j} e_j| \leqslant |P(x_i)|$ for all j. Put

$$v_i = \sum_{|c_{ij} e_j| = |P(x_i)|} c_{ij} e_j.$$

By (4.1), we have $|P(x_i)| = |x_i|$. Therefore

$$\begin{aligned} |x_i - v_i| &= \left|(x_i - P(x_i)) + \sum_{|c_{ij} e_j| < |x_i|} c_{ij} e_j\right| \\ &< |x_i|. \end{aligned}$$

On the other hand, there are constants a_i for $0 \leqslant i \leqslant n$ satisfying

$$\sum_{i=0}^{n} a_i v_i = 0.$$

Thus,

$$\begin{aligned} \left|\sum_{i=0}^{n} a_i x_i\right| &= \left|\sum_{i=0}^{n} a_i x_i - \sum_{i=0}^{n} a_i v_i\right| \\ &\leqslant \max_{0 \leqslant i \leqslant n} |a_i(x_i - v_i)| < \max_{0 \leqslant i \leqslant n} |a_i x_i|. \end{aligned}$$ □

In what follows, we assume that H is a normed K-algebra with an orthonormal Schauder basis (as a normed K-vector space) and that the norm $|\cdot|$ of H is multiplicative. We denote by $\tilde{K}$ the residue field of K.

LEMMA 4.2. *Let* $\|\cdot\|$ *be a multiplicative measurable norm on* H. *Then* $\|x\| \leqslant |x|$ *for all* $x \in H$.

Proof. There is a constant c satisfying $\|x^n\| \leqslant c|x^n|$ for all $x \in H$ and $n \geqslant 1$ by Lemma 3.11. The assertion follows from multiplicativity of $\|\cdot\|$ and $|\cdot|$ □

PROPOSITION 4.3. *Let* $\|\cdot\|$ *be a multiplicative measurable norm on* H. *Put*

$$R = \{x \in H : |x| \leqslant 1\},$$
$$\mathfrak{p} = \{x \in H : |x| < 1\},$$

and

$$\mathfrak{m}(\|\cdot\|) = \{x \in R : \|x\| < 1\}.$$

Then, $\mathfrak{m}(\|\cdot\|)$ *is a maximal ideal of* R *properly containing the prime ideal* $\mathfrak{p}$ *and* $R/\mathfrak{m}(\|\cdot\|)$ *is an algebraic extension of* $\tilde{K}$.

Proof. Take $P \in \mathrm{FOP}(H)$ satisfying $\|x\| \leqslant |x|/2$ for all $x \in \operatorname{Ker} P$. Put $n = \dim P(H)$ and $L = L_{\|\cdot\|} = R/\mathfrak{m}(\|\cdot\|)$. Using Lemma 4.2 and multiplicativity of $\|\cdot\|$, we see that $\mathfrak{m}(\|\cdot\|)$ is a prime ideal of R containing $\mathfrak{p}$. So, L is, at least, an integral domain. We denote the image of $x \in R$ in L by $\tilde{x}$. Let $x \in R - \mathfrak{m}(\|\cdot\|)$. There are $n+1$ constants $a_0, \ldots, a_n \in K$ satisfying $\Sigma_{i=0}^{n} a_i P(x^i) = 0$. Without loss of generality, we assume $|a_i| \leqslant 1$ for $0 \leqslant i \leqslant n$. Put $v = \Sigma_{i=0}^{n} a_i x^i$. Then we see $v \in R \cap \operatorname{Ker} P \subset \mathfrak{m}(\|\cdot\|)$. Hence $\tilde{v} = \Sigma_{i=0}^{n} \tilde{a}_i \tilde{x}^i = 0$. Therefore $\tilde{x}$ is algebraic over $\tilde{K}$, which implies that $\tilde{x}$ is a unit in L. So, L is a field and $\mathfrak{m}(\|\cdot\|)$ is a maximal ideal of R. Finally, we prove $\mathfrak{m}(\|\cdot\|) \neq \mathfrak{p}$. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal Schauder basis of H. By Lemma 4.1, there is a positive integer i satisfying $|(1_H - P)(e_i)| = |e_i| = 1$. By the definition of P, we have $\|(1_H - P)(e_i)\| \leqslant \frac{1}{2}$. Thus $(1_H - P)(e_i) \in \mathfrak{p}^c \cap \mathfrak{m}(\|\cdot\|)$. □

5. Examples

Let T be an indeterminate. For a valued ring A with a norm $|\cdot|$, we put

$$A\langle T\rangle = \left\{\sum_{n=0}^{\infty} a_n T^n : a_n \in A, \text{ and } \lim_{n\to\infty} |a_n| = 0\right\}.$$

Let p be a prime number. We denote by $|\cdot|$ a p-adic valuation of $\mathbf{Q}_p$ normalized as $|p| = p^{-1}$. Let $H = \mathbf{Q}_p\langle T\rangle$. Then, H is a complete normed $\mathbf{Q}_p$-algebra with the norm

$$\left|\sum_{n=0}^{\infty} a_n T^n\right| = \max_{0 \leqslant n < \infty} |a_n|.$$

With respect to this norm, H has an orthonormal Schauder basis $\{T^n\}_{n=0}^{\infty}$.

We fix an integer $k \geqslant 1$. The norm $\|\cdot\|$ defined by

$$\left\|\sum_{n=0}^{\infty} a_n T^n\right\| = \max_{0 \leqslant n < \infty} |a_n| p^{-n/k}$$

is a multiplicative measurable norm. Let B be the completion of H with respect to $\|\cdot\|$. We see that

$$B = \left\{\sum_{n=0}^{\infty} a_n T^n : a_n \in \mathbf{Q}_p, \lim_{n \to \infty} |a_n| p^{-n/k} = 0\right\}.$$

Let ν be an admissible measure on $\mathbf{Q}_p$ defined by $\nu(E) = \mu(E \cap C_1(1))$ where μ is the Haar measure of $\mathbf{Q}_p$ normalized as in (2.8). In this case, we have

$$\mathscr{D}_{\nu,n}(t) = \begin{cases} 1 & (t \leqslant 1) \\ 0 & (t > 1) \end{cases}$$

and

$$\mathscr{D}_{\nu,n}(|(a_1, \ldots, a_n)|) = \prod_{i=1}^{n} \mathscr{D}_{\nu,1}(|a_i|).$$

For simplicity, we denote by W the Wiener measure on B with parameter ν.

The first example is based on a suggestion of Prof. S. Bloch. Let $\bar{\mathbf{Q}}_p$ be the algebraic closure of $\mathbf{Q}_p$ and $\bar{\mathbf{Z}}_p$ the integral closure of $\mathbf{Z}_p$ in $\bar{\mathbf{Q}}_p$. Each $f \in B$ defines a map from D_k to $\bar{\mathbf{Q}}_p$ where

$$D_k = \{x \in \bar{\mathbf{Z}}_p : |x| \leqslant p^{-1/k}\}.$$

Let $z \in D_k$ and put

$$U(z) = \{f \in B : f(z) = 0\}.$$

By the continuity of roots, $U(z)$ is a $\|\cdot\|$-closed set of B. So $U(z)$ is measurable by Theorem 3.18. We show $W(U(z)) = 0$. Let M be an arbitrary positive integer. Put

$$A_N = \left\{\sum_{n=0}^{\infty} a_n T^n \in B : |a_n z^n| < p^{-M} \text{ for all } n > N\right\},$$

$$S_N = \left\{\sum_{n=0}^{\infty} a_n T^n \in B : \left|\sum_{n=0}^{N} a_n z^n\right| < p^{-M}\right\}$$

for an integer $N \geqslant 1$. Since $|z| \leqslant p^{-1/k}$, we have $B = \bigcup_{N=0}^{\infty} A_N$ and $U(z) = \bigcup_{N=0}^{\infty}(U(z) \cap A_N)$. Taking account of $U(z) \cap A_N \subset S_N$, we majorize $W(S_N)$. We note

$$S_N \cap H = P_N^{-1}(K_N)$$

where

$$P_N\left(\sum_{n=0}^{\infty} a_n T^n\right) = \sum_{n=0}^{N} a_n T^n$$

and

$$K_N = \left\{f = \sum_{n=0}^{N} a_n T^n : |f(z)| < p^{-M}\right\}.$$

Since $\{T^n\}_{n=0}^{\infty}$ is an orthonormal Schauder basis of H,

$$\begin{aligned} W(S_N) &= G(S_N \cap H) \\ &= \int_{\mathbf{Q}_p} \cdots \int_{\mathbf{Q}_p} \int_{J(a_N, a_{N-1}, \ldots, a_1)} \prod_{n=0}^{N} \mathscr{D}_{v,1}(|a_n|) d\mu(a_0) d\mu(a_1) \cdots d\mu(a_N) \end{aligned}$$

where

$$J(a_N, a_{N-1}, \ldots, a_1) = \left\{x \in \mathbf{Q}_p : \left|x + \sum_{n=1}^{N} a_n z^n\right| < p^{-M}\right\}.$$

Using the translation invariant property of the Haar measure, we have

$$W(S_N) \leqslant p^{-M}.$$

Since the sequence $\{A_N\}$ is monotone increasing and W is σ-additive, we have

$$\begin{aligned} W(U(z)) &= \lim_{N \to \infty} W(U(z) \cap A_N) \\ &\leqslant \sup_{N \geqslant 1} W(S_N) \leqslant p^{-M}. \end{aligned}$$

Since M is arbitrary, $W(U(z)) = 0$. This value is independent of k.

The second example seems to be rather awkward. For each integer M, we compute measure of

$$K_M = \{x \in B : \|x\| \leqslant p^{-M/k}\}.$$

Since $\{T^n\}_{n=0}^{\infty}$ is an orthogonal Schauder basis of B (with respect to $\|\cdot\|$), we have $K_M = \bigcap_{m=0}^{\infty} E_{m,M}$ where

$$E_{m,M} = \left\{ \sum_{n=0}^{\infty} a_n T^n \in B : |a_m| \leqslant p^{(m-M)/k} \right\}.$$

Using σ-additivity of W, it holds that

$$W(K_M) = \lim_{n\to\infty} W\left(\bigcap_{m=0}^{n} E_{m,M} \right).$$

We have

$$W\left(\bigcap_{m=0}^{n} E_{m,M} \right) = \prod_{m=0}^{n} c_m$$

where

$$c_m = \int_{\{x \in \mathbf{Q}_p : |x| \leqslant p^{(m-M)/k}\}} \mathscr{D}_{v,1}(x) d\mu_{\mathbf{Q}_p}(x)$$
$$= \begin{cases} 1 & (m \geqslant M) \\ p^{[(m-M)/k]} & (0 \leqslant m < M). \end{cases}$$

Hence,

$$W(K_M) = \begin{cases} 1 & (M \leqslant 0) \\ p^{-(q+1)(kq/2+r)} & (M > 0) \end{cases}$$

where $M = kq + r$ with integers q and $0 \leqslant r < k$. For $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 0$ we see that

$$\int_{\{x \in B : \|x\| \leqslant 1\}} \frac{\|x\|^s}{W(\{y \in B : \|y\| \leqslant \|x\|\})} dW(x)$$
$$= \sum_{M=0}^{\infty} p^{-sM/k} \left(1 - \frac{W(K_{M+1})}{W(K_M)} \right)$$
$$= (1 - p^{-1})(1 - p^{-s-1})^{-1}(1 - p^{-s/k})^{-1}. \tag{5.1}$$

Using the notation of Proposition 4.3, we see $R = \mathbf{Z}_p\langle T \rangle$, $\mathfrak{p} = (p)$ and $\mathfrak{m}(\|\cdot\|) = (p, T)$. From a number theoretic point of view, it seems to be necessary to condense all the integrals of type (5.1) arising from a multiplicative measurable norm $\|\cdot\|$ with $\mathfrak{m}(\|\cdot\|) = (p, T)$. It is not clear whether a two-dimensional local field has an orthogonal Schauder basis. Unfortunately, the author does not obtain results of these types.

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