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Holomorphic forms and holomorphic vector fields on compact generalized Hopf manifolds

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1. Introduction

Let M be an $n(\geqslant 2)$-dimensional Hermitian manifold with metric tensor g and complex structure tensor J. For simplicity, all manifolds in this paper are assumed to be connected. If each point $p \in M$ has an open neighborhood U with a differentiable function $\sigma_U: U \to \mathbf{R}$ such that $\tilde{g}_U = e^{-\sigma_U} g$ is a Kähler metric on U, g is called a *locally conformal Kähler* (*l.c.K.*) *metric* and M is called an *l.c.K. manifold* ([7]). If we can take $U = M$, the metric g is called a *globally conformal Kähler* (*g.c.K.*) *metric* and the manifold M is called a *g.c.K. manifold.* If g is an l.c.K. metric, it is easily shown that $d\sigma_U = d\sigma_V$ on $U \cap V$. Therefore the closed 1-form ω ($\omega = d\sigma_U$ on U) is defined globally on M and is called the *Lee form* of an l.c.K. manifold M. The Lee form ω determines the first de Rham cohomology class $[\omega] \in H^1(M;\mathbf{R})$. If the metric g is changed to $g' = e^{\sigma} g$ by a differentiable function σ on M, g' is also an l.c.K. metric and its Lee form ω' is given by $\omega' = \omega + d\sigma$. In particular we have $[\omega] = [\omega']$ in $H^1(M;\mathbf{R})$. It follows that g is g.c.K. if and only if $[\omega] = 0$.

Typical examples of l.c.K. (and not g.c.K.) manifolds are the Hopf manifolds M_α. We set $M_\alpha = \mathbf{C}^n - \{0\}/G_\alpha$, where $\alpha \in \mathbf{C}$, $0 < |\alpha| < 1$ and G_α is the group generated by the holomorphic automorphism $z \to \alpha z$, $z \in \mathbf{C}^n - \{0\}$. Then M_α is a compact complex manifold. On $\mathbf{C}^n - \{0\}$, we consider the Hermitian metric $g = 1/\|z\|^2 \sum_{i=1}^n dz^i\, d\bar{z}^i$ which is conformally related to the flat Kähler metric $\sum_{i=1}^n dz^i\, d\bar{z}^i$. The metric g is invariant under the action of G_α and induces a Hermitian metric on M_α. Then M_α with g is an l.c.K. manifold.

It is an interesting and remarkable fact that compact l.c.K. (and not g.c.K.) manifolds admit no Kähler metrics ([9] Thm. 2.1). Hence the class of compact complex manifolds admitting l.c.K. metrics is essentially different from that of Kähler manifolds. We are interested in topological and complex analytic properties of compact l.c.K. manifolds. Since not much has been known about them, we first focus our study on a special class of l.c.K. manifolds, i.e., *the generalized Hopf* (*g.H.*) *manifolds*. By definition, a g.H. manifold is an l.c.K. manifold whose Lee form ω is parallel, i.e., $\nabla\omega = 0$ ($\omega \neq 0$) with respect to the Riemannian connection ∇ of its Hermitian metric g ([10]). The class of g.H. manifolds contains the Hopf manifolds M_α with g mentioned above. The topology of compact generalized Hopf manifolds has been investigated by Kashiwada ([4]) and Vaisman ([10]). In particular, some remarkable properties about Betti numbers are known (Corollary 2.4 in this paper).

In this paper we shall study holomorphic forms and holomorphic vector fields on compact g.H. manifolds. In Section 3 we deal with harmonic forms of the complex Laplacian $\square$ and show that any holomorphic form is closed (Theorem 3.3). Moreover we obtain the decomposition $b_r(M) = \Sigma_{p+q=r} h^{p,q}(M)$ for the Betti numbers (Theorem 3.5). In Section 4, we investigate holomorphic vector fields and obtain an analogous result to a theorem of Matsushima and Lichnerowicz (Theorem 4.6). In Section 5, we consider the following problem: What domain in $H^1(M;\mathbf{R})$ is occupied by the Lee forms of all l.c.K. metrics on M? (Theorem 5.1). In Section 2 we review basic properties of Kähler foliations which are the key to our study.

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2. Kähler foliations and the decomposition of harmonic forms on a compact g.H. manifold

In this section, we shall show that a foliation $\mathscr{F}$ canonically defined on a g.H. manifold is transversally Kählerian. We review properties of the basic cohomology of Kähler foliations and results about the Betti numbers of a compact g.H. manifold obtained by Kashiwada ([4]) and Vaisman ([10]).

Let (M, g, J) be an l.c.K. manifold with Lee form ω. We denote by Φ the fundamental form defined by $\Phi(X, Y) = g(JX, Y)$. We set $\theta = -\omega \circ J$ and denote by B and A the dual vector fields corresponding to ω and θ, respectively. Then we have $A = JB$. Computing the Riemannian connection $\tilde{\nabla}$ of the local Kähler metric $\tilde{g}_U = e^{-\sigma_U} g$, we get

$$\tilde{\nabla}_X Y = \nabla_X Y - \tfrac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X, Y)B\}. \tag{2.1}$$

Since J is parallel with respect to $\tilde{\nabla}$, the following holds:

$$(\nabla_X J)(Y) = \tfrac{1}{2}\{-\omega(Y)JX - \theta(Y)X + \Phi(X, Y)B + g(X, Y)A\}. \tag{2.2}$$

From now on we assume that (M, g, J) is a g.H. manifold. By a homothetic change of the metric g, we may assume $\|\omega\| = 1$. Since ω is parallel, B is also parallel. Putting $Y = B$ in (2.2), we have

$$\nabla_X A = \tfrac{1}{2}\{-JX - \theta(X)B + \omega(X)A\}. \tag{2.3}$$

From these equations, we obtain the following (for detailed computations see [8]).

LEMMA 2.1 (cf. [8]). *On a g.H. manifold (M, g, J), the following holds:*

(1) *A and B are Killing vector fields, i.e., $L_A g = 0$ and $L_B g = 0$, where L denotes the Lie derivative.*

(2) *A and B are infinitesimal automorphisms of (M, J), i.e., $L_A J = 0$ and $L_B J = 0$. In particular, $V = B - \sqrt{-1}\,A$ is a holomorphic vector field on M.*

(3) *We have $\nabla_A A = \nabla_A B = \nabla_B A = \nabla_B B = 0$ and in particular $[A, B] = 0$.*

By Lemma 2.1(3), the distribution generated by the vector fields A and B is completely integrable and defines a foliation $\mathscr{F}$ whose leaves are 1-dimensional complex submanifolds. Moreover these leaves are totally geodesic and locally flat submanifolds of M. The foliation $\mathscr{F}$ will be called the *canonical foliation* of a g.H. manifold M, which is called the *vertical foliation* in [10]. We shall show that $\mathscr{F}$ is transversally Kählerian.

We review basic definitions and properties of the transversal geometry of foliations. For a general reference, see Tondeur ([6]). Let $\mathscr{F}$ be a foliation on a manifold M. By the transversal geometry, we mean "the differential geometry" of the leaf space $M/\mathscr{F}$. It is given by an exact sequence of vector bundles

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0, \tag{2.4}$$

where L is the tangent bundle and Q the normal bundle of $\mathscr{F}$. We denote by ΓL and ΓQ the spaces of differentiable sections of L and Q, respecively. The action of the Lie algebra ΓL on ΓQ is defined by $L_X s = \pi[X, Y_s]$ for any $X \in \Gamma L$, $s \in \Gamma Q$, where $Y_s \in \Gamma TM$ with $\pi(Y_s) = s$. This action is extended to tensor fields of Q. A tensor field φ of Q is said to be *holonomy invariant* if it satisfies $L_X \varphi = 0$ for any $X \in \Gamma L$. $\mathscr{F}$ is called a *Riemannian foliation* if there exists a holonomy invariant Riemannian metric g_Q on Q. A metric g on M is *bundle-like* if the induced metric g_Q on Q is holonomy invariant. It is known that there is a unique metric and torsion-free connection ∇' in Q for the Riemannian foliation $\mathscr{F}$ with holonomy invariant metric g_Q (cf. [6] Thm. 5.12). In particular, for the holonomy invariant metric g_Q induced by the bundle-like metric g, such a unique connection ∇' is given by

$$\nabla'_X s = \begin{cases} L_X s & \text{for } X \in \Gamma L \\ \pi(\nabla_X Y_s) & \text{for } X \in \Gamma L^{\perp}, \end{cases}$$

where $Y_s \in \Gamma L^{\perp}$ with $\pi(Y_s) = s$. Here we denote by $L^{\perp}$ the vector bundle of orthogonal complements of L in TM.

A foliation $\mathscr{F}$ is *transversally Kählerian* (cf. Nishikawa and Tondeur [5]) if it satisfies the following conditions:

(i) $\mathscr{F}$ is Riemannian with a holonomy invariant metric g_Q on Q,

(ii) there is a holonomy invariant almost complex structure J_Q of Q, with respect to which g_Q is Hermitian, i.e. $g_Q(J_Q s, J_Q t) = g_Q(s, t)$ for $s, t \in \Gamma Q$, and

(iii) J_Q is parallel with respect to the unique metric and torsion-free connection ∇' associated with g_Q.

Returning to g.H. manifolds, we recall the following, which is implicitly proved as Theorem 3.1 in [10].

THEOREM 2.2. *The canonical foliation $\mathscr{F}$ on a g.H. manifold M is Kählerian.*

Proof. Identifying $L^{\perp}$ with Q, we get an almost complex structure J_Q and its Hermitian metric g_Q on Q. By Lemma 2.1(1) and (2), it is easy to see that J_Q and g_Q are holonomy invariant and hence g is a bundle-like metric. We show that it satisfies the third condition of Kähler foliations. For $X \in \Gamma L$, $\nabla'_X J_Q = 0$ holds. In fact this means the holonomy invariance of J_Q. For $X \in \Gamma L^{\perp}$ and $s \in \Gamma Q$, we have

$$\begin{aligned} (\nabla'_X J_Q)(s) &= \nabla'_X(J_Q s) - J_Q(\nabla'_X s) \\ &= \pi(\nabla_X(J Y_s)) - \pi(J(\nabla_X Y_s)) \\ &= \pi((\nabla_X J)(Y_s)) \\ &= \tfrac{1}{2}\pi\{\Phi(X, Y_s)B + g(X, Y_s)A\} \\ &= 0, \end{aligned}$$

where $Y_s \in \Gamma L^{\perp}$ with $\pi(Y_s) = s$. □

Now we return again to a general foliated manifold M with codimension q foliation $\mathscr{F}$. A differential form $\eta \in \Omega^r(M)$ is said to be *basic* if $i(X)\eta = 0$ and $L_X \eta = 0$ for all $X \in \Gamma L$. The exterior derivative d preserves basic forms and the set $\Omega_B^* = \Omega_B^*(\mathscr{F})$ of all basic forms constitutes a subcomplex of the de Rham complex $(\Omega^*(M), d)$. We denote $d|_{\Omega_B} = d_B$. Its cohomology $H_B^*(\mathscr{F})$ is called *the basic cohomology* of $\mathscr{F}$ ([6] Chap. 9).

From now on we assume that the manifold M is compact and oriented, and that $\mathscr{F}$ is transversally oriented and Riemannian with a bundle-like metric g. Moreover it is assumed that $\mathscr{F}$ is *harmonic*, i.e. all leaves of $\mathscr{F}$ are minimal submanifolds of (M, g).

We denote by δ_B the formal adjoint operator of d_B with respect to the natural scalar product $\langle\,,\rangle_B$ in $\Omega_B^r(\mathscr{F})$. We put *the basic Laplacian* $\triangle_B = d_B\delta_B + \delta_B d_B$ and call a basic form η satisfying $\triangle_B\eta = 0$ a *harmonic basic form* ([6] Chap. 12). It is known that the space $\mathscr{H}_B^r$ of harmonic basic r-forms is of finite dimension and that $\mathscr{H}_B^r$ is isomorphic to $H_B^r(\mathscr{F})$ (cf. [2]).

Furthermore suppose that $\mathscr{F}$ is transversally Kählerian with codimension $q = 2m$. The fundamental 2-form Φ' of a Kähler foliation is defined by $\Phi'(X, Y) = g_Q(J_Q\pi X, \pi Y)$ for $X, Y \in \Gamma TM$. Then Φ' is a closed basic 2-form. The complexified normal bundle $Q^{\mathbf{C}} = Q \otimes_{\mathbf{R}} \mathbf{C}$ has the direct sum decomposition:

$$Q^{\mathbf{C}} = Q^+ \oplus Q^-,$$

where Q^+ and Q^- are subbundles associated with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ of J_Q, respectively. According to this decomposition, the complex valued basic r-forms $\Omega_B^r(\mathscr{F})$ are decomposed as follows:

$$\Omega_B^r(\mathscr{F}) = \bigoplus_{s+t=r} \Omega_B^{s,t}(\mathscr{F}),$$

where $\Omega_B^{s,t}(\mathscr{F})$ denotes the space of basic forms of type (s, t). The exterior derivative $d_B : \Omega_B^r(\mathscr{F}) \to \Omega_B^{r+1}(\mathscr{F})$ is decomposed into two operators ∂_B and $\bar\partial_B$ of bidegrees $(1, 0)$ and $(0, 1)$. Then the following differential complex is obtained:

$$0 \longrightarrow \Omega_B^{s,0} \xrightarrow{\bar\partial_B} \Omega_B^{s,1} \xrightarrow{\bar\partial_B} \cdots \xrightarrow{\bar\partial_B} \Omega_B^{s,m} \longrightarrow 0$$

and its cohomology $H_B^{s,t}(\mathscr{F})$ is called *the basic Dolbeault cohomology* of $\mathscr{F}$. Let $\bar\vartheta_B$ be the adjoint operator of $\bar\partial_B$ with respect to $\langle\,,\rangle_B$. We put $\square_B = \bar\partial_B\bar\vartheta_B + \bar\vartheta_B\bar\partial_B$ and

$$\mathscr{H}_B^{s,t} = \{\eta \in \Omega_B^{s,t}(\mathscr{F}) \mid \square_B\eta = 0\}.$$

It is known that $\mathscr{H}_B^{s,t}$ is of finite dimension and that $\mathscr{H}_B^{s,t} \cong H_B^{s,t}(\mathscr{F})$ and $H_B^{s,t} \cong H_B^{t,s}$ ([1]). Similarly to Kähler manifolds, for Kähler foliations $\triangle_B = 2\square_B$ holds. Therefore we have the direct sum decomposition $\mathscr{H}_B^r = \bigoplus_{s+t=r} \mathscr{H}_B^{s,t}$.

We define the operators $L': \Omega_B^{s,t} \to \Omega_B^{s+1,t+1}$ and $\Lambda': \Omega_B^{s,t} \to \Omega_B^{s-1,t-1}$ by $L'\alpha = \Phi' \wedge \alpha$ and $\Lambda'\alpha = i(\Phi')\alpha$. The basic Laplacian $\triangle_B$ commutes with L' and Λ'. Therefore every harmonic basic r-form α $(r \leqslant m + 1)$ admits a unique decomposition into a direct sum of the type:

$$\alpha = \sum_{h=0}^{[r/2]} L'^h \lambda_{r-2h}, \tag{2.5}$$

where λ_{r-2h} are harmonic and *effective* basic $(r - 2h)$-forms, i.e., $\Lambda'\lambda_{r-2h} = 0$.

In general, the basic cohomology $H_B^*(\mathscr{F})$ of $\mathscr{F}$ does not admit remarkable relations to the de Rham cohomology $H^*(M)$ of M. However, for the canonical foliation $\mathscr{F}$ on a compact g.H. manifold M there are beautiful relations between the basic cohomology of $\mathscr{F}$ and the de Rham cohomology of M ([4], [10]). Let ω and B be the Lee form and its dual vector field on a g.H. manifold M. Since $\triangle(\omega \wedge \alpha) = \omega \wedge \triangle\alpha$ and $\triangle i(B)\alpha = i(B)\triangle\alpha$, every harmonic r-form λ has a unique decomposition of the following type:

$$\lambda = \alpha + \omega \wedge \beta,$$

where α, β are harmonic forms and $i(B)\alpha = 0$, $i(B)\beta = 0$. Now we recall the following.

THEOREM 2.3 ([4], [10]). *Let M be an n-dimensional compact g.H. manifold. For an r-form λ $(0 \leqslant r \leqslant n - 1)$ on M, the following two conditions are equivalent*:

(i) *λ is harmonic, i.e., $\triangle\lambda = 0$;*

(ii) *λ has the decomposition $\lambda = \alpha + \omega \wedge \beta$, where α and β are basic forms of the canonical foliation $\mathscr{F}$ and satisfy $\triangle_B\alpha = 0$, $\triangle_B\beta = 0$, $\Lambda'\alpha = 0$, $\Lambda'\beta = 0$ (that is, α and β are effective harmonic basic forms of $\mathscr{F}$).*

We denote by e_h the dimension of the basic cohomology $H_B^h(\mathscr{F})$. Then, by the decomposition (2.5) of harmonic basic forms, the following holds.

COROLLARY 2.4 ([4], [10]). *On an n-dimensional compact g.H. manifold M, the Betti numbers $b_h(M)$ are given by*

$$b_h = e_h + e_{h-1} - e_{h-2} - e_{h-3} \quad (0 \leqslant h \leqslant n-1)$$

$$b_h = e_{h-2} + e_{h-1} - e_h - e_{h+1} \quad (n + 1 \leqslant h \leqslant 2n)$$

$$b_n = 2(e_{n-1} - e_{n-3}).$$

In particular the first Betti number $b_1(M)$ is odd.

3. Holomorphic forms

In this section we shall study holomorphic forms on compact g.H. manifolds. We keep the notation in Section 2.

Let M be an n-dimensional compact g.H. manifold. We denote by $\Omega^r(M)$ the space of complex valued r-forms on M with the scalar product $\langle\,,\rangle$. We consider the following differential operators acting on forms: d, δ, ∂, $\bar\partial$, ϑ, $\bar\vartheta$, $\triangle$, $\square$. Here $d = \partial + \bar\partial$ is the exterior derivative with its decomposition into two operators of bidegrees $(1, 0)$ and $(0, 1)$. δ, ϑ, and $\bar\vartheta$ are defined by $\delta = -*d*$, $\vartheta = -*\bar\partial*$, and $\bar\vartheta = -*\partial*$, respectively. Then we have

$$\langle d\alpha, \beta\rangle = \langle\alpha, \delta\beta\rangle,\ \langle\partial\alpha, \beta\rangle = \langle\alpha, \vartheta\beta\rangle \quad\text{and}\quad \langle\bar\partial\alpha, \beta\rangle = \langle\alpha, \bar\vartheta\beta\rangle.$$

$\triangle = d\delta + \delta d$ and $\square = \bar\partial\bar\vartheta + \bar\vartheta\bar\partial$ are the Laplacians of d and $\bar\partial$, respectively. We denote by $e(F)$ and $i(F)$ the exterior product and the interior product by the k-form F, respectively. In particular, we use the following notation: $L = e(\Phi)$, $\Lambda = i(\Phi)$, $L' = e(\Phi')$, $\Lambda' = i(\Phi')$, where Φ and Φ' are the fundamental 2-forms of M and of the canonical Kähler foliation $\mathscr{F}$, respectively. We recall the Lee form ω and $\theta = -\omega\circ J$. From (2.3), it follows that $d\theta = -\Phi + \omega\wedge\theta = -\Phi'$ and $\delta\theta = 0$. We define the differential form φ of type $(1,0)$ by $\varphi = \omega + \sqrt{-1}\,\theta$. Then we see that $\partial\varphi = 0$, $\vartheta\varphi = 0$ and $\bar\partial\varphi = -\sqrt{-1}\,\Phi'$. In fact, since $0 = d\omega = \frac{1}{2}(\partial\varphi + \bar\partial\varphi + \partial\bar\varphi + \bar\partial\bar\varphi)$, we have $\partial\varphi = \bar\partial\bar\varphi = 0$ and $\bar\partial\varphi + \partial\bar\varphi = 0$ because of types of forms. Since

$$-\Phi' = d\theta = -\frac{\sqrt{-1}}{2}(\bar\partial\varphi - \partial\bar\varphi) = -\sqrt{-1}\,\bar\partial\varphi,$$

and

$$\delta\varphi = \delta\omega + \sqrt{-1}\,\delta\theta = 0,$$

we obtain $\bar\partial\varphi = -\sqrt{-1}\,\Phi'$ and $\vartheta\varphi = 0$.

By straightforward computation, we obtain the following:

LEMMA 3.1. *The following commutation formulas hold for r-forms on an n-dimensional g.H. manifold*:

$$\partial e(\varphi) + e(\varphi)\partial = 0$$

$$\partial e(\bar\varphi) + e(\bar\varphi)\partial = \sqrt{-1}\,L'$$

$$\bar\partial e(\varphi) + e(\varphi)\bar\partial = -\sqrt{-1}\,L'$$

$$\bar\partial e(\bar\varphi) + e(\bar\varphi)\bar\partial = 0$$

$$\partial i(\bar\varphi) + i(\bar\varphi)\partial = -(\bar\vartheta e(\bar\varphi) + e(\bar\varphi)\bar\vartheta)$$

$$\bar\partial i(\bar\varphi) + i(\bar\varphi)\bar\partial = -(\vartheta e(\bar\varphi) + e(\bar\varphi)\vartheta) = 0$$

$$\partial i(\varphi) + i(\varphi)\partial = -(\bar\vartheta e(\varphi) + e(\varphi)\bar\vartheta) = 0$$

$$\bar\partial i(\varphi) + i(\varphi)\bar\partial = -(\vartheta e(\varphi) + e(\varphi)\vartheta)$$

$$\vartheta i(\bar\varphi) + i(\bar\varphi)\vartheta = 0$$

$$\vartheta i(\varphi) + i(\varphi)\vartheta = -\sqrt{-1}\,\Lambda'$$

$$\bar\vartheta i(\bar\varphi) + i(\bar\varphi)\bar\vartheta = \sqrt{-1}\,\Lambda'$$

$$\bar\vartheta i(\varphi) + i(\varphi)\bar\vartheta = 0$$

$$\partial L - L\partial = \tfrac{1}{2}e(\varphi)L$$

$$\bar\partial L - L\bar\partial = \tfrac{1}{2}e(\bar\varphi)L$$

$$\partial = -\sqrt{-1}\,(\bar\vartheta L - L\bar\vartheta) + \tfrac{1}{2}(r + 1 - n)e(\varphi) + \frac{\sqrt{-1}}{2}\,Li(\varphi)$$

$$\bar\partial = \sqrt{-1}\,(\vartheta L - L\vartheta) + \tfrac{1}{2}(r + 1 - n)e(\bar\varphi) - \frac{\sqrt{-1}}{2}\,Li(\bar\varphi)$$

$$\vartheta = -\sqrt{-1}\,(\bar\partial\Lambda - \Lambda\bar\partial) + \tfrac{1}{2}(r - n)i(\bar\varphi) - \frac{\sqrt{-1}}{2}\,e(\bar\varphi)\Lambda$$

$$\bar\vartheta = \sqrt{-1}\,(\partial\Lambda - \Lambda\partial) + \tfrac{1}{2}(r - n)i(\varphi) + \frac{\sqrt{-1}}{2}\,e(\varphi)\Lambda$$

$$\vartheta\Lambda - \Lambda\vartheta = -\tfrac{1}{2}\Lambda i(\bar\varphi)$$

$$\bar\vartheta\Lambda - \Lambda\bar\vartheta = -\tfrac{1}{2}\Lambda i(\varphi)$$

We shall investigate harmonic forms with respect to $\Box$ and get the complex version of Theorem 2.3. Namely we show the following theorem.

THEOREM 3.2. *Let M be an n-dimensional compact g.H. manifold and λ a differential form of type (p, q) with $p + q \leqslant n - 1$ on M. Then the following two conditions for λ are equivalent*:

(1) *λ satisfies $\Box\lambda = 0$*,

(2) *λ has the decomposition $\lambda = \alpha + \bar\varphi \wedge \beta$, where α and β are basic forms of the canonical foliation $\mathscr{F}$ and satisfy $\triangle_B\alpha = 0$, $\triangle_B\beta = 0$, $\Lambda'\alpha = 0$, $\Lambda'\beta = 0$ (i.e. α and β are effective harmonic basic forms of $\mathscr{F}$).*

Proof. We first note that any differential form λ has a unique decomposition of the following type:

$$\lambda = \alpha + \bar{\varphi} \wedge \beta,$$

where $i(\varphi)\alpha = 0$, $i(\varphi)\beta = 0$. In the decomposition above, α and β are given by $\beta = \frac{1}{2}i(\varphi)\lambda$ and $\alpha = \lambda - \bar{\varphi} \wedge \beta$. From Lemma 3.1, it follows that $\square e(\bar{\varphi}) = e(\bar{\varphi})\square$ and $\square i(\varphi) = i(\varphi)\square$. In fact we have

$$\begin{aligned}
\bar{\partial}\bar{\vartheta}e(\bar{\varphi}) &= \bar{\partial}(-e(\bar{\varphi})\bar{\vartheta} - \partial i(\bar{\varphi}) - i(\bar{\varphi})\partial) \\
&= e(\bar{\varphi})\bar{\partial}\bar{\vartheta} - \bar{\partial}\partial i(\bar{\varphi}) - \bar{\partial} i(\bar{\varphi})\partial \\
&= e(\bar{\varphi})\bar{\partial}\bar{\vartheta} - \partial i(\bar{\varphi})\bar{\partial} + i(\bar{\varphi})\bar{\partial}\partial \\
\bar{\vartheta}\bar{\partial}e(\bar{\varphi}) &= -\bar{\vartheta}e(\bar{\varphi})\bar{\partial} \\
&= e(\bar{\varphi})\bar{\vartheta}\bar{\partial} + \partial i(\bar{\varphi})\bar{\partial} + i(\bar{\varphi})\partial\bar{\partial}
\end{aligned}$$

and hence $\square e(\bar{\varphi}) = e(\bar{\varphi})\square$. Similarly we can prove $\square i(\varphi) = i(\varphi)\square$. Accordingly, for the decomposition $\lambda = \alpha + \bar{\varphi} \wedge \beta$, we see that $\square\lambda = 0$ if and only if $\square\alpha = 0$ and $\square\beta = 0$.

Proof of the implication (2) $\to$ (1). If λ has the decomposition $\lambda = \alpha + \bar{\varphi} \wedge \beta$ satisfying (2), by Theorem 2.3 α and β are harmonic forms with respect to $\triangle$ and hence they satisfy $d\alpha = 0$, $d\beta = 0$ and $\delta\alpha = 0$, $\delta\beta = 0$. Therefore α and β are also harmonic forms with respect to $\square$. Consequently we have $\square\lambda = 0$.

Proof of the implication (1) $\to$ (2). First we give an outline of the proof. Let α be a differential form of type (p, q) $(p + q \leqslant n - 1)$ which satisfies $\square\alpha = 0$ and $i(\varphi)\alpha = 0$. Then we shall show that such an α satisfies $d\alpha = 0$, $\delta\alpha = 0$ and hence $\triangle\alpha = 0$ and that α is a basic form of the canonical foliation $\mathscr{F}$. By this fact and Theorem 2.3, we see that α is an effective harmonic basic form of $\mathscr{F}$. Thus combining this with the preceding argument on the decomposition $\lambda = \alpha + \bar{\varphi} \wedge \beta$, we can prove our assertion (1) $\to$ (2).

For our purpose, we shall prove the following key formula for a differential form α of type (p, q) $(p + q = r)$ which satisfies $\square\alpha = 0$ and $i(\varphi)\alpha = 0$:

$$\langle\partial\alpha, \partial\alpha\rangle = -\tfrac{1}{2}(n-r-1)\langle\Lambda'\alpha, \Lambda'\alpha\rangle - \tfrac{1}{4}(n-r)(n-r-1)\langle i(\bar{\varphi})\alpha, i(\bar{\varphi})\alpha\rangle. \tag{3.1}$$

We note that $\square\alpha = 0$ if and only if $\bar{\partial}\alpha = 0$ and $\bar{\vartheta}\alpha = 0$.

By Lemma 3.1, we have

$$
\begin{aligned}
\langle \partial\alpha, \partial\alpha\rangle &= \langle \vartheta\partial\alpha, \alpha\rangle \\
&= \left\langle -\sqrt{-1}\,(\bar\partial\Lambda-\Lambda\bar\partial)\partial\alpha+\tfrac{1}{2}(r+1-n)i\,(\bar\varphi)\partial\alpha-\frac{\sqrt{-1}}{2}\,e(\bar\varphi)\Lambda\partial\alpha,\ \alpha\right\rangle \\
&= -\sqrt{-1}\,\langle \Lambda\partial\alpha, \bar\vartheta\alpha\rangle+\tfrac{1}{2}(r+1-n)\langle i\,(\bar\varphi)\partial\alpha, \alpha\rangle \\
&\quad -\frac{\sqrt{-1}}{2}\,\langle \Lambda\partial\alpha, i(\varphi)\alpha\rangle \\
&= -\tfrac{1}{2}(r+1-n)\langle \partial i\,(\bar\varphi)\alpha+\bar\vartheta e(\bar\varphi)\alpha+e(\varphi)\bar\vartheta\alpha, \alpha\rangle \\
&= -\tfrac{1}{2}(r+1-n)\{\langle i\,(\bar\varphi)\alpha, \vartheta\alpha\rangle+\langle e(\bar\varphi)\alpha, \bar\partial\alpha\rangle\} \\
&= -\tfrac{1}{2}(r+1-n)\langle i\,(\bar\varphi)\alpha, \vartheta\alpha\rangle \\
&= -\tfrac{1}{2}(r+1-n)\left\langle i(\bar\varphi)\alpha,\ -\sqrt{-1}\,\bar\partial\Lambda\alpha+\tfrac{1}{2}(r-n)i\,(\bar\varphi)\alpha-\frac{\sqrt{-1}}{2}\,e(\bar\varphi)\Lambda\alpha\right\rangle \\
&= -\frac{\sqrt{-1}}{2}\,(r+1-n)\langle i\,(\bar\varphi)\alpha, \bar\partial\Lambda\alpha\rangle \\
&\quad -\tfrac{1}{4}(n-r)(n-r-1)\langle i\,(\bar\varphi)\alpha, i(\bar\varphi)\alpha\rangle-\frac{\sqrt{-1}}{4}\langle i(\bar\varphi)\alpha, e(\bar\varphi)\Lambda\alpha\rangle.
\end{aligned}
$$

Calculating the first term of the last equation, we obtain

$$
\begin{aligned}
\langle i(\bar\varphi)\alpha, \bar\partial\Lambda\alpha\rangle &= \langle \bar\vartheta i(\bar\varphi)\alpha, \Lambda\alpha\rangle \\
&= \langle -i(\bar\varphi)\bar\vartheta\alpha+\sqrt{-1}\,\Lambda'\alpha, \Lambda\alpha\rangle \\
&= \sqrt{-1}\,\langle \Lambda'\alpha, \Lambda'\alpha+\frac{\sqrt{-1}}{2}\,i(\bar\varphi)i(\varphi)\alpha\rangle \\
&= \sqrt{-1}\,\langle \Lambda'\alpha, \Lambda'\alpha\rangle.
\end{aligned}
$$

Calculating the third term, we have

$$
\begin{aligned}
\langle i(\bar\varphi)\alpha, e(\bar\varphi)\Lambda\alpha\rangle &= \langle i(\varphi)i\,(\bar\varphi)\alpha, \Lambda\alpha\rangle \\
&= -\langle i(\bar\varphi)i\,(\varphi)\alpha, \Lambda\alpha\rangle \\
&= 0.
\end{aligned}
$$

Therefore (3.1) is proved. From (3.1), it follows that if $p+q=r\leqslant n-1$, $\partial\alpha=0$ and hence $d\alpha=0$. In particular, if $p+q=r\leqslant n-2$, (3.1) implies that $\Lambda'\alpha=0$ and $i(\bar\varphi)\alpha=0$. Next we shall prove that $\Lambda'\alpha=0$ and $i(\bar\varphi)\alpha=0$ hold if $p+q=r\leqslant n-1$. By Lemma 3.1, we have

$$\begin{aligned}\partial i(\bar\varphi)\alpha &= -\sqrt{-1}\,(\bar\vartheta L - L\bar\vartheta)i(\bar\varphi)\alpha + \tfrac{1}{2}(r-n)e(\varphi)i(\bar\varphi)\alpha + \frac{\sqrt{-1}}{2}Li(\varphi)i(\bar\varphi)\alpha\\ &= -\sqrt{-1}\,\bar\vartheta Li(\bar\varphi)\alpha + \sqrt{-1}\,L\{-i(\bar\varphi)\bar\vartheta\alpha + \sqrt{-1}\,\Lambda'\alpha\}\\ &\quad + \tfrac{1}{2}(r-n)e(\varphi)i(\bar\varphi)\alpha - \frac{\sqrt{-1}}{2}Li(\bar\varphi)i(\varphi)\alpha\\ &= -\sqrt{-1}\,\bar\vartheta Li(\bar\varphi)\alpha - L\Lambda'\alpha + \tfrac{1}{2}(r-n)e(\varphi)i(\bar\varphi)\alpha\end{aligned}$$

and hence

$$\begin{aligned}\langle \partial i(\bar\varphi)\alpha,\ \alpha\rangle &= -\sqrt{-1}\,\langle Li(\bar\varphi)\alpha,\ \bar\partial\alpha\rangle - \langle \Lambda'\alpha,\ \Lambda\alpha\rangle\\ &\quad + \tfrac{1}{2}(r-n)\langle i(\bar\varphi)\alpha,\ i(\bar\varphi)\alpha\rangle\\ &= -\langle \Lambda'\alpha,\ \Lambda'\alpha\rangle - \tfrac{1}{2}(n-r)\langle i(\bar\varphi)\alpha,\ i(\bar\varphi)\alpha\rangle.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle \partial i(\bar\varphi)\alpha,\ \alpha\rangle &= -\langle i(\bar\varphi)\partial\alpha + \bar\vartheta e(\bar\varphi)\alpha + e(\bar\varphi)\bar\vartheta\alpha,\ \alpha\rangle\\ &= -\langle e(\bar\varphi)\alpha,\ \bar\partial\alpha\rangle = 0\end{aligned}$$

and hence

$$\langle \Lambda'\alpha,\ \Lambda'\alpha\rangle + \tfrac{1}{2}(n-r)\langle i(\bar\varphi)\alpha,\ i(\bar\varphi)\alpha\rangle = 0.$$

Consequently, if $p+q=r\leqslant n-1$, we have $\Lambda'\alpha=0$ and $i(\bar\varphi)\alpha=0$. Since $i(\varphi)\alpha=0$, $i(\bar\varphi)\alpha=0$ and $d\alpha=0$, α is a basic form of the canonical foliation $\mathscr{F}$. Since

$$\vartheta\alpha = -\sqrt{-1}\,(\bar\partial\Lambda - \Lambda\bar\partial)\alpha + \tfrac{1}{2}(r-n)i(\varphi)\alpha - \frac{\sqrt{-1}}{2}e(\bar\varphi)\Lambda\alpha = 0,$$

we have $\delta\alpha = 0$. □

As to holomorphic forms, the following holds.

THEOREM 3.3. *On an n-dimensional compact g.H. manifold, every holomorphic p-form α satisfies $d\alpha=0$ and $\triangle\alpha=0$. Moreover if $p\leqslant n-1$, α is a basic form of the canonical foliation $\mathscr{F}$.*

Proof. If $p\leqslant n-1$, we have already shown that α satisfies $d\alpha=0$, $\triangle\alpha=0$ and that α is a basic form of $\mathscr{F}$. For a holomorphic n-form α, $\partial\alpha=0$ holds trivially and hence $d\alpha=0$. Moreover since

$$\vartheta\alpha = -\sqrt{-1}(\bar{\partial}\Lambda - \Lambda\bar{\partial})\alpha - \frac{\sqrt{-1}}{2}e(\bar{\varphi})\Lambda\alpha = 0,$$

we have $\delta\alpha = 0$ and hence $\triangle\alpha = 0$. □

COROLLARY 3.4. *On an n-dimensional compact g.H. manifold, we have*

$$H_{\bar{\partial}}^{n,0}(M) \cong H_{\bar{\partial}}^{n-1,0}(M).$$

Proof. The interior product $i(V)$ by the holomorphic vector field V is an injective homomorphism of $H_{\bar{\partial}}^{n,0}(M)$ into $H_{\bar{\partial}}^{n-1,0}(M)$. Conversely for $\alpha \in H_{\bar{\partial}}^{n-1,0}(M)$ $\varphi \wedge \alpha$ is a holomorphic n-form. In fact, we have $\bar{\partial}(\varphi \wedge \alpha) = -\sqrt{-1}\,\Phi' \wedge \alpha$ and since $\Phi' \wedge \alpha$ is a basic form of type $(n, 1)$, we get $\Phi' \wedge \alpha = 0$ because of types of basic forms. It is easily seen that the map $\alpha \mapsto \frac{1}{2}\varphi \wedge \alpha$ of $H_{\bar{\partial}}^{n-1,0}(M)$ into $H_{\bar{\partial}}^{n,0}(M)$ is the inverse of $i(V)$. □

We denote by $H_{\bar{\partial}}^{p,q}(M)$ the Dolbeault cohomology group of type (p, q) and put $h^{p,q}(M) = \dim H_{\bar{\partial}}^{p,q}(M)$. Combining Theorem 2.3 and Theorem 3.2, we get the following.

THEOREM 3.5. *On a compact g.H. manifold M, we have $b_r(M) = \Sigma_{p+q=r}h^{p,q}(M)$.*

Proof. We set:

$$S_B^r(\mathscr{F}) = \{\alpha \in \Omega_B^r(\mathscr{F}) \mid \triangle_B\alpha = 0,\ \Lambda'\alpha = 0\}$$

$$S_B^{p,q}(\mathscr{F}) = \{\alpha \in \Omega_B^{p,q}(\mathscr{F}) \mid \triangle_B\alpha = 2\square_B\alpha = 0,\ \Lambda'\alpha = 0\}.$$

Then we have $S_B^r(\mathscr{F}) = \bigoplus_{p+q=r}S_B^{p,q}(\mathscr{F})$. From Theorem 2.3 and Theorem 3.2, it follows that $H^r(M; \mathbf{C}) \cong S_B^r(\mathscr{F}) \oplus S_B^{r-1}(\mathscr{F})$ for $r \leqslant n-1$ and that $H_{\bar{\partial}}^{p,q}(M) \cong S_B^{p,q}(\mathscr{F}) \oplus S_B^{p,q-1}(\mathscr{F})$ for $p + q \leqslant n - 1$. Hence for $r \leqslant n - 1$, we have $b_r(M) = \Sigma_{p+q=r}h^{p,q}(M)$. By Poincaré duality and Serre duality, the same relation holds for $r \geqslant n + 1$. Using a result of Frölicher ([3]) which states that

$$\chi(M) = \sum_{r=0}^{2n}(-1)^r b_r(M) = \sum_{p+q=0}^{2n}(-1)^{p+q}h^{p,q}(M),$$

we can prove $b_n(M) = \Sigma_{p+q=n}h^{p,q}(M)$. □

Remark 3.6., Theorem 3.3 and Theorem 3.5 show that the same results as in the case of compact Kähler manifolds hold on compact g.H. manifolds. But the relation $h^{p,q}(M) = h^{q,p}(M)$ is not true on a compact g.H. manifold. For example we have $h^{0,p} = h^{p,0} + h^{p-1,0}$ for $p \leqslant n - 1$. In particular, we have $h^{1,0} = \frac{1}{2}(b_1 - 1)$ and $h^{0,1} = \frac{1}{2}(b_1 + 1)$.

4. Holomorphic vector fields

In this section, we shall investigate holomorphic vector fields on a compact g.H. manifold.

Let $\mathscr{F}$ be the canonical foliation of a g.H. manifold M with tangent bundle L and normal bundle Q of $\mathscr{F}$ and π denote the projection $TM \to Q$. Usually we identify Q with the orthogonal complement $L^{\perp}$ of L. By Lemma 2.1(2), $V = B - \sqrt{-1}\,A$ is a holomorphic vector field with no zero points and hence L^{+} is a holomorphic subbundle of TM^{+}. Thus the quotient bundle $Q^{+} = TM^{+}/L^{+}$ is a holomorphic vector bundle and $\pi : TM^{+} \to Q^{+}$ is a bundle homomorphism between holomorphic vector bundles. If X is a holomorphic vector field on M, then $\pi(X)$ is a holomorphic section of Q^{+}.

Now, we shall discuss holomorphic sections of Q^{+}. Let us recall the unique metric and torsion-free connection ∇' induced in Q and also in Q^{+}. Then ∇' is *a connection of type* (1, 0) on the holomorphic vector bundle Q^{+}, i.e., a connection which maps local holomorphic sections in Q^{+} onto Q^{+}-valued forms of type (1, 0). In particular, ∇' is a Hermitian connection of the holomorphic Hermitian vector bundle (Q^{+}, g_Q), where g_Q is an induced holonomy invariant metric on Q. Thus for a section X of Q^{+}, X is holomorphic if and only if $\nabla'_{\bar{Y}}X = 0$ for any $Y \in \Gamma TM^{+}$. Let ξ be the corresponding complex differential 1-form to $X \in \Gamma Q^{+}$ defined by $\xi(Y) = g_Q(X, Y)$ for $Y \in \Gamma Q^{C}$. Then ξ is of type (0, 1) and we see that X is a holomorphic section of Q^{+} if and only if $\nabla'_{\bar{Y}}\xi = 0$ for any $Y \in \Gamma TM^{+}$. We denote simply by ξ the 1-form $\pi^{*}\xi$ of type (0, 1) on M.

For later convenience, we present the relation between the connections ∇', ∇, and $\tilde{\nabla}$, where ∇ and $\tilde{\nabla}$ denote the Riemannian connection and the connection defined by (2.1) respectively:

$$
\begin{aligned}
\nabla_X Y &= \nabla'_X Y + \frac{\sqrt{-1}}{4}\,\Phi(X, Y)(V - \bar{V}) \\
\nabla_V X &= \nabla'_V X + \frac{\sqrt{-1}}{2}\,JX \\
\nabla_{\bar{V}} X &= \nabla'_{\bar{V}} X - \frac{\sqrt{-1}}{2}\,JX \\
\nabla_X V &= \frac{\sqrt{-1}}{2}\,JX \\
\nabla_X \bar{V} &= -\frac{\sqrt{-1}}{2}\,JX
\end{aligned}
\tag{4.1}
$$

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla'_X Y + \tfrac{1}{4}\{g(X, Y) + \sqrt{-1}\,\Phi(X, Y)\}V \\
&\qquad + \tfrac{1}{4}\{g(X, Y) - \sqrt{-1}\,\Phi(X, Y)\}\bar{V} \\
\tilde{\nabla}_V Y &= \nabla'_V Y - \tfrac{1}{2}(Y - \sqrt{-1}\,JY) \\
\tilde{\nabla}_{\bar{V}} Y &= \nabla'_{\bar{V}} Y - \tfrac{1}{2}(Y + \sqrt{-1}\,JY) \\
\tilde{\nabla}_X V &= -\tfrac{1}{2}(X - \sqrt{-1}\,JX) \\
\tilde{\nabla}_X \bar{V} &= -\tfrac{1}{2}(X + \sqrt{-1}\,JX) \\
\tilde{\nabla}_V V &= -V \\
\tilde{\nabla}_{\bar{V}} \bar{V} &= -\bar{V} \\
\tilde{\nabla}_V \bar{V} &= \tilde{\nabla}_{\bar{V}} V = 0
\end{aligned} \tag{4.2}$$

where X, $Y \in \Gamma(L^{\perp})^{\mathrm{C}}$ and $V = B - \sqrt{-1}\,A$. In the above, we identify Q with $L^{\perp}$.

PROPOSITION 4.1. *If ξ is the 1-form of type* (0, 1) *corresponding to a holomorphic section $X \in \Gamma Q^{+}$, we have $\bar{\partial}\xi = 0$.*

Proof. Using (4.1), we have

$$\begin{aligned}
\bar{\partial}\xi(\bar{Y}, \bar{Z}) &= d\xi(\bar{Y}, \bar{Z}) = (\nabla_{\bar{Y}}\xi)(\bar{Z}) - (\nabla_{\bar{Z}}\xi)(\bar{Y}) \\
&= (\nabla'_{\bar{Y}}\xi)(\bar{Z}) - (\nabla'_{\bar{Z}}\xi)(\bar{Y}) = 0
\end{aligned}$$

and

$$\begin{aligned}
\bar{\partial}\xi(\bar{Y}, \bar{V}) &= d\xi(\bar{Y}, \bar{V}) = (\nabla_{\bar{Y}}\xi)(\bar{V}) - (\nabla_{\bar{V}}\xi)(\bar{Y}) \\
&= -(\nabla'_{\bar{V}}\xi)(\bar{Y}) = 0,
\end{aligned}$$

for $Y, Z \in \Gamma(L^{\perp})^{+}$. □

THEOREM 4.2. *Any holomorphic section X of Q^{+} on a compact g.H. manifold M is a holonomy invariant section with respect to $\mathscr{F}$.*

Proof. Let ξ be the 1-form of type (0, 1) corresponding to X. Then it is sufficient to prove that ξ is a basic form of $\mathscr{F}$. By Proposition 4.1, we can write ξ as $\xi = \xi_0 + \bar{\partial}f$, where ξ_0 satisfies $\Box\xi_0 = 0$ and f denotes a complex-valued function on M. Since $\Box i(\varphi)\xi_0 = i(\varphi)\Box\xi_0 = 0$, $i(\varphi)\xi_0 = \xi_0(\bar{V})$ is a constant function on M. Thus we have

$$i(\varphi)\xi_0 = \frac{1}{\mathrm{vol}(M)}\int_M i(\varphi)\xi_0 * 1 = \frac{1}{\mathrm{vol}(M)}\langle \xi_0, \bar{\varphi}\rangle,$$

where vol(M) denotes the volume of M. Since

$$\langle \xi_0, \bar{\varphi} \rangle = \langle \xi - \bar{\partial} f, \bar{\varphi} \rangle = \langle i(\varphi)\xi, 1 \rangle - \langle f, \bar{\vartheta}\bar{\varphi} \rangle = 0,$$

we obtain $i(\varphi)\xi_0 = 0$. From Theorem 3.2, it follows that ξ_0 is a basic form of $\mathscr{F}$. Moreover we have $\bar{V}f = i(\varphi)\xi - i(\varphi)\xi_0 = 0$. This means that the function f is holomorphic on each leaf of $\mathscr{F}$. By Lemma 2.1, the universal covering of each leaf is biholomorphic to $\mathbf{C}$. Since f is bounded, f is a constant function on each leaf and then f is a basic function of $\mathscr{F}$. Thus it has been proved that ξ is a basic form of $\mathscr{F}$. □

From the preceding theorem, it follows that the 1-form ξ corresponding to a holomorphic section X of Q^+ determines a basic Dolbeault cohomology class $[\xi] \in H_B^{0,1}(\mathscr{F})$. For a holomorphic section X of Q^+, Vaisman in [10] constructed a $\bar{\partial}$-closed 1-form $\kappa(X)$ of type (0, 1), which is related to our 1-form ξ by $-2[\kappa(X)] = [\xi]$ in $H_{\bar{\partial}}^{0,1}(M)$. Moreover he answered the question whether there exists a holomorphic vector field $\tilde{X}$ of M such that $\pi(\tilde{X}) = X$ in terms of its Dolbeault cohomology class $[\kappa(X)]$ (Theorem 4.5 in [10]). Now we shall answer the same question as above in terms of its basic Dolbeault cohomology class $[\xi]$.

THEOREM 4.3. *Let X be a holomorphic section of Q^+ with the corresponding 1-form ξ on a compact g.H. manifold M. Then there exists a holomorphic vector field $\tilde{X}$ on M such that $\pi(\tilde{X}) = X$ if and only if $[\xi] = 0$ in $H_B^{0,1}(\mathscr{F})$.*

Proof. For a given holomorphic section $X \in \Gamma Q^+$, any vector field $\tilde{X}$ of type $(1,0)$ on M such that $\pi(\tilde{X}) = X$ is written as $\tilde{X} = X + fV$, where f denotes some differentiable function on M. Here we identified Q^+ with $(L^\perp)^+$. Noticing that the connection $\tilde{\nabla}$ is of type $(1,0)$, we see that $\tilde{X}$ is holomorphic if and only if $\tilde{\nabla}_{\bar{Y}}\tilde{X} = 0$ for any $Y \in \Gamma(L^\perp)^+$ and $\tilde{\nabla}_{\bar{V}}\tilde{X} = 0$. Using (4.2), we calculate

$$\tilde{\nabla}_{\bar{Y}}\tilde{X} = \nabla'_{\bar{Y}}X + \{\tfrac{1}{2}\xi(\bar{Y}) + \bar{Y}f\}V = \tfrac{1}{2}(\xi + 2\bar{\partial}f)(\bar{Y})V,$$

for $Y \in \Gamma(L^\perp)^+$ and

$$\tilde{\nabla}_{\bar{V}}\tilde{X} = \nabla'_{\bar{V}}X + (\bar{V}f)V = \bar{\partial}f(\bar{V})V.$$

If $\tilde{X}$ is a holomorphic vector field, by the second equation in the above, we obtain $\bar{\partial}f(\bar{V}) = 0$ and by the argument in the proof of Theorem 4.2, we see that f is a basic function of $\mathscr{F}$. From the first equation, it follows that $\xi = -2\bar{\partial}_B f$ and hence $[\xi] = 0$ in $H_B^{0,1}(\mathscr{F})$. The converse is easily seen. □

COROLLARY 4.4. *Let M be a compact g.H. manifold with $b_1(M) = 1$. Then* *for any holomorphic section* $X \in \Gamma Q^+$, *there exists a holomorphic vector field* $\tilde{X}$ *such that* $\pi(\tilde{X}) = X$.

Proof. If $b_1(M) = 1$, a result obtained in Section 3 implies that $H_B^{0,1}(\mathscr{F}) = \{0\}$. □

We denote by $\mathfrak{A}$ the complex Lie algebra consisting of all holomorphic vector fields on M. Then on a compact g.H. manifold M, we have

$$\{cV \mid c \in \mathbf{C}\} \subset \mathfrak{A}, \quad V = B - \sqrt{-1}\,A.$$

Moreover the following holds:

COROLLARY 4.5. $\{cV \mid c \in \mathbf{C}\}$ *is contained in the center of* $\mathfrak{A}$.

Proof. We note that $[V, X] \in \Gamma(L^{\perp})^+$ for any $X \in \Gamma(L^{\perp})^+$. From the argument in the proof of Theorem 4.3, it follows that for any $\tilde{X} \in \mathfrak{A}$ $\tilde{X}$ is written as $\tilde{X} = X + fV$, where X is a holomorphic section of $(L^{\perp})^+ \cong Q^+$ and f is a basic function of $\mathscr{F}$. By Theorem 4.2, X is a holonomy invariant section of $\mathscr{F}$. Therefore we obtain $[V, \tilde{X}] = 0$. □

Now we shall show an analogous result to a theorem of Matsushima and Lichnerowicz. We denote by $\mathfrak{K}$ the real Lie subalgebra of $\mathfrak{A}$ consisting of holomorphic vector fields whose associated real vector fields are Killing vector fields. Then we have the following.

THEOREM 4.6. *Let* M *be a compact g.H. manifold with constant scalar curvature. Then we have*

$$\mathfrak{A} = \mathfrak{K} + \sqrt{-1}\,\mathfrak{K} \qquad \mathfrak{K} \cap \sqrt{-1}\,\mathfrak{K} = \{cV \mid c \in \mathbf{C}\},$$

where $V = B - \sqrt{-1}\,A$ *is a vector field given in Lemma* 2.1(2).

Proof. In the proof of this theorem, the canonical foliation also plays an important role.

First, we review basic properties of transversally holomorphic sections of Kähler foliations (cf. [5]). Let $\mathscr{F}$ be a Kähler foliation on a manifold M. We denote by $V(\mathscr{F})$ and ΓQ^L the Lie algebra of infinitesimal automorphisms of $\mathscr{F}$ and that of holonomy invariant sections of the normal bundle Q, respectively. Then we have an exact sequence of Lie algebras:

$$0 \to \Gamma L \to V(\mathscr{F}) \xrightarrow{\pi} \Gamma Q^L \to 0,$$

which is associated with (2.4) (cf. [6] Chap. 9). For Kähler foliations, the space $\Gamma(Q^+)^L$ of holonomy invariant sections of type $(1,0)$ is a Lie subalgebra of $\Gamma(Q^{\mathbf{C}})^L$. A section $s \in \Gamma(Q^+)^L$ is said to be *transversally holomorphic* if its associated real section $u = s + \bar{s}$ satisfies $L_{Y_u} J_Q = 0$, where $\pi(Y_u) = u$. We denote by $\mathfrak{A}'$ the complex Lie subalgebra of $\Gamma(Q^+)^L$ consisting of transversally holomorphic sections. Given $s \in \Gamma(Q^+)^L$, we see that $s \in \mathfrak{A}'$ if and only if $\nabla'_{\bar{Y}} s = 0$ for any $Y \in \Gamma(L^\perp)^+$. A section $s \in \Gamma Q^L$ is said to be *transversally Killing* if $L_{Y_s} g_Q = 0$ holds. Then $s \in \Gamma Q^L$ is transversally Killing if and only if it satisfies $g_Q(\nabla'_X s, \pi(Y)) + g_Q(\pi(X), \nabla'_Y s) = 0$ for any X, $Y \in \Gamma L^\perp$. We denote by $\mathfrak{K}'$ the real Lie subalgebra of ΓQ^L consisting of transversally Killing sections. From now on, we assume that M is compact and orientable and that $\mathscr{F}$ is harmonic. Then similarly to the case of compact Kähler manifolds, we see that if u is a transversally Killing section, its associated section $s = \frac{1}{2}(u - \sqrt{-1}\, J_Q u)$ of type $(1,0)$ is transversally holomorphic (cf. Theorem B in [5]). In particular, $\mathfrak{K}'$ is identified with a real Lie subalgebra of $\mathfrak{A}'$. Moreover, Nishikawa and Tondeur generalized a theorem of Lichnerowicz to the foliation context. To state this result, we recall complex Lie subalgebras $\mathfrak{B}'$ and $\mathfrak{C}'$ of $\mathfrak{A}'$: $\mathfrak{B}'$, by definition, is the ideal of $\mathfrak{A}'$ consisting of transversally holomorphic sections whose corresponding basic (0, 1)-forms vanish in $H_B^{0,1}(\mathscr{F})$; $\mathfrak{C}'$ is the Lie subalgebra of $\mathfrak{A}'$ consisting of transversally holomorphic sections which are parallel with respect to ∇'. Then the following holds.

THEOREM 4.7 (Theorem D in [5]). *Let $\mathscr{F}$ be a harmonic Kähler foliation on a compact orientable manifold M with constant transversal scalar curvature. Then we have*

$$\mathfrak{A}' = \mathfrak{B}' + \mathfrak{C}' \quad (\textit{Lie algebra direct sum})$$

$$\mathfrak{B}' = \mathfrak{K}' \cap \mathfrak{B}' + \sqrt{-1}(\mathfrak{K}' \cap \mathfrak{B}') \quad (\textit{direct sum})$$

Now we return to the proof of Theorem 4.6. Let $\mathscr{F}$ be the canonical foliation of a compact g.H. manifold M. Let X be a holomorphic section of the holomorphic vector bundle Q^+. From Theorem 4.2, it follows that X is a holonomy invariant section. Moreover since $\nabla'_{\bar{Y}} X = 0$ for any $Y \in \Gamma TM^+$, X is transversally holomorphic. Conversely if X is a transversally holomorphic section of Q^+, we have $\nabla'_{\bar{V}} X = 0$ $(V = B - \sqrt{-1}\, A)$ and $\nabla'_{\bar{Y}} X = 0$ for any $Y \in \Gamma(L^\perp)^+$. Consequently X is a holomorphic section of Q^+. Therefore $\mathfrak{A}'$ coincides with the space of holomorphic sections of Q^+.

The projection π is a Lie algebra homomorphism of the complex Lie algebra $\mathfrak{A}$ of all holomorphic vector fields on M into $\mathfrak{A}'$. Let X be a holomorphic vector field which satisfies $\pi(X) = 0$. Then X is written as $X = fV$, where f is a holomorphic function on M. Since M is compact, f is constant. Therefore we have $\ker \pi = \{cV \mid c \in \mathbf{C}\}$. From Theorem 4.3, it follows that $\pi(\mathfrak{A}) = \mathfrak{B}'$.

Next we shall prove the following.

LEMMA 4.8. *For $X \in \mathfrak{A}$, X belongs to $\mathfrak{K}$ if and only if $\pi(X)$ belongs to $\mathfrak{K}'$.*

Proof. We denote by $\tilde{W}$ the real vector field associated to X. $\tilde{W}$ is decomposed into $\tilde{W} = W + f_1 B + f_2 A$, where $W \in \Gamma L^{\perp}$. Let us denote by $\tilde{\alpha}$ and α the dual 1-forms corresponding to $\tilde{W}$ and W, respectively. Then we have $\tilde{\alpha} = \alpha + f_1 \omega + f_2 \theta$. From the argument in the proof of Theorem 4.3, it follows that f_1 and f_2 are basic functions of $\mathscr{F}$ and that $\alpha = -df_1 + df_2 \circ J$ holds. $\tilde{W}$ is a Killing vector field if and only if the following equations hold:

$$(\nabla_A \tilde{\alpha})(A) = 0, (\nabla_B \tilde{\alpha})(B) = 0, (\nabla_A \tilde{\alpha})(B) + (\nabla_B \tilde{\alpha})(A) = 0 \tag{1}$$

$$(\nabla_A \tilde{\alpha})(Y) + (\nabla_Y \tilde{\alpha})(A) = 0 \tag{2}$$

$$(\nabla_B \tilde{\alpha})(Y) + (\nabla_Y \tilde{\alpha})(B) = 0 \tag{3}$$

$$(\nabla_Y \tilde{\alpha})(Z) + (\nabla_Z \tilde{\alpha})(Y) = 0, \tag{4}$$

for any $Y, Z \in \Gamma L^{\perp}$. Since f_1 and f_2 are basic, the equation (1) always holds. The equations (2) and (3) hold if and only if f_1 is constant on M. In fact, we have

$$(\nabla_A \tilde{\alpha})(Y) + (\nabla_Y \tilde{\alpha})(A) = df_2(Y) + \alpha(JY) = -df_1(JY)$$

$$(\nabla_B \tilde{\alpha})(Y) + (\nabla_Y \tilde{\alpha})(B) = df_1(Y).$$

The equation (4) is equivalent to $(\nabla'_Y \alpha)(Z) + (\nabla'_Z \alpha)(Y) = 0$ for $Y, Z \in \Gamma L^{\perp}$. In fact we have

$$\begin{aligned}
&(\nabla_Y \tilde{\alpha})(Z) + (\nabla_Z \tilde{\alpha})(Y) \\
&\quad = (\nabla'_Y \alpha)(Z) - \tfrac{1}{2}\Phi(Y, Z) f_2 + (\nabla'_Z \alpha)(Y) - \tfrac{1}{2}\Phi(Z, Y) f_2 \\
&\quad = (\nabla'_Y \alpha)(Z) + (\nabla'_Z \alpha)(Y).
\end{aligned}$$

Therefore if $X \in \mathfrak{K}$, the real vector field W associated to $\pi(X)$ is transversally Killing. Conversely suppose that W is transversally Killing. Then $(\nabla'_Y \alpha)(Z) + (\nabla'_Z \alpha)(Y) = 0$ for any $Y, Z \in \Gamma L^{\perp}$ and hence $\delta_B \alpha = 0$. Since $\delta_B(df_2 \circ J) = \delta_B(d_B f_2 \circ J) = 0$, we have $\triangle_B f_1 = \delta_B d_B f_1 = -\delta_B \alpha + \delta_B(df_2 \circ J) = 0$ and hence f_1 is a constant function. Therefore $\tilde{W}$ is a Killing vector field. □

We continue the proof of Theorem 4.6. We have the following relation between the scalar curvature τ and the transversal scalar curvature τ' of $\mathscr{F}$: $\tau = -\frac{1}{2}(n - 1) + \tau'$. Applying Theorem 4.7, we complete the proof of Theorem 4.6.

THEOREM 4.9. *Let M be an n-dimensional compact g.H. manifold with* $\dim_{\mathbf{C}} \mathfrak{A} \geqslant 2$. *Then M admits no non-zero holomorphic n-forms and* $(n-1)$-*forms.*

Proof. Since $\dim_{\mathbf{C}} \mathfrak{A} \geqslant 2$, there exists a holomorphic vector field X such that $\pi(X)$ is not identically zero. We set $U = \{p \in M \mid \pi(X_p) \neq 0\}$. Evidently U is a non-empty open set. Let α be a holomorphic n-form on M. By Theorem 3.3, $i(X)\alpha$ is a basic $(n-1)$-form of $\mathscr{F}$. This implies that $i(V)i(X)\alpha = 0$. Since V_p and X_p are linearly independent at $p \in U$, α vanishes on U and hence also on M. This, with Corollary 3.4, implies that M admits no non-zero holomorphic $(n-1)$-forms. □

COROLLARY 4.10. *If an n-dimensional compact g.H. manifold M admits a non-zero holomorphic n-form, all the leaves of the canonical foliation $\mathscr{F}$ are compact.*

Proof. We denote by $\mathrm{Aut}_0(M, g)$ the identity component of the group of isometrically holomorphic transformations of a compact g.H. manifold (M, g). $Aut_0(M, g)$ is a compact Lie transformation group acting on M and its Lie algebra is identified with $\mathfrak{K}$. By Theorem 4.9, we have $\mathfrak{A} = \mathfrak{K} = \{cV \mid c \in \mathbf{C}\}$. Therefore each leaf of $\mathscr{F}$ is an orbit of $\mathrm{Aut}_0(M, g)$. Hence our assertion holds. □

5. l.c.K. metrics on compact g.H. manifolds

In this section we consider the following problem: We fix a compact complex manifold M which admits at least one l.c.K. (and not g.c.K.) metric. Then how many l.c.K. metrics do there exist on M? One way of answering this problem is as follows: Let us denote by $\mathscr{M}$ the set of all l.c.K. metrics on M and for $g \in \mathscr{M}$, $l(g)$ denotes the de Rham cohomology class in $H^1(M; \mathbf{R})$ to which the Lee form of the l.c.K. metric g belongs. Hence we obtain a map l of $\mathscr{M}$ into $H^1(M; \mathbf{R})$.

PROBLEM. What domain in $H^1(M; \mathbf{R})$ is occupied by the image $l(\mathscr{M})$ of the map l?

We denote by $\tilde{\mathscr{H}}$ the subspace of $H^{1,0}_{\bar\partial}(M)$ consisting of d-closed holomorphic 1-forms. For $\alpha \in \tilde{\mathscr{H}}$, $j(\alpha)$ denotes the de Rham cohomology class in $H^1(M; \mathbf{R})$ to which the real component of α belongs. Then j is a real linear injective map of $\tilde{\mathscr{H}}$ into $H^1(M; \mathbf{R})$. We denote by $\mathscr{H}$ the image $j(\tilde{\mathscr{H}})$. From the proof of Theorem 2.1 in [9], it follows that the intersection of $l(\mathscr{M})$ and $\mathscr{H}$ is empty.

Now we suppose that M is a compact g.H. manifold with g.H. metric g_0 and Lee form ω_0. By Theorem 3.3, all holomorphic 1-forms are closed, i.e. $\tilde{\mathscr{H}} = H^{1,0}_{\bar\partial}(M)$. Therefore we have $\dim_{\mathbf{R}} \mathscr{H} = 2 \dim_{\mathbf{C}} H^{1,0}_{\bar\partial} = 2h^{1,0}$. By Remark 3.6, $b_1 = 2h^{1,0} + 1$ and hence $\dim \mathscr{H} = \dim H^1(M; \mathbf{R}) - 1$. On the other hand, $[\omega_0] \notin \mathscr{H}$. Therefore an arbitrary $\xi \in H^1(M;\mathbf{R})$ is written as $\xi = t[\omega_0] + \eta$, $t \in \mathbf{R}$, $\eta \in \mathscr{H}$.

THEOREM 5.1. *Let M be a compact g.H. manifold with g.H. metric g_0 and Lee form ω_0. Then the image $l(\mathscr{M})$ is given by*

$$l(\mathscr{M}) = \{t[\omega_0] + \eta \mid t > 0\ \eta \in \mathscr{H}\}.$$

We shall prove this theorem dividing into two parts. The first part is to construct l.c.K. metrics. The second part is to prove non-existence of l.c.K. metrics.

First part of proof. The length of Lee form ω_0 is assumed to be equal to 1. We denote by Φ_0 the fundamental form given by the g.H. metric g_0. We put $\theta_0 = -\omega_0 \circ J$ and denote by B_0 and A_0 the dual vector fields corresponding to ω_0 and θ_0 with respect to g_0. Given a holomorphic 1-form λ, we put $\alpha =$ the real component of λ and $\beta =$ the imaginary component of λ, i.e. $\alpha = \frac{1}{2}(\lambda + \bar{\lambda})$, $\beta = 1/2\sqrt{-1}(\lambda - \bar{\lambda})$. Then we have $\beta = -\alpha \circ J$ and $d\alpha = 0$, $d\beta = 0$. By Theorem 3.3, α and β are basic 1-forms of the foliation $\mathscr{F}$ generated by A_0 and B_0. Now we define a new metric g by

$$\begin{aligned} g = t(g_0 - \omega_0 \otimes \omega_0 - \theta_0 \otimes \theta_0) + (t\omega_0 + \alpha) \otimes (t\omega_0 + \alpha) \\ + (t\theta_0 + \beta) \otimes (t\theta_0 + \beta) \end{aligned} \tag{5.1}$$

for an arbitrary $t > 0$. We shall show that g is an l.c.K. metric with parallel Lee form $\omega = t\omega_0 + \alpha$. It is easily checked that g is a Hermitian metric and that its fundamental form Φ is given by

$$\begin{aligned} \Phi &= t(\Phi_0 - \omega_0 \wedge \theta_0) + (t\omega_0 + \alpha) \wedge (t\theta_0 + \beta) \\ &= -td\theta_0 + (t\omega_0 + \alpha) \wedge (t\theta_0 + \beta). \end{aligned}$$

We put $\omega = t\omega_0 + \alpha$ and $\theta = t\theta_0 + \beta$. Then ω is a closed 1-form and $\theta = -\omega \circ J$ holds. Moreover we have $d\Phi = \omega \wedge \Phi$. This implies that g is an l.c.K. metric with Lee form ω (cf. [7]). Finally we shall prove that ω is parallel with respect to the Riemannian connection of g. Let us denote by B the dual vector field corresponding to ω with respect to g. Then B is given by $B = (1/t)B_0$. Noticing that α and β are basic forms of the foliation generated by A_0 and B_0, we can prove B is a Killing vector field with respect to g. Therefore ω is parallel. From these arguments, it follows that $l(\mathscr{M}) \supset \{t[\omega_0] + \eta \mid t > 0\ \eta \in \mathscr{H}\}$.

Second part of proof. We shall prove that for an arbitrary $t > 0$ and $\eta \in \mathscr{H}$, $-t[\omega_0] + \eta$ does not belong to $l(\mathscr{M})$. By virtue of the first part of proof, it is sufficient to prove that there does not exist an l.c.K. metric g whose Lee form ω is given by $\omega = -\omega_0$. Suppose that g is such an l.c.K. metric on M. We denote by Φ the fundamental form with respect to g. We put $\theta = -\omega \circ J = -\theta_0$ and denote by B and A the dual vector fields corresponding to ω and θ with respect to g. Then we have

$$L_B\Phi = d\theta + \|\omega\|^2\Phi - \omega \wedge \theta = -d\theta_0 + \|\omega_0\|^2\Phi - \omega_0 \wedge \theta_0.$$

We shall show that the inner product $(L_B\Phi, \Phi)$ is positive at every point of M. In fact, we have

$$\begin{aligned}
(L_B\Phi, \Phi) &= \frac{1}{2}\sum_{i,j=1}^{2n} (L_B\Phi)(e_i, e_j)\Phi(e_i, e_j) \\
&= \frac{1}{2}\sum_{i=1}^{2n} (L_B\Phi)(e_i, Je_i) \\
&= -\frac{1}{2}\sum_{i=1}^{2n} d\theta_0(e_i, Je_i) + (n-1)\|\omega_0\|^2,
\end{aligned}$$

where $\{e_1, \dots, e_{2n}\}$ is an orthonormal basis with respect to g. Now we recall $-d\theta_0 = \Phi_0 - \omega_0 \wedge \theta_0$ because of a g.H. metric g_0. Since

$$\begin{aligned}
-d\theta_0(x, Jy) &= \Phi_0(x, Jy) - \omega_0 \wedge \theta_0(x, Jy) \\
&= (g_0 - \omega_0 \otimes \omega_0 - \theta_0 \otimes \theta_0)(x, y),
\end{aligned}$$

$-\frac{1}{2}\Sigma_{i=1}^{2n} d\theta_0(e_i, Je_i)$ is positive and hence $(L_B\Phi, \Phi)$ is positive. Therefore we have

$$\begin{aligned}
\int_M L_B\Phi^n &= n\int_M L_B\Phi \wedge \Phi^{n-1} \\
&= n!\int_M (L_B\Phi, \Phi) * 1 > 0.
\end{aligned}$$

On the other hand, we have

$$\int_M L_B\Phi^n = \int_M di(B)\Phi^n = 0,$$

which is a contradiction.

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